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Quiver asymptotics: $\mathcal{N} = 1$ free chiral ring

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Abstract

The large N generating functions for the counting of chiral operators in $\mathcal{N} = 1$, four-dimensional quiver gauge theories have previously been obtained in terms of the weighted adjacency matrix of the quiver diagram. We introduce the methods of multi-variate asymptotic analysis to study this counting in the limit of large charges. We describe a Hagedorn phase transition associated with the asymptotics, which refines and generalizes known results on the 2-matrix harmonic oscillator. Explicit results are obtained for two infinite classes of quiver theories, namely the generalized clover quivers and affine $\mathbb{C}^3/\hat{A}_n$ orbifold quivers.

Keywords: quiver gauge theory, asymptotic analysis, Hagedorn phase transition

(Some figures may appear in colour only in the online journal)

1. Introduction

The Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence gives an equivalence between four dimensional gauge theories and ten dimensional string theories [21]. Generalizations of the correspondence involve four dimensional quiver gauge theories [11] and a six-dimensional non-compact Calabi–Yau (CY) space in the transverse directions. The dictionary between $\mathcal{N}=1$ four-dimensional quiver gauge theories and the Calabi–Yau geometry has been developed, in the case of toric CY [15], using brane tilings [14, 17, 19].

Chiral gauge invariant operators, which are annihilated by supercharges of one chirality, play a central role in identifying the CY space for a given quiver gauge theory. These operators form a chiral ring and their expectation values are independent of the positions of insertion of the operators [7].

The combinatorics of the chiral ring at non-zero super-potential has been studied using generating functions and Hilbert series in the plethystic program [2, 12, 18]. The asymptotics of the counting formulae have also been studied [2, 9, 12, 20]. These studies have primarily used one-variable methods appropriate for special cases of the asymptotics, although a few results in the multi-variable case are also available [9, 20].

The holographic duals of free scalar field theories have been conjectured to be higher spin gravity theories [10, 30]. The chiral rings of free quiver gauge theories provide observables for a large class of theories, which can be studied as instances of this type of holography. Some special quivers, associated to 2-matrix systems, arise as sectors of $\mathcal{N}=4$ SYM, and have been recently discussed in connection with black holes in the dual of $AdS_5\times S^5$ [5, 6]. For quivers corresponding to orbifolds of $AdS_5\times S^5$, the free limit has a known holographic dual as a strongly coupled limit from the gravity side. For the case of conifold, it is known that the free UV fixed point theory flows to a known limit of the IR moduli space of conformal theories, which also includes the semiclassical gravity limit [24, 29]. For general quivers corresponding to toric CY spaces, the relation between the free UV fixed points at zero potential to the moduli space of IR fixed points is in general not well understood. These IR moduli spaces have been studied for example in [3]. The zero coupling quiver gauge theory chiral rings have also been discussed in connection with free fermions and generalized oscillators in [4].

General results on the counting of chiral ring operators in the free limit of zero superpotential were obtained in [22, 24]. The generating function for chiral operators in this free limit is an infinite product of inverse determinants involving the weighted adjacency matrix of the quiver diagram. This weighted adjacency matrix is a function of multiple variables associated with the edges of the quiver, and corresponding charges. The asymptotic behaviour of the free quiver counting in the limit of large charges naturally requires multi-variate complex analysis. In this paper, we will study the asymptotics of these generating functions using the methods of multi-variate asymptotics recently developed by Pemantle and Wilson [26, 27]. These methods allow systematic algorithmic derivation of asymptotics associated to a rational generating function in several variables, and apply directly to the generating function of chiral operators.

We find that the asymptotic counting of the chiral operators for any free quiver gauge theory is given by a compact general formula (20). To make this general formula more explicit for different quivers requires, at present, symbolic computer tools. For two infinite classes of examples, we analytically derive the explicit asymptotic results. These two classes are the generalized clover quivers and the affine $\mathbb{C}^3/\hat{A}_n$ orbifold quivers. These results are the asymptotic formulas (24) and (39).

