

## Note

# Difficulties of teleparallel theories of gravity with local Lorentz symmetry

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## Abstract

A brief discussion is made about the relevance of surface terms in the Lagrangian and Hamiltonian formulations of theories of gravity. These surface terms play an important role in the variation of the action integral and in the definition of field quantities such as the gravitational energy-momentum. Then we point out several inconsistencies of a recently proposed formulation of teleparallel theories of gravity with local Lorentz symmetry.

Keywords: teleparallel equivalent of general relativity, local Lorentz symmetry, surface terms in the action integral

## 1. Introduction

The search for alternative descriptions of Einstein's general relativity was initiated right after the advent of general relativity, with the proposal of the Kaluza–Klein theory. Teleparallel theories were considered by Einstein in 1928 as a possible geometrical set up for the unification of the electromagnetic and gravitational fields. Nowadays, extended or alternative formulations of general relativity are investigated with the purpose of solving cosmological problems, or establishing a possible quantum theory of gravity, or even to address unsolved issues of the standard formulation of general relativity. The teleparallel equivalent of general relativity (TEGR) is one such formulation which, among other features, allows the definitions of new geometrical quantities, constructed out of the torsion tensor, and provides a framework for defining energy, momentum and angular momentum of the gravitational field. In this article we address the issue of covariance of the TEGR under Lorentz transformations. In order to analyse this issue, we briefly review in section 3 the importance of surface terms in the action integral of gravitational theories. The discussion in section 3 will clarify the analysis of the problems regarding a surface term that appears in a recently proposed teleparallel theory with local Lorentz invariance.

## 2. The TEGR

The TEGR is a geometrical formulation of the relativistic theory of gravity in terms of tetrad fields, which allows the notion of distant parallelism of vector or tensor fields. The theory has a long history, which includes contributions from Hayashi and Shirafuji [1], Hehl [2], Cho [3] and Nester [4]. See [5] for additional relevant references. The theory is defined by the field equations, which are equivalent to the Einstein's field equations of the metric formulation of general relativity. The set of tetrad fields is interpreted as a frame adapted to observers in space-time, and allows the projection of vector or tensors on the frame of an observer. The projection of a vector field  $V^\mu(x^\alpha)$  on a certain frame, in the tangent space at the position  $x^\alpha$ , is given by  $V^a(x^\alpha) = e^a{}_\mu(x^\alpha) V^\mu(x^\alpha)$ . This is one of the main geometrical properties of the set of tetrad fields.

The equivalence of the TEGR with the metric formulation of general relativity is based on a geometrical identity between the scalar curvature  $R(e)$ , constructed out of the tetrad fields, and an invariant combination of quadratic terms in the torsion tensor, given by

$$eR(e) \equiv -e \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) + 2\partial_\mu (eT^\mu), \quad (1)$$

where  $T_a = T^b{}_{ba}$ ,  $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$  and  $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$ . The latter is the torsion tensor, which is the anti-symmetric part of the Weitzenböck connection  $\Gamma_{\mu\nu}^\lambda = e^{a\lambda} \partial_\mu e_{a\nu}$ , i.e.  $T_{a\mu\nu} = e_{a\lambda} T^\lambda{}_{\mu\nu} = e_{a\lambda} (\Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda)$ .

(Notation: space-time indices  $\mu, \nu, \dots$  and SO(3,1) (Lorentz) indices  $a, b, \dots$  run from 0 to 3. The flat space-time metric tensor raises and lowers tetrad indices, and is fixed by  $\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (-1, +1, +1, +1)$ . The frame components are given by the inverse tetrads  $\{e_a{}^\mu\}$ . The determinant of the tetrad fields is written as  $e = \det(e^a{}_\mu)$ .)

