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SU(4) description of bilayer skyrmion-antiskyrmion pairs

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Abstract – The antiferromagnetic coupling and entanglement between skyrmion lattices are treated in magnetic bilayer systems. We first formulate the problem of large bilayer skyrmions using the $\mathbb{CP}^1 \otimes \mathbb{CP}^1$-theory. We have considered bilayer skyrmions under the presence of Dzyaloshinskii-Moriya (DMI) and Zeeman interactions confined in a two-dimensional chiral magnet such as $Fe_{0.5}Co_{0.5}Si$. We parametrize bilayer skyrmions using the $SU(4)$ representation, and represent each skyrmion and antiskyrmion using the Schmidt decomposition. The reduced density matrices for skyrmion and antiskyrmion have been calculated. The conditions for maximal, partial entanglement and separable bilayer skyrmions are presented. Our results can be used for generating entanglement in systems with a large number of spins.

Introduction. – Magnetic skyrmions are microscopic topological defects in spin textures that are characterized by the charge [1]

$$Q = \frac{1}{4\pi} \int \mathrm{d}^2\mathbf{r}\, \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x} \times \frac{\partial \mathbf{n}}{\partial y} \right) = \pm 1. \qquad (1)$$

In the mathematical literature, Q is a topologically invariant quantity known as Pontryagin number. It counts how many times $\mathbf{n}(\mathbf{r}) = \mathbf{n}(x, y)$ wraps the unit sphere [2]. Skyrmions were first introduced by Skyrme [3] to explain hadrons in nuclei. Interestingly they have turned out to be relevant in other condensed matter systems such as chiral magnets [4]. Theoretically, magnetic skyrmions were introduced and investigated by Bogdanov and his collaborators in [5,6]. Skyrmions can be driven by charge or spin currents in confined geometries [7]. In general, skyrmions are subject to the skyrmion Hall effect (SkHE) caused by the Magnus force. SkHE was predicted theoretically in [8] and has been observed experimentally [9]. The Magnus force is the force acting transversely to the skyrmion velocity in the medium and can be interpreted as a manifestation of the real-space Berry phase [10,11].

SkHE is a detrimental effect since the skyrmions experiencing it will deviate from following a straight path. As a result, moving skyrmions can be damaged or even destroyed at the edges of a thin film sample. One way of suppressing SkHE is to consider two perpendicular chiral thin films strongly coupled via antiferromagnetic (AFM) exchange coupling. It is expected that when skyrmion lattice is formed at the bottom thin film, simultaneously another skyrmion lattice is created at the top thin film with opposite topological charge. In this case, the SkHE vanishes since the Magnus force acting on the top skyrmion (antiskyrmion) is equal to the Magnus force that acts on the bottom antiskyrmion (skyrmion) with opposite sign leaving us with zero net force. An analogous scheme was proposed to suppress SkHE in nanoscale Néel skyrmions by considering two perpendicular ferromagnetic films separated by an insulator with a heavy metal underneath the second ferromagnetic film [12].

Quantum signatures for large skyrmions can emerge at the phase boundary between skyrmion crystal phase (SkX) and ferromagnetic phase at zero temperature like skyrmions in $Fe_{0.5}Co_{0.5}Si$. During this phase transition a quantum liquid phase is expected to emerge [13]. In this case, the classical LLG [14] and Thiele equation [15] break down due to quantum fluctuations. The full quantum theory of bilayer skyrmions is out of the scope of this work and it can be recovered under some circumstances. As an example, for sufficiently weak antiferromagnetic exchange coupling between thin films, bilayer skyrmion (antiferromagnetically coupled skyrmion-antiskyrmion pair) can be seen as two separate skyrmions and the quantum dynamics is already known for a single large skyrmion [13]. In this work, we give a detailed theory of large bilayer skyrmions (with sizes of the order of 100 nm) using the HDMZ (Heisenberg exchange + Dzyaloshinskii-Moriya interaction + Zeeman interaction) model. We study the problem of entanglement in large bilayer skyrmions from a general perspective using our developed continuum theory of bilayer skyrmions and $SU(4)$ representation. In the final section, we study the geometry of quantum states in bilayer skyrmions.

The $\mathbb{CP}^1 \otimes \mathbb{CP}^1$-theory of large bilayer skyrmions. – We consider two thin films of chiral magnets separated by an insulating spacer with antiferromagnetic coupling between chiral films. We assumed each film to host Bloch skyrmions under certain ranges of temperature and external magnetic field determined by the film parameters. Skyrmions in the first thin film are equal in size to skyrmions in the second thin film but with opposite topological charge. For our model to hold, we assume temperatures lower than the magnon gap and skyrmions with large radius [13]. Fortunately, skyrmions in $Fe_{0.5}Co_{0.5}Si$ support these assumptions [16]. We present a detailed theory of bilayer skyrmions written with respect to $\mathbb{CP}^1 \otimes \mathbb{CP}^1$-theory. The HDMZ Hamiltonian density for each chiral magnet layer is

$$\mathcal{H}_i = \frac{J}{2}\,(\partial_\mu \mathbf{n}^i)\cdot(\partial_\mu \mathbf{n}^i) + D\,\mathbf{n}^i\cdot(\nabla\times\mathbf{n}^i) - \mathbf{B}\cdot\mathbf{n}^i. \quad (2)$$

