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Lorentz-violating gravity and the bootstrap procedure

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Abstract

In conventional gravitational physics, the so-called 'bootstrap procedure' can be used to extrapolate from a linear model of a rank-2 tensor to a full non-linear theory of gravity (i.e. general relativity) via a coupling to the stress-energy of the model. In this work, I extend this procedure to a 'Lorentz-violating' gravitational model, in which the linear tensor field and the matter fields 'see' different metrics due to a coupling between the tensor field and a background vector field. The resulting model can be thought of as a generalized Proca theory with a non-minimal coupling to conventional matter. It has a similar linearized limit to the better-known 'bumblebee model', but differs at higher orders in perturbation theory. Its effects are unobservable in FRW spacetimes, but are expected to be important in anisotropic cosmological spacetimes.

Keywords: Lorentz symmetry, linearized gravity, standard model extension, bootstrap

1. Introduction

Lorentz symmetry and general relativity have been intimately related ever since their inception. The idea of Lorentz symmetry between locally defined reference frames is inherent in the Einstein equivalence principle, and in the description of gravity as due to the effects of a spacetime metric [1].

However, in recent years some physicists have started to questions whether Lorentz symmetry is in fact an exact symmetry of nature, or whether it could be broken and how such a breaking would manifest itself. One of the major frameworks for these investigations is the *standard model extension* (SME) [2], which 'extends' the standard model Lagrangian by relaxing the restriction that the operator combinations appearing in the Lagrangian be Lorentz scalars. The coefficients of these operators are then Lorentz tensors, and it becomes an experimental question to measure or constrain the components of these tensor coefficients in a particular reference frame. A wide variety of experiments have been performed over the past two decades in order to constrain these coefficients [3].

Given its roots in particular physics, the picture underlying the SME is that of fields propagating on flat spacetime. For this reason, research into gravitational phenomenology in the context of the SME has almost entirely focused on the description of metric perturbations about flat spacetime [4–6], and almost entirely on the linearized equations of motion for these perturbations. (However, see [7] for a case where second-order perturbation theory can be applied in a Lorentz-violating gravitational context.) In some models involving a 'Lorentz-violating' tensor field (i.e. a tensor field whose dynamics give it a non-zero vacuum expectation value), the perturbations of the Lorentz-violating tensor field effectively decouple from the linearized Einstein equation, and the linearized Einstein equation can therefore be put into a standard form involving the linearized Riemann tensor (and its derivatives) and various contractions of the background value of the Lorentz-violating tensor.

These investigations provide valuable constraints on the possible behavior of Lorentz-violating gravity models. However, they cannot access the full range of phenomenology that one could describe as 'Lorentz-violating gravity', for the simple reason that they are confined to perturbations of a flat spacetime background. Some of the most fascinating behavior in general relativity, as well as some of the most sensitive constraints on it, come not from the weak-gravity limit but from situations that cannot be viewed as 'close' to flat spacetime: black hole physics and cosmology. To model these situations, we require a full non-linear model of gravity in which Lorentz symmetry is broken.

The question then arises how to construct such a model in a well-motivated way. Ideally, we would like this model to in some sense extrapolate from the linearized Lorentz-violating gravity picture of the SME to a fully dynamical Lorentz-violating version of general relativity. In the case of Lorentz-invariant gravity, there is a known technique to make this extrapolation: the so-called 'bootstrap' procedure. (See [8–10] among others). One starts with a model containing a massless symmetric rank-2 tensor field h_{ab} in flat spacetime, along with some other matter sources. One then adds terms to the Lagrangian that couple h_{ab} to the total stress-energy tensor of the model, including its own. These new terms may themselves contribute to the stress-energy tensor, so we must then insert couplings between h_{ab} and these new contributions. Iterating this procedure generates an infinite series of terms in the action; and the infinite series of terms involving h_{ab} alone can be shown to converge to the Einstein–Hilbert action, with R being the Ricci scalar of the metric $g_{ab} = \eta_{ab} + h_{ab}$. Moreover, if the matter sector is not too complicated, the infinite series of terms coupling h_{ab} and the matter fields will simply have the effect of replacing the flat spacetime matter Lagrangian with a 'minimally coupled' version of the matter Lagrangian, substituting $\eta_{ab} \to g_{ab}$ and $\partial_a \to \nabla_a$. In effect, this procedure 'bootstraps' a linear model into a non-linear one.

It is natural to ask whether this elegant procedure can be applied if we relax some of the underlying assumptions. In particular, if we start with a linear field theory that violates Lorentz symmetry in some way, is it still possible to apply the bootstrap procedure? Is there a mathematical impediment to this process? Is the interpretation of the resulting model the same? In this work, I show that the bootstrap procedure can in fact be applied even if Lorentz symmetry is violated in the linear field theory for the tensor field h_{ab}. The result is a bimetric model, in which the Ricci curvature appearing in the Einstein–Hilbert action is associated with an effective metric $\tilde{g}_{ab}$ constructed in a non-linear way from the metric g_{ab} that is 'seen' by matter and a dynamical Lorentz-violating vector field A_a.

The paper is structured as follows. In section 2, I will discuss what it means for a linear gravity model to be Lorentz-violating, and how such a model can be constructed in the presence of a background vector field. Section 3 reviews the Lorentz-invariant bootstrap procedure, and then applies it to the Lorentz-violating models constructed in section 2. Finally, section 4 briefly discusses two simple applications of the model constructed in section 3: the SME coefficients of the resulting model, and the application to FRW universes.

We will use units where $c = \hbar = 1$ throughout; the sign convention will be $(-, +, +, +)$. Symmetrizations and antisymmetrizations of tensors over n indices will be weighted by a factor of $1/n!$, e.g. $\nabla_{(a}A_{b)} = (\nabla_a A_b + \nabla_b A_a)/2!$.

2. Linearized gravity without Lorentz symmetry

2.1. Defining 'Lorentz violation'

Before discussing the construction of a linear gravity model that 'violates Lorentz symmetry', it is important to state clearly what we mean by the phrase. As a toy model, consider two versions of the massless Klein–Gordon equation:

$$-\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{1}$$

$$-\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{4}\frac{\partial^2 \phi}{\partial z^2} = 0. \tag{2}$$

Which of these equations is 'Lorentz-invariant'? Since Lorentz symmetry includes rotations, and since the speed of waves in the x-direction and z-direction are different in (2), one might conclude that only (1) is Lorentz-invariant. However, it is not hard to see that (1) and (2) are equivalent if we have the freedom to redefine our coordinates; one merely needs to rescale $z \to z/2$ in (2) to obtain (1).

More generally, given a quadratic form α^{ab} with signature $(-, +, +, +)$, it is always possible to find some set of coordinates in which the equation

$$\alpha^{ab}\partial_a\phi\partial_b\phi = 0 \tag{3}$$

takes the form (1). In such a coordinate system, the components of α^{ab} will simply be the familiar components of the (inverse) Minkowski metric η^{ab}. This allows us to define an 'inertial coordinate system' to be one in which the equation of motion for ϕ takes the form (1). Such sets of coordinates are not unique, of course; the set of linear coordinate transformations that leave the wave equation in this form will simply be a subgroup of $GL(4)$ that is isomorphic to $SO(3, 1)$, and will be the 'Lorentz transformations' between our inertial coordinate systems.

In a real sense, then, it is not possible to define a 'violation' of Lorentz symmetry in the context of a model containing only one scalar field obeying a wave equation. We can always use the behavior of this field to define our clocks and metersticks, and a preferred set of transformations of coordinates between observers, in such a way that the speed of wave propagation is the same in all directions and for all observers. Where a notion of Lorentz violation can arise is when we have multiple fields which propagate with respect to different metrics. For example, if our Universe contains two massless scalar fields ϕ and ψ, with ϕ propagating according to (3) and ψ propagating according to

$$\tilde{\alpha}^{ab}\partial_a\psi\partial_b\psi = 0 \tag{4}$$

with $\alpha^{ab} \neq \tilde{\alpha}^{ab}$, then generically we cannot define a set of coordinates so that the equations of motion for both ϕ and ψ are both of the form (1). (The exception being if $\alpha^{ab} = \lambda\tilde{\alpha}^{ab}$ for some

$\lambda \neq 1$; but in this case (4) is equivalent to (3).) In mathematical terms, the $SO(3,1)$ subgroups of $GL(4)$ which leave α^{ab} and $\tilde{\alpha}^{ab}$ invariant are not necessarily the same. We are free to use either one of these fields to define what we mean by clocks, metersticks, and transformations between 'inertial reference frames'; but once we have done so, the other field will 'violate Lorentz symmetry' according to this description.

More generally, if our model contains several 'sectors', it is sometimes possible to define Lorentz transformations in such a way that one of the sectors is Lorentz-invariant. From this perspective, it is not particularly miraculous that a 'privileged' speed exists in our universe; we could simply define our notion of Lorentz transformations in such a way that the speed of light was the same for all inertial observers. What is remarkable, rather, is that this privileged speed appears to be the same for all polarizations of all fundamental fields: electromagnetic fields, fermion fields, and gravitational fields. Even within a sector, it is not always possible to choose coordinates for a sector such that it becomes Lorentz-invariant. For example, in minimal Lorentz-violating electrodynamics, an arbitrary Lorentz-violating Lagrangian contains nineteen free coefficients for the photon sector, of which only nine can be shifted to the matter sector [11]. The remaining ten coefficients cause light to have a polarization-dependent speed (i.e. birefringence), and so cannot be removed via a simple coordinate redefinition. Similar redefinitions can shift nine of the coordinates in the gravitational SME into the photon sector or vice versa [12].

In the context of this work, we will assume that this choice has already been made in some portion of the 'matter sector', which propagates according to some metric η^{ab}; our notion of Lorentz transformations will be those transformations which leave this matter metric unchanged. I will call this metric the *fiducial metric*. I will remain agnostic as to whether all parts of the matter sector propagate according to the fiducial metric, though the simplest choice (see below) would be that all of them do.

2.2. Constructing the propagator

Assuming that we have defined a fiducial metric and a notion of Lorentz symmetry with reference to some portion of the matter sector, the question then arises what sorts of phenomenology can arise if the 'linearized gravity sector' is not Lorentz-symmetric. In general, one can imagine two broad classes of Lorentz-violating effects: direction-dependent propagation speeds, and polarization-dependent propagation speeds (i.e. 'gravitational birefringence'.) While this question was addressed in a group-theoretic context in [6], it is instructive to take an axiomatic approach to this matter: if we make a certain set of assumptions about the propagation of the linearized gravity field, which types of effects are allowed?