The identity (1) allows the definition of the Lagrangian density for the gravitational field in the TEGR, which reads (see [5] for a review, and [6, 7] for recent analyses and criticisms)

$$\begin{aligned} L(e) &= -k e \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) - \frac{1}{c} L_M \\ &\equiv -k e \Sigma^{abc} T_{abc} - \frac{1}{c} L_M, \end{aligned} \quad (2)$$

where  $k = c^3/(16\pi G)$ ,  $L_M$  represents the Lagrangian density for the matter fields, and  $\Sigma^{abc}$  is defined by

$$\Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^b - \eta^{ab} T^c). \quad (3)$$

Thus, the Lagrangian density is geometrically equivalent to the scalar curvature density. The variation of  $L(e)$  with respect to  $e^{a\mu}$  yields the field equations

$$e_{a\lambda} e_{b\mu} \partial_\nu (e \Sigma^{b\lambda\nu}) - e (\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd}) = \frac{1}{4kc} e T_{a\mu}, \quad (4)$$

where  $T_{a\mu}$  is defined by  $\delta L_M / \delta e^{a\mu} = e T_{a\mu}$ . As expected, the field equations are equivalent to Einstein's equations. It is possible to verify by explicit calculations that the equations above can be rewritten as

$$\frac{1}{2} [R_{a\mu}(e) - \frac{1}{2} e_{a\mu} R(e)] = \frac{1}{4kc} T_{a\mu}. \quad (5)$$

The field equations (4) are covariant under local Lorentz transformations (LLT). Obviously, this property can be verified more easily from equations (5). The meaning of this symmetry is that the theory can be formulated in any reference frame, adapted to any observer in space-time. Thus, there are no privileged frames on which one can construct the theory. In equation (4), one does not need a connection to ensure the covariance of the equations under LLT, although in equation (5) there appears the Levi-Civita connection  ${}^0\omega_{\mu ab}(e)$ . The meaning of LLT in a theory is precisely this: the theory can be formulated in any frame. Therefore, the theory is valid for all observers in space-time. This is the relevance and main feature of the local Lorentz symmetry. Finally, the inertial frames may be characterised by the acceleration tensor, that yields the inertial accelerations (accelerations that are not due to the gravitational field) that act on a given frame in space-time.

### 3. Surface terms in the action for the gravitational field

In this section we recall some very interesting issues discussed by Faddeev [8], with respect to the action integral of the gravitational field in the context of Einstein's formulation of general relativity. In the consideration of Hamilton's variational principle that leads to Einstein's equations, one normally starts with the density  $\sqrt{-g}R(g)$ , where  $R(g)$  is the scalar curvature constructed out of the metric tensor  $g_{\mu\nu}$ . In similarity to Faddeev's article, we will restrict the considerations to asymptotically flat space-times. In the limit  $r \rightarrow \infty$ , and for finite  $x^0 = t$  (we will now adopt  $c = 1$ ), the asymptotically flat limit is characterised by

$$g_{\mu\nu} = \eta_{\mu\nu} + O(1/r), \quad \partial_\lambda g_{\mu\nu} = O(1/r^2), \quad {}^0\Gamma_{\mu\nu}^\lambda = O(1/r^2), \quad (6)$$

where  $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ ,  $\eta_{\mu\nu} = (-1, +1, +1, +1)$ , and  ${}^0\Gamma_{\mu\nu}^\lambda$  are the Christoffel symbols. The energy-momentum tensor  $T_{\mu\nu}$  for the matter fields must be of the order  $T_{\mu\nu} = O(1/r^4)$ . This condition ensures that the matter fields are effectively localized in a compact region of the space.

For large values of the radial coordinate  $r$ , the asymptotic form of the coordinate transformations is taken to be

$$x'^\mu = \eta^\mu(x), \quad (7)$$

where

$$\begin{aligned} \eta^\mu(x) &= \Lambda^\mu{}_\nu x^\nu + a^\mu + O(1/r), \\ \partial_\nu \eta^\mu &= \Lambda^\mu{}_\nu + O(1/r^2). \end{aligned} \quad (8)$$

The quantities  $\Lambda^\mu{}_\nu$  are matrices of the Lorentz transformations, and  $a^\mu$  is an arbitrary constant vector (we are adopting Faddeev's original notation). Faddeev assumes that these transformations act on the metric tensor and on the connection referred to a fixed coordinate system. The resulting infinitesimal transformations are given by