We adopted the Einstein summation notation for repeated indices $\partial_\mu \mathbf{n}\cdot\partial_\mu\mathbf{n} \equiv \Sigma_\mu \partial_\mu\mathbf{n}\cdot\partial_\mu\mathbf{n}$. Since we are interested in two-dimensional thin films $\mu = x, y$. The index $i = S, A$ labels the skyrmion and antiskyrmion respectively and $\mathbf{n}^i = (\sin\theta^i\cos\phi^i, \sin\theta^i\sin\phi^i, \cos\theta^i)^T$ gives the transpose magnetic moment unit written in the $O(3)$ representation with a unit modulus constraint $|\mathbf{n}^i|^2 = 1$. The first term in the Hamiltonian represents the exchange interaction with exchange constant J, the second term is the DMI term with D being the Dzyaloshinskii-Moriya (DM) vector constant. DMI term is a manifestation of chirality in the system since it has a vanishing value for centrosymmetric crystal structures. The last term is the Zeeman interaction. According to the Derrick-Hobart theorem, the Hamiltonian (2) supports the emergence of large skyrmions [17,18]. Suppose there exists a skyrmion solution $\mathbf{n}^0$ to the system. We compute each contribution in the energy functional as E^0_H, E^0_{DM} and E^0_Z, where H, DM and Z denote Heisenberg exchange, Dyzaloshinskii-Moriya and Zeeman terms. Now we consider the scaling $\mathbf{n} = \mathbf{n}^0(\lambda x)$. Substituting this scaled solution into each term in the energy functional gives

$$E(\lambda) = E^0_H - \lambda^{-1}|E^0_{DM}| + \lambda^{-2}E^0_Z. \quad (3)$$

This has a unique minimum point which could be found by the relation $\lambda = \frac{2E^0_Z}{|E^0_{DM}|}$. It is safe to choose $\lambda = 1$ for consistency throughout our argument. From eq. (3), it is not difficult to observe that the skyrmion is stabilized by the DMI term. When $\lambda\to\infty$, eq. (3) implies that a skyrmion shrinks to zero without the DMI term. The perpendicular magnetic anisotropy (PMA) term was ignored since such a term does not play an important role in $Fe_{0.5}Co_{0.5}Si$ [16]. The total energy is the spatial integral of $\mathcal{H}_i$: $H_i = \int d^2r\ \mathcal{H}_i$. The bilayer skyrmion can be described by the following Hamiltonian: $H_{tot} = H_S + H_A + H_{int}$. The term H_{int} is assumed to contain the AFM exchange coupling between the two chiral magnets,

$$H_{inter} = -J_{int}\int d^2x\ \mathbf{n}^{i=S}\cdot\mathbf{n}^{i=A}. \quad (4)$$

The AFM interaction term is responsible for the coupling between spin degrees of freedom in the skyrmion and spin degrees of freedom in the antiskyrmion. AFM-coupled spins are in opposite alignment with each others.

We will use a purely geometric approach in our investigation of quantum entanglement. Thus, it is more convenient to work in the equivalent $\mathbb{CP}^1$ formulation of the nonlinear sigma model NLσM [19,20]. This can be achieved using the Hopf map $\mathbf{n}^i = (\mathbf{z}^i)^\dagger\boldsymbol{\sigma}^i\mathbf{z}^i$. This mapping connects the classical object $\mathbf{n}^i$ with spinor $\mathbf{z}^i = \left[\begin{smallmatrix}\cos\frac{\theta^i}{2}\\ \sin\frac{\theta^i}{2}e^{i\phi^i}\end{smallmatrix}\right]$. The spinor $\mathbf{z}^i$ can be interpreted as the coherent-state wavefunction of spin-$\frac{1}{2}$ particles. The DMI term can be phrased in term of the spinor $\mathbf{z}_i$ as follows:

$$\begin{aligned}\mathbf{n}^i\cdot(\nabla\times\mathbf{n}^i) &= \sin\theta^i\cos\theta^i(\cos\phi_i\ \partial_x\phi^i + \sin\phi\ \partial_y\phi^i)\\ &\quad +(\sin\phi^i\ \partial_x\theta^i - \cos\phi^i\ \partial_y\theta^i - \sin^2\theta^i\partial_z\phi^i)\\ &= -2\mathbf{n}^i\cdot\mathbf{a}^i - i(\mathbf{z}^i)^\dagger(\boldsymbol{\sigma}^i\cdot\nabla)\,\mathbf{z}^i\\ &\quad +i(\nabla(\mathbf{z}^i)^\dagger)\cdot\ \boldsymbol{\sigma}^i\mathbf{z}^i. \end{aligned} \quad (5)$$

Equation (2) can be re-expressed in terms of the spinor $\mathbf{z}^i$ as

$$\begin{aligned}\mathcal{H}_i &= \frac{J}{2}\,(\partial_\mu\mathbf{n}^i)\cdot(\partial_\mu\mathbf{n}^{\ i}) + D\,\mathbf{n}^i\cdot(\nabla\times\mathbf{n}^i) - \mathbf{B}\cdot\mathbf{n}^i\\ &= 2J\left(\partial_\mu(\mathbf{z}^i)^\dagger + ia^i_\mu\ \mathbf{z}^\dagger_i - i\kappa\ (\mathbf{z}^i)^\dagger\sigma^i_\mu\right)\\ &\quad\left(\partial_\mu\mathbf{z}^i - ia^i_\mu\ \mathbf{z}^i + i\kappa\ \sigma^i_\mu\mathbf{z}^i\right) - \mathbf{B}\ (\mathbf{z}^i)^\dagger\sigma^i\mathbf{z}^i\\ &= 2J(D^i_\mu\mathbf{z}^i)^\dagger D^i_\mu\mathbf{z}^i - \mathbf{B}\ (\mathbf{z}^i)^\dagger\sigma^i\mathbf{z}^i, \end{aligned} \quad (6)$$