In choosing my axioms, I will adopt a philosophy of *ceteris paribus*: I will attempt to construct a model that preserves as many key features of conventional linearized gravity as possible, while relaxing the assumption of Lorentz symmetry. These features include:

(i) Being described by a rank-2 symmetric tensor h_{ab}.
(ii) Being expressible in terms of an action principle.
(iii) Having second-order questions of motion.
(iv) Being coupled to a conserved stress-energy tensor.

Under criteria (i), (ii), and (iii), the Lagrange density for the free field h_{ab} must be of the form

$$\mathcal{L} = \frac{1}{2}\left[\mathcal{P}^{abcdef}\partial_a h_{bc}\partial_d h_{ef} + \mathcal{R}^{abcef}(\partial_a h_{bc})h_{ef} + \mathcal{Q}^{bcef}h_{bc}h_{ef}\right] \tag{5}$$

for some tensors $\mathcal{P}^{abcdef}$, $\mathcal{R}^{abcde}$, and $\mathcal{Q}^{abcd}$. These tensors are assumed to be constant in spacetime, but they will in general involve some additional 'background geometry': they should not be expected to remain invariant under the Lorentz transformations that keep η^{ab} fixed.

From (5), we can see that $\mathcal{P}^{abcdef}$ can be taken to be symmetric under the simple exchanges $b \leftrightarrow c$ and $e \leftrightarrow f$, and under the simultaneous exchange $\{abc\} \leftrightarrow \{def\}$. Similarly, $\mathcal{Q}^{bcef}$ can be taken to be symmetric under the exchanges $b \leftrightarrow c$, $e \leftrightarrow f$, and $\{bc\} \leftrightarrow \{ef\}$. Finally, since we can write

$$\left(\mathcal{R}^{abcef} + \mathcal{R}^{aefbc}\right)(\partial_a h_{bc})h_{ef} = \partial_a\left(\mathcal{R}^{abcef} h_{bc} h_{ef}\right), \tag{6}$$

it follows that the part of $\mathcal{R}^{abcef}$ that is symmetric under the exchange $\{bc\} \leftrightarrow \{ef\}$ does not contribute to the equations of motion. We can thus take $\mathcal{R}^{abcef}$ to be antisymmetric under the exchange $\{bc\} \leftrightarrow \{ef\}$, as well as being symmetric under the exchanges $b \leftrightarrow c$ and $e \leftrightarrow f$.

The equations of motion that arise from (5) are

$$-\mathcal{P}^{abcdef}\partial_a\partial_d h_{ef} - \mathcal{R}^{abcef}\partial_a h_{ef} + \mathcal{Q}^{bcef} h_{ef} = 0, \tag{7}$$

or, in momentum space,

$$\mathcal{P}^{abcdef} k_a k_d h_{ef} - \mathrm{i}\mathcal{R}^{abcef} k_a h_{ef} + \mathcal{Q}^{bcef} h_{ef} = 0. \tag{8}$$

We note from this equation that we can take $\mathcal{P}^{abcdef}$ to be symmetric under the exchange $a \leftrightarrow d$.

We can now apply criterion (iv) to this equation. We will eventually want to couple (7) to the matter stress-energy tensor T^{bc}. In the linearized limit about flat spacetime, we expect this tensor to be identically conserved: $\partial_b T^{bc} = 0$. This implies that the divergence of (7) must also vanish identically; in momentum space, this means that

$$\mathcal{P}^{abcdef} k_a k_b k_d - \mathrm{i}\mathcal{R}^{abcef} k_a k_b + \mathcal{Q}^{bcef} k_b = 0 \tag{9}$$

for any four-vector k_b. Note that given the symmetries of $\mathcal{P}^{abcdef}$, $\mathcal{R}^{abcef}$, and $\mathcal{Q}^{bcef}$, the condition (9) is equivalent to (7) being invariant under the customary gauge transformation $h_{ab} \to h_{ab} + \partial_{(a}\xi_{b)}$.

In principle, any set of tensors $\mathcal{P}^{abcdef}$, $\mathcal{R}^{abcef}$, and $\mathcal{Q}^{bcef}$ with the appropriate symmetries and satisfying (9) would provide a Lorentz-violating equation of motion for h_{ab}. However, the underlying picture we have is that this tensor is due to a coupling between h_{ab} and some new fundamental field that spontaneously breaks Lorentz symmetry. The simplest choice for such a field is a Lorentz vector A_a; the propagator tensor $\mathcal{P}^{abcdef}$ must then be constructed locally out of tensor products of A_a and the fiducial metric η^{ab}. The question then becomes how many distinct tensors there are which can be so constructed and which satisfy the desired symmetry properties and the contraction identity (9).

To answer this question, we can simply write down a list of all tensors of a given rank, without any particular symmetry properties, that can be so constructed. The most general such tensor of a given rank must be a linear combination of these; and the symmetry requirements and the contraction identity (9) will then place constraints on the coefficients of each tensor in this linear combination. For example, suppose we want to construct a rank-6 tensor from the metric alone. such a tensor must be constructed from three 'copies' of the metric, and so the six indices must be paired off; there are fifteen such pairings. The most general rank-6 tensor that can be constructed from the metric is then

$$C_1\eta^{ab}\eta^{cd}\eta^{ef} + C_2\eta^{ab}\eta^{ce}\eta^{df} + C_3\eta^{ab}\eta^{cf}\eta^{de} + \ldots, \tag{10}$$

where the C_i are arbitrary coefficients. However, if we require that this expression be symmetric under the exchanges $b \leftrightarrow c$, $e \leftrightarrow f$, and $\{abc\} \leftrightarrow \{def\}$, and require it to obey the contraction identity (9), then it is straightforward (if a bit tedious) to show that there is only one such tensor:

$$\begin{aligned}\mathcal{P}_{(0)}^{abcdef} &= \eta^{a(b}\eta^{c)d}\eta^{ef} + \eta^{a(e}\eta^{f)d}\eta^{bc} - \eta^{a(b}\eta^{c)(e}\eta^{f)d} \\ &\quad - \eta^{a(e}\eta^{f)(b}\eta^{c)d} - \eta^{ad}\eta^{bc}\eta^{ef} + \eta^{ad}\eta^{b(e}\eta^{f)c}.\end{aligned} \tag{11}$$

This expression, when contracted with $\partial_a\partial_d h_{ef}$ as in (7), yields the standard linearized Einstein equation.

Similarly, there are 45 tensors that can be constructed from two copies of the metric and two copies of the vector $A^a \equiv \eta^{ab}A_b$; and it is also straightforward (if rather more tedious) to show that there is also only one possible combination of them that satisfies the desired symmetries and the contraction identity:

$$\begin{aligned}\mathcal{P}_{(1)}^{abcdef} &= \eta^{a(b}\eta^{c)d}A^eA^f + \eta^{a(e}\eta^{f)d}A^bA^c - 2A^{(a}\eta^{d)(b}\eta^{c)(e}A^{f)} - 2A^{(a}\eta^{d)(e}\eta^{f)(b}A^{c)} \\ &\quad - 2A^{(b}\eta^{c)(a}\eta^{d)(e}A^{f)} + 2\eta^{bc}A^{(a}\eta^{d)(e}A^{f)} + 2\eta^{ef}A^{(a}\eta^{d)(b}A^{c)} + 2\eta^{ad}A^{(b}\eta^{c)(e}A^{f)} \\ &\quad - \eta^{ad}\eta^{bc}A^eA^f - \eta^{ad}\eta^{ef}A^bA^c - \eta^{bc}\eta^{ef}A^aA^d + \eta^{b(e}\eta^{f)c}A^aA^d.\end{aligned} \tag{12}$$

There are fifteen rank-6 tensors that can be constructed from four copies of A^a and one copy of the metric; however, it can be shown via similar techniques that any linear combination of these tensors with the desired symmetry properties must vanish. The tensor $A^aA^bA^cA^dA^eA^f$ must also be excluded from our expression for $\mathcal{P}^{abcdef}$ by a similar logic.

We can also apply the same logic to the rank-5 tensor $\mathcal{R}^{abcef}$ and the rank-4 tensor $\mathcal{Q}^{bcef}$; when we do, however, we find that these tensors must vanish. This implies that the dispersion relation (8) for wave solutions is homogeneous, i.e. if a plane wave of the form

$$h_{ab}(x^c) = h_{ab}^{(0)}e^{ik_cx^c} \tag{13}$$

is a solution of the equations of motion, then it remains a solution under the substitution $k_\mu \to \lambda k_\mu$ for any λ. This ensures (among other things) that the speed of a wave is independent of its frequency for a given polarization and a given direction of propagation.

Since the Lagrange density is only defined up to an overall factor, this means that the most general possible expression for our propagator tensor is

$$\mathcal{P}^{abcdef} = \mathcal{P}_{(0)}^{abcdef} + \xi\mathcal{P}_{(1)}^{abcdef}, \tag{14}$$

where ξ is a free parameter. However, it is not hard to show (albeit, again, tedious) that this expression is equivalent to taking the Lorentz-symmetric graviton propagator (11) and substituting

$$\eta^{ab} \to \tilde{\eta}^{ab} \equiv \eta^{ab} + \xi A^aA^b. \tag{15}$$

In other words: in the presence of a background vector field A_a, and assuming the criteria listed above, the only modification to the linearized Einstein equation that is possible is to change the effective metric that governs the propagation of the waves and their polarization states. (The usual 'transverse traceless gauge' for these waves would be defined via $\tilde{\eta}^{ab}\partial_a h_{bc} = 0$ and $h \equiv \tilde{\eta}^{ab}h_{ab} = 0$, rather than the equivalent expressions with η^{ab}.) It is not possible to define a propagator which allows for 'gravitational birefringence', i.e. a polarization-dependent speed of gravity. This is consistent with the results found in [6]; criterion (iii) above restricts us to what were called '$d = 4$ operators' in that work.

3. Bootstrapping Lorentz-violating linearized gravity

3.1. Deser bootstrap procedure for conventional gravity

In section 2.2, we found that under certain assumptions, the only way to modify the linearized Einstein equation to include a coupling to a 'Lorentz-violating' vector field A_a is by modifying the metric that appears in the Lorentz-invariant equation of motion for h_{ab} (11). This modification of the linearized Einstein equation is, in itself, self-consistent. However, we also know that it is possible to extend Lorentz-invariant linearized gravity to a non-linear theory (namely, conventional general relativity) by self-consistently coupling h_{ab} to all sources of stress-energy in the theory, including its own. The question then arises whether a similar procedure can be applied to a model in which h_{ab}'s propagation is governed by the effective metric (15), or whether there is an impediment to this.