$$\begin{aligned} \delta g_{\mu\nu} &= -\partial_\mu \varepsilon^\lambda g_{\lambda\nu} - \partial_\nu \varepsilon^\lambda g_{\lambda\mu} - \varepsilon^\lambda \partial_\lambda g_{\mu\nu}, \\ \delta ({}^0\Gamma_{\mu\nu}^\lambda) &= -\partial_\mu \varepsilon^\sigma ({}^0\Gamma_{\sigma\nu}^\lambda) - \partial_\nu \varepsilon^\sigma ({}^0\Gamma_{\sigma\mu}^\lambda) + \partial_\sigma \varepsilon^\lambda ({}^0\Gamma_{\mu\nu}^\sigma) \\ &\quad - \varepsilon^\sigma \partial_\sigma ({}^0\Gamma_{\mu\nu}^\lambda) - \partial_\mu \partial_\nu \varepsilon^\lambda, \end{aligned} \quad (9)$$

where  $\varepsilon^\lambda$  is an infinitesimal vector field that in the limit  $r \rightarrow \infty$  has the asymptotic form given by equation (8).

Faddeev considered the action integral constructed out of the Lagrangian density

$$L = \sqrt{-g}R(g) - \partial_\mu(\sqrt{-g}g^{\nu\sigma}{}^0\Gamma_{\nu\sigma}^\mu - \sqrt{-g}g^{\mu\nu}{}^0\Gamma_{\nu\sigma}^\sigma), \quad (10)$$

which differs from  $\sqrt{-g}R(g)$  by a total divergence, and argued that it is the action constructed out of  $L$ ,

$$S = \int L d^3x dt, \quad (11)$$

and not the one constructed out of  $\sqrt{-g}R(g)$ , that is invariant under the infinite dimensional group  $G$  generated by the transformations (9). In view of the asymptotic behaviour given by (6), one can verify that in the limit  $r \rightarrow \infty$  we have

$$\sqrt{-g}R(g) = O(1/r^3), \quad L = O(1/r^4). \quad (12)$$

There is an additional essential feature of the action integral (11). In the process of varying the action in order to obtain the field equations, all surface terms that arise in the variation of (11), via integration by parts, vanish in the limit  $r \rightarrow \infty$ , whereas in the variation of the action constructed out of  $\sqrt{-g}R(g)$ , several of these terms do not vanish in the same asymptotic limit. Thus, the variation of the action constructed out of  $\sqrt{-g}R(g)$  only is not well defined. Moreover, if one establishes the Hamiltonian formulation starting from (11), the standard ADM Hamiltonian is obtained together with the correct surface terms that define the total ADM energy-momentum, i.e. one does not need to add any surface term by hand.

Similar considerations were made earlier in 1974 in the famous Lecture Notes by Hanson *et al* [9] on constrained Hamiltonian systems. These authors attempted to write the field equations of general relativity in Hamiltonian form, i.e. the standard Hamilton's equations in the phase space of the theory. Besides the constraint equations, there are the evolution equations for the spatial metric  $g_{ij}$ , and for the canonically conjugated momenta  $\Pi^{\tilde{ij}}$ , generated by the total Hamiltonian. Hanson, Regge and Teitelboim noted that one needs to add suitable surface terms to the total Hamiltonian, so that the variation of the total Hamiltonian is well defined<sup>1</sup>. These surface terms are precisely the terms that yield the total energy-momentum and angular momentum at spatial infinity. Without these surface terms, the variation of the total Hamiltonian is not well defined, because of the non-vanishing of several terms that arise via integration by parts. The improved Hamiltonian, including the surface terms, has well defined functional derivatives.

In the context of tetrad fields  $e^a{}_\mu$  and of the spin connection  $\omega_{\mu ab}$ , one also needs to add surface terms to the action integral in order to have well behaved functional derivatives [11]. The Lagrangian density is normally considered to be  $eR(e, \omega)$ . This framework is mandatory in the case of Einstein–Cartan type theories, or when one needs to couple Dirac spinor fields to the gravitational field. Again, the variation of the action integral must be well defined, so that all surface terms that arise via integration by parts vanish at spacelike infinity. As above, we consider the space-time to be asymptotically flat, and assume the asymptotic behaviour

$$e_{a\mu} \simeq \eta_{a\mu} + O(1/r), \quad \omega_{\mu ab} = O(1/r^2), \quad (13)$$

in the limit  $r \rightarrow \infty$ . The Lagrangian density that is well defined with respect to functional derivatives is

<sup>1</sup> See pages 111–3 of [9].