where $D^i_\mu = \partial_\mu - ia^i_\mu + i\kappa\sigma^i_\mu$ denotes the covariant derivative for the thin film i, $\kappa = \frac{D}{2J}$, and $a^i_\mu = -i(\mathbf{z}^i)^\dagger\partial_\mu\mathbf{z}^i$ is the emergent gauge field. Each magnetic layer carries a $\mathbb{CP}^1$- field which is responsible for the magnetization. The $\mathbb{CP}^1$-field is a two-component normalized vector with complex entries such that each field is being represented using the $SU(2)$ representation. The inclusion of the DMI term in the effective Hamiltonian (6) is done simply by adding a non-Abelian gauge field proportional to Pauli matrices σ_μ. The emergent gauge field a^i_μ is usually called the real-space Berry connection. It is synthesized by adiabatically varying the spin texture sufficiently slowly in time. The real-space Berry phase connection can give rise to the skyrmion Hall effect, unlike the momentum-space Berry connection which gives rise to the anomalous Hall effect [11]. Although the non-Abelian gauge field is non-dynamic (constant), it has an associated flux with it. The covariant derivative commutator gives the field tensor,

$$F^i_{\mu\nu} = i[D^i_\mu, D^i_\nu] = f^i_{\mu\nu} + 2\kappa^2\epsilon_{\mu\nu\lambda}\ \sigma^i_\lambda. \quad (7)$$

where the Abelian part of the flux is $f^i_{\mu\nu} = \partial_\mu a^i_\nu - \partial_\nu a^i_\mu$.

The two-dimensional emergent vector potential for a single magnetic skyrmion is

$$\mathbf{a}^i = -i(\mathbf{z}^i)^\dagger \nabla_2 \mathbf{z}^i = \frac{\hat{\phi}^i}{2r}(1-\cos\theta^i(r)) = \frac{\hat{\phi}^i}{r}\sin^2\frac{\theta^i}{2}. \quad (8)$$

Since $\hat{\phi}^i = (\sin\phi^i, \cos\phi^i, 0)$, the gauge field $\mathbf{a}^i$ turns out to be a two-dimensional object. The magnetic flux originating from this gauge field is

$$\nabla_2 \times \mathbf{a}^i = \frac{1}{2r}\sin\theta^i(r)\ (\theta^i)'(r), \quad (9)$$

where $\nabla_2 \equiv (\partial_x, \partial_y, 0)$.

The local spin orientation (θ^i, ϕ^i) is related to the local coordinate system of a single skyrmion (r, φ) such that $\theta_i = \theta_i(r)$ and $\phi_i = \varphi_i - \frac{\pi}{2}$. For the sake of simplicity, we assume $\mathbf{B} = B\hat{z} > 0$. The geometric considerations on skyrmions impose the following boundary conditions on θ_i: a) $\theta_i(\infty) = 0$ and b) $\theta_i(0) = \pi$. The total energy of a single skyrmion (antiskyrmion) reads [20]

$$E_{SK} = 4\pi J \int_0^\infty r\mathrm{d}r \Bigg[\left(\frac{1}{2}\frac{\mathrm{d}\theta_i}{\mathrm{d}r} + \kappa\right)^2 -\kappa^2 + \frac{\kappa}{r}\sin\theta_i\cos\theta_i + \frac{1}{4r^2}\sin^2\theta_i - \gamma(\cos\theta_i - 1)\Bigg], \quad (10)$$

where $\gamma = \frac{B}{2J}$. The total energy of the large bilayer skyrmion is

$$E_{tot} = E_{Sk}(\theta_S) + E_{Sk}(\theta_A) + E_{int}(\theta_S, \theta_A). \quad (11)$$

In the $\mathbb{CP}^1$-formulation, the AFM interaction term takes the form

$$E_{int} = -2\pi J_{int} \int_o^\infty r\mathrm{d}r \cos\theta_S \cdot \cos\theta_A. \quad (12)$$

For sufficient AFM interaction, we have the case where each spin in the first film is coupled with another opposite spin in the second film. This allows us to write $\theta_A = \pi - \theta_S$ and express the total energy functional (11) in terms of a single angle θ_S or θ_A. The total energy functional (11) simplifies for fixed values of DM interaction constants D, exchange couplings J and magnetic fields B in both skyrmion and its AFM-coupled antiskyrmion. It takes the following simple form:

$$\begin{aligned} E_{tot} &= 4\pi J \int_0^\infty r\mathrm{d}r \left[\left(\frac{1}{2}\frac{\mathrm{d}\theta^S}{\mathrm{d}r}\right)^2 + 2\gamma + \frac{1}{2r^2}\sin^2\theta^S \right] \\ &\quad + 2\pi J_{int}\int_0^\infty r\mathrm{d}r\cos^2\theta^S. \end{aligned} \quad (13)$$

Realistically, in order for eq. (13) to make sense, we have to introduce a hard cutoff r_{Sk} such that $\theta_{S,A}(r) = 0$ for $r \geq r_{Sk}$. Physically, r_{Sk} is a half-skyrmion distance in the skyrmion phase crystal or the size of skyrmion.