To frame this discussion, it will help to first review the bootstrap procedure proposed by Deser [13]. (See also [14] for a more detailed explanation of this procedure.) In this procedure, the fundamental fields are a tensor density $\mathfrak{h}^{ab}$ and a rank-(1,2) undensitized tensor $C^a{}_{bc} = C^a{}_{(bc)}$. The linear Lagrange density is written in first-order form:

$$\mathcal{L} = \mathcal{L}^{LI}_{G,lin} + \mathcal{L}_{\mathrm{mat}}[\eta, \Psi^A], \tag{16}$$

where

$$\mathcal{L}^{LI}_{G,lin} = \kappa \left[2\mathfrak{h}^{ab}\partial_{[c}C^c{}_{b]a} + 2\eta^{ab}C^c{}_{a[b}C^d{}_{d]c} \right] \tag{17}$$

with $\kappa \equiv 1/(16\pi G)$, and $\mathcal{L}_{\mathrm{mat}}[\eta, \Psi^A]$ is the Lagrange density governing the 'matter sector' of the action. The matter sector is assumed to depend on the fiducial metric density η^{ab} as well as some collection of matter fields Ψ^A. Note that for consistency in what follows, we will need to view η^{ab} as a tensor density rather than as a simple tensor.

The equations of motion derived from (17) are then

$$\partial_c C^c{}_{ab} - \partial_{(a}C^c{}_{b)c} = 0 \tag{18}$$

and

$$\partial_c \mathfrak{h}^{ab} - \partial_d \mathfrak{h}^{d(a}\delta^{b)}{}_c = \eta^{ab}C^d{}_{dc} + \eta^{de}C^{(a}{}_{de}\delta^{b)}{}_c - 2\eta^{d(a}C^{b)}{}_{cd}. \tag{19}$$

Some algebra can then show that (19) is equivalent to

$$C^c{}_{ab} = -\frac{1}{2}\left[\eta_{da}\partial_b\bar{\mathfrak{h}}^{cd} + \eta_{db}\partial_a\bar{\mathfrak{h}}^{cd} - \eta_{ad}\eta_{be}\eta^{cf}\partial_f\bar{\mathfrak{h}}^{de}\right] \equiv \Gamma^c{}_{ab}, \tag{20}$$

where we have defined $\bar{\mathfrak{h}}^{ab} \equiv \mathfrak{h}^{ab} - \frac{1}{2}\eta^{ab}\eta_{cd}\mathfrak{h}^{cd}$. For future reference, the right-hand side of this equation is defined to be $\Gamma^c{}_{ab}$; the notation is intentionally suggestive.

We now define a tensor density $\mathfrak{g}^{ab} = \eta^{ab} + \mathfrak{h}^{ab}$. This tensor density will be related to an inverse metric g^{ab} by $\mathfrak{g}^{ab} = \sqrt{-g}g^{ab}$, with $g \equiv \det(g_{\mu\nu})$ according to the flat spacetime coordinates. This implies that $\det(\mathfrak{g}_{\mu\nu}) = 1/\det(g_{\mu\nu})$, and so we have

$$g^{ab} = \sqrt{-\mathfrak{g}}\mathfrak{g}^{ab}. \tag{21}$$

But if we define $h^{ab} = g^{ab} - \eta^{ab}$, then it is not hard to show that to linear order in $\mathfrak{h}^{ab}$,

$$h^{ab} \approx \bar{\mathfrak{h}}^{ab}. \tag{22}$$

In other words, (20) implies that $C^c{}_{ab}$ is the linearized Christoffel symbol $\Gamma^c{}_{ab}$ associated with the inverse metric g^{ab} and this coordinate basis. Equation (18), meanwhile, says that the linearized Ricci tensor associated with this Christoffel symbol is zero. Thus, at linear order, the free gravitational action (17) yields equations of motion that are equivalent to the linearized Einstein equations.

We now wish to couple $\mathfrak{h}^{ab}$ to the stress-energy of the model. As the left-hand side of (18) is equal to the Ricci tensor, we expect the source on the right-hand side to be the trace-reversed stress-energy tensor τ_{ab}. This can be accomplished by adding the term $-\mathfrak{h}^{ab}t_{ab}$ to the action (17). The trace-reversed stress energy tensor τ_{ab} can be found via the Rosenfeld prescription by 'promoting' the fiducial metric density η^{ab} to an auxiliary metric density ψ^{ab}, differentiating the Lagrange density $\mathcal{L}^{LI}_{(0)}$ with respect to this auxiliary density, and then setting $\psi^{ab} \to \eta^{ab}$ in the result; this will yield $-\frac{1}{2}\tau_{ab}$. (In this process, factors of $\sqrt{-\eta}$ may need to be inserted into the action to ensure that various sums in our expressions have a definite weight.) The derivatives in the first term of (17) must also be 'promoted' to covariant derivatives, which are then varied along with ψ^{ab}. The result is

$$-\tau_{ab} = 2\kappa\left[C^c{}_{a[b}C^d{}_{d]c} + \sigma_{ab}\right], \tag{23}$$

where σ_{ab} is a total derivative[1]:

$$\sigma_{ab} = -\frac{1}{2}\partial_c\Big[2\mathfrak{h}^{cd}\eta_{d(a}C_{b)} - \eta_{ad}\eta_{be}\eta^{cf}\mathfrak{h}^{de}C_f + 2\eta_{e(a}\eta_{b)f}\eta^{gc}\mathfrak{h}^{de}C^f{}_{dg} - 2\eta_{e(a}\mathfrak{h}^{cd}C^e{}_{b)d}$$
$$-2\eta_{e(a}\delta_{b)}{}^f\mathfrak{h}^{de}C^c{}_{df} - \eta_{ab}\eta_{df}\eta^{cg}\mathfrak{h}^{de}C^f{}_{eg} + \frac{1}{2}\eta_{ab}\eta_{de}\eta^{cf}\mathfrak{h}^{de}C_f\Big], \tag{24}$$

where we have defined $C_a \equiv C^b{}_{ba}$. I have explicitly written out the fiducial metrics used to raise and lower indices in this expression in order to illustrate a point that will arise later in the Lorentz-violating case.

In Deser's procedure, the gravitational sector of the action is completed by adding the *non-derivative* portion of (23) to the action, coupled to $\mathfrak{h}^{ab}$:

$$\mathcal{L}^{LI}_G = \kappa\left[2\mathfrak{h}^{ab}\partial_{[c}C^c{}_{b]a} + 2(\eta^{ab} + \mathfrak{h}^{ab})C^c{}_{a[b}C^d{}_{d]c}\right]. \tag{25}$$

Importantly, these new terms do not refer to the fiducial metric density η^{ab} in any way; thus, this additional term will not contribute to τ_{ab}. This is the signal advantage of Deser's choice to use the tensor density as the fundamental field; it does not require a further infinite series of terms in the gravitational action, as is necessary when viewing the metric perturbation h_{ab} as the fundamental field.

If we define $\mathfrak{g}^{ab} = \eta^{ab} + \mathfrak{h}^{ab}$, the resulting equations of motion from this non-linear action can then be written as

$$\partial_c C^c{}_{ab} - \partial_{(a}C^c{}_{b)c} + C^c{}_{ab}C^d{}_{dc} - C^c{}_{ac}C^d{}_{bd} = 0 \tag{26}$$

and

$$\partial_c\mathfrak{g}^{ab} - \partial_d\mathfrak{g}^{d(a}\delta^{b)}{}_c = \mathfrak{g}^{ab}C^d{}_{dc} + \mathfrak{g}^{de}C^{(a}{}_{de}\delta^{b)}{}_c - 2\mathfrak{g}^{d(a}C^{b)}{}_{cd}. \tag{27}$$

By taking various traces and linear combinations of this second equation, it can be shown to be equivalent to

[1] Note that the expression for this quantity in [14] contains a sign error in one term.

$$C^c{}_{ab} = -\frac{1}{2}\left[2\mathfrak{g}_{d(a}\partial_{b)}\mathfrak{g}^{cd} - \mathfrak{g}_{ad}\mathfrak{g}_{be}\mathfrak{g}^{cf}\partial_f\mathfrak{g}^{de} + \mathfrak{g}_{de}\left(-\delta^c{}_{(a}\partial_{b)}\mathfrak{g}^{de} + \frac{1}{2}\mathfrak{g}_{ab}\mathfrak{g}^{cf}\partial_f\mathfrak{g}^{de}\right)\right], \quad (28)$$

where $\mathfrak{g}_{ab}$ is the inverse of $\mathfrak{g}^{ab}$ (i.e. $\mathfrak{g}_{ab}\mathfrak{g}^{bc} = \delta_a{}^c$.) In other words, $C^c{}_{ab}$ is equal to the Christoffel symbol associated with the metric $g^{ab} = \sqrt{-\mathfrak{g}}\mathfrak{g}^{ab}$ (with $\mathfrak{g} \equiv \det(\mathfrak{g}_{ab})$), and in the absence of other matter, the Ricci tensor (26) associated with these Christoffel symbols vanishes.

On the other hand, it turns out that (26) and (27) are equivalent to coupling (18) to the *full* stress-energy of the action (23), including the derivative term σ_{ab}. To show this, rewrite (27) as

$$\begin{aligned}&\eta^{ab}C^d{}_{dc} + \eta^{de}C^{(a}{}_{de}\delta^{b)}{}_c - 2\eta^{d(a}C^{b)}{}_{cd}\\ &\quad = \partial_c\mathfrak{h}^{ab} - \partial_d\mathfrak{h}^{d(a}\delta^{b)}{}_c - \left[\mathfrak{h}^{ab}C^d{}_{dc} + \mathfrak{h}^{de}C^{(a}{}_{de}\delta^{b)}{}_c - 2\mathfrak{h}^{d(a}C^{b)}{}_{cd}\right].\end{aligned} \quad (29)$$