$$\begin{aligned}
L(e, \omega) &= eR(e, \omega) - \partial_\mu (e e^{a\mu} e^{b\nu} \omega_{\nu ab} - e e^{a\nu} e^{b\mu} \omega_{\nu ab}) \\
&= -\partial_\mu (e e^{a\mu} e^{b\nu}) \omega_{\nu ab} + \partial_\nu (e e^{a\mu} e^{b\nu}) \omega_{\mu ab} \\
&\quad + (e e^{a\mu} e^{b\nu}) (\omega_{\mu ac} \omega_{\nu}{}^c{}_b - \omega_{\nu ac} \omega_{\mu}{}^c{}_b).
\end{aligned} \tag{14}$$

In the variation of  $eR(e, \omega)$  alone, one finds, via integration by parts, the term

$$\int dt d^3x \partial_\nu (e e^{a\mu} e^{b\nu} \delta \omega_{\mu ab}) \neq 0, \tag{15}$$

which does not vanish in general when integrated over a spatial surface at spacelike infinity. For a vector field  $V^\alpha$  whose asymptotic behaviour in the limit  $r \rightarrow \infty$  is  $V^\alpha = O(1/r^2)$ , we have

$$\begin{aligned}
\int d^4x \partial_\alpha (\sqrt{-g} V^\alpha) &= \oint_{S \rightarrow \infty} dS_\alpha (\sqrt{-g} V^\alpha) \\
&\simeq \oint_{r \rightarrow \infty} dt r^2 d\Omega [O(1/r^2)] \neq 0,
\end{aligned} \tag{16}$$

where  $d\Omega = \sin \theta d\theta d\phi$  and,  $S$  is a surface of constant radius. The action integral constructed out of the Lagrangian density (14) is not affected by this problem. In the analysis above, it makes no difference whether we use an arbitrary spin connection in the Palatini variational principle, which is eventually determined by the field equations, or the Levi-Civita connection  ${}^0\omega_{\mu ab}(e)$ .

The field equations derived from the Lagrangian (14) are precisely equations (5). Therefore, the theory determined by (14) is covariant under LLT.

If the action integral is defined on a manifold with boundary, one may use the ordinary Hilbert–Einstein action for the gravitational field, plus the Gibbons–Hawking surface term [10], determined by the integration of the trace of the extrinsic curvature over the boundary. A recent discussion on the relevance and necessity of boundary terms (and their variations) in the Hilbert–Einstein action is given in [13] (see also the references therein).

We note finally that the action integral constructed out of the Lagrangian density (2) is not affected by the emergence of non-vanishing surface terms, in the variation of the action. In the analysis carried out in [12], special attention was paid to the need of surface terms in the action. The total divergence in equation (1) cancels the total divergence in equation (14).

#### 4. Teleparallel gravity with local Lorentz symmetry

A recent article [14] summarizes a formulation of the TEGR that attempts to exhibit local Lorentz symmetry (see also [15]). The local Lorentz symmetry is achieved by introducing a flat space-time connection  $\Omega^a{}_{b\mu}$ , that corresponds to equation (16) of [14], and which is given by

$$\Omega^a{}_{b\mu} = \Lambda^a{}_c(x) \partial_\mu \Lambda_b{}^c(x), \tag{17}$$

where  $\Lambda^a{}_b(x)$  are matrices of the local Lorentz group, and therefore these quantities are space-time dependent functions. The torsion tensor, that in section 2 was written as  $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$  (the notation for the tetrad fields in [14] is different from ours), is now considered to be

$$T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu} + \Omega_{ac\mu} e^c{}_\nu - \Omega_{ac\nu} e^c{}_\mu. \tag{18}$$

Although the connection (17) is not linked to any field quantity that has a clear transformation property, it is assumed to transform as a standard spin connection, so that equation (18) eventually transforms as a tensor under LLT. Except for satisfying  $\Lambda^a{}_c \Lambda^b{}_d \eta_{ab} = \eta_{cd}$ , the matrices  $\Lambda^a{}_b$  are arbitrary. The authors of [14] argue that the Lagrangian density (2), constructed in terms of (18), is invariant under LLT. We refer to [14] for additional details.

The flat spin connection (17) is the Levi-Civita connection of the flat space-time (see equation (112) of [14]). This connection is irrelevant to the dynamics of the tetrad field, which is the quantity that yields physical results. This fact was already noted in [12] (see equation (9) of [12]).