It is not difficult to show that our system has a vanishing skyrmion Hall effect. For each chiral magnet film, the Thiele equation can be written as [12,15]

$$\mathbf{F} = \mathbf{G} \times (\mathbf{v}_s - \dot{\mathbf{R}}) + \Gamma_{ij}\ (\beta\ \mathbf{v}_s - \alpha\ \dot{\mathbf{R}}), \quad (14)$$

where $\mathbf{v}_s$ denotes the velocity of spin polarized current, α is the Gilbert damping term, β is the non-adiabatic damping constant, $\mathbf{R}$ represents the center of mass coordinates, $\mathbf{G}$ and Γ_{ij} are the gyromagnetic vector and dissipative tensor respectively given by

$$\begin{aligned} G_i &= \varepsilon_{ijk} \int \mathrm{d}^2\mathbf{r}\ (\mathbf{n}, \partial_i\mathbf{n}, \partial_j\mathbf{n}), \\ \Gamma_{ij} &= \int \mathrm{d}^2\mathbf{r}\ \partial_i\mathbf{n}\ \partial_j\mathbf{n}. \end{aligned} \quad (15)$$

We introduced the non-adiabatic spin transfer torque with parameter β in eq. (14) to account for small dissipative forces that break the conservation of spin in the spin-transfer process. Since we have considered an external magnetic field parallel to the z-direction and an in-plane spin polarized current, by symmetry considerations, the dissipation tensor has the following simple form:

$$\mathbf{\Gamma} = \Gamma \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (16)$$

and the gyromagnetic vector takes the form

$$\mathbf{G} = 4\pi Q \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (17)$$

Note that $\mathbf{F}$ in eq. (14) vanishes since the HDMZ action is translationally invariant $\mathbf{r} \to \mathbf{r} + \delta\mathbf{r}$ provided that the Zeeman field is uniform. Thus, we obtain the following coupled equations:

$$\begin{pmatrix} \alpha\Gamma & -4\pi Q \\ 4\pi Q & \alpha\Gamma \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \beta\Gamma & -4\pi Q \\ 4\pi Q & \beta\Gamma \end{pmatrix}. \quad (18)$$

This system is non-singular and always has a solution of the form

$$\begin{aligned} \mathbf{v}_{Sk} &= \dot{\mathbf{R}} = (\dot{X}, \dot{Y}) \\ &= \frac{\beta}{\alpha}\mathbf{v}_s + \frac{\alpha - \beta}{\alpha^3(\frac{\Gamma}{4\pi Q^2}) + \alpha}\left(\mathbf{v}_s + \frac{\alpha\Gamma}{4\pi Q}\hat{z} \times \mathbf{v}_s\right). \end{aligned} \quad (19)$$

From the last result, we observe that the skyrmion (antiskyrmion) velocity $\mathbf{v}_{Sk}$ is a combination of drag velocity $\mathbf{v}_s$ and Magnus term proportional to the topological charge Q. Since we dealt with two thin films of chiral magnets where skyrmions in one layer have opposite charge with respect to the other, *i.e.*, $Q_S = -Q_A$, this implies the vanishing of the Magnus term contribution for the whole system.

$SU(4)$ **parametrization of bilayer skyrmion.** – At the perfect coupling between skyrmion-antiskyrmion pairs such that each spin in the skyrmion is AFM-coupled to a spin in the antiskyrmion, the system skyrmion-antiskyrmion pair can be described by a four-component wave function. Thus we can represent the spin degrees of freedom in bilayer skyrmions using $SU(4)$ symmetry. The $SU(4)$ skyrmions were studied before in the multi-component quantum Hall system [21] and graphene [22]. It was found that skyrmions in these systems are stabilized mainly by the competition between Zeeman and Coulomb interactions, unlike skyrmions in chiral magnets. However, both skyrmions share the same topological properties in common regardless of the system details. Since we have AFM coupled skyrmion-antiskyrmion pairs, our system resembles the spin-pseudospin skyrmions in term of parametrization despite the fact that we now have two skyrmions instead of one. For large bilayer skyrmions, we consider the properties of $SU(2) \otimes SU(2)$ skyrmion-antiskyrmion pairs under the presence of DM and Zeeman interactions (HDMZ model). We do this from a perspective of entanglement between the spin degrees of freedom in the skyrmion and its AFM-coupled antiskyrmion. Because of the Zeeman interaction term, the full $SU(4)$ symmetry breaks down to $U(1) \otimes U(1)$ symmetry where each symmetry group corresponds to a rotation of spin in the skyrmion or antiskyrmion along the applied magnetic field direction (in our case, the z-direction). Thus, we have the symmetry breaking sequence $SU(4) \to SU(2) \otimes SU(2) \to U(1) \otimes U(1)$. Interestingly, the DMI term written in terms of spinors $\mathbf{z}^i$ preserves the full $SU(2) \otimes SU(2)$ symmetry. This is due to the embedding of the DMI term in the covariant derivative that acts on the spinor $\mathbf{z}^i$ as a non-dynamic term.

