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Dynamic transition in a Brownian fluid: role of fluctuation–dissipation constraints

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Abstract. In this paper we study equations of fluctuating non-linear hydrodynamics (FNH) for a liquid in which the constituent particles follow Brownian dynamics (BD). Here the microscopic-level dynamics are dissipative as compared to the case of fluids with reversible Newtonian dynamics (ND). The implications of non-linearities in FNH equations for an ND liquid on its long-time dynamics and the possibility of an ideal ergodicity–non-ergodicity (ENE) transition at high density have been widely investigated in literature. It is known that in an ND fluid, dynamics described by FNH equations do not support a sharp ENE transition. In the present paper we demonstrate that, as a consequence of the fluctuation–dissipation constraints, an ideal ENE transition is also not supported from the FNH equations for BD fluid.

Keywords: Brownian motion, fluctuating hydrodynamics, glasses (structural), mode-coupling theory

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Contents

1. Introduction

In understanding the development of extremely slow dynamics in a many-particle system, the self-consistent mode-coupling theory (MCT) [1–3] has played an important role. Central to the MCT for liquids is a non-linear feedback mechanism due to the coupling of slowly decaying density fluctuations in the supercooled state. In the simplest version proposed initially [4–6] a sharp ergodic to non-ergodic (ENE) transition of the supercooled liquid into a glassy phase was predicted. This transition occurs at a critical density or the corresponding value of another suitable thermodynamic parameter. Beyond the ENE transition point the auto-correlation fluctuations freeze at a nonzero value over a long time. Hence the order parameter in this transition is taken to be the long time limit $f(q)$ of the density correlation function [7] $\psi(q,t)$. In the ergodic liquid state $f(q) = 0$ for all q. The packing fraction at which all the $f(q)$'s jump discontinuously to a nonzero value [8] marks the transition point. Subsequently it was demonstrated that the sharp transformation to a non-ergodic state at a critical density as

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predicted in the original MCT [9] is rounded. The absence of a sharp ENE transition in supercooled liquids was supported by the work in [10–12] using similar theoretical models. These models are generally termed as extended MCT.

The original treatment of the extended MCT presented in [9] was based on equations of fluctuating non-linear hydrodynamics (FNH) [13, 14]. This formulation was developed for a fluid in which the particles follow reversible Newtonian dynamics (ND) [9, 15, 16]. The same FNH equations were recently obtained [17] for a system of particles following Brownian dynamics (BD), which is dissipative [18]. The respective forms of the FNH equations for fluids in which the particles follow ND and BD are similar with some subtle differences. The set of hydrodynamic variables considered here in FNH formulation are $\{\rho, \mathbf{g}\}$, respectively denoting mass and momentum densities. The equation of motion for mass density $\rho(\mathbf{x}, t)$ is the continuity equation. The equation for the momentum density $\mathbf{g}(\mathbf{x}, t)$ is a generalised Langevin equation with noise. For ND fluid, the dissipative term of this generalised Navier–Stokes equation involves $1/\rho$ non-linearity. The stochastic term represents simple noise and the noise correlation is defined with constant bare transport coefficients [19]. On the other hand, for the Brownian system, with intrinsically dissipative dynamics, the noise is multiplicative [20] and the corresponding bare transport coefficient [21] is dependent on the fluctuating density. However, in this case, the dissipative term of the $\mathbf{g}$ equation is linear in $\mathbf{g}$. It does not contain the $1/\rho$ non-linearity which is essential for removing the ideal ENE transition of MCT [9, 22] in the Newtonian case. In the present work we analyse implications of these FNH equations for the Brownian system on the asymptotic behaviour of the order parameter (density–density correlation function) for the ENE transition and compare the results with those for Newtonian systems.

The paper is organised as follows. In the next section we summarise the deductions and basic features of the FNH equations corresponding to the collective hydrodynamic modes $\{\rho, \mathbf{g}\}$ respectively for fluids with ND and BD. In section 3 we summarise the non-perturbative formulation of the Martin–Siggia–Rose (MSR) field theory [23–25] and renormalisation scheme for a Brownian system. In section 4 we analyse the corresponding set of fluctuation–dissipation relations (FDRs) between correlation and response functions of MSR field theory. In section 5 we study the implications for multiplicative noise on the asymptotic dynamics of the fluid. In particular our focus is on the feasibility of an ENE transition at high density within the constraints of the FDRs. We end the paper with a brief discussion and summary of conclusions. The present analysis in terms of the MSR field theories is technically rather involved. A lot of the machinery used here is already available in the literature and hence we have kept the description to a minimum level and shifted the technical parts in the appendices which are therefore long. The informed reader can skip those. We briefly describe in appendix A the deduction of the corresponding FNH equations for the BD system. In appendix B the formulation of the renormalised MSR field theory is described for FNH equations with multiplicative noise. Here we discuss deduction of the appropriate MSR action functional, calculation of renormalised correlations functions using the so-called Dyson equation and finally the time reversal operator that keeps the MSR action invariant.

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2. Fluctuating nonlinear hydrodynamics: Newtonian versus Brownian fluid

In the FNH description the dynamics of a system are formulated in terms of the time evolution of a set of local densities $\{\psi_a(\mathbf{x},t)\}$. These equations of dynamics originate from a corresponding set of underlying conservation laws, or as a representation of Nambu–Goldstone modes which occur due to the breaking of continuous symmetry in the system [26]. In some situations the slow mode can result due to some specific property, like the high inertia of a Brownian particle. For an isotropic fluid in which the particles follow reversible ND, conservation laws of mass, momentum and energy in the fluid give rise to a corresponding set of balance equations for microscopically defined collective densities $\{\tilde{\psi}_a\}$.

$$\frac{\partial \tilde{\psi}_a}{\partial t} + \nabla \cdot \tilde{\mathbf{j}}_a = 0, \tag{1}$$

with $\tilde{\mathbf{j}}_a$ being the current density corresponding to $\tilde{\psi}_a$. The index a runs over a set of conserved properties. For example, in a fluid, at a microscopic level the mass density $\tilde{\rho}(\mathbf{x},t)$ and momentum density $\tilde{\mathbf{g}}(\mathbf{x},t)$ are defined in terms of the phase space coordinates of the individual particles $\{\mathbf{x}_\alpha, \mathbf{p}_\alpha\}$.

$$\tilde{\rho}(\mathbf{x},t) = \sum_{\alpha=1}^{N} \delta(\mathbf{x} - \mathbf{x}_\alpha(t)) \tag{2}$$

$$\tilde{g}_i(\mathbf{x},t) = \sum_{\alpha=1}^{N} p_\alpha^i \delta(\mathbf{x} - \mathbf{x}_\alpha(t)). \tag{3}$$

We have adopted the usual notation that the Greek indices correspond to the particle labels while the Roman indices are for the Cartesian components of a vector. In dealing with the non-linear dynamics of the density field we take the mass m of the particles as unity to keep the notations simple. This makes the number density $\tilde{n}(\mathbf{x})$ and mass density $\tilde{\rho}(\mathbf{x})$ the same. The microscopic densities defined above are averaged over all possible phase space coordinates for a suitable non-equilibrium ensemble to obtain the corresponding coarse-grained functions with smooth spatio-temporal dependence. Thus

$$\langle \tilde{\psi}_a(\mathbf{x},t)\rangle_{\text{n.e}} = \psi_a(\mathbf{x},t). \tag{4}$$

The set of balance equations (1), when averaged over a suitable non-equilibrium distribution, obtains the corresponding equations for the coarse-grained densities $\{\psi_a(\mathbf{x},t)\}$. These equations with smooth spatio-temporal dependence are obtained using standard methods of statistical mechanics [27, 28]. With an intrinsic time scale separation in the dynamics, divergence of the averaged current $\mathbf{j}_a$ is split into a regular (slow) and a stochastic (fast) part. The time evolution of the coarse-grained local density $\psi_a(\mathbf{x},t)$ in an FNH description is obtained in terms of a generalised Langevin equation [29] consisting of regular parts and noise with the two parts having widely different time scales [20]. These stochastic partial differential equations are treated as the actual time evolution of the respective local densities and a plausible generalisation of the standard hydrodynamic laws.

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For both ND and BD the coarse-grained mass density $\rho(\mathbf{x},t)$ satisfies the continuity equation (5) with the momentum density $\mathbf{g}(\mathbf{x},t)$ as its current.

$$\frac{\partial\rho}{\partial t}+\nabla.\mathbf{g}=0. \tag{5}$$

Note that this equation is invariant under time reversal and does not contain a dissipative term or noise. The equation of motion for the momentum density $\mathbf{g}$ has important differences for ND and BD. We describe the two cases below.

2.1. Newtonian dynamics

The Langevin equation for the density ψ_a is obtained in the generalised form

$$\frac{\partial\psi_a}{\partial t}+\sum_{\{\psi_b\}}\left[Q_{ab}+L^0_{ab}\right]\frac{\delta F}{\delta\psi_b}=\zeta_a. \tag{6}$$

Q_{ab} is the classical Poisson bracket (PB) between the densities ψ_a and ψ_b.

$$Q_{ab}=\{\tilde{\psi}_a,\tilde{\psi}_b\}, \tag{7}$$

and finally the $\{\tilde{\psi}_a\}$ is replaced by the corresponding coarse-grained quantity $\{\psi_a\}$. The noise ζ_a in the generalised Langevin equation is assumed to be Gaussian with the following FDR with the bare transport coefficients L^0_{ij},

$$\langle\zeta_a(\mathbf{x},t)\zeta_b(\mathbf{x}',t')\rangle=2\beta^{-1}L^0_{ab}(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{8}$$

Here $\beta=1/(k_BT)$ is the Boltzmann factor. Both Q_{ij} and L^0_{ij} are symmetric under exchange of its indices. The Langevin equation (6) in the equivalent Fokker–Planck description [30, 31] has a stationary solution $\sim\exp[-F]$, where F is identified as a coarse-grained free energy functional [32].