The counting of degrees of freedom of the flat spin connection  $\Omega^a{}_{b\mu}$  is absolutely not clear in [14] (a recent discussion on the degrees of freedom of the TEGR has been given in [16]), and in the context of  $f(T)$  gravity in [17–19]. This issue is important, because when a certain gauge is fixed, the number of degrees of freedom of the connection should be decreased. The vector potential  $A_\mu$  in electrodynamics, for instance, has initially 4 degrees of freedom at each space-time event. After fixing all gauges, the number of degrees of freedom is reduced to 2 at each space-time event. A similar situation does not occur in the context of [14].

The presentation of [14] is subject to at least 5 major criticisms.

1. The first criticism is that a flat space-time connection (the local  $\text{SO}(3,1)$  group in [14] is restricted to the flat space-time) is added to the non-flat space-time torsion of the Weitzenböck connection, as displayed in equation (23) of [14]. This procedure is inconsistent. A consistent procedure would be to consider a standard, ordinary affine connection subject to the condition of zero curvature, i.e. a regular flat connection of the local  $\text{SO}(3,1)$  group in the curved space-time. This would be achieved by introducing into the Lagrangian density Lagrange multipliers  $\lambda^{ab\mu\nu}$ , in order to ensure the vanishing of the curvature tensor constructed out of an arbitrary connection  $\omega^a{}_{b\mu}$ , i.e.  $\lambda^{ab\mu\nu} R_{ab\mu\nu}(\omega)$ . Of course, further consequences would result from the introduction of the Lagrange multipliers.
2. The second criticism is related to the variation of the action integral in the context of [14]. According to the authors, the Lagrangian density that they consider,  $L(e^a{}_\mu, \Omega^a{}_{b\mu})$ , may be rewritten as

$$L(e^a{}_\mu, \Omega^a{}_{b\mu}) = L(e^a{}_\mu) - \frac{1}{8\pi G} \partial_\mu (e \Omega^\mu), \quad (19)$$

where  $\Omega^\mu = \Omega^a{}_{b\nu} e_a{}^\nu e^{b\mu}$ , and  $L(e^a{}_\mu)$  is precisely equation (2). It is argued in [14] that since  $\Omega^a{}_{b\mu}$  ‘enters the Lagrangian as a total derivative, the variation with respect to the spin connection vanishes identically’. However, the whole discussion in section 3 was intended to show that such variation is not trivial and non vanishing, in general. Since the variation of the flat connection alone is given by  $\delta(\Omega^a{}_{b\mu}) = \partial_\alpha(\Omega^a{}_{b\mu})\delta x^\alpha$ , the integral

$$\int d^4x \partial_\mu [e e_a{}^\nu e^{b\mu} \delta(\Omega^a{}_{b\nu})] = \oint_{S \rightarrow \infty} dS_\mu [e e_a{}^\nu e^{b\mu} \delta(\Omega^a{}_{b\nu})] \neq 0, \quad (20)$$

does not vanish, in general. In fact, if the connection  $\Omega^a{}_{b\mu}$  is constructed out of the Lorentz transformations given by the matrices in equation (121) of [14], for instance, then  $\delta(\Omega^a{}_{b\mu}) = O(1/r^0)$  everywhere in space-time. The variation above would vanish only if  $\delta(\Omega^a{}_{b\mu}) = O(1/r^3)$  in the asymptotic limit  $r \rightarrow \infty$ . Otherwise, the variation of the surface term may diverge when integrated in the limit  $r \rightarrow \infty$ .

Variations of surface terms do not vanish, in general. As an example, let us consider the surface term that determines the total ADM mass, and which depends on the parameter  $m$  that represents the total mass of a gravitational system. By varying  $m$ ,  $m \rightarrow m + \delta m$ , for instance, the resulting variation of the surface integral obviously does not vanish. In electrodynamics, the variation of the Poynting vector, integrated over a surface of constant radius, is also non vanishing.