We parametrize the $SU(4)$ bilayer skyrmion using a Schmidt decomposition [23]. According to the Schmidt decomposition, every pure state in the Hilbert space $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$ can be written in the form

$$|\psi\rangle = \Sigma_{i=0}^{N-1} \lambda_i |e_i\rangle \otimes |f_i\rangle, \tag{20}$$

where $\{|e_i\rangle\}_{i=0}^{N_1-1}$ is an orthonormal basis for $\mathcal{H}_1$, $\{|f_i\rangle\}_{i=0}^{N_2-1}$ is an orthonormal basis for $\mathcal{H}_2$, $N \leq \min\{N_1, N_2\}$, and λ_i are non-negative real numbers such that $\Sigma_{i=0}^{N-1} \lambda_i^2 = 1$. Thus, we can express the wave function as

$$|\Psi(\mathbf{r})\rangle = \cos\frac{\alpha}{2} \mid \varphi^S\rangle \otimes |\varphi^A\rangle + \sin\frac{\alpha}{2} e^{i\beta} |\chi^S\rangle \otimes |\chi^A\rangle$$
$$= \begin{pmatrix} \cos\frac{\alpha}{2}\cos\frac{\theta^A}{2}\cos\frac{\theta^S}{2} + \sin\frac{\theta^A}{2}\sin\frac{\theta^S}{2} e^{i(\beta-\phi^A-\phi^S)} \\ \cos\frac{\alpha}{2}\sin\frac{\theta^A}{2}\cos\frac{\theta^S}{2} e^{i\phi^A} - \sin\frac{\alpha}{2}\sin\frac{\theta^S}{2}\cos\frac{\theta^A}{2} e^{i(\beta-\phi^S)} \\ \cos\frac{\alpha}{2}\cos\frac{\theta^A}{2}\sin\frac{\theta^S}{2} e^{i\phi^S} - \sin\frac{\alpha}{2}\sin\frac{\theta^A}{2}\cos\frac{\theta^S}{2} e^{i(\beta-\phi^A)} \\ \cos\frac{\alpha}{2}\sin\frac{\theta^A}{2}\sin\frac{\theta^S}{2} e^{i(\phi^A+\phi^S)} + \sin\frac{\alpha}{2}\cos\frac{\theta^A}{2}\cos\frac{\theta^S}{2} e^{i\beta} \end{pmatrix}, \tag{21}$$

where $\alpha \,\epsilon\, [0,\pi]$ and $\beta \,\epsilon\, [0,2\pi]$ are functions of $\mathbf{r}$, and the local two-component spinors $|\varphi^S\rangle$, $|\chi^S\rangle$, $|\varphi^A\rangle$ and $|\chi^A\rangle$ are constructed as follows: $|\varphi^i\rangle = \begin{pmatrix} \cos\frac{\theta^i}{2} \\ \sin\frac{\theta^i}{2} e^{i\phi^i} \end{pmatrix}$ and $|\chi^i\rangle = \begin{pmatrix} -\sin\frac{\theta^i}{2} e^{-i\phi^i} \\ \cos\frac{\theta^i}{2} \end{pmatrix}$, where $\theta^i \,\epsilon\, [0,\pi]$ and $\phi^i \,\epsilon\, [0,2\pi]$ are the usual polar angles defining the vector $\mathbf{n}^i$. We can read off directly the reduced density matrices using the Schmidt decomposition. The reduced density matrices for spins in skyrmion and antiskyrmion are

$$\begin{aligned} \rho_S &= \mathrm{Tr}_A(|\Psi(\mathbf{r})\rangle\langle\Psi(\mathbf{r}|) \\ &= \cos^2\frac{\alpha}{2}|\varphi^S\rangle\langle\varphi^S| + \sin^2\frac{\alpha}{2} \mid \chi^S\rangle\langle\chi^S|, \end{aligned} \tag{22}$$

$$\begin{aligned} \rho_A &= \mathrm{Tr}_S(|\Psi(\mathbf{r})\rangle\langle\Psi(\mathbf{r}|) \\ &= \cos^2\frac{\alpha}{2} \mid \varphi^A\rangle\langle\varphi^A| + \sin^2\frac{\alpha}{2}|\chi^A\rangle\langle\chi^A|. \end{aligned} \tag{23}$$

It is convenient to express the wave function (21) as $|\Psi(\mathbf{r})\rangle = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$ such that the entanglement measure can be written easily as [23]

$$\mathfrak{E} = 4|z_1 z_4 - z_2 z_3|^2. \tag{24}$$

For maximal entangled states we have $z_1 = z_2 = \frac{1}{\sqrt{2}}$ and $z_2 = z_3 = 0$ while for separable (factorisable) states we have $z_1 z_4 = z_2 z_3$. Consider for simplicity the case when spins in skyrmion and antiskyrmion are perfectly entangled. Let $|\varphi_S\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\varphi_A\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $|\chi_S\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|\chi_A\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Clearly when $\alpha = \frac{\pi}{2}$, the off-diagonal terms of the density matrix vanish and the diagonal terms become 1. This verify the maximal entanglement condition $\rho^A_{ik} = \frac{1}{2}\mathbb{I}_2$.

The local transformation operators U of the density matrices form a six-dimensional subgroup $SU(2) \otimes SU(2)$ of the full unitary group $U(4) = U(1) \otimes SU(4)$. The local transformation operators U are parametrized by six arbitrary real variables such that $U(\theta_S, \phi_S, \theta_A, \phi_A, \alpha, \beta)^\dagger U(\theta_S, \phi_S, \theta_A, \phi_A, \alpha, \beta) = \mathbb{I}_4$ (4×4 identity matrix). Without loss of generality, we can use $\mathbb{I}_2 \otimes \sigma_\mu$ and $\sigma_\mu \otimes \mathbb{I}_2$ as Hermitian $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$ Lie algebra basis of the full $SU(4)$ bilayer skyrmion theory. Here, σ_μ and $\mathbb{I}_2$ denote the Pauli matrices and the two-dimensional identity matrix, respectively.