For the ND case, the equations for microscopic dynamics are reversible. The Poisson brackets Q_{ab} are calculated [33, 34] in terms of the fundamental PB, $\{x^i_\alpha,p^j_\nu\}=\delta_{\alpha\nu}\delta_{ij}$. The generalised Langevin equation (6) for momentum density g_i is written in the useful form

$$\frac{\partial g_i}{\partial t}+\nabla_j\left[\frac{g_ig_j}{\rho}\right]+\rho\nabla_i\frac{\delta F_{\mathrm{U}}}{\delta\rho}-L^0_{ij}\frac{g_j}{\rho}=\theta_i. \tag{9}$$

$F[\rho,g]$ appearing is expressed as a sum of two parts

$$F[\rho,g]=F_{\mathrm{K}}[\rho,g]+F_{\mathrm{U}}[\rho]. \tag{10}$$

The so-called kinetic part $F_{\mathrm{K}}[\rho,g]$ [35] is

$$F_{\mathrm{K}}[\rho,g]=\int\mathrm{d}\mathbf{x}\left[\frac{g^2(\mathbf{x})}{2\rho(\mathbf{x})}\right]. \tag{11}$$

The momentum density equations include dissipation in a phenomenological way. The dissipative terms are obtained by maintaining positive entropy production [28]. The bare transport matrix is L^0_{ij} for the isotropic system and is obtained as

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$$L^0_{ij} = (\zeta_0 + \frac{\eta_0}{3})\nabla_i\nabla_j + \eta_0\delta_{ij}\nabla^2, \tag{12}$$

where ζ_0, η_0 etc are the corresponding bulk and shear viscosities of ND fluid. The noise $\theta_i \equiv \zeta_{g_i}$ relates to the bare dissipative matrix L^0_{ij} through the FDR:

$$\langle \theta_i(\mathbf{x}, t)\theta_j(\mathbf{x}', t')\rangle = 2\beta^{-1}L^0_{ij}(\mathbf{x} - \mathbf{x}')\delta(t - t'). \tag{13}$$

2.2. Brownian dynamics

In the case of BD, the equation of motion of a single constituent particle is dissipative and has an intrinsic noise component. For position variables the time derivative is $\dot{\mathbf{x}}_\alpha = \mathbf{p}_\alpha$. For the ith component of $\mathbf{p}_\alpha$ the equation of motion is

$$\frac{\mathrm{d}p^i_\alpha(t)}{\mathrm{d}t} = f^i_\alpha - \gamma^0_{ij}p^j_\alpha + \xi^i_\alpha(t). \tag{14}$$

The force on particle α (where $\alpha = 1, ., N$) has a regular part f^i_α due to interaction with other particles and a stochastic part described with a random noise ξ^i_α. The frictional force is $\gamma^0_{ij}p^j_\alpha$ where the kinetic coefficient γ^0_{ij} [36] has the dimension of inverse time. We treat the dynamics in Markovian approximation of widely different time scales for the regular and stochastic components of equation (14). The fast part constitutes the noise ξ^α_i which is assumed to be white and Gaussian. Correlation of the noise is related to the dissipative coefficient γ_{ij} through the FDR:

$$\langle \xi^i_\alpha(t)\xi^j_\nu(t')\rangle = 2m\beta^{-1}\gamma^0_{ij}\delta_{\alpha\nu}\delta(t - t'), \tag{15}$$

where m is the mass of the particle, and set to unity. In appendix A we discuss how by averaging equation (14) over a local equilibrium distribution [37] we obtain the coarse-grained equation of FNH for the momentum density $\mathbf{g}(\mathbf{x}, t)$:

$$\frac{\partial g_i}{\partial t} + \Gamma_i[\rho, \mathbf{g}] - \gamma^0_{ij}g_j = \vartheta_i. \tag{16}$$

The reversible component of the regular part of the equation of motion is

$$\Gamma_i[\rho, \mathbf{g}] \equiv \nabla_j\left[\frac{g_i g_j}{\rho}\right] + \rho\nabla_i\frac{\delta F_{\mathrm{U}}}{\delta\rho}. \tag{17}$$

The bare transport matrix of friction coefficients γ^0_{ij} thus represents a tensor kinematic viscosity and has to be multiplied with a factor of density to obtain the corresponding bare viscosity tensor involving ζ_0 and η_0, respectively denoting the bulk and shear viscosities of BD fluid. Note that in order to conserve momentum the dissipative coefficient matrix γ^0_{ij} has $\nabla_i\nabla_j$ operators associated with it.

2.3. Primary distinctions

We summarise here the primary differences between the respective FNH equations for the ND and BD systems. In both cases, the corresponding FNH equations for the local densities $\psi_a \in \{\rho, \mathbf{g}\}$ have the same generalised form described in (6)–(8). In the case of BD, the bare transport coefficient is γ^0_{ij} and the frictional term in equation (16) is $L^0_{ij}(\delta F/\delta g_j)$, where the kinetic coefficient L^0_{ij} is dependent on the local densities $\rho(\mathbf{x},t)$:

$$L^0_{ij} = \gamma^0_{ij}\rho(\mathbf{x},t). \tag{18}$$

Equations (9) and (16) are stochastic partial differential equations of FNH, respectively corresponding to fluids following ND and BD. The term $\nabla_j[g_i g_j/\rho]$ present in both equations is essential for Galilean invariance of the FNH equations and is termed as a convective non-linearity. For the ND case however, the $1/\rho$ non-linearity also appears in the dissipative term. In order to take into account this non-linearity for a fluid with ND, another fluctuating field $\mathbf{v}$ has been introduced [9, 22] and was linked to $\{\rho,\mathbf{g}\}$ through the non-linear constraint

$$\mathbf{g} = \rho\mathbf{v}. \tag{19}$$

This makes the dissipative term, the last one on the left-hand side of equation (9), linear in the $\mathbf{v}(\mathbf{x},t)$ field, i.e. $L^0_{ij}v_j$. For BD, unlike ND, in the corresponding generalised Langevin equation (16) for g_i, the dissipative term is linear in $\mathbf{g}$. The noise in this case is multiplicative and so the noise correlation involves the density factor $\rho(\mathbf{x},t)$.

For an ND system the MCT of glassy dynamics has been obtained from the corresponding set of FNH equations which includes the continuity equation (5), the generalised Langevin equation (9) and the non-linear constraint (19). In the BD case, equation (16) replaces equation (9) for the momentum density. Both these equations include the key non-linear couplings of the hydrodynamic modes which give rise to the slow dynamics of MCT. Partial treatment of the non-linearities predicts an ENE transition in the metastable liquid. A detailed analysis [9, 22] of the renormalised theory involving FNH equations (5) and (9) for an ND system showed that, unlike the predictions of simple MCT, a sharp ENE transition is not supported in the full analysis. This holds within requirements of the existing FDRs [38–40] in the model. However, this was primarily a result for the FNH equation described above for fluids in which the particles follow reversible ND. In this paper we focus on the FNH equation (16) for BD fluids.

3. Field theoretic model for renormalised dynamics

The time correlation of density fluctuations or the dynamic structure factor is a key quantity studied in liquid-state physics. This correlation function is treated as the order parameter for ENE transition predicted in self-consistent MCT of glassy dynamics [3, 5]. It is conveniently obtained from FNH formulation by performing averages over the associated noise. The renormalised perturbation theory for the dynamics is developed following the standard approach of the MSR field theory [23]. It is particularly suitable for the discussion of MCT since the renormalised dynamics are formulated in a self-consistent manner in the MSR approach. The related methodology provides a recipe for calculating noise-averaged correlation and response functions involving a set of fields $\{\psi_a\}$ whose dynamics are given by stochastic partial differential equations of FNH. In the MSR field theory a set of conjugate or hatted fields $\{\hat{\psi}_a\}$ are introduced such that

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the matrix of two point correlation functions between original and hatted fields represent response functions. The response functions are by definition time-ordered.

We focus here on the formulation of renormalised dynamics for a Brownian system, based on the FNH equations (5) and (16). The associated noise in this case is multiplicative and the noise correlation is proportional to the fluctuating density field ρ. In the MSR theory correlations of the various fields $\{\psi_a\}$ involve averages defined in terms of the action $\mathcal{A}$, which is a functional of the field variables $\{\psi_a\}$ and the corresponding conjugate hatted fields $\{\hat{\psi}_a\}$. Generally, the MSR action $\mathcal{A}$ is evaluated for the case of additive noise. For the case of multiplicative noise [41, 42] it has been shown [43] that the form of the action remains similar to that obtained in the case in which the bare transport coefficients are field-dependent. With the choice of proper time discretisation, it is possible to treat the Jacobian of transformation between variables involving noise and fields as constant [43]. Formulation of the MSR theory corresponding to the FNH equations following a path integral approach is briefly outlined in the appendix B. We list the specific result for the BD system here. The action $\mathcal{A}$ is obtained as a functional of the fields $\psi_a \in \{\rho, \mathbf{g}\}$ and their hatted conjugates [9, 43],

$$\mathcal{A}[\rho, \mathbf{g}, \hat{\rho}, \hat{\mathbf{g}}] = \int \mathrm{d}t \int \mathrm{d}\mathbf{x} \Bigg\{ \beta^{-1} \hat{g}_i L^0_{ij}[\rho] \hat{g}_j + i\hat{g}_i \left(\frac{\partial g_i}{\partial t} + \Gamma_i[\rho, \mathbf{g}] - \gamma^0_{ij} g_j \right) + i\hat{\rho} \left(\frac{\partial \rho}{\partial t} + \nabla_j g_j \right) \Bigg\}. \tag{20}$$

The noise is multiplicative, or in other words, the transport coefficient L^0_{ij} which determines the noise correlation depends on the hydrodynamic field $\rho(\mathbf{x}, t)$ as shown in equation (18).