3. Gauge theories are normally understood as constrained Hamiltonian systems, as formulated by Dirac and summarised in [9]. The set of first class constraints generate the gauge transformations. This feature is connected to our third criticism. In [14] there are no fields that would define first class constraints and that would yield a transformation law for  $\Omega^a{}_{b\mu}$ . The gauge transformations in [14] are not generated by any kind of first class constraints, they are ‘generated’ by hand. Suppose one fixes a gauge in the context of [14]. What would prevent the reappearance of the connection after the gauge fixing?
4. According to [14], (i) inertial effects are represented by a spin connection, (ii) there exists a class of Lorentz frames, the proper frames, in which there are no inertial effects, since the spin connection vanishes in these frames, (iii) the flat spin connection (17) is the Levi-Civita connection constructed out of the reference tetrads (see equation (112) of [14]). The reference tetrads are obtained by demanding the vanishing of all physical parameters of the metric tensor. These are tetrads for the flat space-time. In [14], the proper frames are characterised by the vanishing of the flat spin connection.

One would expect that for a given space-time metric tensor there would exist a unique proper frame, i.e. a unique set of tetrad fields that is associated to a vanishing spin connection, where no ‘spurious inertial effects’ exist. However, it is easy to show that there may exist several distinct, physically inequivalent tetrad fields, for a given metric tensor (specially with off diagonal components), that yield the same reference tetrads. Consequently, all these tetrads are associated to a vanishing flat spin connection, and all of them are supposed to be exempt of the so called spurious inertial effects. This feature is an inconsistency, because the spurious inertial effects, whatever it means, should vanish in the context of only one frame, the proper frame, or in a class of frames related by global Lorentz transformations. (The definition of inertial effects is not given in [14].)

In order to understand this problem, let us consider the metric tensor for the Kerr space-time in Boyer–Lindquist coordinates  $(t, r, \theta, \phi)$ . It is given by

$$\begin{aligned} ds^2 = & -\frac{\psi^2}{\rho^2} dt^2 - \frac{2\chi \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 \\ & + \rho^2 d\theta^2 + \frac{\Sigma^2 \sin^2 \theta}{\rho^2} d\phi^2, \end{aligned} \quad (21)$$

with the following definitions:

$$\begin{aligned} \Delta &= r^2 + a^2 - 2mr, \\ \rho^2 &= r^2 + a^2 \cos^2 \theta, \\ \Sigma^2 &= (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta, \\ \psi^2 &= \Delta - a^2 \sin^2 \theta, \\ \chi &= 2amr. \end{aligned} \quad (22)$$

The set of tetrad fields that defines a static frame in the space-time, i.e. that is adapted to static observers, is given by

$$e_{a\mu} = \begin{pmatrix} -A & 0 & 0 & -B \\ 0 & C \sin \theta \cos \phi & \rho \cos \theta \cos \phi & -D \sin \theta \sin \phi \\ 0 & C \sin \theta \sin \phi & \rho \cos \theta \sin \phi & D \sin \theta \cos \phi \\ 0 & C \cos \theta & -\rho \sin \theta & 0 \end{pmatrix}, \quad (23)$$

where

$$\begin{aligned} A &= \frac{\psi}{\rho}, \\ B &= \frac{\chi \sin^2 \theta}{\rho \psi}, \\ C &= \frac{\rho}{\sqrt{\Delta}}, \\ D &= \frac{\Lambda}{\rho \psi}. \end{aligned} \quad (24)$$

In the expression of  $D$  we have  $\Lambda = (\psi^2 \Sigma^2 + \chi^2 \sin^2 \theta)^{1/2}$ .

A different set of tetrad fields that satisfies Schwinger's time gauge condition, and which is defined from spacelike infinity up to the external surface of the ergosphere, reads

$$e_{a\mu} = \begin{pmatrix} -A' & 0 & 0 & 0 \\ B' \sin \theta \sin \phi & C' \sin \theta \cos \phi & D' \cos \theta \cos \phi & -E' \sin \theta \sin \phi \\ -B' \sin \theta \cos \phi & C' \sin \theta \sin \phi & D' \cos \theta \sin \phi & E' \sin \theta \cos \phi \\ 0 & C' \cos \theta & -D' \sin \theta & 0 \end{pmatrix}, \quad (25)$$

where

$$\begin{aligned} A' &= \frac{1}{\rho} \sqrt{\psi^2 + \frac{\chi^2}{\Sigma^2} \sin^2 \theta}, \\ B' &= \frac{\chi}{\Sigma \rho}, \\ C' &= \frac{\rho}{\sqrt{\Delta}}, \\ D' &= \rho, \\ E' &= \frac{\Sigma}{\rho}. \end{aligned} \quad (26)$$