Geometry of quantum states. – We give a geometric description to the problem of entanglement in bilayer skyrmions based on complex projective geometry [24]. $\mathbb{CP}^{\mathcal{N}}$ is the space of rays in $\mathbb{C}^{\mathcal{N}+1}$, or equivalently the space of equivalence classes of $\mathcal{N}+1$ complex numbers, at least one of them is non-zero, under $(Z^0, Z^1, \ldots, Z^{\mathcal{N}}) \sim \lambda(Z^0, Z^1, \ldots, Z^{\mathcal{N}})$, where $\lambda \varepsilon\ \mathbb{C}$ and $\lambda \neq 0$. In quantum theory, a $\mathbb{CP}^{\mathcal{N}-1}$-field corresponds to an N-component normalized spinor $z = (z_1, z_2, z_3, \ldots z_{\mathcal{N}})^T$ such that the two vectors z and $e^{i\varphi} z$ are equivalent for arbitrary $\varphi \in \mathbb{R}$. The normalization of the $\mathbb{CP}^{\mathcal{N}-1}$ spinor takes away two real parameters (or one complex parameter) which explains why the space $\mathbb{CP}^{\mathcal{N}-1}$ corresponds to $\mathbb{C}^{\mathcal{N}}$. As we have seen in the previous section, the skyrmion-antiskyrmion coupled pair can be described using a four-component spinor which lives in a $\mathbb{CP}^3$-manifold. Any $\mathbb{CP}^3$-manifold is isomorphic to $\frac{U(4)}{[U(3)\otimes U(1)]} \cong \frac{SU(4)}{[SU(3)\otimes U(1)]}$, therefore the second homotopy group is $\pi_2(\mathbb{CP}^3) = \pi_2\{\frac{SU(4)}{[SU(3)\otimes U(1)]}\} = \pi_1[SU(3)\otimes U(1)]$. Using the fact that the homotopy group for the product manifold factorizes as $\pi_k(\mathrm{g}\otimes\mathcal{H}) = \pi_k(\mathrm{g})\otimes\pi_k(\mathcal{H})$ alongside with the fact that any simple Lie group g has a vanishing fundamental homotopy group (*i.e.*, $\pi_1(\mathrm{g}) = 0$) give $\pi_2(\mathbb{CP}^3) = \pi_1[SU(3)]\otimes\pi_1[U(1)] = \mathbb{Z}$ [2,25].

The pure state for each spin-$\frac{1}{2}$ represents a vector in the two-dimensional complex vector space. In the Dirac notation, this vector can be expressed as $|\Psi\rangle = \Sigma_{i=0}^{N-1} Z^i|i\rangle$, where $|i\rangle$ is a given orthonormal basis. The distance D_{FS} between two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ is given by the Fubini-Study distance [26]

$$\cos^2 D_{FS} = \frac{|\langle\Psi_1|\Psi_2\rangle|^2}{\langle\Psi_1|\Psi_1\rangle\langle\Psi_2|\Psi_2\rangle} = \frac{|Z_1\cdot\overline{Z}_2|^2}{(Z_1\cdot\overline{Z}_1)(Z_2\cdot\overline{Z}_2)}, \tag{25}$$

where $\overline{Z}_i$ is the row vector whose entries are the complex conjugates of entries in the column vector $\overline{Z}^i$. The Fubini-Study metric measures the distinguishability of pure quantum states. In quantum communication theory, the Fubini-Study distance is known as fidelity function. Since we have considered a continuum theory for describing large bilayer skyrmions in this letter, the distinguishability of any two arbitrary states in a large skyrmions or antiskyrmions is supposed to be difficult to observe. The infinitesimal form of the Fubini-Study distance approaches the Fubini-Study metric tensor

$$\mathrm{d}s^2 = \frac{Z\cdot\overline{Z}\mathrm{d}Z\cdot\mathrm{d}\overline{Z} - Z\cdot\mathrm{d}\overline{Z}\mathrm{d}Z\cdot\overline{Z}}{(Z\cdot\overline{Z})(Z\cdot\overline{Z})}. \tag{26}$$

Here $Z\cdot\overline{Z} = Z^i\overline{Z}_i$. From the Fubini-Study metric, a time-energy uncertainty relation can be derived directly for each single spin [24]. As a spin-coherent state goes through a closed loop, it will gain the phase $\gamma = \oint\langle\psi(s)\mid\frac{\mathrm{d}}{\mathrm{d}s}\mid\psi(s)\rangle$. It was found that this phase is equal to the Riemannian curvature $K = \frac{1}{2S}$ of the phase space of spin-coherent state up to a constant. When $S = 1/2$ (like large 2D skyrmions), the curvature is equal to its maximum value $K = 1$ [26].