We denote the matrix of correlation functions involving the combined group of fields $\{\psi_a, \hat{\psi}_a\}$ as $\mathbf{G}$ and it is calculated in terms of the action functional $\mathcal{A}[\psi_a, \hat{\psi}_a]$. For completely linear dynamics the functional $\mathcal{A}$ is quadratic in the fields and the corresponding $\mathbf{G}$ matrix is denoted as $\mathbf{G}_0$. The correction due to non-linearities (in the dynamics) are included in the theory in terms of the self-energy matrix Σ introduced through the Dyson equation:

$$\mathbf{G}^{-1} = \mathbf{G}_0^{-1} - \Sigma. \tag{21}$$

For the present formulation, the action $\mathcal{A}$ gives $\mathbf{G}^{-1}$ and $\mathbf{G}_0^{-1}$ in a block diagonal form. In appendix B.1, the inverse of $\mathbf{G}_0$ is obtained for the isotropic system in terms of matrices $\mathcal{B}_0$, $\mathcal{C}_0$ introduced in equation (B.21). In the present case of BD with fields $\{\rho, \mathbf{g}\}$ the matrices $\mathcal{B}_0$, $\mathcal{C}_0$ are of size 2×2. Using (20) for MSR action, we obtain the matrix $\mathcal{B}_0$ from the equations of linear dynamics as

$$\mathcal{B}_0(q, \omega) = \begin{bmatrix} \omega & -q \\ -qc_0^2 & \omega + iL_0 \end{bmatrix} \tag{22}$$

where $L_0 = q^2 \rho_0 \gamma_0$. We have taken here the γ_{ij} as diagonal, $\gamma^0_{ij} = \rho_0 \gamma_0 \delta_{ij}$, where γ_0 is a kinematic viscosity. Next, we consider the corresponding self-energy matrix Σ. Using basic symmetries and the tensorial structure of the various elements for Σ, we identify their respective wave vector dependences. The self-energies $\Sigma_{\hat{g}_i\rho}$ and $\Sigma_{\hat{g}_i g_j}$ are respectively identified as

$$\Sigma_{\hat{g}_i\rho} = q_i\gamma_{\hat{g}\rho}\ , \quad \Sigma_{\hat{g}_i g_j} = q_i q_j\gamma_{\hat{g}g}, \quad \Sigma_{\hat{g}_i\hat{g}_j} = q_i q_j\gamma_{\hat{g}\hat{g}}. \tag{23}$$

Since there is no non-Gaussian term in the MSR action functional (20) with a $\hat{\rho}$ field, the corresponding vertex functions with this field are absent and hence Σ_{ab} is zero if either of the a or b indices correspond to $\hat{\rho}$. Hence the matrix $\mathcal{C}_0$ has only one nonzero element corresponding to both indices being $\hat{g}$, and is equal to $2\beta^{-1}L_0$. The corresponding renormalised matrix element of $\mathcal{C}$ is

$$\mathcal{C}_{\hat{g}g} = 2\beta^{-1}L_0 - \Sigma_{\hat{g}g}\equiv 2\beta^{-1}\tilde{\gamma}_R(q,\omega). \tag{24}$$

The inverse of the full Green's function matrix is obtained using the Dyson equation (21) in the block diagonal form with matrices (22) and (24). The correlation and functions between two unhatted fields are obtained in the symmetric form

$$G_{\alpha\beta} = -\sum_{\mu\nu} G_{\alpha\hat{\mu}}\mathcal{C}_{\hat{\mu}\hat{\nu}}G_{\hat{\nu}\beta}, \tag{25}$$

which is obviously real. The response functions are expressed in the general form,

$$G_{a\hat{b}}(q,\omega) = \frac{N_{a\hat{b}}(q,\omega)}{\mathcal{D}(q,\omega)}, \tag{26}$$

where $N_{a\hat{b}}$ and $\mathcal{D}$ are respectively the co-factors and determinant of matrix $\mathcal{B}$ defined in appendix B.2. The matrix $N_{\hat{\alpha}\beta}$ in the expression (26) for the response function is obtained as

$$N_{\alpha\hat{\beta}}(q,\omega) = \begin{bmatrix} \rho_0\omega + iq^2\gamma_R(q,\omega) & -\rho_0 q \\ -\rho_0 qc^2 & \rho_0\omega \end{bmatrix}. \tag{27}$$

The denominator $\mathcal{D}$ of the expression given in (26) is obtained as

$$\mathcal{D}(q,\omega) = \rho_0(\omega^2 - q^2c^2) + iq^2\omega\gamma_R(q,\omega). \tag{28}$$

We have in terms of single-hatted or response self-energies, the following renormalised quantities to leading order in wave number q:

$$\gamma_R(q,\omega) = \rho_0\gamma_0 + i\gamma_{\hat{g}g}(q,\omega) \tag{29}$$

$$c^2(q,\omega) = c_0^2(q) + \gamma_{\hat{g}\rho}(q,\omega). \tag{30}$$

Note that γ_R has the dimension of viscosity while that of γ_0 is of kinetic viscosity. $c_0(q)$ represents the speed of sound in the equations for linear dynamics of density fluctuations and is related to the static structure factor $S(q)$ of the fluid.

4. Fluctuation dissipation relations

A set of relations between the correlation and response functions is reached by considering how the MSR action (20) changes under time reversal [9, 40]. This is closely

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linked to the time reversal property of the associated Langevin equations of the FNH description. We define the time reversal operation on the field ψ_a as

$$\mathcal{T}\psi_a(\mathbf{x},t) = \epsilon_a \psi_i(\mathbf{x},-t) \tag{31}$$

where $\epsilon_a = \pm 1$. The MSR action functional $\mathcal{A}[\psi,\hat{\psi}]$ is invariant [40] under the transformation $\mathcal{T}$ defined as

$$\psi_a(\mathbf{x},-t) \to \epsilon_a \psi_a(\mathbf{x},t) \tag{32}$$

$$\hat{\psi}_a(\mathbf{x},-t) \to -\epsilon_a \left[\hat{\psi}_a(\mathbf{x},t) - i\beta \frac{\delta F}{\delta \psi_b(\mathbf{x},t)}\right]. \tag{33}$$

This time invariance property leads to FDRs between the response and correlation functions. Using (33) and (32) for the fields $\{\hat{\psi}_a, \psi_b\}$ we obtain

$$\langle \hat{\psi}_a(\mathbf{x},t)\psi_b(\mathbf{x}',t')\rangle - i\beta\langle \kappa_a(\mathbf{x},t)\psi_b(\mathbf{x}',t')\rangle = 0, \tag{34}$$

where the field κ_a is defined as

$$\kappa_a(\mathbf{x},t) = \frac{\delta F}{\delta \psi_a(\mathbf{x},t)}. \tag{35}$$

The response function is time-ordered and is nonzero for $t > t'$. The FDT which follows from equation (34) is expressed as

$$G_{\hat{\psi}_a\psi_b}(t,t') = i\beta\Theta(t-t')G_{\kappa_a\psi_b}(t,t'), \tag{36}$$

where $\Theta(t)$ is the step function. Note that the FDT relation (36) provides a closed set of relations if newly defined (see equation (35) above) $\kappa_b \in \{\psi_a\}$. For compressible liquid this is achieved by extending the set of fluctuating fields to include the current field $\mathbf{v}$ to be specified below. The set of field variables involved in the FDT (36) for the MSR theory goes beyond $\{\rho, \mathbf{g}\}$ and includes a new fluctuating field $\mathbf{v}$ defined with the nonlinear constraint (19).

In the appendix B.3 we show that for the MSR action $\mathcal{A}$ in equation (20), the invariant properties (33) and (32) are maintained even if the transport coefficients L^0_{ij} are field-dependent, as is the case for multiplicative noise. For $\psi_a \equiv g_i$, the functional derivative of the driving free energy is

$$\kappa_a = \frac{\delta F}{\delta \psi_a} \equiv \frac{\delta F}{\delta g_i} = \frac{g_i}{\rho}. \tag{37}$$

This follows from the kinetic part $F_{\mathrm{K}}[\rho,\mathbf{g}]$ given in equation (11) for the driving free energy functional. It is important to note here that the specific form F_{K} leads to the continuity equation (5) for $\rho(\mathbf{x},t)$ and it also gives rise to the term $\nabla_j\{g_i g_j/\rho\}$ which is essential for maintaining the Galilean invariance of the equation (9). Both these are key aspects of hydrodynamic description. The response functions involving $\hat{g}$ links through the FDT to correlations with the current field $\mathbf{v}$:

$$G_{\mathrm{v}_i\psi_a}(q,\omega) = -2\beta^{-1}\mathrm{Im}G_{\hat{g}_i\psi_a}(q,\omega). \tag{38}$$

It is important to note that the fluctuating field $\mathbf{v}(\mathbf{x}, t)$ only enters the theory through a constraint and does not have an independent FNH equation describing its dynamics.

5. The ergodicity nonergodicity transition

Using the above expressions for the correlation and response functions we consider the validity of an ENE transition in the present model for a BD system. Let us first consider the basic requirements of the ENE transition on the correlation and corresponding memory functions or the generalised transport coefficient for the fluid. In the following we suppress the position or wave vector arguments as a simplification of notations. A key point of this analysis is the self-consistent feedback mechanism which makes respective Fourier transforms of both the density auto-correlation $\tilde{G}_{\rho\rho}(\omega)$ and the corresponding transport coefficient $\tilde{\gamma}_R(\omega)$ diverge at the ENE transition. The integral relation between $\gamma_R(\omega)$ and $G_{\rho\rho}(t)$ gives rise to the non-linear feedback mechanism referred to above. At the ENE transition in MCT the long time limit of the density auto-correlation function $G_{\rho\rho}(t)$ changes discontinuously to a nonzero value. In this case the one-sided Laplace transform of $G_{\rho\rho}(t)$ follows a singular behaviour $G_{\rho\rho}(z) \sim 1/z$ in the small z limit. Hence the two-sided Fourier transform $\tilde{G}_{\rho\rho}(\omega) \sim \delta(\omega)$. The definition of $G_{\rho\rho}(z)$ in terms of the generalised transport coefficient $\gamma_R(z)$ will imply that the latter will also have a $1/z$ pole or equivalently, $\tilde{\gamma}_R(\omega)$ have a δ function peak. Therefore an ENE transition implies a diverging viscosity or equivalently the blowing up of the self-energy $\Sigma_{\hat{g}\hat{g}}$ at small frequencies. This conforms to the physics of viscosity blowing up as one enters the non-ergodic phase.