These two sets of tetrad fields are adapted to observers in space-time that have quite different 4-velocities. In equation (23), the observers are static in space-time (with respect to observers at infinity). In the equation (25), the observers have rotational motion because of the dragging effects of the rotating black hole. These two sets of tetrad field have quite different inertial properties (see the appendix). However, both of them lead to the same reference tetrads,

$$e_{a\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \sin \theta \cos \phi & r \cos \theta \cos \phi & -\sin \theta \sin \phi \\ 0 & \sin \theta \sin \phi & r \cos \theta \sin \phi & \sin \theta \cos \phi \\ 0 & \cos \theta & -r \sin \theta & 0 \end{pmatrix}, \quad (27)$$

for which the flat spin connection vanishes.

Thus, from the point of view of [14], both sets of tetrad fields above are proper frames, and both should be free of the so called spurious inertial effects. But it does not make sense to have two (or more) physically inequivalent frames, for a given space-time metric tensor, that are free of the same inertial effects. (Again, disregarding the class of frames related by a global Lorentz transformation.) Whatever is the definition of inertial effects in [14], two physically different sets of tetrad fields should exhibit different inertial effects. From an alternative perspective, it does not make sense that the same flat spin connection removes the inertial effects of two physically inequivalent frames.

5. In addition, these frames, equations (23) and (25), are related by a local Lorentz transformation that depends on the parameters  $m$  and  $a$ , and thus cannot be given by the matrices  $\Lambda^a_b$  of the flat spin connection (17), since the latter matrices do not depend on physical parameters of the metric tensor, as we conclude from equation (112) of [14]. The flat spin connection (17) does not provide a realisation of the full local Lorentz group  $SO(3,1)$ , since it does not generate all possible 4-rotations of the local Lorentz group. This is one further difficulty of the formalism presented in [14].

In view of all considerations above, we are led to conclude that the formulation of the TEGR endowed with LLT, as presented in [14], is inconsistent.

## 5. Conclusions

A theory is defined by the field equations, and by a set of assumptions and interpretations of the field quantities. In this sense, the theory determined by equation (2) is invariant under LLT, as well as the theory determined by equation (14). One important theoretical requirement is that the action integral of a theory must be well defined under functional derivatives. One has to pay attention to surface terms that arise via integration by parts when varying the action. Surface terms may carry important information about the total energy, momentum and angular momentum of the theory.

The action integral of the TEGR with local Lorentz symmetry, presented in [14], is not well defined under variations of the flat spin connection  $\Omega^a_{b\mu}$ . The variation of the action with respect to this connection does not lead to an identically vanishing result, as the authors argue. This issue is a serious inconsistency. As it stands, equation (94) of [14] is wrong. Furthermore, the LLT of [14] are not generated by first class constraints, as is usual in ordinary gauge theories.

The two sets of tetrads considered in the previous section, equations (23) and (25), are not related by a local Lorentz transformation generated by the flat spin connection (17). These two sets of tetrads are physically distinct. We have shown that the flat spin connection does not generate all possible 4-rotations of the Lorentz group. Purely inertial effects of the flat space-time cannot generate gravitational effects.

It is curious to note that in the analysis of the problem of localizability of the gravitational energy, Møller investigated the formulation of a tetrad theory of gravity, and already advocated the establishment of an energy-momentum complex that is invariant under global Lorentz transformations [20, 21]. In fact, he argued that not only the energy-momentum complex would be invariant under global (constant) Lorentz transformations, but also the Lagrangian density itself (the particular one that he addressed) would be invariant under transformations with constant matrices of the Lorentz group. These considerations make sense, to a certain extent, because the energy-momentum and angular momentum of the gravitational field, as

well as of any other physical system or fields, are not invariant under LLT, but covariant under global Lorentz transformations.

The physical relevance of LLT as a symmetry of a theory of gravity is that it ensures that the theory is valid in the frame of any observer in space-time. This is an issue of consistency of the theory, that is verified in the context of the TEGR discussed in section 2.