Any arbitrary state vector for a bipartite composite system reads

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\Sigma_{i=0}^{N-1}\Sigma_{j=0}^{N-1}C_{ij}|i\rangle\otimes|j\rangle, \tag{27}$$

where C_{ij} is an $N\times N$ matrix with complex entries. The density matrix for the composite system is given by $\rho_{ij,kl} = \frac{1}{N}C_{ij}C^\star_{kl}$. Since the system is in pure state, its density matrix has rank one. Now suppose we perform experiment in one of the two chiral magnets, the reduced density matrix for this subsystem is the partially traced density matrix $\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}\rho := \mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}}\rho$ which equals $\rho^{\mathcal{A}}_{ik} = \Sigma_{j=0}^{N-1}\rho_{ij,kj}$. The rank of this subsystem density matrix may be greater than one. The global state of the chiral magnets system may be written as a product state spanned in the total Hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{A}}\otimes\mathcal{H}_{\mathcal{B}}$

$$\mid\Psi\rangle = \mid\mathcal{A}\rangle\otimes\mid\mathcal{B}\rangle = \Sigma_{i=0}^{N-1}\Sigma_{j=0}^{N-1}(a_i\mid i\rangle)\otimes(b_j\mid j\rangle), \tag{28}$$

so the matrix $C_{ij} = a_ib_j$ is the dyadic product of two vectors a and b. It is not difficult to notice that a global state of this kind is disentangled or separable since the partially traced matrix and the matrix C_{ij} have rank one and the subsystems are in pure states of their own. On the other hand, the maximally entangled state can be identified using the condition $\rho^{\mathcal{A}}_{ik} = \frac{1}{N}\mathbb{I}_N$ which corresponds to $\Sigma_{j=0}^{N-1}C_{ij}C^\star_{kj} = \delta_{ik}$. It means that we know nothing at all about the state of the subsystems even though the global state is precisely determined. The maximally entangled states form an orbit of the group of local unitary transformations. In our case, this group is $\frac{SU(2)}{\mathbb{Z}} = SO(3)$ which is identical to the real projective space $\mathbb{RP}^3$. In general, the group $\frac{U(\mathcal{N})}{U(1)} = \frac{SU(\mathcal{N})}{\mathbb{Z}^{\mathcal{N}}}$ is a Lagrangian submanifold of $\mathbb{CP}^{\mathcal{N}^2-1}$. Between these two cases, the separable and maximally entangled cases, the Von Neumann entropy $S = -\mathrm{Tr}(\rho_{\mathcal{A}}\ \mathrm{ln}\rho_{\mathcal{A}})$ takes some intermediate value with a possibility of partial entanglement between the spins.

Conclusion. – In this letter, the problem of antiferromagnetically coupled skyrmions (bilayer skyrmion) has been studied in detail using the continuum theory approach. This was done by considering two thin films formed from the same chiral magnet separated by an insulating spacer with antiferromagnetic coupling between chiral films. We assumed each chiral film to host Bloch skyrmions under a certain range of temperatures and external magnetic fields determined by the film parameters. Skyrmions in the first thin film are equal in size to skyrmions in the second thin film but with opposite topological charge. We give a representation for the spin degrees of freedom based on $SU(4)$ Lie algebra. Moreover, we have computed the density matrices for the spin degrees of freedom in a skyrmion and its AFM-coupled antiskyrmion using the Schmidt decomposition. Using the computed density matrices, we found the conditions for maximal or partial entanglement and separability within bilayer skyrmions. The full $SU(4)$ symmetry is broken to $SU(2)\otimes SU(2)$ symmetry during the Schmidt decomposition process while the Zeeman interaction term causes the breaking to $U(1)\otimes U(1)$ symmetry. Interestingly the DMI term preserves the $SU(2)\otimes SU(2)$ symmetry.

The geometry of quantum states in bilayer skyrmions can be described using a complex projective space $\mathbb{CP}^3$ endowed with a unitary-invariant Fubini-Study metric. Geometrically, the entangled states can be described naturally using a $\mathbb{CP}^3$ space. We have two extreme cases corresponding to maximally entangled and separable states. The space of maximally entangled states happens to be the real projective space $\mathbb{RP}^3$ while for separable states is simply the space $\mathbb{CP}^1\otimes\mathbb{CP}^1$. The entanglement in skyrmion-antiskyrmion pairs can be extended to the whole skyrmion lattice SkX. We can have maximally entangled, partially entangled and separable cases for each coupled pairs. However, for uncoupled skyrmion-antiskyrmion pairs, the formalism studied in this letter cannot be accurate in terms of entanglement conditions. This is mainly because of the fact that skyrmion-antiskyrmion pairs are treated as antiferromagnetically coupled pairs throughout our formalism. This allowed us to impose conditions on energy eigenstates of skyrmion and antiskyrmion accordingly. We need to consider each skrymion and antiskyrmion as XXZ spin chains and calculate the corresponding entanglement entropy [27]. Other possible scenarios such as having saturated ferromagnetic phase or general helical spin phase in one layer and skyrmion lattice in the second layer are theoretically possible. However we do not find these structures to be of great interest at least in terms of having a vanishing Magnus force for the whole system.

In comparison with graphene and multicomponent Hall systems, an intimate relation between the entanglement conditions in large bilayer skyrmions and $SU(4)$-skyrmions has been found. However the system which has been investigated in this letter is different from that studied in graphene and multicomponent quantum Hall systems. For example, Lian *et al.* dealt in the graphene case with spin-valley pseudospin degrees of freedom in a single skyrmion [28]. In contrast, we have considered two skyrmions with AFM coupling between its internal spins. This is the reason why we used $\mathbb{CP}^1 \otimes \mathbb{CP}^1$-theory instead of $\mathbb{CP}^3$-theory. However, the space of entangled states is a $\mathbb{CP}^3$-manifold as expected [24,29].

As a last comment, we propose the usage of entanglement in skyrmion-antiskyrmion lattices for probing the geometric nature of quantum entanglement. This will help in turn to further understand and possibly manipulate skyrmion lattices in performing quantum technological tasks such as generating entanglement in systems with a large number of spins.