The rest of the paper is organized as follows. In section 2, we define a thermodynamics of the chiral operators in a free quiver gauge theory, based on the counting problem of the chiral operators. We discuss associated phase transitions and a generalized Hagedorn hyper-surface related to the asymptotic analysis. In section 3, we present an adapted version of multivariate asymptotic techniques for our quiver gauge theory problem. In section 4, we apply the developed method to two (infinite) classes of examples and obtain explicit results for the asymptotics. Section 5 discusses some possible directions for future studies.

2. Thermodynamics of chiral ring in free CY3 quiver theories

In this section, we will consider generating functions for the counting of chiral ring operators in free quiver gauge theories as generalizations of thermal partition functions in AdS/CFT. We will define a generalized Hagedorn hyper-surface. We will observe that it controls the asymptotics of these generating functions, a point which will be developed in more detail in subsequent sections.

In the context of the AdS/CFT correspondence, type IIB string theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ Super Yang–Mills (SYM) theory on four dimensional Minkowski space $\mathbb{R}^{1,3}$. Conformal field theories have a symmetry of scaling of the space-time coordinates. Their quantum states are characterized by a scaling dimension, which is the eigenvalue of a scaling operator. In the AdS/CFT correspondence, the scaling operator of the CFT corresponds to the Hamiltonian for global time translations in AdS [31]. The eigenvalues of this Hamiltonian are the energies of quantum states obtained from the quantum theory of gravity in Anti-de-Sitter space.

For free CFTs in four space-time dimensions, the scaling dimension for any scalar field is 1. For a composite field (also called composite operator), which is a monomial function of the elementary scalar fields, the scaling dimension is the number of constituent scalar fields.

The thermal partition function of the AdS theory is a function of β, the inverse temperature, given by

$$F(\beta) = \mathrm{Tr}\, \mathrm{e}^{-\beta H}. \tag{1}$$

For a system with a discrete spectrum of energies, as in the case at hand, this is a sum over energy eigenvalues

$$F(\beta) = \sum_E a(E)\, \mathrm{e}^{-\beta E}, \tag{2}$$

where $a(E)$ is the number of states of energy E. For a free CFT, the partition function becomes

$$F(x) = \sum_r a_r\, x^r, \tag{3}$$

where $x = \mathrm{e}^{-\beta}$, a_r is the number of composite fields with r constituent elementary scalars and r is being summed over the natural numbers. In a case where we have multiple types of scalar fields, as in quiver gauge theories, the above partition function can be generalized to a multi-variable function

$$F(x_1, \cdots, x_d) = \sum_{r_1, \cdots, r_d} a_{r_1, \cdots, r_d}\, x_1^{r_1} \cdots x_d^{r_d}, \tag{4}$$

where $x_i = \mathrm{e}^{-\beta_i}$ and $a_{r_1, \cdots, r_d}$ is the multiplicity of composite operators with specified numbers $r_1, \cdots, r_d$ of the different types of scalar fields. The multivariate generating functions (4) are refined versions of the partition function (3). We will refer to r_i as charges and the variables x_i as fugacity factors for each field. Using the vector notation $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_d)$ and $\mathbf{r} = (r_1, \cdots, r_d)$, with $|\mathbf{r}| = \sqrt{r_1^2 + \cdots + r_d^2}$ and $\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$, we define

$$\mathbf{x}^{\mathbf{r}} := \prod_{j=1}^{d} x_j^{r_j}. \tag{5}$$

We can then write equation (4) as

$$F(\mathbf{x}) = \sum_{\mathbf{r}} a_{\mathbf{r}}\, \mathbf{x}^{\mathbf{r}}. \tag{6}$$

The 1-variable partition function (3), for systems which have an exponential growth of the number of states $a_r \sim \mathrm{e}^{\alpha r}$ in the large r limit, has regions of convergence and divergence meeting at a critical $\beta = \alpha$. The partition function converges for $\beta > \alpha$ and diverges for $\beta < \alpha$. This type of behaviour occurs in string theory, where it is associated with the Hagedorn phase transition [16].