In MCT, correlation functions are expressed in terms of generalised transport coefficients. Perturbative expressions for renormalised transport coefficients due to nonlinear couplings of the hydrodynamics modes are obtained using diagrammatic methods of quantum field theory [44]. In the following we will consider the implications of mode couplings from a non-perturbative approach. The renormalised transport coefficients are obtained in terms of the so-called self-energy matrix Σ introduced through the Dyson equation [9]. By setting both α and β respectively equal to ρ in equation (25) it follows that the correlation function $G_{\rho\rho}$ contains a δ function contribution through couplings to $\Sigma_{\hat{g}\hat{g}}$. Thus, the divergence of viscosity or that of the self-energy $\gamma_{\hat{g}\hat{g}}$ at small frequencies implies

$$\gamma_{\hat{g}\hat{g}} = -A\delta(\omega) + \{\text{Regular Terms for } \omega \to 0\}. \tag{39}$$

This is also supported from a perturbative approach, since one loop contribution to $\Sigma_{\hat{\mathbf{g}}\hat{\mathbf{g}}}$ involves convolution of $G_{\rho\rho}(t)$. The singular contribution of $G_{\rho\rho}$ comes from $\Sigma_{\hat{g}\hat{g}}$ in the form

$$G_{\rho\rho} = G_{\rho\hat{g}}\Sigma_{\hat{g}\hat{g}}G_{\hat{g}\rho} + \{\text{Regular Terms for } \omega \to 0\}. \tag{40}$$

Therefore for an ENE transition to occur it is necessary that the response function $G_{\rho\hat{g}}$ does not vanish as $\omega \to 0$. We test these conditions for Brownian dynamics fluids for which the renormalised model was discussed above in terms of the FNH equations for the field's mass and momentum densities $\{\rho, \mathbf{g}\}$. The matrix element $N_{\rho\hat{\mathbf{g}}} = -\rho_0 q$ given

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Table 1. Elements of matrix $[G^{-1}]_{\hat{\alpha}\beta}$ defined in terms of the associated parameters.

	ρ	g	v
$\hat{\rho}$	ω	$-q$	0
$\hat{\mathrm{g}}$	$-qc_R^2$	$\omega + i\gamma_R q^2$	$-\gamma_{\hat{\mathrm{g}}\mathrm{v}} q^2$
$\hat{\mathrm{v}}$	$-q\gamma_{\hat{\mathrm{v}}\rho}$	i	$-i\rho_\mathrm{L}$

in equation (27) leads to a singular term involving $\delta(\omega)$ in the correlation $G_{\rho\rho}$. On the other hand, since $N_{\mathrm{g}\hat{\mathrm{g}}} = \rho_0\omega$, the corresponding correlation $G_{gg}(\omega)$ or $G_{\rho g}(\omega)$ involving a $\mathbf{g}$ field does not have any divergence by linking with $\Sigma_{\hat{g}\hat{g}}$ as shown in equation (40). The ENE transition therefore would appear to be supported in this model for BD fluid. However, a crucial quantity in calculating $G_{\rho\rho}$ starting from equation (25) is $G_{\hat{g}\rho}$ and the FDR (38) links it to $G_{v\rho}$. The behaviour of the latter has not been considered in the matrix of correlations involving the set $\{\rho,\mathbf{g}\}$. To include the FDR requirements in the analysis, we need to extend the set of slow modes in the formulation. This is discussed next.

5.1. Implications of fluctuation dissipation relations

In order to check implications of the FDR of equation (38), we extend in the MSR theory with fields $\{\rho,\mathbf{g}\}$ to include the field $\mathbf{v}(\mathbf{x},t)$ through the non-linear constraint (19). The action functional $\mathcal{A}$ of equation (20) has an additional term

$$\tilde{\mathcal{A}}[\rho,\mathbf{g},\mathbf{v}] = \mathcal{A}[\rho,\mathbf{g}] + \int \mathrm{d}t \int \mathrm{d}\mathbf{x}\, \{i\hat{\mathbf{v}}.(\mathbf{g} - \rho\mathbf{v})\}\,, \tag{41}$$

where the constraint (19) is implemented with inclusion of a new hatted field $\hat{\mathbf{v}}$ in the theory. For this definition of the MSR action, the corresponding Gaussian part, denoted with the inverse matrix $\mathbf{G}_0^{-1}$, is given in table 1. For the self-energy matrix Σ, the additional elements, over those described in equation (23) are obtained as follows:

$$\Sigma_{\hat{\mathrm{v}}_i\rho} = q_i\gamma_{\hat{\mathrm{v}}\rho}, \quad \Sigma_{\hat{\mathrm{g}}_i\mathrm{v}_j} = q_iq_j\gamma_{\hat{\mathrm{g}}\mathrm{v}}, \quad \Sigma_{\hat{\mathrm{v}}\mathrm{v}} = i\gamma_{\hat{\mathrm{v}}\mathrm{v}}. \tag{42}$$

To simplify matters we have replaced in all non-linear terms in equation (16) involving the g_i fields with ρv_i and hence the self-energy matrix Σ_{ab}, vanishes if at least on of the indices a or b is the momentum density g_i. Using the above described form of the self-energy matrix in Dyson equation (21), the inverse of the full Green's function matrix $\mathbf{G}$ with the extended set of slow modes $\{\rho,\mathbf{g},\mathbf{v}\}$ is obtained. The correlation and response functions are respectively obtained using the relations (25) and (26). The matrix $N_{\hat{\alpha}\beta}$ in the expression (26) for the response function is given in table 2. The denominator $\mathcal{D}$ in the right-hand side of equation (26) involves the matrix element $\gamma_{\hat{\mathrm{v}}\rho}$ and is obtained as

$$\mathcal{D}(q,\omega) = \rho_\mathrm{L}(\omega^2 - q^2c^2) + iq^2\gamma_R(\omega + iq^2\gamma_{\hat{\mathrm{v}}\rho}). \tag{43}$$

The various quantities in the right-hand side of equation (43) are organised similar to the linear theory. Thus ρ_L, c^2 and γ_R are identified as the respective renormalised quantities for the bare density ρ_0, speed of sound squared c_0^2 and longitudinal (kinematic) viscosity $\rho_0\gamma_0$. Equations (29) and (30) respectively define γ_R and c_0^2. In writing

Table 2. The matrix of the coefficients $N_{\alpha\hat{\beta}}$ in the numerator on right-hand side of equation (26) for the response functions.

	$\hat{\rho}$	$\hat{\mathrm{g}}$	$\hat{\mathrm{v}}$
ρ	$\omega\rho_{\mathrm{L}} + iq^2\gamma_R$	$\rho_{\mathrm{L}} q$	$iq^3\gamma_{\hat{\mathrm{g}}\mathrm{v}}$
g	$q(\rho_{\mathrm{L}} c^2 + i\gamma_{\hat{\mathrm{v}}\rho} q^2 \gamma_{\hat{\mathrm{g}}\mathrm{v}})$	$\rho_{\mathrm{L}}\omega$	$i\omega q^2\gamma_{\hat{\mathrm{g}}\mathrm{v}}$
v	$q(c^2 + i\gamma_{\hat{\mathrm{v}}\rho}(\omega + iq^2\gamma_R))$	$\omega + iq^2\gamma_{\hat{\mathrm{v}}\rho}$	$i(\omega^2 - q^2c^2 + i\omega q^2\gamma_R)$

the renormalised expression for $c^2(q,\omega)$ order q^2 contributions in (30) are ignored. The average density is renormalised in terms of single-hatted or the response self-energies. To leading order in wave number q, we write

$$\rho_{\mathrm{L}}(q,\omega) = \rho_0 + \gamma_{\hat{\mathrm{v}}\mathrm{v}}(q,\omega). \tag{44}$$

Using the above expressions for the correlation and response functions, we now consider the validity of the basic feedback mechanism of MCT that drives the ENE transition.

In the previous section we have already discussed the self-consistent feedback mechanism by which respective Fourier transforms of both the density auto-correlation $G_{\rho\rho}$ and the corresponding self-energy matrix element $\gamma_{\hat{\mathrm{g}}\hat{\mathrm{g}}}$ develop their respective singular contributions involving a $\delta(\omega)$ term. Setting both α and β in equation (25) respectively equal to ρ, v, etc it follows that all three correlation functions $G_{\rho\rho}$, $G_{\rho\mathrm{v}}$ and G_{vv} contain a δ function contribution through couplings to $\Sigma_{\hat{g}\hat{g}}$. On examining (25) it is clear that the self-consistent mechanism is viable only if $G_{\rho\hat{\mathrm{g}}}$ and $G_{\mathrm{v}\hat{\mathrm{g}}}$ are both nonzero. Using the explicit form of $G_{\mathrm{v}\hat{\mathrm{g}}}$ from table 2, this requires $\gamma_{\hat{\mathrm{v}}\rho}$ to be nonzero in the $\omega \to 0$ limit. However, the FDR (38) implies that if $G_{\mathrm{v}\rho}$ and G_{vv} blow up, then the imaginary parts of the response functions $G_{\hat{\mathrm{g}}\rho}$ and $G_{\hat{\mathrm{g}}\mathrm{v}}$ should respectively blow up. For this to happen, considering the explicit form of these response functions, as presented in table 2, $\mathcal{D}$ must blow up in the small ω limit. But in that case $G_{\rho\hat{\mathrm{g}}}$ will go to zero, which is a contradiction of the initial assumption. Thus, maintaining a nonzero $G_{\hat{\mathrm{g}}\rho}$ is not compatible with preserving the fluctuation–dissipation constraints while having a non-vanishing $\Sigma_{\hat{\mathrm{v}}\rho}$.