(Note that $\partial_a\mathfrak{h}^{bc} = \partial_a\mathfrak{g}^{bc}$.) Via the same algebraic procedure used to obtain (28) from (27), we find that

$$\begin{aligned}C^c{}_{ab} = &-\frac{1}{2}\left[2\eta_{d(a}\partial_{b)}\bar{\mathfrak{h}}^{cd} - \eta_{ad}\eta_{be}\eta^{cf}\partial_f\bar{\mathfrak{h}}^{de}\right]\\ &+\frac{1}{2}\left[2\eta_{d(a}\mathfrak{h}^{cd}C_{b)} - 2\eta_{e(a}\mathfrak{h}^{de}C^c{}_{b)d} - 2\eta_{e(b}\mathfrak{h}^{cd}C^e{}_{c)d} - \eta_{ae}\eta_{bf}\eta^{cd}\mathfrak{h}^{ef}C_d\right.\\ &+2\eta_{e(a}\eta_{b)f}\eta^{cg}\mathfrak{h}^{de}C^f{}_{dg} - \eta_{de}\delta^c{}_{(a}C_{b)}\mathfrak{h}^{de} + 2\eta_{df}\delta^c{}_{(a}\mathfrak{h}^{de}C^f{}_{b)e}\\ &\left.+\frac{1}{2}\eta_{ab}\eta^{cf}\eta_{de}C_f\mathfrak{h}^{de} - \eta_{ab}\eta^{cg}\eta_{df}\mathfrak{h}^{de}C^f{}_{eg}\right]\end{aligned} \quad (30)$$

where we have defined $C_a \equiv C^b{}_{ab}$. The first set of terms can be seen to be $\Gamma^c{}_{ab}$. Taking the appropriate derivatives and contractions of (30), and after a fair amount of algebra, we find that (27) implies that

$$\partial_c C^c{}_{ab} - \partial_{(a}C^c{}_{b)c} = \partial_c\Gamma^c{}_{ab} - \partial_{(a}\Gamma^c{}_{b)c} + \sigma_{ab}, \quad (31)$$

with σ_{ab} (remarkably) defined as in (24). Combining this with (26), we obtain

$$\partial_c\Gamma^c{}_{ab} - \partial_{(a}\Gamma^c{}_{b)c} = -\left[C^c{}_{ab}C^d{}_{dc} - C^c{}_{ad}C^d{}_{bc} + \sigma_{ab}\right] = \frac{1}{2\kappa}\tau_{ab}, \quad (32)$$

with τ_{ab} defined as in (23). In other words, the non-linear equations of motion (26) and (27) are equivalent to the equations (18) and (19) from the linear action (17), with the full stress-energy of the linear action acting as a source. Note that even though the derivative portion of the stress-energy σ_{ab} is not explicitly coupled to the density $\mathfrak{h}^{ab}$ in the non-linear action (25), the terms corresponding to it still arise in (32) so long as the full non-linear equations of motion (26) and (27) hold. I will return to this point when we pass to the Lorentz-violating version of the theory.

To include the effects of matter, one must also apply the bootstrap procedure to the matter Lagrange density $\mathcal{L}_{\text{mat}}$. So long as the matter action $\mathcal{L}_{\text{mat}}[\eta^{ab}, \Psi^A]$ only depends on the fiducial metric density η^{ab} itself, and not on its derivatives, it can be shown [14] that the net effect of applying the bootstrap procedure to the matter action is simply to replace η^{ab} with $\mathfrak{g}^{ab}$.[2] Any matter action only containing Lorentz scalars will satisfy this condition, as well as any n-form

[2] In the Deser procedure, this process *does* sometimes require an infinite series of terms that might not be required if we view h_{ab} as the fundamental field. One cannot always escape both Scylla and Charybdis.

field whose kinetic terms depend only on that field's exterior derivative. In particular, the Maxwell kinetic term $-\frac{1}{4}F_{ab}F^{ab}$, with $F_{ab} = 2\partial_{[a}A_{b]}$, is independent of the choice of derivative operator and so only depends on the metric itself and not on the metric derivatives. The trace-reversed stress-energy will then appear on the right-hand side of (26), while (27) will be unaffected.

3.2. Deser bootstrap procedure for Lorentz-violating gravity

In section 2.2, it was shown that under certain assumptions, the only possible modification of the linearized Einstein equation which couples the linearized metric perturbations to a background vector A_a is equivalent to replacing the fiducial metric η^{ab} with an effective metric:

$$\tilde{\eta}^{ab} = \eta^{ab} + \xi\eta^{ac}\eta^{bd}A_cA_d. \tag{33}$$

Since we are starting off from the context of flat spacetime, we will assume that A_a is a constant background vector field; only later will we ascribe dynamics to it.

We now wish to do two things. First, we wish to modify the linear action (17) so that its equations of motion are equivalent to (7), with $\mathcal{P}^{abcdef}$ given by (14) and $\mathcal{Q}^{abcd} = \mathcal{R}^{abcef} = 0$. Second, we want to self-consistently couple the stress-energy of the resulting action to itself, to obtain a non-linear model of Lorentz-violating gravity.

There are two possible terms we can add to the action (17) to couple the fields to a constant background vector field A_a:

$$\begin{aligned}\mathcal{L}^{LV}_{G,lin} =\kappa \big[&2(\mathfrak{h}^{ab} + \xi_1\eta^{ae}\eta^{bf}A_eA_f)\partial_{[c}C^c{}_{b]a} \\ &+ 2(\eta^{ab} + \xi_2\eta^{ae}\eta^{bf}A_eA_f)C^c{}_{a[b}C^d{}_{d]c}\big] ,\end{aligned} \tag{34}$$

where ξ_1 and ξ_2 are arbitrary coupling constants. The first term, with coupling constant ξ_1, does not affect the linear equations of motion at all; since A_a is a constant vector field, this term is a total derivative. It does, however, change the stress-energy of the model, and it will become important for our interpretation of the non-linear model. The second term, with coupling constant ξ_2, basically replaces the fiducial metric density with the 'effective metric density' given by (33) under the substitution $\xi \to \xi_2$. At this point, there is no particular relationship between ξ_1 and ξ_2; however, we will find that the interpretation of the model is much more compelling and elegant when they are equal.

In the interests of compactness, I will need to define various versions of tensors and tensor densities that depend on ξ_1 and ξ_2. I will use $[i]$ as a prepended superscript to denote a version of a quantity defined in section 3.1 (or subsequently) under the substitution $\xi \to \xi_i$. For example, the addition of the ξ_2 term in (34) effectively replaces

$$\eta^{ab} \to {}^{[2]}\tilde{\eta}^{ab} = \eta^{ab} + \xi_2\eta^{ac}\eta^{bd}A_bA_d. \tag{35}$$

For convenience in what follows, I will also define

$${}^{[i]}\tilde{\mathfrak{h}}^{ab} = \mathfrak{h}^{ab} + \xi_i\eta^{ae}\eta^{bd}A_bA_d. \tag{36}$$

It is not hard to see that under these modifications, the linear equations of motion for $C^c{}_{ab}$ and $\mathfrak{h}^{ab}$ are simply (18), unchanged, and (19) under the substitution $\eta^{ab} \to {}^{[2]}\tilde{\eta}^{ab}$. Thus, the linear Lagrange density (34) yields the desired Lorentz-violating linearized Einstein equation given by (7) and (14), with the substitution $\xi \to \xi_2$.

We now wish to apply the bootstrap procedure to the action (34). The stress-energy in (23) will become

$$-\tau_{ab} = 2\kappa \left[C^c{}_{a[b} C^d{}_{d]c} + {}^{[1]}\sigma_{ab} \right] - \kappa(\tau_\xi)_{ab}, \tag{37}$$

where $(\tau_\xi)_{ab}$ is the contribution to the stress-energy tensor coming from the *algebraic* appearances of η^{ab} in the new coupling terms in (34), and ${}^{[1]}\sigma_{ab}$ is obtained from (24) under the substitution $\mathfrak{h}^{ab} \to {}^{[1]}\tilde{\mathfrak{h}}^{ab}$.

In parallel with Deser's procedure, the method will be to again couple the non-derivative portion of the stress-energy (37) to the field $\mathfrak{h}^{ab}$, and to see whether these equations are equivalent to the linear equations (18), with $-\frac{1}{2}\tau_{ab}$ included as a source, and (19), with $\eta^{ab} \to {}^{[2]}\tilde{\eta}^{ab}$. The $(\tau_\xi)_{ab}$ piece of the new stress-energy tensor can be viewed as new contribution from the 'matter sector'. It depends on the metric η^{ab} itself, and so we will need to apply the procedure to this new term, adding in the non-derivative portion of the stress-energy from this term. Iterating this procedure, this will generate an infinite sum of terms as we add the higher-order contributions of these terms to the stress-energy. The procedure is analogous to the infinite series of terms that arises from the matter sector in the Lorentz-invariant bootstrap procedure; and as in that case, the resulting terms can be resummed, with the net effect of replacing

$$\xi_i \eta^{ae} \eta^{bf} A_e A_f \to \xi_i \sqrt{-\mathfrak{g}}\, \mathfrak{g}^{ae} \mathfrak{g}^{bf} A_e A_f \tag{38}$$

in (34). Including this modification, along with the coupling between $\mathfrak{h}^{ab}$ and the first term of (37), the full non-linear action becomes

$$\mathcal{L}_G^{LV} = 2\kappa \left[({}^{[1]}\tilde{\mathfrak{g}}^{ab} - \eta^{ab}) \partial_{[c} C^c{}_{b]a} + {}^{[2]}\tilde{\mathfrak{g}}^{ab} C^c{}_{a[b} C^d{}_{d]c} \right], \tag{39}$$

where we have defined

$${}^{[i]}\tilde{\mathfrak{g}}^{ab} = \mathfrak{g}^{ab} + \xi_i \sqrt{-\mathfrak{g}}\, \mathfrak{g}^{ae} \mathfrak{g}^{bf} A_e A_f. \tag{40}$$

Note that in contrast with the Lorentz-invariant definition,

$${}^{[i]}\tilde{\mathfrak{g}}^{ab} \neq \eta^{ab} + {}^{[i]}\tilde{\mathfrak{h}}^{ab} = \eta^{ab} + \mathfrak{h}^{ab} + \xi_i \eta^{ae} \eta^{bd} A_b A_d. \tag{41}$$

This difference will become important in what follows.

The equations of motion derived from (39) can be obtained by viewing $\mathfrak{g}^{ab}$ and $C^c{}_{ab}$ as the configuration variables. In performing the variation with respect to $\mathfrak{g}^{ab}$, it is useful to note that

$$\begin{aligned} \frac{\delta \left({}^{[i]}\mathfrak{g}^{cd} \right)}{\delta \mathfrak{g}^{ab}} &= \delta^c{}_{(a} \delta^d{}_{b)} + \xi_i \sqrt{-\mathfrak{g}} \left[2A_{(a} \delta_{b)}{}^{(c} \mathfrak{g}^{d)e} A_e - \frac{1}{2} \mathfrak{g}_{ab} \mathfrak{g}^{ce} \mathfrak{g}^{df} A_e A_f \right] \\ &= \delta^c{}_{(a} \delta^d{}_{b)} + \xi_i \left[2A_{(a} \delta_{b)}{}^{(c} g^{d)e} A_e - \frac{1}{2} g_{ab} g^{ce} g^{df} A_e A_f \right], \end{aligned} \tag{42}$$

and so the equation of motion associated with $\mathfrak{g}^{ab}$ becomes

$$\begin{aligned} &\partial_c C^c{}_{ab} - \partial_{(a} C_{b)} + 2C^c{}_{a[b} C^d{}_{d]c} \\ &\quad + \xi_1 \left[2A_{(a} \delta_{b)}{}^{(c} A^{d)} - \frac{1}{2} g_{ab} A^c A^d \right] \left(\partial_e C^e{}_{cd} - \partial_{(c} C_{d)} \right) \\ &\quad + 2\xi_2 \left[2A_{(a} \delta_{b)}{}^{(c} A^{d)} - \frac{1}{2} g_{ab} A^c A^d \right] C^e{}_{c[d} C^f{}_{f]c} = \frac{1}{2\kappa} (\tau_{\text{mat}})_{ab}, \end{aligned} \tag{43}$$

where all indices are raised and lowered with g_{ab} and its inverse.