Similarly, in the multi-variable case, assuming that the function $F(x_1, \cdots, x_d)$ has an exponential growth of the multiplicity factor $a_{r_1,\cdots,r_d}$ for large r_i, then there is a hyper-surface separating convergence and divergence regions for the multi-variable partition function. This hyper-surface is given by the equation

$$\frac{1}{|\mathbf{r}|} \log a_{\mathbf{r}} = \boldsymbol{\beta} \cdot \hat{\mathbf{r}} \quad . \tag{7}$$

In the quiver examples we will be studying in this paper, there is indeed this type of exponential behaviour and a corresponding hyper-surface. This may be viewed as a generalized Hagedorn hyper-surface.

This Hagedorn hyper-surface was studied in the case of a 2-matrix model in [1]. The 2-matrix model problem is associated with a quiver consisting of a single node and 2-directed edges. We consider this model and the generalised Hagedorn hyper-surface for more general partition functions associated with quivers [24].

A *quiver diagram* is a directed graph $G = (V, E)$ with a set V of *nodes* and a set E of directed *edges*; self-loops at a node are explicitly allowed. A quiver gauge theory has a product gauge group of the form $\prod_a U(N_a)$, where each $U(N_a)$ is associated with a node, and matter fields in the bi-fundamental representation of the gauge group are associated with the edges. The interactions between the matter fields are described by a *superpotential* W which is a gauge-invariant polynomial in the matter fields. In our study, we focus on the *zero superpotential* case $W = 0$.

The observables of quiver gauge theories are *gauge invariant operators* and their correlation functions. An interesting class of observables is formed by chiral operators, which form a ring, called the chiral ring. For the definition and properties of the chiral ring, see for example [7, 18]. The space of chiral operators in the zero-superpotential limit is typically much larger than that at non-zero superpotential.

The generating function for the chiral operators in the large N limit in an arbitrary free (with zero superpotential $W = 0$) quiver gauge theory was derived in [22–24] and is given by

$$F(\mathbf{x}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{x}^{\mathbf{r}} = \prod_{i=1}^{\infty} \det(\mathbb{I} - X(\mathbf{x}^i))^{-1}, \tag{8}$$

where $\mathbf{x} = (x_1, ..., x_d)$ is the fugacity factor, $\mathbf{r} = (r_1, ..., r_d)$ is the charge associated with the edges, $X(\mathbf{x}) = X(x_1, \cdots, x_d)$ is the weighted adjacency matrix of the quiver and $X(\mathbf{x}^i) = X(x_1^i, \cdots, x_d^i)$. The multivariate generating function (8) is an example of equation (6), i.e. a refined version of the partition function (3). The *degeneracy* $a_{\mathbf{r}}$ is the number of chiral operators with *charge vector* $\mathbf{r}$ in the chiral ring of the free quiver gauge theory. In this paper, we will be interested in the asymptotic behaviour of $a_{\mathbf{r}}$ for large $(r_1, r_2, \cdots, r_d)$.

In order to study the asymptotics, with the methods of [25–27], it will be useful to write F as a ratio

$$F(\mathbf{x}) = \prod_{i=1}^{\infty} \det(\mathbb{I} - X(\mathbf{x}^i))^{-1} = \frac{G(\mathbf{x})}{H(\mathbf{x})},$$

where

$$H(\mathbf{x}) = \det(\mathbb{I} - X(\mathbf{x})), \quad G(\mathbf{x}) = \prod_{i=2}^{\infty} \det(\mathbb{I} - X(\mathbf{x}^i))^{-1}. \tag{9}$$

In fact, we observe that, with the parameterization $\mathbf{x} = \exp(-\boldsymbol{\beta})$, $F(\mathbf{x})$ is convergent in the domain $H(\mathbf{x}) \geqslant 0$ for small enough (and positive) $x_i > 0$ for all i . The boundary between the convergence and divergence domains is the phase transition hyper-surface, characterized by $H(\mathbf{x}) = 0$. We will see some examples of this in section 4. The asymptotic regime of the generating function is obtained by approaching the $H = 0$ hyper-surface inside the domain of convergence. Thus this hyper-surface controls the phase structure of the theory and also determines the leading asymptotic behavior of $a_{\mathbf{r}}$. The asymptotic analysis will be developed in sections 3 and 4.