The self-energy element $\Sigma_{\hat{\mathrm{v}}\rho}$ originates from the non-linear constraint (19) included in the field theoretic formulation and is the key quantity in this analysis. If for some reason this self-energy $\Sigma_{\hat{\mathrm{v}}\rho}$ vanishes at zero frequency, then equation (25) implies that $G_{\rho\mathrm{v}}$ and G_{vv} vanish as ω goes to zero and do not show a $\delta(\omega)$ component. Then the determinant $\mathcal{D}$ can be finite. In this case one may have an ENE transition in this model. Alternatively, if the validity of an FDR like (38) is ignored, then the element $\gamma_{\hat{\mathrm{v}}\rho}$ is not involved, and all three correlations $G_{\rho\rho}$, G_{vv} and G_{vv} respectively develop a finite long time limit which is characteristic of the non-ergodic state. In this case the ENE transition is characterised by the divergence of viscosity and vanishing of the self-diffusion coefficient [45–47]. Hence for a BD system, if the FDRs are enforced then the ENE transition is not supported since there is no *a priori* reason for the self-energy $\Sigma_{\rho\hat{\mathrm{v}}}$ to become zero.

6. Discussion

This present work focuses on the collective behaviour of a fluid in which the constituent particles follow BD. The microscopic-level time evolution is described in terms of dissipative equations. We analyse the viability of a sharp transition of this liquid into a non-ergodic state as a consequence of mode-coupling effects. The modes here refer to collective processes which signify conservation laws for a many-particle system. The non-ergodic state of the fluid is characterised by the property that the time correlation function of collective density fluctuations remain nonzero in the long time limit. Correlations functions for the fluid are calculated here using equations of motion for the local densities $\{\rho, \mathbf{g}\}$ of mass and momentum, respectively. These equations have smooth spatio-temporal dependence and contain stochastic noise. The equation for ρ is the continuity equation, which is invariant under time reversal. The equation for momentum density $\mathbf{g}$ for the BD fluid has a dissipative term which is linear in $\mathbf{g}$. The associated noise in this generalised Langevin equation for the time evolution of $\mathbf{g}$ is multiplicative. Time correlation of density fluctuations is calculated by averaging over the noise. The renormalised theory is formulated in terms of correlation functions which include the role of non-linear couplings of the hydrodynamic modes in the FNH equations.

We make a non-perturbative analysis of the renormalised model using the Dyson–Schwinger equations of the associated MSR field theory that is based on the equations of FNH. The corresponding MSR action functional obeys time reversal symmetry rules that give rise to FDRs between correlation and response functions. These are given in equation (38). In the present case of BD, the kinetic coefficients appearing in the MSR action functional have linear dependence [39] on the fluctuating field ρ, the associated noise being multiplicative. However, the set of time reversal transformations for invariance of the MSR action is the same as that for ND fluid. Hence the associated FDR stated in equation (38) is also the same for the two dynamics.

In the model involving $\{\rho, \mathbf{g}\}$ fields, the response functions like $G_{\rho\hat{g}}$ etc with one hatted field are related to correlation functions between two unhatted fields through fluctuation–dissipation constraints (38). However, within the set $\{\rho, \mathbf{g}\}$ these are not linear relations. To preserve the FDR in a *self-consistent* manner, the MSR theory is formulated with the extended set of fields $\{\rho, \mathbf{g}, \mathbf{v}\}$, where the current field $\mathbf{v}$ is related to the primary fields through the non-linear constraint (19). In the corresponding MSR action functional, a new conjugate field $\hat{\mathbf{v}}$ is introduced to enforce the non-linear constraint (19) between the fluctuating fields and as a result elements like $\Sigma_{\hat{\mathrm{v}}\rho}$ of the self-energy matrix Σ enter the model. Our analysis shows that nonzero values of $\Sigma_{\hat{\mathrm{v}}\rho}$ are not compatible with the self-consistent feedback mechanism essential for driving the fluid from an ergodic to a non-ergodic state. Since there is no *a priori* reason for $\Sigma_{\hat{\mathrm{v}}\rho}$ to become zero, the dynamics which preserve FDR self-consistently do not support an ENE transition in the model.

The present work treats the dynamics in terms of collective modes, which includes both fluctuating variables $\{\rho, \mathbf{g}\}$. It is useful to discuss here the equivalent description in terms of only ρ. For Brownian systems, the equation of motion for a single particle is obtained in the over-damped limit with a first-order stochastic partial differential equation, known as a Smoulchowski equation [48], which involves only the position variables. An exact representation of these dynamics has been obtained [49–51] in terms of a balance equation for the microscopic density $\tilde{\rho}(\mathbf{x}, t)$. The corresponding FNH description for the coarse-grained density involves a single Langevin equation for $\rho(\mathbf{x}, t)$ [17]. This stochastic partial differential equation has a noise which is multiplicative and a driving free energy $F[\rho]$ which is same as the standard Ramakrishnan–Youssuff functional [63] in terms of direct correlation functions [7]. For the case of an ND fluid with reversible dynamics at the microscopic level, the same equation for ρ is obtained by applying the over-damping approximation to FNH equations with $\{\rho, \mathbf{g}\}$ and integrating out the $\mathbf{g}$ field [52]. This formulation of the dynamics in terms of ρ constitutes the so-called dynamic density functional model (DDFT) [53–59]. The effects of non-linearities and the feasibility of the ENE transition in this DDFT model have been studied using MSR field theory [68] and it was shown that ENE transition [69] is not supported in the reduced description as well. For the DDFT model, the role of $1/\rho$ terms in the field theory gives rise to similar constraints to those we discussed for the $\{\rho, \mathbf{g}\}$ case in the present work. The feasibility of an ENE transition in the DDFT model would require ignoring the presence of $1/\rho$ non-linearities [60]. It will also be useful to note here that in both types of model, in terms of ρ and $\{\rho, \mathbf{g}\}$, the presence of a functional derivative term $\rho\nabla_i\{\delta F[\rho]/\delta\rho\}$ in the respective FNH equations constitutes the key non-linearity driving the slow dynamics. Even for a $F[\rho]$, quadratic in density fluctuations, there is a dynamic non-linearity in the respective FNH equations. For simpler dynamics in which the factor of ρ in front of the functional derivative $(\delta F[\rho]/\delta\rho)$ is simply changed to ρ_0, non-Gaussian Hamiltonians are required for generating the crucial coupling for slow dynamics. Moreover, in DDFT models, it has been demonstrated that $F[\rho]$, which is non-linear only in the ideal gas term (with the standard logarithmic term), proves insufficient [70] for producing the very slow dynamics typical of ideal MCT. However, a similar replacement (of ρ with ρ_0) in the free energy functional (11), making it completely Gaussian in the $\{\rho, \mathbf{g}\}$ formulation, does not affect the basic conclusions of the original model [9, 11, 12].

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Appendix A. Fluctuating nonlinear hydrodynamics equations for fluids with Brownian dynamics

We present here the fluids with interacting Brownian particles. The force on each particle has a regular part due to interaction with other particles in the fluid through a two-body potential $U(\mathbf{r})$ and a stochastic part described with a random noise. In the present work we apply the Markovian approximation of widely different time scales of the two dynamics. The fast part constitutes the noise, which is assumed to be white and Gaussian. The microscopic dynamics are described in terms of the time evolution of the position $\mathbf{x}_\alpha$ and that of the momentum $\mathbf{p}_\alpha$ of the αth $(\alpha = 1, ...N)$ particle. For $\mathbf{p}_\alpha$, the equation of motion has a regular component due to the interaction potential, a dissipative or frictional term and noise, respectively.

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$$\frac{\mathrm{d}\mathbf{p}_\alpha^i(t)}{\mathrm{d}t} = -\sum_{\nu=1}^{N} \nabla_\alpha^i U(\mathbf{x}_\alpha(t) - \mathbf{x}_\nu(t)) + \gamma_{ij}^0 \mathbf{p}_\alpha^j + \boldsymbol{\xi}_\alpha^i(t), \tag{A.1}$$

where γ_{ij}^0 is a dissipative coefficient having the dimension of inverse time. A realistic example of the system is a set of solute particles with effective interaction $U(\mathbf{x}_\alpha - \mathbf{x}_\nu)$. The solvent is a liquid at temperature T, which produces the noise $\boldsymbol{\xi}$. Correlation of the noise is related to the dissipative coefficient γ_{ij} through the FDR:

$$\langle \xi_\alpha^i(t)\xi_\nu^j(t')\rangle = 2\beta^{-1}\gamma_{ij}^0\delta_{\alpha\nu}\delta(t-t'). \tag{A.2}$$

A.1. Exact balance equations

The balance equations for the collective densities $\{\tilde{\rho}, \tilde{\mathbf{g}}\}$ obtained above are exact representations of the microscopic dynamics. Deduction of the balance equations for $\{\tilde{\rho}, \tilde{\mathbf{g}}\}$ has been described in [61]. For mass density we have the continuity equation

$$\frac{\partial\tilde{\rho}}{\partial t} + \nabla.\tilde{\mathbf{g}} = 0. \tag{A.3}$$

For momentum density $\mathbf{g}$ is a generalised Langevin equation:

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{g}_i(\mathbf{x},t) + \sum_\alpha \frac{p_\alpha^i p_\alpha^j}{m}\nabla^j\tilde{\rho}(\mathbf{x},t) + \tilde{\rho}(\mathbf{x},t)\nabla_x^i \int dx' U(\mathbf{x}-\mathbf{x}')\tilde{\rho}(\mathbf{x}',t) \\ &\qquad - \gamma_{ij}^0\tilde{g}_i(\mathbf{x},t) = \tilde{\vartheta}_i(\mathbf{x},t). \end{aligned} \tag{A.4}$$

The noise $\tilde{\vartheta}_i$ is defined in terms of the noise ξ_α in the micro-dynamic equations:

$$\tilde{\vartheta}_i(\mathbf{x},t) = \sum_\alpha \delta(\mathbf{x} - \mathbf{x}_\alpha^i(t))\xi_\alpha^i(t). \tag{A.5}$$

The correlation of noise $\tilde{\vartheta}_i(\mathbf{x},t)$ is obtained using the definition (A.2) of the average of noise $\boldsymbol{\xi}_\alpha$ in the microscopic equations of motion.

$$\langle\tilde{\vartheta}_i(\mathbf{x},t)\tilde{\vartheta}_j(\mathbf{x}',t)\rangle_{\mathrm{T}} = 2\beta^{-1}\gamma_{ij}^0\tilde{\rho}(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{A.6}$$

The pair of angular brackets $\langle..\rangle_{\mathrm{T}}$ in equation (A.6) represents an average over the bath variables which maintain the microscopic dynamics of the Brownian particles, given by stochastic equation (A.1) at a constant temperature T. A typical example of such a case are the stochastic dynamics of the solute particles in a solvent, which are in equilibrium at temperature T. The average in equation (A.6) is implied over the positions and momenta of the solvent particles [62].