Meanwhile, the only change for the $C^c{}_{ab}$ equation of motion, relative to the Lorentz-invariant version (27), is that we must substitute $\mathfrak{g}^{ab} \to {}^{[i]}\tilde{\mathfrak{g}}^{ab}$ appropriately:

$$\partial_c {}^{[1]}\tilde{\mathfrak{g}}^{ab} - \partial_d {}^{[1]}\tilde{\mathfrak{g}}^{d(a}\delta^{b)}{}_c = {}^{[2]}\tilde{\mathfrak{g}}^{ab} C^d{}_{dc} + {}^{[2]}\tilde{\mathfrak{g}}^{de} C^{(a}{}_{de}\delta^{b)}{}_c - 2{}^{[2]}\tilde{\mathfrak{g}}^{d(a} C^{b)}{}_{cd}. \quad (44)$$

I have been unable to find an elegant interpretation of the equations of motion (43) and (44) in the general $\xi_1 \neq \xi_2$ case. Similar to the Lorentz-invariant case, (44) can still be inverted to obtain an expression for $C^c{}_{ab}$:

$$C^c{}_{ab} = -\frac{1}{2}\left[2{}^{[2]}\tilde{\mathfrak{g}}_{d(a}\partial_{b)}{}^{[1]}\tilde{\mathfrak{g}}^{cd} - {}^{[2]}\tilde{\mathfrak{g}}_{ad}{}^{[2]}\tilde{\mathfrak{g}}_{be}{}^{[2]}\tilde{\mathfrak{g}}^{cf}\partial_f{}^{[1]}\tilde{\mathfrak{g}}^{de} + {}^{[2]}\tilde{\mathfrak{g}}_{de}\left(-\delta^c{}_{(a}\partial_{b)}{}^{[1]}\tilde{\mathfrak{g}}^{de} + \frac{1}{2}{}^{[2]}\tilde{\mathfrak{g}}_{ab}{}^{[2]}\tilde{\mathfrak{g}}^{cf}\partial_f{}^{[1]}\tilde{\mathfrak{g}}^{de}\right)\right], \quad (45)$$

where ${}^{[2]}\tilde{\mathfrak{g}}_{ab}$ is the inverse of ${}^{[2]}\tilde{\mathfrak{g}}^{ab}$. If $\xi_1 \neq \xi_2$, $C^c{}_{ab}$ can no longer be interpreted as the Christoffel symbol associated with either of the metrics ${}^{[1]}\tilde{g}_{ab}$ or ${}^{[2]}\tilde{g}_{ab}$. Even if $C^c{}_{ab}$ could be interpreted as the Christoffel symbol for some third metric, the fact that $\xi_1 \neq \xi_2$ in (43) prevents us from interpreting that equation in terms of the curvature of that metric.

However, if $\xi_1 = \xi_2 \equiv \xi$, then neither of these problems arise. In this case, $C^c{}_{ab}$ is simply the Christoffel symbol associated with the inverse metric density

$$\tilde{\mathfrak{g}}^{ab} = \tilde{\mathfrak{g}}^{ab} = \mathfrak{g}^{ab} + \xi\sqrt{-\mathfrak{g}}\mathfrak{g}^{ae}\mathfrak{g}^{bf}A_eA_f, \quad (46)$$

and (43) simply becomes

$$\left[\delta^c{}_{(a}\delta^d{}_{b)} + \xi\left(2A_{(a}\delta_{b)}{}^{(c}g^{d)e}A_e - \frac{1}{2}g_{ab}g^{ce}g^{df}A_eA_f\right)\right]\left(\partial_e C^e{}_{cd} - \partial_{(c}C_{d)} + 2C^e{}_{c[d}C^f{}_{f]c}\right) = \frac{1}{2\kappa}(\tau_{\text{mat}})_{ab}. \quad (47)$$

The second factor on the left-hand side of (47) is then equal to $\tilde{R}_{cd}$, the Ricci tensor of the gravitational metric $\tilde{g}_{ab}$ given implicitly by the relationship $\tilde{\mathfrak{g}}^{ab} = \sqrt{-\tilde{g}}\tilde{g}^{ab}$. The equations of motion from the action (39) (with $\xi_1 = \xi_2 = \xi$) are therefore equivalent to

$$\tilde{R}_{ab} + 2\xi A_{(a}\tilde{R}_{b)c}A^c - \frac{1}{2}\xi g_{ab}\tilde{R}_{cd}A^cA^d = 8\pi G(\tau_{\text{mat}})_{ab}, \quad (48)$$

where all indices are raised and lowered with g^{ab} and its inverse. Given the relative straightforwardness of this particular case, I will be assuming that $\xi_1 = \xi_2 = \xi$ for the remainder of this work.

It can be shown that if $\tilde{\mathfrak{g}}^{ab}$ and $\mathfrak{g}^{ab}$ are related by (46), then the corresponding undensitized inverse metrics are related by

$$\tilde{g}^{ab} = \frac{1}{\sqrt{1+\xi A^2}}\left(g^{ab} + \xi g^{ae}g^{bf}A_eA_f\right), \quad (49)$$

and the metrics themselves are related by

$$\tilde{g}_{ab} = \sqrt{1+\xi A^2}\,g_{ab} - \frac{\xi}{\sqrt{1+\xi A^2}}A_aA_b \quad (50)$$

with $A^2 \equiv A_aA_bg^{ab}$. The determinants of these metrics, meanwhile, are related by

$$\det(\tilde{g}_{\mu\nu}) = (1+\xi A^2)\det(g_{\mu\nu}). \quad (51)$$

This action can be viewed as a natural generalization of Deser's bootstrap procedure to a Lorentz-violating gravity model. However, the interpretation of this procedure as 'coupling $\mathfrak{h}^{ab}$ to its own stress-energy', a particularly elegant feature of the Lorentz-invariant model, does not carry over nicely to the present case. (I will ignore the matter sector for the remainder of this section, as it does not affect the following argument.) Recall that in the Lorentz-invariant case, we added a coupling between $\mathfrak{h}^{ab}$ and the *non-derivative* part of the stress-energy tensor. We were then able to interpret the resulting non-linear equations of motion (26) and (27) in terms of the the linear equations of motion coupled to the *full* stress-energy tensor of the linear model; the derivation from equations (29) to (32) showed how the derivative portion of the stress-energy tensor σ_{ab} emerged from the non-linear model naturally.

The presence of the Lorentz-violating terms disrupts this line of logic severely. In the case where $\xi_1 = \xi_2 = \xi \neq 0$, the derivative portion of the stress-energy is now

$$\sigma_{ab} = -\frac{1}{2}\partial_c\Big[2\tilde{\mathfrak{h}}^{cd}\eta_{d(a}C_{b)} - \eta_{ad}\eta_{be}\eta^{cf}\tilde{\mathfrak{h}}^{de}C_f + 2\eta_{e(a}\eta_{b)f}\eta^{gc}\tilde{\mathfrak{h}}^{de}C^f{}_{dg} - 2\eta_{e(a}\tilde{\mathfrak{h}}^{cd}C^e{}_{b)d} \\ -2\eta_{e(a}\delta_{b)}{}^f\tilde{\mathfrak{h}}^{de}C^c{}_{df} - \eta_{ab}\eta_{df}\eta^{cg}\tilde{\mathfrak{h}}^{de}C^f{}_{eg} + \frac{1}{2}\eta_{ab}\eta_{de}\eta^{cf}\tilde{\mathfrak{h}}^{de}C_f\Big], \tag{52}$$

and the analogue of (29) becomes

$$\tilde{\eta}^{ab}C^d{}_{dc} + \tilde{\eta}^{de}C^{(a}{}_{de}\delta^{b)}{}_c - 2\tilde{\eta}^{d(a}C^{b)}{}_{cd} \\ = \left(\partial_c\mathfrak{h}^{ab} - \partial_d\mathfrak{h}^{d(a}\delta^{b)}{}_c\right) - \left(\mathfrak{h}^{ab}C^d{}_{dc} + \mathfrak{h}^{de}C^{(a}{}_{de}\delta^{b)}{}_c - 2\mathfrak{h}^{d(a}C^{b)}{}_{cd}\right) \\ + \left(\partial_c\mathfrak{k}^{ab} - \partial_d\mathfrak{k}^{d(a}\delta^{b)}{}_c\right) - \left(\mathfrak{l}^{ab}C^d{}_{dc} + \mathfrak{l}^{de}C^{(a}{}_{de}\delta^{b)}{}_c - 2\mathfrak{l}^{d(a}C^{b)}{}_{cd}\right) \tag{53}$$

where I have defined

$$\mathfrak{k}^{ab} = \xi\sqrt{-\mathfrak{g}}\mathfrak{g}^{ac}\mathfrak{g}^{bd} \tag{54}$$

and

$$\mathfrak{l}^{ab} = \xi\left[\sqrt{-\mathfrak{g}}\mathfrak{g}^{ac}\mathfrak{g}^{bd} - 2\eta^{ac}\eta^{bd}\right]A_cA_d. \tag{55}$$

It does not seem possible to parallel the Lorentz-invariant derivation any further from this point. The next step would be to isolate $C^c{}_{ab}$ from (53), to obtain an analog of (30). However, this would lead to terms involving $\mathfrak{k}^{ab}$ and $\mathfrak{l}^{ab}$ (arising from the third and fourth sets of terms on the right-hand side of (53), respectively) that do not have an analog in the linear equations of motion. Moreover, the process of isolating $C^c{}_{ab}$ from (53) in the present case would involve raising and lowering indices with the metric $\tilde{\eta}^{ab}$, rather than the fiducial metric η^{ab}. The terms that enter into the derivative portion of the stress-energy σ_{ab}, however, have their indices raised and lowered with η^{ab}. There do not appear to be any fortuitous cancellations in all of these extra terms. It seems that the equations of motion for the non-linear model (39) cannot easily be interpreted as 'the linear field $\mathfrak{h}^{ab}$ coupled to its own stress-energy.'

The presence of a fixed background vector field A_a on the underlying flat spacetime means that the equations of motion will not have diffeomorphism invariance; the fixed background vector field explicit breaks this symmetry. While some recent work has explored the possibilities of Lorentz symmetry violation via an explicitly breaking of diffeomorphism invariance [15], a more common tactic is to restore diffeomorphism invariance to these equations by 'promoting' the background vector field A_a to a dynamical field [16]. In particular, this ensures that so long as the equations of motion for A_a are satisfied, the stress-energy of associated with A_a will be conserved (as we assumed at the outset.)