* * *

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Appendix: $SU(4)$ representation. – The special unitary group $SU(N)$ has (N^2-1) generators, where -1 is because of the condition $\det(M)=1$, where M is any element from $SU(N)$. We denote the generators as λ_A, $A=1,2,\ldots,N^2-1$. We choose the following normalization condition between generators: $\mathrm{Tr}(\lambda_A\lambda_B)=2\delta_{AB}$. Their commutator and anti-commutator are [30]

$$[\lambda_A,\lambda_B] = 2i\, f_{ABC}\lambda_C, \tag{A.1}$$

$$\{\lambda_A,\lambda_B\} = \frac{4}{N} + 2\, d_{ABC}\lambda_C, \tag{A.2}$$

where f_{ABC} and d_{ABC} are the structure constants of $SU(N)$. When $\lambda_A=\sigma_A$ (Pauli matrix) we have $f_{ABC}=\varepsilon_{ABC}$ and $d_{ABC}=0$ in the case of $SU(2)$.

Since our developed model of bilayer skyrmions in chiral magnets is based on $SU(4)$ we will give a specific attention to this group. $SU(4)$ has 15 generators while $SU(2)\otimes SU(2)$ has 6 generators in total. Embedding $SU(2)\otimes SU(2)$ into $SU(4)$ we find the matrix representation for skyrmion S and its AFM-coupled antiskyrmion A

$$\tau_x^S = \begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix},\quad \tau_y^S = \begin{pmatrix}\sigma_y & 0\\ 0 & \sigma_y\end{pmatrix},\quad \tau_z^S = \begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix}, \tag{A.3}$$

$$\tau_x^A = \begin{pmatrix}0 & \mathbb{I}_2\\ \mathbb{I}_2 & 0\end{pmatrix},\quad \tau_y^A = \begin{pmatrix}0 & -i\mathbb{I}_2\\ i\mathbb{I}_2 & 0\end{pmatrix},\quad \tau_z^A = \begin{pmatrix}\mathbb{I}_2 & 0\\ 0 & -\mathbb{I}_2\end{pmatrix}. \tag{A.4}$$

REFERENCES

[1] Nagaosa N. and Tokura Y., *Nat. Nanotechnol.*, **8** (2013) 899.

[2] Nash C. and Sen S., *Topology and Geometry for Physicists* (Academic Press) 1983.

[3] Skyrme T. H. R., *Proc. R. Soc. London A*, **260** (1961) 127.

[4] Mühlbauer S. *et al.*, *Science*, **323** (2009) 915.

[5] Bogdanov A. N. and Yablonskii D. A., *Sov. Phys. JETP*, **68** (1989) 101.

[6] Bogdanov A. N. and Hubert A., *J. Magn. & Magn. Mater.*, **138** (1994) 255.

[7] Jonietz F. *et al.*, *Science*, **330** (2010) 1648.

[8] Zang J., Mostovoy M., Han J. H. and Nagaosa N., *Phys. Rev. Lett.*, **107** (2011) 136804.

[9] Jiang W. *et al.*, *Nat. Phys.*, **13** (2017) 162.

[10] Sonin E.B., *Phys. Rev. B*, **55** (1997) 485.

[11] Freimuth F., Bamler R., Mokrousov Y. and Rosch A., *Phys. Rev. B*, **88** (2013) 214409.

[12] Zhang X., Zhou Y. and Ezawa M., *Nat. Commun.*, **7** (2016) 10293.

[13] Takashima R., Ishizuka H. and Balents L., *Phys. Rev. B*, **94** (2016) 134415.

[14] Gilbert T. L., *Phys. Rev.*, **100** (1955) 1243.

[15] Thiele A. A., *Phys. Rev. Lett.*, **30** (1973) 230.

[16] Yu X. Z., Onose Y., Kanazawa N., Park J. H., Han J. H., Matsui Y., Nagaosa N. and Tokura Y., *Nature*, **465** (2010) 901.

[17] Derrick G. H., *J. Math. Phys.*, **5** (1964) 1252.

[18] Hobart R. H., *Proc. Phys. Soc. London*, **82** (1963) 201.

[19] Belavin A. A. and Polyakov A. M., *JETP Lett.*, **22** (1975) 10.

[20] Han J. H., Zang J., Yang Zhihua, Park J. and Nagaosa N., *Phys. Rev. B*, **82** (2010) 094429.

[21] Ezawa Z. F. and Tsitsishvili G., *Phys. Rev. B*, **70** (2004) 125304.

[22] Yang K., Das Sarma S. and MacDonald A. H., *Phys. Rev. B*, **74** (2006) 075423.

[23] Doucot B., Goerbig M. O., Lederer P. and Moessner R., *Phys. Rev. B*, **78** (2008) 195327.

[24] Bengtsson I. and Życzkowski K., *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press) 2017.
[25] Nakahara M., *Geometry, Topology, and Physics*, second edition (CRC Press) 2003.
[26] Provost J. P. and Vallee G., *Commun. Math. Phys.*, **76** (1980) 289.
[27] Amico L., Fazio R., Osterloh A. and Vedral V., *Rev. Mod. Phys.*, **80** (2008) 517.
[28] Lian Y., Rosch A. and Goerbig M. O., *Phys. Rev. Lett.*, **117** (2016) 056806.
[29] Brody D. C. and Hughston L. P., *J. Geom. Phys.*, **38** (2001) 19.
[30] Gell-Mann M. and Neéman Y., *The Eight-Fold Way* (Benjamin, New York) 1964.