A.2. Fluctuating nonlinear hydrodynamics with multiplicative noise

Averaging the exact balance equations we obtain the time evolution of the coarse-grained densities $\{\rho(\mathbf{x},t), \mathbf{g}(\mathbf{x},t)\}$. For the mass density we obtain the continuity equation (5) with the momentum density $\mathbf{g}$ as its current. For the momentum density $\mathbf{g}$ we average the equation (A.4) reducing it to a stochastic partial differential equation which involves the fields $\rho(\mathbf{x},t)$ and $\mathbf{g}(\mathbf{x},t)$ and a stochastic component called noise. The primary task in obtaining the coarse-grained equations is to evaluate the non-equilibrium averages of the last two terms on the left-hand side of (A.4). This has been presented in [17] and for the sake of completeness we briefly review it below.

The second and third terms on the left-hand side of equation (A.4) are evaluated by replacing the non-equilibrium average with that over the local equilibrium ensemble. We transform to a frame moving with the fluid. This co-moving frame (denoted by prime) has the local velocity $\mathbf{v}(\mathbf{r}, t)$ in a continuum description and obeys the following rules of transformation:

$$\mathbf{x}_\alpha = \mathbf{x}'_\alpha, \quad \text{and} \quad \mathbf{p}_\alpha = \mathbf{p}'_\alpha + m\mathbf{v}(\mathbf{x}'_\alpha). \tag{A.7}$$

In a co-moving frame the fluid is locally at rest. Using the concept of Gibbsian ensemble, the distribution is written as

$$f_{\mathrm{le}}(\Gamma'_N, t) = Q_l^{-1} \exp\Big[-\beta\Big\{H' - \int \mathrm{d}\mathbf{x}\mu(\mathbf{x}, t)\tilde{\rho}'(\mathbf{x})\Big\}\Big] \equiv Q_l^{-1} \exp\Big(-\beta\tilde{H}'\Big), \tag{A.8}$$

where Γ'_N symbolises the phase space coordinates and H' is a Hamiltonian in the local rest frame in terms of primed coordinates $\mathbf{p}'_\alpha$. The temperature T is taken to be constant and the chemical potential $\mu(\mathbf{x}, t)$ represents the local thermodynamic property in the local equilibrium ensemble. We obtain the average with respect to the local equilibrium distribution as

$$\nabla_j \left\langle \sum_\alpha \Big\{\frac{p_\alpha^i p_\alpha^j}{m}\delta(\mathbf{x} - \mathbf{x}_\alpha(t))\Big\} \right\rangle_{\mathrm{l.e}} = \beta^{-1}\nabla_i \rho(\mathbf{x}, t) + \nabla_j \left[\frac{g_i(\mathbf{x}, t)g_j(\mathbf{x}, t)}{\rho(\mathbf{x}, t)}\right]. \tag{A.9}$$

Substituting the above result in the coarse-grained equation (A.4), we obtain the equation of motion for the coarse-grained momentum density $g_i(\mathbf{x}, t)$ as

$$\frac{\partial g_i}{\partial t} + \gamma^0_{ij} g_j + \nabla_j \left[\frac{g_i g_j}{\rho}\right] + \mathcal{I}[\rho] = \vartheta_i \tag{A.10}$$

where the term $\mathcal{I}[\rho]$ is defined as

$$\mathcal{I}[\rho] = \beta^{-1}\nabla^i \rho(\mathbf{x}, t) + \left\langle \tilde{\rho}(\mathbf{x}, t)\nabla^i_x \int dx' U(\mathbf{x} - \mathbf{x}')\tilde{\rho}(\mathbf{x}', t) \right\rangle_{\mathrm{le}}. \tag{A.11}$$

The integral $\mathcal{I}[\rho]$ through some simple algebra reduces to $\langle -i\mathcal{L}\tilde{g}'(\mathbf{x}, t)\rangle_{l.e.}$ where $\mathcal{L}$ is the Liouville operator [17]. Using the derivative form of operator $\mathcal{L}$, the integral $\mathcal{I}$ further simplifies [17]).

$$\begin{aligned}\mathcal{I}[\rho] = \langle -i\mathcal{L}\tilde{g}'_i(\mathbf{x})\rangle_{\mathrm{l.e}} &= \beta \int \mathrm{d}\mathbf{x}'\mu(\mathbf{x}')\langle \tilde{g}'_i(\mathbf{x}) i\mathcal{L}\tilde{\rho}(\mathbf{x}')\rangle_{\mathrm{l.e}} \\ &= -\beta \int \mathrm{d}\mathbf{x}'\mu(\mathbf{x}')\langle \tilde{g}'_i(\mathbf{x})\nabla'_j \tilde{g}'_j(\mathbf{x}')\rangle_{\mathrm{l.e}} \\ &= \rho(\mathbf{x})\nabla_i \mu(\mathbf{x}).\end{aligned} \tag{A.12}$$

The right-hand side of equation (A.12) is expressed in terms of the hydrodynamic fields by appealing to a corresponding thermodynamic relation. We make note here that the Helmholtz free energy F is expressed as a functional of the inhomogeneous density

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$\rho(\mathbf{x})$. Using the equilibrium relation $F - G = \Omega \equiv -PV$, where Ω is the thermodynamic potential, we have in the density functional formalism [28, 63]

$$F_{\mathrm{U}}[\rho(\mathbf{x})] \equiv \Omega[\rho(\mathbf{x})] + \int \mathrm{d}\mathbf{x}\rho(\mathbf{x})\mu(\mathbf{x}). \tag{A.13}$$

$\Omega[\rho]$ is a functional of the density obtained from the equivalent result of grand canonical ensemble partition function $\Omega[\rho(\mathbf{r})] \equiv -k_B T \ln \Xi$. The density functional theory identifies the equilibrium density by minimising the grand potential $[\delta\Omega/\delta\rho(\mathbf{x})] = 0$. Using the above relations it then follows that the corresponding Helmholtz free energy functional satisfies

$$\frac{\delta F_{\mathrm{U}}[\rho]}{\delta\rho(\mathbf{x})} = \mu(\mathbf{x}). \tag{A.14}$$

Using the results (A.12) and (A.14) in equation (A.11), we obtain an equation for the momentum density $\mathbf{g}$ with a coupling to the collective density fluctuations.

$$\frac{\partial g_i}{\partial t} + \nabla_j \left[\frac{g_i g_j}{\rho}\right] + \rho\nabla_i \frac{\delta F_{\mathrm{U}}}{\delta\rho} + \gamma_{ij}^0 g_j = \vartheta_i. \tag{A.15}$$

The noise $\vartheta_i(\mathbf{x}, t)$ in the right-hand side of the generalised Langevin equation (A.15) is obtained by coarse-graining of the noise $\tilde{\vartheta}_i(\mathbf{x}, t)$ defined in equation (A.5). Obtaining the correlation of coarse-grained noise ϑ_i at two different points requires consideration of the following two steps. First, we average ξ_α^i in the microscopic equations over the different configurations of the Brownian particles so as to obtain the coarse-grained noise $\vartheta_i(\mathbf{x}, t)$. Second, the product of the respective noise at two different space time points is averaged over states in which the equilibrium temperature of the bath is maintained at T. These two averages are respectively denoted as $\langle ...\rangle_{\mathrm{C}}$ and $\langle ...\rangle_{\mathrm{T}}$. The noise correlation in the present case is estimated by interchanging the order of the two operations stated above in the following way:

$$\begin{aligned}
&\langle \vartheta_i(\mathbf{x}, t)\vartheta_j(\mathbf{x}', t')\rangle = \left\langle \left\langle \tilde{\vartheta}_i(\mathbf{x}, t)\right\rangle_{\mathrm{C}} \left\langle \tilde{\vartheta}_j(\mathbf{x}', t')\right\rangle_{\mathrm{C}} \right\rangle_{\mathrm{T}} \\
&\approx \left\langle \left\langle \tilde{\vartheta}_i(\mathbf{x}, t)\tilde{\vartheta}_j(\mathbf{x}', t')\right\rangle_{\mathrm{C}} \right\rangle_{\mathrm{T}} = 2k_B T\gamma_{ij}^0 \langle\tilde{\rho}(\mathbf{x}, t)\rangle_{\mathrm{C}}\delta(\mathbf{x} - \mathbf{x}')\delta(t - t'). \\
&\equiv 2\beta^{-1} L_{ij}^0[\rho]\delta(\mathbf{x} - \mathbf{x}')\delta(t - t').
\end{aligned} \tag{A.16}$$

We have defined $L_{ij}^0[\rho] = \gamma_{ij}^0 \rho(\mathbf{x}, t)$. In the Markovian approximation for large separation of time scales we assume that correlating the noise is independent of the coarse-graining process. To summarise, the equations (5), (A.15) and (A.16) constitute the basic set of FNH equations for a fluid whose microscopic-level dynamics are Brownian.

Appendix B. The Martin-Siggia-Rose action with multiplicative noise

We briefly sketch developments of the MSR field theory for the FNH equations for the BD system in this appendix.