The simplest way to do this is by assuming that A_a is governed by the action

$$\mathcal{L}_A = -\frac{1}{4}\eta^{ac}\eta^{bd}F_{ab}F_{cd} - V(A_aA_b\eta^{ab}) \tag{56}$$

in flat spacetime, where $V(A_aA_b\eta^{ab})$ is a Higgs-like potential energy for A_a that is responsible for the breaking of Lorentz symmetry. When we apply the bootstrap procedure, this will simply replace the inverse metric density η^{ab} with $\mathfrak{g}^{ab} = \eta^{ab} + \mathfrak{h}^{ab}$ throughout; the result will then be

$$\begin{aligned}\mathcal{L}_A &= -\frac{1}{4}\sqrt{-\mathfrak{g}}\mathfrak{g}^{ac}\mathfrak{g}^{bd}F_{ab}F_{cd} - \frac{1}{\sqrt{-\mathfrak{g}}}V\left(\sqrt{-\mathfrak{g}}\mathfrak{g}^{ab}A_aA_b\right)\\ &= \sqrt{-g}\left[-\frac{1}{4}g^{ac}g^{bd}F_{ab}F_{cd} - V(A_aA_bg^{ab})\right].\end{aligned} \tag{57}$$

3.3. Formalisms and frames

3.3.1. Palatini & metric formalisms. All told, our bootstrapped Lorentz-violating action is

$$\begin{aligned}\mathcal{L} = {}& 2\kappa\tilde{\mathfrak{g}}^{ab}\left[\partial_{[c}C^c{}_{b]a} + C^c{}_{a[b}C^d{}_{d]c}\right]\\ &- \frac{1}{4}\sqrt{-\mathfrak{g}}\mathfrak{g}^{ac}\mathfrak{g}^{bd}F_{ab}F_{cd} - \frac{1}{\sqrt{-\mathfrak{g}}}V\left(\sqrt{-\mathfrak{g}}\mathfrak{g}^{ab}A_aA_b\right) + \mathcal{L}_{\mathrm{mat}}.\end{aligned} \tag{58}$$

In terms of the metrics $\tilde{g}^{ab}$ and g^{ab}, this becomes

$$\begin{aligned}\mathcal{L} = {}& 2\kappa\sqrt{-\tilde{g}}\tilde{g}^{ab}\left[\partial_{[c}C^c{}_{b]a} + C^c{}_{a[b}C^d{}_{d]c}\right]\\ &+ \sqrt{-g}\left[-\frac{1}{4}g^{ac}g^{bd}F_{ab}F_{cd} - V(A_aA_bg^{ab})\right] + \mathcal{L}_{\mathrm{mat}}.\end{aligned} \tag{59}$$

Thus far, we have effectively been using a 'Palatini' (first-order) form of the gravitational action; the resulting equations of motion are given by (48). However, one can also obtain the same equations of motion in a more familiar 'metric' (second-order) formalism, where the connection is viewed as a function of the metric rather than as an independent field:

$$\mathcal{L} = \kappa\sqrt{-\tilde{g}}\tilde{g}^{ab}\tilde{R}_{ab} + \sqrt{-g}\left[-\frac{1}{4}g^{ac}g^{bd}F_{ab}F_{cd} - V(A_aA_bg^{ab})\right] + \mathcal{L}_{\mathrm{mat}}. \tag{60}$$

To see that these are equivalent, it suffices to vary the gravitational portion of the action. We can view g^{ab} and A_a as our fundamental fields, with variations δg^{ab} and δA_a. The gravitational metric $\tilde{g}^{ab}$ will then change by $\delta\tilde{g}^{ab}$ under these variations. Thus, the variation of the gravitational portion of the action (60) is

$$\begin{aligned}\int d^4x\,\delta(\sqrt{-\tilde{g}}\tilde{g}^{ab}\tilde{R}_{ab}) &= \int d^4x\left[\delta(\sqrt{-\tilde{g}}\tilde{g}^{ab})\tilde{R}_{ab} + \sqrt{-\tilde{g}}\tilde{g}^{ab}\delta(\tilde{R}_{ab})\right]\\ &= \int d^4x\left\{\delta(\sqrt{-\tilde{g}}\tilde{g}^{ab})\tilde{R}_{ab} + \sqrt{-\tilde{g}}\tilde{\nabla}_a\left[\left(-\tilde{g}^{ab}\tilde{g}^{cd} + \tilde{g}^{ad}\tilde{g}^{bc}\right)\tilde{\nabla}_b\delta\tilde{g}_{cd}\right]\right\},\end{aligned} \tag{61}$$

where $\tilde{\nabla}_a$ is the covariant derivative defined by $\tilde{\nabla}_a\tilde{g}_{bc} = 0$. The second term can be seen to be a total derivative, and so it will not contribute to the equations of motion. Moreover, from (49) and (51) it can be seen that

$$\sqrt{-\tilde{g}}\tilde{g}^{ab} = \sqrt{-g}\left(g^{ab} + \xi A^aA^b\right), \tag{62}$$

where $A^a \equiv g^{ab} A_b$. Thus,

$$\delta(\sqrt{-\tilde{g}}\tilde{g}^{ab})\tilde{R}_{ab} = \sqrt{-g}\left[\delta g^{ab} + 2\xi\left(\delta g^{ac}A^{bd}A_c + A^a g^{bd}\delta A_d\right) \right. \\ \left. - \frac{1}{2}g_{cd}\delta g^{cd}\left(g^{ab} + \xi A^a A^b\right)\right]\tilde{R}_{ab}. \tag{63}$$

The equation of motion arising from the variation of the metric g^{ab} in this formalism is thus

$$\tilde{R}_{ab} - \frac{1}{2}g_{ab}g^{cd}\tilde{R}_{cd} + 2\xi A_{(a}\tilde{R}_{b)c}A^c - \frac{1}{2}\xi g_{ab}A^c A^d \tilde{R}_{cd} = 8\pi G T_{ab}, \tag{64}$$

where T_{ab} here includes the stress-energy contributions of both the vector field A_a and any other matter sources present; in particular, we have for the vector field

$$T_{ab} = F_a{}^c F_{bc} - \frac{1}{4}g_{ab}F_{cd}F^{cd} + g_{ab}V(A^2) + 2A_a A_b V'(A^2), \tag{65}$$

where $A^2 \equiv g^{ab}A_a A_b$, and all indices have been raised and lowered with g^{ab}. Equation (64) can be seen to be equivalent to (48) by taking a contraction of that equation with g^{ab}. (For a more general discussion of when the Palatini and metric formalisms are equivalent, see [17, 18].)

The equation of motion arising from the variation of A_a in (60), meanwhile, is

$$\nabla_b F^{ba} - V'(A^2)A^a + 2\xi A^b \tilde{R}_b{}^a = 0, \tag{66}$$

where all indices have been raised and lowered with g^{ab}.

Finally, it should be noted that the Ricci tensor associated with the gravitational metric $\tilde{g}^{ab}$ is related to the Ricci tensor of the fiducial metric by

$$\tilde{R}_{ab} = R_{ab} + \nabla_c C^c{}_{ab} - \nabla_a C^c{}_{cb} + C^c{}_{cd}C^d{}_{ab} - C^c{}_{\mathrm{d}a}C^d{}_{cb}, \tag{67}$$

where

$$C^c{}_{ab} \equiv \frac{1}{2}\tilde{g}^{cd}\left[\nabla_a \tilde{g}_{bd} + \nabla_b \tilde{g}_{ad} - \nabla_d \tilde{g}_{ab}\right]. \tag{68}$$

These relationships would allow the equations of motion (64) and (66) to be fully expressed in terms of the variables g_{ab} and A_a. However, the relationships between $\tilde{g}_{ab}$ and g_{ab}, as given in equations (49) and (50), result in expressions that are rather complicated, and so we will not exhibit them explicitly here.

3.3.2. Jordan & Einstein frames. In taking the equations of motion from the action (60), we can make a choice of which fields we view as fundamental. In particular, we can either choose to view g^{ab} or $\tilde{g}^{ab}$ as the fundamental metric in the theory, and vary the action with respect to one or the other along with the vector field A_a. Since the relationship between the sets of field variables $\{g^{ab}, A_a\}$ and $\{\tilde{g}^{ab}, A_a\}$ is invertible, we will obtain equivalent sets of equations of motion with either choice. Using terminology from scalar-tensor gravity theories, viewing g^{ab} as the fundamental field (as was done in the previous subsection) corresponds to working in the *Jordan frame*, while viewing $\tilde{g}^{ab}$ as the fundamental field corresponds to working in the *Einstein frame* [19].

When working in the Einstein frame, the variation of the gravitational portion of the action is straightforward and familiar. However, the variation of the vector portion of the action (as well as any other matter sources that might be present) is complicated by the fact that they depend on the fiducial metric g^{ab} rather than the gravitational metric $\tilde{g}^{ab}$. To derive the equations of motion in the Einstein frame, we begin by contracting (49) with $A_a A_b$, yielding

$$\tilde{A}^2 \equiv \tilde{g}^{ab} A_a A_b = A^2 \sqrt{1 + \xi A^2}. \tag{69}$$

We can then invert (49) and (50) to obtain

$$g^{bc} = \sqrt{1 + \xi A^2} \tilde{g}^{bc} - \frac{\xi}{1 + \xi A^2} \tilde{g}^{be} \tilde{g}^{cf} A_e A_f. \tag{70}$$

Note that (69) could in principle be inverted to yield a closed-form expression for A^2 in terms of $\tilde{A}^2$; but this involves taking the root of a cubic polynomial, leading to complicated expressions. Instead, we can simply view A^2 as an function of $\tilde{g}^{ab}$ and A_a, defined implicitly by (69). In particular, varying both sides of (69) with respect to $\tilde{g}^{ab}$ yields the relation

$$\mathcal{N}_{ab} = \frac{\delta(A^2)}{\delta \tilde{g}^{ab}} = \frac{\sqrt{1 + \xi A^2}}{1 + \frac{3}{2}\xi A^2} A_a A_b. \tag{71}$$