## Appendix. The acceleration tensor

The inertial properties of any frame in space-time may be characterized by the acceleration tensor. Here, we recall very briefly the properties of this tensor. Let us consider that the trajectory  $C$  of an observer in space-time is given by  $x^\mu(\tau)$ , where  $\tau$  is the proper time of the observer. The 4-velocity of the observer on  $C$  reads  $u^\mu = dx^\mu/d\tau$ . We identify the observer's velocity with the  $a = (0)$  component of  $e_a{}^\mu$ :  $u^\mu(\tau) = e_{(0)}{}^\mu$ . The observer's acceleration  $a^\mu$  is given by the absolute derivative of  $u^\mu$  along  $C$  [22],

$$a^\mu = \frac{Du^\mu}{d\tau} = \frac{De_{(0)}{}^\mu}{d\tau} = u^\alpha \nabla_\alpha e_{(0)}{}^\mu, \quad (\text{A.1})$$

where the covariant derivative is constructed out of the Christoffel symbols. Thus,  $e_a{}^\mu$  determines the velocity and acceleration along the worldline of an observer adapted to the frame. The set of tetrad fields for which  $e_{(0)}{}^\mu$  describes a congruence of timelike curves is adapted to a class of observers characterized by the velocity field  $u^\mu = e_{(0)}{}^\mu$  and by the acceleration  $a^\mu$ .

We may consider not only the acceleration of observers along trajectories whose tangent vectors are given by  $e_{(0)}{}^\mu$ , but the acceleration of the whole frame along  $C$ . The acceleration of the frame is determined by the absolute derivative of  $e_a{}^\mu$  along the path  $x^\mu(\tau)$ . Assuming that the observer carries an orthonormal tetrad frame  $e_a{}^\mu$ , the acceleration of the latter along the path is given by [23, 24]

$$\frac{De_a{}^\mu}{d\tau} = \phi_a{}^b e_b{}^\mu, \quad (\text{A.2})$$

where  $\phi_{ab}$  is the antisymmetric acceleration tensor. According to [23], in analogy with the Faraday tensor we can identify  $\phi_{ab} \rightarrow (\mathbf{a}, \boldsymbol{\Omega})$ , where  $\mathbf{a}$  is the translational acceleration ( $\phi_{(0)(i)} = a_{(i)}$ ) and  $\boldsymbol{\Omega}$  is the angular velocity of the local spatial frame with respect to a nonrotating (Fermi–Walker transported) frame. It follows that

$$\phi_a{}^b = e^b{}_\mu \frac{De_a{}^\mu}{d\tau} = e^b{}_\mu u^\lambda \nabla_\lambda e_a{}^\mu. \quad (\text{A.3})$$

Therefore, given any set of tetrad fields for an arbitrary gravitational field configuration, its geometrical interpretation may be obtained by suitably interpreting the velocity field  $u^\mu = e_{(0)}{}^\mu$  and the acceleration tensor  $\phi_{ab}$ . The acceleration vector  $a^\mu$  defined by equation (A.1) may be projected on a frame in order to yield

$$a^b = e^b{}_\mu a^\mu = e^b{}_\mu u^\alpha \nabla_\alpha e_{(0)}{}^\mu = \phi_{(0)}{}^b. \quad (\text{A.4})$$

Thus,  $a^\mu$  and  $\phi_{(0)(i)}$  are not different accelerations of the frame. Along a geodesic trajectory, we have  $a^\mu = 0$ .

It is possible to show that the acceleration tensor  $\phi_{ab}$  may be rewritten as [25, 26]

$$\phi_{ab} = \frac{1}{2} [T_{(0)ab} + T_{a(0)b} - T_{b(0)a}], \quad (\text{A.5})$$

where the torsion tensor is given by  $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$ .

The expression above is not invariant under local  $SO(3,1)$  transformations, and for this reason the values of  $\phi_{ab}$  characterize the frame. However, equation (A.5) is invariant under coordinate transformations. We interpret  $\phi_{ab}$  as the inertial (i.e. non gravitational) accelerations along the trajectory  $C$ .

The set of tetrad fields (23) and (25) clearly yield different values for the acceleration tensor  $\phi_{ab}$ . This fact demonstrates that the two frames are physically inequivalent, since they are subject to different inertial accelerations. The values of  $\phi_{ab}$  for both sets of tetrads are very long, and for this reason we will not present them here.

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