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B.1. Martin-Siggia-Rose action functional

We consider a functional $f[\psi]$ of the set of fields $f[\psi]$ written in the form

$$f[\psi] = \int D\psi' \delta(\psi - \psi')\, f[\psi']. \tag{B.1}$$

The functional integral with $D\psi'$ is defined as the multiple integral over the set $\{\psi_i\}$ at all space points on a grid of size ϵ

$$D(\psi) \equiv \lim_{\epsilon\to 0} \prod_i \int d\psi(i). \tag{B.2}$$

The functional δ function is defined as

$$\delta(\psi - \psi') = \lim_{\epsilon\to 0} \prod_i \delta(\psi(i) - \psi'(i)) \tag{B.3}$$

where $i \in \Lambda^{d+1}$ belongs to a $(d+1)$-dimensional lattice (including $d =$ number of spatial dimensions and time). If ψ corresponds to the set of hydrodynamic fields $\psi \equiv \{\rho, \mathbf{g}\}$ for the fluids then they satisfy the equations of FNH written in the form

$$\frac{\partial \psi(1)}{\partial t_1} + \Gamma[\{\psi\}] = \theta(1). \tag{B.4}$$

The quantity $\Gamma[\psi]$ represents the deterministic part of the equation of motion and is a shorthand way of writing $[Q_{ab} + L^0_{ab}](\delta F/\delta\psi_b)$ in the equation of motion (6). The random part or noise is denoted as θ and its correlation is related to the bare transport coefficient $\{L^0_{ab}\}$ as indicated in equation (13). Since the noise in this case is multiplicative, bare transport coefficients are dependent on the local field variables ψ. Since ψ is a solution of the equation of motion (B.4) we replace the delta function in the right-hand side of equation (B.1) with a change of coordinates to

$$f[\psi] = \int D\psi J[\psi, \theta]\, f[\psi] \delta\left(\partial_1 \psi(1) + \Gamma[\psi] - \theta(1)\right) \tag{B.5}$$

where ∂_1 refers to the time derivative with respect to t_1 and J is the Jacobian of the transformation, due to the change of argument of the delta function.

$$J[\psi, \theta] = \det \left| \frac{\delta\theta}{\delta\psi} \right|. \tag{B.6}$$

Using a causal connection in the time discretisation the Jacobian of this transformation is treated as a constant C_0. Finally, replacing the δ function in the right-hand side of (B.6) by its functional Fourier transform in terms of a conjugate field, we obtain

$$f[\psi] = C_0 \int D\psi \int D\hat{\psi} f[\psi] \exp\left[-i \int \hat{\psi}(1) \left\{ \frac{\partial \psi(1)}{\partial t_1} + \Gamma[\psi] - \theta(1) \right\} \right] \tag{B.7}$$

.

Next, we average over the randomness and obtain the corresponding noise-averaged quantity as

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$$\langle f[\psi]\rangle = C_0 \int D\psi' \int D\hat{\psi} f[\psi] \left\langle \exp\left[-\mathrm{i}\int d1 \hat{\psi}(1) \left(\frac{\partial \psi(1)}{\partial t_1} + \Gamma[\psi] - \theta(1)\right)\right]\right\rangle \tag{B.8}$$

Further simplification of the right-hand side will require the details of the nature of randomness described by θ. This should lead to an action functional $\mathcal{A}$ for developing an appropriate field theoretic model which takes into account the role of the non-linearities in the equation of motion. Equation (B.8) is expressed in the form

$$\langle f[\psi]\rangle = C_0 \int D\psi \int D\hat{\psi} f[\psi] \exp\left[-\mathcal{A}[\psi, \hat{\psi}]\right] \tag{B.9}$$

where the action functional $\mathcal{A}$ is obtained for the case of compressible fluid involving equations (5) and (A.15) as

$$\begin{aligned}\mathcal{A}[\psi,\hat{\psi}] = \int \mathrm{d}t \int \mathrm{d}\mathbf{x} \Big\{ & i\hat{g}_i \left[\frac{\partial g_i}{\partial t} + \rho \nabla_i \frac{\delta F_u}{\delta \rho} + \nabla_j \left(\frac{g_i g_j}{\rho}\right) - \gamma^0_{ij} g_j\right] \\ & + i\hat{\rho}\left[\frac{\partial \rho}{\partial t} + \nabla_j g_j\right]\Big\} \quad + C[\hat{\psi}],\end{aligned} \tag{B.10}$$

where the cumulant $C[\hat{\psi}]$ is obtained as

$$C[\hat{\psi}] = \sum_{n=1}^{\infty} \frac{1}{n} \left[\hat{\psi}(1) \ldots . \hat{\psi}(n)\right] \langle\langle \theta(1) \ldots . \theta(n)\rangle\rangle \tag{B.11}$$

where $\langle\langle \ldots \rangle\rangle$ is the cumulant average of the random force. For Gaussian random forces we obtain

$$C[\hat{\psi}] = \hat{\psi}(1) \langle\langle \theta(1)\rangle\rangle + \frac{1}{2}\hat{\psi}(1)\hat{\psi}(2) \langle\langle \theta(1)\theta(2)\rangle\rangle . \tag{B.12}$$

Applying the result (13) for noise correlation we obtain the MSR action function

$$\begin{aligned}\mathcal{A}[\psi,\hat{\psi}] = \int \mathrm{d}t \int \mathrm{d}\mathbf{x} \Big\{ & \beta^{-1} \hat{g}_i L^0_{ij}[\rho] \hat{g}_j + i\hat{g}_i \left[\frac{\partial g_i}{\partial t} + \rho \nabla_i \frac{\delta F_u}{\delta \rho} + \nabla_j \left(\frac{g_i g_j}{\rho}\right) - \gamma^0_{ij} g_j\right] \\ & + i\hat{\rho}\left[\frac{\partial \rho}{\partial t} + \nabla_j g_j\right]\Big\}.\end{aligned} \tag{B.13}$$

The quantity C_0 in the right-hand side of equation (B.9) gives a normalisation constant which is fixed to ensure that $\langle 1\rangle = 1$.

$$\langle f[\psi]\rangle = \frac{\int D\psi \int D\hat{\psi} f[\psi] \exp\left[-\mathcal{A}[\psi,\hat{\psi}]\right]}{\int D\psi \int D\hat{\psi} \exp\left[-\mathcal{A}[\psi,\hat{\psi}]\right]}. \tag{B.14}$$

B.2. Renormalised correlation functions

In this section, we briefly discuss the necessary MSR field theoretic analysis developed along the lines of earlier works [22] for ND fluids. In the following discussion we use a functional integral formulation [43, 64–67] of the MSR theory. We adopt a compact notation in which the spatial coordinate $\mathbf{x}_1$ and time t_1 for both the hatted and unhatted fields are all incorporated into one single index 1 of the vector field variable $\Psi(1)$. The partial differential equations of the FNH involve only the unhatted fields, and include the non-linear couplings of the slow modes. The dynamics go beyond the linear level and the field theory is renormalised due to non-linear couplings of the fields Ψ in the equations of motion. Correlation of the slow variables averaged over the noise is computed from the functional derivatives of the generating functional Z_{U} obtained in terms of an action A_{U} as

$$Z_{\mathrm{U}} = I_0 \int D\psi \int D\hat{\psi} \exp[-A_{\mathrm{U}}(\psi, \hat{\psi})] \tag{B.15}$$

with I_o being a constant. The deduction of the MSR action functional A_{U} involve [9] enforcing the FNH equations for the time evolution of the hydrodynamic fields ψ through introduction of the corresponding set of hatted fields $\hat{\psi}$. In order to facilitate the discussion of the renormalisation scheme, the action functional is written in a polynomial form,

$$A_{\mathrm{U}}[\Psi] = \frac{1}{2}\sum_{1,2} \Psi(1) G_o^{-1}(12)\Psi(2) + \frac{1}{3}\sum_{1,2,3} V(123)\Psi(1)\Psi(2)\Psi(3) - \sum_{1} \Psi(1)\mathrm{U}(1). \tag{B.16}$$

The renormalised correlation functions are obtained in a systematic manner using the non-Gaussian part of the action functional presented in equation (B.16). Going by the form of non-linearities in equation (A.15) for a compressible liquid we have only indicated a cubic term here in the action functional. The vertex functions $V(123)$ in the MSR action functional are defined in a way so that they are symmetric under exchange of the indices. The one-point function $G(1) = \langle\Psi(1)\rangle$ is obtained from the generating function Z_{U} in terms of the derivative

$$\langle\Psi(1)\rangle = \frac{\delta}{\delta \mathrm{U}(1)} [\ln Z_{\mathrm{U}}]\,. \tag{B.17}$$

Including the density variable in the set of slow variables Ψ as $\delta\rho(1) = \rho(1) - \langle\rho(1)\rangle$, it follows directly that $G(1)$ vanishes as $\mathrm{U} \to 0$. The two-point function $G(12)$ is given by

$$G(12) = \frac{\delta}{\delta \mathrm{U}(2)} G(1) = \langle\delta\Psi(1)\delta\Psi(2)\rangle, \tag{B.18}$$

where $\delta\Psi(1) = \Psi(1) - \langle\Psi(1)\rangle \equiv \Psi(1)$. The inverse of the two-point correlation matrix $G(12)$ is defined through the relation

$$\sum_{3} G^{-1}(13)G(32) = \delta(12). \tag{B.19}$$

The simplest level form of the correlation functions are zeroth-order quantities denoted by the matrix $\mathbf{G}_0$. The Gaussian part of the action which is quadratic order in the fields is expressed with $\mathbf{G}_0$. Contributions from all higher order terms in the action or so-called vertices are expressed with the self-energy matrix $\mathbf{\Sigma}$. The full correlation functions, including effects of the non-linearities, are denoted as the matrix $\mathbf{G}$. The inverse of the full Green's function matrix is expressed in terms of the Dyson equation

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$$\mathbf{G}^{-1} = \mathbf{G}_0^{-1} - \Sigma. \tag{B.20}$$

We note here the properties which follow from the general structure of $\mathbf{G}_0^{-1}$.

(a) $[\mathbf{G}_0{}^{-1}]_{ab} = 0$, which follows from the action functional obtained in the MSR field theory [28]. In addition we have the property that $[\mathbf{G}_0{}^{-1}]_{a\hat{b}} = -[\mathbf{G}_0{}^{-1}]_{\hat{a}b}$. The structure of matrix $\mathbf{G}_0^{-1}$ involving the whole set of actual and hatted fields $\{\psi_a, \psi_{\hat{a}}\}$ is of the form:

$$G_0^{-1} = \begin{bmatrix} \bigcirc & \mathcal{B}_0^\dagger \\ \mathcal{B}_0 & \mathcal{C}_0 \end{bmatrix} \tag{B.21}$$

where the matrix $\mathcal{B}_0^\dagger$ is the transpose and complex conjugate of the matrix $\mathcal{B}_0$. $\mathcal{C}_0$ presents the elements of $\mathbf{G}_0^{-1}$ whose indices are both hatted fields. The $\bigcirc$ in the right-hand side of equation (B.21) represents the null matrix with all its elements equal to zero.