With all of this in hand, we can write out the full equations of motion for this model. When we vary A_a, we obtain

$$\begin{aligned}
(\mathcal{E}_A)^a &\equiv \frac{1}{\sqrt{-\tilde{g}}} \frac{\delta \mathcal{L}}{\delta A_a} \\
&= \tilde{\nabla}_b \left(\sqrt{\mathcal{Q}} g^{bc} g^{ad} F_{cd} \right) - \sqrt{\mathcal{Q}} \mathcal{M}^{abc} g^{de} F_{bd} F_{ce} \\
&\quad - \frac{g^{ab} A_b}{\mathcal{Q}^{3/2}} \left(-\frac{\xi}{4} g^{ac} g^{bd} F_{ab} F_{cd} - \xi V(A^2) + 2\mathcal{Q} V'(A^2) \right),
\end{aligned} \tag{72}$$

where we have defined $\mathcal{Q} \equiv 1 + \xi A^2$ and

$$\mathcal{M}^{abd} = \frac{1}{2} \frac{\delta g^{bc}}{\delta A_a} = \frac{\xi}{\mathcal{Q}} \left(-\tilde{A}^{(b} g^{c)a} + \frac{1}{\mathcal{Q}} \tilde{A}^b \tilde{A}^c g^{ad} A_d \right) \tag{73}$$

with $\tilde{A}^a \equiv \tilde{g}^{ab} A_b$. The equation of motion obtained when we vary $\tilde{g}^{ab}$, meanwhile, is

$$\begin{aligned}
(\mathcal{E}_{\tilde{g}})_{ab} &\equiv \frac{1}{\sqrt{-\tilde{g}}} \frac{\delta \mathcal{L}}{\delta \tilde{g}^{ab}} \\
&= \kappa \tilde{G}_{ab} - \frac{1}{2} \sqrt{\mathcal{Q}} F_{ac} F_{bd} \tilde{g}^{cd} + \frac{\xi}{\mathcal{Q}} \left(A_{(a} F_{b)c} F_{de} \tilde{g}^{ce} \tilde{A}^d + \frac{1}{2} \tilde{A}^c \tilde{A}^d F_{c(a} F_{b)d} \right) \\
&\quad - \frac{1}{2} \tilde{g}_{ab} \left(-\frac{1}{4} g^{ac} g^{bd} F_{ab} F_{cd} - V(A^2) \right) \\
&\quad + \frac{1}{\sqrt{\mathcal{Q}}} \left(-\frac{\xi}{8} F_{ab} \tilde{F}^{ab} - \frac{\xi}{2\mathcal{Q}^{3/2}} \tilde{A}^a \tilde{A}^c \tilde{g}^{bd} F_{ab} F_{cd} + \frac{\xi}{2\mathcal{Q}} V(A^2) - V'(A^2) \right) \mathcal{N}_{ab},
\end{aligned} \tag{74}$$

with $\mathcal{N}_{ab}$ defined as in (71) and $\tilde{F}^{ab} \equiv \tilde{g}^{ac} \tilde{g}^{bd} F_{bd}$.

4. Applications & connections

4.1. SME coefficients

The primary motivation of this work was to extend a Lorentz-violating model of linearized gravity [4] to a fully non-linear model. Still, this model should still have a linearized limit, and should be able to make predictions about the behavior of objects moving under the influence of gravity in (for example) the solar system. Within the SME, a substantial machinery has been developed to address such questions. In the gravity sector [4], the observational effects will be parametrized by a set of ten coefficients: a scalar $\bar{u}$ and a trace-free tensor $\bar{s}^{ab}$. (If one allows higher-order equations of motion, the situation is more complicated [6]. However, since we required in section 2.2 that the equations of motion only be second-order in derivatives of the metric, such effects will not be present in this model.)

In general, to find the SME coefficients for a general gravitational model, one must take the Euler–Lagrange equations, linearize them, and combine them to yield an effective equation for the linearized Ricci tensor. In the process, one can typically only work to first order in the parameter which controls Lorentz violation (ξ in the present case); there are often terms of order ξ^2 which are discarded in the process, under the assumption that they will be negligible. It is therefore legitimate to expand (60) to $\mathcal{O}(\xi)$, with the understanding that the effective SME gravity equation will only be accurate to this order in any event. More pragmatically, we will see that the action simplifies greatly in this limit.

The Ricci tensor corresponding to $\tilde{g}^{ab}$ is given by (67) and (68). We have, to $\mathcal{O}(\xi)$,

$$\tilde{g}_{ab} = g_{ab} + \xi\left(\frac{1}{2}A^2 g_{ab} - A_a A_b\right) + \mathcal{O}(\xi^2), \tag{75}$$

and so

$$\mathcal{C}^c{}_{ab} = \frac{\xi}{2}\left[\delta^c{}_{(a}\nabla_{b)}\left(A^2\right) - 2\nabla_{(a}\left(A_{b)}A^c\right) - \frac{1}{2}g_{ab}\nabla^c\left(A^2\right) + \nabla^c\left(A_a A_b\right)\right] + \mathcal{O}(\xi^2). \tag{76}$$

In other words, $\mathcal{C}^c{}_{ab}$ is $\mathcal{O}(\xi)$, and so the gravitational portion of the action is

$$\sqrt{-\tilde{g}}\tilde{g}^{ab}\tilde{R}_{ab} = \sqrt{-g}\left[R + \xi A^a A^b R_{ab} + \nabla_a\left(g^{bc}\mathcal{C}^a{}_{bc} - g^{ab}\mathcal{C}^c{}_{cb}\right)\right] + \mathcal{O}(\xi^2). \tag{77}$$

The terms involving the derivatives of $\mathcal{C}^c{}_{ab}$ are total derivatives and so will not contribute to the equations of motion, while the terms quadratic in $\mathcal{C}^c{}_{ab}$ are higher-order in ξ.

Since we can ignore these terms, and using the relation (62) derived earlier, the non-linear action is

$$\mathcal{L} \approx \sqrt{-g}\left[\kappa(R + \xi A^a A^b R_{ab}) - \frac{1}{4}g^{ac}g^{bd}F_{ab}F_{cd} - V(A^2)\right] \tag{78}$$

to this order in ξ. This can be recognized as the action for the so-called *bumblebee model* [16]. The SME analysis for this equation was carried out in [4], with the results

$$\bar{s}^{\mu\nu} = \xi\left(A^\mu A^\nu - \frac{1}{4}\eta^{\mu\nu}A^2\right), \qquad \bar{u} = -\frac{1}{12}\xi A^2. \tag{79}$$

The bootstrapped Lorentz-violating model (60) will therefore have these same SME coefficients when we look at linearized solutions about a background where the matter metric is Minkowski ($g^{ab} = \eta^{ab}$) and the vector field A_a is constant.

The components of the tensor $\bar{s}^{\mu\nu}$ in the Sun-centered Frame [20] have been bounded, directly or indirectly, by a variety of experiments [3]. For experiments within the Solar System, the magnitudes of these components are 10^{-5} for $\bar{s}^{TT}$, 10^{-8} for $\bar{s}^{TI}$ ($I \in \{X, Y, Z\}$), and 10^{-10} for $\bar{s}^{IJ}$; these bounds come from a combination of precision gravimetry [21] and lunar laser ranging measurements [22]. These constraints can be viewed as bounding various combinations of the coupling constant ξ and the components of the vector field A_μ in the current cosmological environment.

More stringent bounds on the components $\bar{s}^{\mu\nu}$, down to the 10^{-14} level, have also been inferred from sources outside the solar system, from observations of cosmic rays [5]. While these latter bounds are indirect, requiring some assumptions about the origin and nature of cosmic rays, they do imply that ξ and/or the components A^μ must be quite small in the current epoch. The gravitational wave event GW170817, which was observed in conjunction with a gamma-ray burst, also bounded certain combinations of the $\bar{s}^{\mu\nu}$ components to the 10^{-15} level [23]. While this bound is direct, it is worth noting that a single event such as GW170817 only bounds the difference between the speeds of electromagnetic and gravitational waves for one particular direction of propagation, and thus only places bounds on a single combination of the components $\bar{s}^{\mu\nu}$. At the present time, the region of parameter space consistent with these observations is still unbounded; but as LIGO sees more such events, we would expect this region to become rather stringently bounded.

4.2. Generalized Proca theory

The bootstrapped Lorentz-violating model (60) can be connected to *generalized Proca theory* [24, 25]. Such models were constructed as a Galileon-like generalization of Proca theory, and generically include derivative self-interactions. The general form of the kinetic terms for the vector field in a generalized Proca theory is

$$\mathcal{L}_K = -\frac{1}{4}F_{ab}F^{ab} + \sum_{n=2}^{6}\beta_n \mathcal{L}_n, \tag{80}$$

where the β_i's are arbitrary coefficients, and $\mathcal{L}_2$ is an arbitrary algebraic function G_2 of F_{ab} and A^a. The terms $\mathcal{L}_i$ ($3 \leqslant i \leqslant 6$) depend on the symmetric part $\nabla_{(a}A_{b)}$ of the derivative of the vector field and on arbitrary algebraic functions G_3 through G_6. The precise form of these terms can be found in the above references; however, we will see shortly that these terms vanish in the present case.

To cast bootstrapped Lorentz-violating gravity into the above form, we can rewrite (60) in the Einstein frame using (49) and (51). The result is

$$\begin{aligned}\mathcal{L} = \kappa\sqrt{-\tilde{g}}\Bigg[&\tilde{g}^{ab}\tilde{R}_{ab} - \frac{1}{4}\sqrt{Q}\tilde{g}^{ac}\tilde{g}^{bd}F_{ab}F_{cd} \\ &+ \frac{1}{2}\frac{\xi}{Q}\tilde{A}^a\tilde{A}^c\tilde{g}^{bd}F_{ab}F_{cd} - V(A^2)\Bigg] + \mathcal{L}_{\text{mat}}[g].\end{aligned} \tag{81}$$

The kinetic term for A_a can clearly be seen to be a function of F_{ab} and A^a only; in other words, we have

$$\mathcal{L}_2 = -\frac{\sqrt{Q}-1}{4}\tilde{g}^{ac}\tilde{g}^{bd}F_{ab}F_{cd} + \frac{1}{2}\frac{\xi}{Q}\tilde{A}^a\tilde{A}^c\tilde{g}^{bd}F_{ab}F_{cd}, \tag{82}$$

and the remaining terms in (80) vanish. Since this model is a special case of generalized Proca theory, this implies that the bootstrapped Lorentz-violating model has the same desirable properties as generalized Proca theory; in particular, it is free of ghost instabilities and has only three propagating degrees of freedom associated with the vector field.

Finally, I will make two related observations concerning this connection. First, in most work involving generalized Proca theory, it is assumed that the 'matter sector' couples minimally to the gravitational metric. This is not the case here; to put the action in the form (80), it was necessary to work in the Einstein frame. This means that the cosmological solutions found in [26] (and similar works) would not necessarily be solutions of the current model, since the matter couples to the gravitational fields differently. This is an example of an observation made (and exploited) in a recent work by Gümrükçüoğlu and Koyama [27]: while it is possible to find equivalent descriptions of a 'pure gravity' action in terms of different frames, the choice of coupling between 'conventional matter' and the gravitational fields can break this equivalency.