(b) Correlation functions of the Gaussian theory correspond to linearised dynamics of the fields. Effects of non-linear dynamics are expressed in terms of the so-called 'self-energy' matrix $\mathbf{\Sigma}$, which is defined through the Schwinger–Dyson equation, (B.20). The so-called self-energy $\mathbf{\Sigma}$ is expressed in a perturbation series expansion in terms of the corresponding vertices which appear in the non-Gaussian terms of the MSR action. Renormalised transport coefficients in the model are obtained from the respective self-energy matrix elements, and correlation and response functions of the fully non-linear theory, i.e. elements of the $\mathbf{G}$ matrix are expressed in terms of these renormalised transport coefficients. For the action functional (B.16) involving cubic vertices $V(123)$, the self- energy matrix $\mathbf{\Sigma}$ is self-consistently expressed in terms of the correlation functions

$$\Sigma(12) = \sum_{3,4,5,6} V(134)G(35)G(46)V(526). \tag{B.22}$$

From the causal nature of the response functions in MSR field theory, it follows that if both indices of Σ are unhatted, $\mathbf{\Sigma}_{\alpha\beta} = 0$.

Inverting the matrix $\mathbf{G}^{-1}$ having the above structure, we obtain for the correlation functions of the physical, unhatted field variables

$$G_{\alpha\beta} = -\sum_{\mu\nu} G_{\alpha\hat{\mu}}\mathcal{C}_{\hat{\mu}\hat{\nu}}G_{\hat{\nu}\beta} \tag{B.23}$$

where $\mathcal{C}_{\hat{a}\hat{b}}$ denotes the elements of the matrix $\mathcal{C}$ defined in terms of $\mathcal{C}_0$ and the corresponding block of the Σ matrix.

$$\mathcal{C}_{\hat{a}\hat{b}} = [\mathcal{C}_0]_{\hat{a}\hat{b}} - \Sigma_{\hat{a}\hat{b}}. \tag{B.24}$$

The response functions are expressed in the general form

$$G_{a\hat{b}}(q,\omega) = \frac{N_{a\hat{b}}(q,\omega)}{\mathcal{D}(q,\omega)}, \tag{B.25}$$

where $N_{a\hat{b}}$ and $\mathcal{D}$ are respectively the co-factor and the determinant of matrix $\mathcal{B}$ which is defined in terms of the matrix $\mathcal{B}_0$ introduced in equation (B.21) and response block of the self-energy matrix Σ.

$$\mathcal{B}_{\hat{a}b} = [\mathcal{B}_0]_{\hat{a}b} - \Sigma_{\hat{a}b}. \tag{B.26}$$

The various renormalised transport coefficients which appear in the right-hand side of (B.25) are expressed in terms of the corresponding response elements $\Sigma_{\hat{a}b}$ of the self-energy matrix.

For the case of compressible liquids, the bare viscosities $\{L_{ij}^0\}$ are renormalised due to non-linear coupling of the hydrodynamic modes in the FNH equations. The corresponding renormalising contributions are obtained in terms of the elements of the self-energy matrix Σ. The latter is defined in the Schwinger–Dyson equation (B.20). Thus the renormalised longitudinal viscosity $L(\omega)$ is obtained from the self-energy $\Sigma_{\hat{g}\hat{g}}$. In this work we discuss the structure of the renormalised theory from a non-perturbative approach.

B.3. Invariance of the Martin-Siggia-Rose action

Consider the MSR action for a fluid with BD in the following form:

$$\mathcal{A}[\psi,\hat{\psi}] = \int_{t_1}^{t_2} \mathrm{d}t \Big[\hat{\psi}_i(t)\beta^{-1}L_{ij}^0\hat{\psi}_j(t) + i\hat{\psi}_i(t)\left(\frac{\partial\psi_i}{\partial t} + \left\{Q_{ij}[\psi] - L_{ij}^0\right\}\frac{\delta F_t}{\delta\psi_j(t)}\right)\Big] \tag{B.27}$$

where $L_{ij}^0 = \rho(\mathbf{x},t)\gamma_{ij}^0$ is the bare transport coefficient with index i and j referring to fields g_i and g_j. It is also dependent on the fluctuating density ρ. The time reversal property of the MSR action $\mathcal{A}[\psi,\hat{\psi}]$ depends on how the fields ψ and $\hat{\psi}$ change under $\mathcal{T}$. We take

$$\mathcal{T}\psi_a(t) = \epsilon_a\psi_a(-t). \tag{B.28}$$

We note that the action functional $\mathcal{A}$ involves other than ψ_a and $\hat{\psi}_a$ the functional derivative $\delta F/\delta\psi_a$. Therefore we use the following prescription for $\hat{\psi}_a$:

$$\mathcal{T}\hat{\psi}_a(t) = -\epsilon_a\left[\hat{\psi}_i(-t) - i\beta\left(\frac{\delta F}{\delta\psi_i}\right)_{-t}\right]. \tag{B.29}$$

We now consider using (32) and (33) the effect of time reversal on the MSR action. As for the equations of motion, the continuity equation (5) is invariant under time reversal while equation (9) for momentum density has both reversible and dissipative parts. The transformations for these are controlled by the following relations for the PB Q_{ab} and bare transport matrix L_{ij}^0:

$$Q_{ab}[\psi(-t)] = -\epsilon_a\epsilon_b Q_{ab}[\psi(t)] \tag{B.30}$$

$$L_{ij}^0(t)[\psi(-t)] = \epsilon_a\epsilon_b L_{ij}[\psi(-t)]. \tag{B.31}$$

Using the above rules in the MSR action functional (B.27) we obtain the corresponding result for time-reversed functions as

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$$\mathcal{A}[\psi(-t),\hat{\psi}(-t)] = \beta^{-1}\int_{t_1}^{t_2} \mathrm{d}t \left[\hat{\psi}_i(-t) - i\beta\left(\frac{\delta F}{\delta\psi_i}\right)_{-t}\right] L^0_{ij}\left[\hat{\psi}_j(-t) - i\beta\left(\frac{\delta F}{\delta\psi_j}\right)_{-t}\right]$$
$$+ i\epsilon_i\int_{t_1}^{t_2}\mathrm{d}t\left[\hat{\psi}_i(-t) - i\beta\left(\frac{\delta F}{\delta\psi_i}\right)_{-t}\right]\left[\epsilon_i\frac{\partial\psi_i(-t)}{\partial(-t)} + \epsilon_i\epsilon_j\left\{Q_{ij}[\psi(-t)] + L^0_{ij}\right\}\epsilon_j\left(\frac{\delta F}{\delta\psi_j}\right)_{-t}\right]$$
$$\equiv \mathcal{I}_1 + \mathcal{I}_2. \tag{B.32}$$

The two integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ are evaluated in the following forms

$$\mathcal{I}_1 = \int_{t_1}^{t_2}\mathrm{d}t L^0_{ij}\left[\hat{\psi}_i\beta^{-1}\hat{\psi}_j - 2i\hat{\psi}_i\left(\frac{\delta F}{\delta\psi_j}\right) - \beta\left(\frac{\delta F}{\delta\psi_i}\right)\left(\frac{\delta F}{\delta\psi_j}\right)\right]_{-t} \tag{B.33}$$

$$\mathcal{I}_2 = i\int_{t_1}^{t_2}\mathrm{d}t\left[\hat{\psi}_i\left\{\frac{\partial\psi_i}{\partial t} + \left(Q_{ij}[\psi] + L^0_{ij}\right)\frac{\delta F}{\delta\psi_j}\right\}\right]_{-t}$$
$$+\beta\int_{t_1}^{t_2}\mathrm{d}t\left[\left(\frac{\delta F}{\delta\psi_i}\right)\frac{\partial\psi_i}{\partial t}\right]_{-t} + \int_{t_1}^{t_2}\mathrm{d}t\left[\beta\frac{\delta F}{\delta\psi_i}\left\{Q_{ij}[\psi] + L^0_{ij}\right\}\frac{\delta F}{\delta\psi_j}\right]_{-t}. \tag{B.34}$$

We have denoted in the right-hand side of both the equations with subscript '$-t$', that the fields ψ and $\hat{\psi}$ within the square bracket are evaluated at time $-t$. Since Q_{ij} is odd under the exchange of indices i and j both of which are summed over, it follows that

$$\frac{\delta F}{\delta\psi_i}Q_{ij}[\psi]\frac{\delta F}{\delta\psi_j} = 0. \tag{B.35}$$

After some simple algebra we obtain from expression (B.33) and finally taking the limit $t_1 = -t_2 \to \infty$:

$$\mathcal{A}[\psi(-t),\hat{\psi}(-t)] = \int_{t_1}^{t_2}\mathrm{d}t\left[\hat{\psi}_i\beta^{-1}\hat{\psi}_j + i\hat{\psi}_i\left(\frac{\partial\psi_i}{\partial t} + \{Q_{ij}[\psi] - L^0_{ij}\}\frac{\delta F}{\delta\psi_j}\right)\right]_{-t}$$
$$+\beta\left[F_{-t_1} - F_{-t_2}\right]_{-t}$$
$$= \mathcal{A}[\psi,\hat{\psi}] + \beta\left[F_{-t_1} - F_{-t_2}\right]. \tag{B.36}$$

Hence the time reversal transformation $\mathcal{T}$ described by equations (B.28) and (B.29) leaves the MSR action invariant. If F is a non-Gaussian functional of the fields $\{\psi_i\}$ then the above transformations are non-linear. The important point to note here is that invariance of the MSR action does not change even when the bare transport matrix L^0_{ij} is dependent on the fluctuating fields. The latter is the case with multiplicative noise. Hence the corresponding FDRs in the model are preserved even with multiplicative noise.

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