Second, consider again this model in the Jordan frame. While the action (60) would be rather complicated if written entirely in terms of g_{ab}, ∇_a, and A_a, it is obvious that it would not fit naturally into the class of theories described in [24, 25]. In particular, the resulting action in our current model would contain a term of the form $A^a A^b R_{ab}$, which is not included in any of the $\mathcal{L}_i$ terms in the generalized Proca Lagrangian. But we know that (at least in the absence of matter) this model is equivalent to a special case of generalized Proca theory. This suggests that there may be more models having the desirable properties of the 'original' generalized Proca theories [24, 25] that have not yet been described. Such models could be obtained using similar techniques to those described in [27]: a change of frame in the gravitational sector, followed by a minimal coupling between the 'new' metric and conventional matter.

4.3. FRW cosmology

As a simple illustration of a non-linear solution of this model, we ask what a dark-energy-dominated FRW spacetime would look like in this case. If we assume that our solution is spatially homogenous and isotropic, the form of the gravitational metric must be of the standard FRW form

$$d\tilde{s}^2 = \tilde{g}^{\mu\nu} \mathrm{d}x_\mu \mathrm{d}x_\nu = -\mathrm{d}t^2 + a^2(t) d\Sigma^2, \tag{83}$$

where $\mathrm{d}\Sigma^2$ is the metric on surfaces of constant t, which are assumed to be maximally symmetric (S^3, $\mathbb{R}^3$, or H^3.) The vector field A_a, meanwhile, must simply be

$$A_a = A_t (\mathrm{d}t)_a \tag{84}$$

in order to respect the symmetry of the solution. This later condition is quite restrictive, since it implies that $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} = 0$. The equation of motion (72) then simplifies drastically:

$$\frac{1}{\mathcal{Q}^{3/2}} \left(\xi V(A^2) - 2\mathcal{Q} V'(A^2) \right) = -2 \frac{d}{d(A^2)} \left[\frac{V(A^2)}{\sqrt{\mathcal{Q}}} \right] = 0. \tag{85}$$

In other words, the norm of the vector field A, as measured with respect to the physical metric g^{ab}, is not found at the minimum of the potential $V(A^2)$, but instead at the minimum of an effective potential defined by

$$V_{\mathrm{eff}}(A^2) = \frac{V(A^2)}{\sqrt{1 + \xi A^2}}. \tag{86}$$

We can define b^2 such that $V'(-b^2) = 0$, and $\bar{b}^2$ such that $V'_{\mathrm{eff}}(-\bar{b}^2) = 0$. Note that in general, these two quantities will differ. For example, suppose the potential is of the form $V(A^2) = \frac{\beta}{4}(A^2 + b^2)^2 + \Lambda$: a 'Higgs-like' potential plus a cosmological constant term. It is then the case that

$$-\bar{b}^2 = \frac{1}{3\xi} \left[2\sqrt{(1 - b^2\xi)^2 + \frac{3\Lambda\xi^2}{\beta}} - b^2\xi - 2 \right] = -b^2 + \frac{\xi\Lambda}{\beta} + \mathcal{O}(\xi^2). \tag{87}$$

The equation of motion for the gravitational metric $\tilde{g}^{ab}$ is then simply

$$\tilde{G}_{ab} + 8\pi G \tilde{\Lambda} \tilde{g}_{ab} = 0 \tag{88}$$

where $\tilde{\Lambda} \equiv V(-\bar{b}^2)$. This implies that the gravitational metric is de Sitter, anti-de Sitter, or Minkowski, depending on the sign of $\tilde{\Lambda}$ and the Gaussian curvature $\tilde{k}$ of the spatial hypersurfaces; the scale factor $a(t)$ will simply obey the Friedmann equation

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 - \frac{8\pi G\tilde{\Lambda}}{3}a^2 = -\tilde{k}. \tag{89}$$

To find the matter metric g^{ab}—which is, after all, what would be measurable via observations of 'normal matter' in such a Universe—we first note that since $A^2 = g^{ab}A_aA_b = -\bar{b}^2$, we have

$$A_t = \frac{\bar{b}}{\sqrt{-g^{tt}}}. \tag{90}$$

Recalling (49), this implies that

$$\begin{aligned} \tilde{g}^{\mu\nu}\mathrm{d}x_\mu\mathrm{d}x_\nu &= \frac{1}{\sqrt{1-\xi\bar{b}^2}}\left(g^{\mu\nu} + \xi g^{\mu\rho}g^{\nu\sigma}A_\rho A_\sigma\right)\mathrm{d}x_\mu\mathrm{d}x_\nu \\ &= \sqrt{1-\xi\bar{b}^2}g^{tt}\mathrm{d}t^2 + \frac{1}{\sqrt{1-\xi\bar{b}^2}}g^{ij}(x^\mu)\mathrm{d}x_i\mathrm{d}x_j. \end{aligned} \tag{91}$$

Comparing this to (83), we conclude that

$$\mathrm{d}s^2 = g^{\mu\nu}\mathrm{d}x_\mu\mathrm{d}x_\nu = -\frac{\mathrm{d}t^2}{\sqrt{1-\xi\bar{b}^2}} + \sqrt{1-\xi\bar{b}^2}a^2(t)\mathrm{d}\Sigma^2. \tag{92}$$

The constant in front of the spatial part of the metric can be absorbed into $a(t)$, and we can rescale our time coordinate $\bar{t} = t/(1-\xi\bar{b}^2)^{1/4}$. In terms of this coordinate, the Friedmann equation (89) becomes

$$\left(\frac{\mathrm{d}a}{\mathrm{d}\bar{t}}\right)^2 - \frac{8\pi G\sqrt{1-\xi\bar{b}^2}\tilde{\Lambda}}{3}a^2 = -\tilde{k}\sqrt{1-\xi\bar{b}^2}. \tag{93}$$

In other words, the matter metric is (like the gravitational metric) de Sitter, anti-de Sitter, or Minkowski; however, the measurable values of the cosmological constant $\bar{\Lambda}$ and the Gaussian curvature $\bar{k}$ would be rescaled:

$$\bar{\Lambda} = \tilde{\Lambda}\sqrt{1-\xi\bar{b}^2}, \qquad \bar{k} = \tilde{k}\sqrt{1-\xi\bar{b}^2}. \tag{94}$$

The effects of this model would therefore not be distinguishable from a conventional FRW cosmology with $\Lambda \neq 0$; the net effect of the non-trivial couplings between A_a and the metric in (60) is simply to rescale the 'bare' values of the cosmological constant and the spatial curvature.

The lack of directly observable effects in this simplistic spacetime does not necessarily imply that physically meaningful effects do not exist in other circumstances. The spatial isotropy of this spacetime makes it a particularly poor test bed for the effects of a fundamental non-zero vector field, since it implies that the field strength vanishes identically. A more reasonable assumption in the context of Lorentz symmetry violation would be a spacetime that is homogeneous but anisotropic, with a local two-dimensional rotational symmetry at every point corresponding to rotations keeping the spatial part of A_a fixed. Metrics with this symmetry structure have been previously examined in the context of perfect-fluid solutions [28–30], as well as in the context of more recent vector-tensor models [26, 31]. In the Lorentz-violating bootstrap model, this would lead to non-trivial dynamics for A_a, since we would now have $F_{ab} \neq 0$; the equations of motion (72) and (74) would also become significantly more complicated. Work on the evolution of such spacetimes in this model is ongoing, and will be described in a future paper.

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References

[1] Will C M 2014 *Living Rev. Relativ.* **17** 4
[2] Colladay D and Kostelecký V A 1998 *Phys. Rev.* D **58** 116002
[3] Kostelecký V A and Russell N 2011 *Rev. Mod. Phys.* **83** 11–31 (Current data tables available at arxiv:0801.0287)
[4] Bailey Q and Kostelecký V A 2006 *Phys. Rev.* D **74** 045001
[5] Kostelecký V A and Tasson J D 2015 *Phys. Lett.* B **749** 551–9
[6] Kostelecký V A and Mewes M 2016 *Phys. Lett.* B **757** 510–4
[7] Bailey Q G 2016 *Phys. Rev.* D **94** 065029
[8] Kraichnan R H 1955 *Phys. Rev.* **98** 1118–22
[9] Gupta S N 1952 *Proc. Phys. Soc.* A **65** 608–19
[10] Thirring W 1961 *Ann. Phys., NY* **16** 96–117
[11] Bailey Q and Kostelecký V A 2004 *Phys. Rev.* D **70** 1–10
[12] Kostelecký V A and Tasson J D 2011 *Phys. Rev.* D **83** 016013
[13] Deser S 1970 *Gen. Relativ. Gravit.* **1** 9–18
[14] Kostelecký V A and Potting R 2009 *Phys. Rev.* D **79** 065018
[15] Bluhm R 2017 *Symmetry* **9** 230
[16] Kostelecký V A 2004 *Phys. Rev.* D **69** 105009
[17] Iglesias A, Kaloper N, Padilla A and Park M 2007 *Phys. Rev.* D **76** 104001
[18] Lanczos C 1957 *Rev. Mod. Phys.* **29** 337–50
[19] Flanagan É É 2004 *Class. Quantum Grav.* **21** 3817–29
[20] Bluhm R, Kostelecký V A, Lane C D and Russell N 2003 *Phys. Rev.* D **68** 125008
[21] Shao C G, Chen Y F, Sun R, Cao L S, Zhou M K, Hu Z K, Yu C and Müller H 2018 *Phys. Rev.* D **97** 24019
[22] Bourgoin A, Le Poncin-Lafitte C, Hees A, Bouquillon S, Francou G and Angonin M C 2017 *Phys. Rev. Lett.* **119**
[23] Abbott B P *et al* 2017 *Astrophys. J.* **848** L13
[24] Heisenberg L 2014 *J. Cosmol. Astropart. Phys.* JCAP05(2014)015
[25] Heisenberg L 2017 *Proc. 52nd Rencontres Moriond Gravitation (Moriond Gravitation 2017)* (arxiv:1705.05387)
[26] Heisenberg L, Kase R and Tsujikawa S 2016 *J. Cosmol. Astropart. Phys.* JCAP11(2016)008
[27] Gümrükçüolu A E and Koyama K 2019 *Phys. Rev.* D **99** 084004
[28] Kantowski R and Sachs R K 1966 *J. Math. Phys.* **7** 443
[29] Ellis G F R and MacCallum M A H 1969 *Commun. Math. Phys.* **12** 108–41
[30] King A R and Ellis G F R 1973 *Commun. Math. Phys.* **31** 209–42
[31] Artymowski M and Lalak Z 2012 *Phys. Lett.* B **707** 203–8