2.1. Entropy

The logarithm of the degeneracy $a_{\mathbf{r}}$ is the *thermodynamic entropy* $S_{\mathbf{r}} := \log a_{\mathbf{r}}$. For convenience, sometimes we consider the leading term of the entropy, which we denote by $S^*_{\mathbf{r}}$.

Following the general result (7), in the chiral ring of the free quiver gauge theories, the Hagedorn-type transition can be seen as a result of the competition between the leading term of the entropy $S^*_{\mathbf{r}}$ obtained from the logarithm of the multiplicity and the temperature term $-\boldsymbol{\beta} \cdot \mathbf{r}$ in the generating function (8). The generalized Hagedorn hyper-surface is given by

$$S^*_{\mathbf{r}} - \boldsymbol{\beta} \cdot \mathbf{r} = 0. \tag{10}$$

Using equation (10), the critical couplings can be simply obtained as $\beta_i = \partial_i S^*_{\mathbf{r}}$. We will see explicit equations for this hyper-surface in some classes of examples in section 4.

3. Method of asymptotic analysis

In this part we adopt a novel technique of asymptotic analysis of the multivariate generating functions, and apply it to the counting problem for the corresponding quiver gauge theories. First, we review some known material from the asymptotic analysis of multivariate generating functions and then in the second part, we present an ongoing development on the evaluation of certain Hessian determinants specified at some critical points. In the third part, the phase structure of the quiver theories is explained and the relations between the entropy and critical couplings are discussed.

3.1. Multivariate asymptotic counting

In this section, we answer the question of asymptotic counting for the multivariate generating functions that appear in the chiral ring of the quiver gauge theories. We will not review the details of the proofs from multivariate asymptotic analysis in this article and only present the main result in the following. For a comprehensive presentation of such analysis see [25–27].

We now briefly summarize the general results for asymptotics of multivariate generating functions obtained by Pemantle and Wilson [26], applied to our situation of interest. In the present paper, we encounter generating functions that only require the *smooth point* analysis of [26]. We present an adapted and extended version of these results, in four steps suitable for quiver gauge theories. In the next section we apply these results to some infinite classes of examples of quivers.

The basic steps of the analysis of [26] are as follows.

(i) We consider a generating function in d variables $\mathbf{x} = (x_1, ..., x_d)$,

$$F(\mathbf{x}) = \frac{G(\mathbf{x})}{H(\mathbf{x})} \tag{11}$$

where G and H are holomorphic in some neighbourhood of the origin, $H(\mathbf{0}) \neq 0$, and all coefficients are nonnegative.

(ii) For a given $\mathbf{r}$ we find the *contributing points* to the asymptotics in direction parallel to $\mathbf{r}$. To do this we first find the *critical points* of H. A smooth critical point $\mathbf{x}^*$ is a solution of the following set of equations

$$H(\mathbf{x}^*) = 0, \quad r_d x_j^* \partial_j H(\mathbf{x}^*) = r_j x_d^* \partial_d H(\mathbf{x}^*) \qquad \text{for } (1 \leqslant j \leqslant d-1). \tag{12}$$

From general theory [27] there is a solution $\mathbf{x}^*$ to these equations that has only positive coordinates and which is a contributing point. Generically, this is the unique solution to the critical point equations. Furthermore all other contributing points, if they exist, lie on the same torus. We require $F = G/H$ to be meromorphic in a neighbourhood of the *closed polydisk* containing $\mathbf{x}^*$, defined by the condition that $|x_i| \leqslant x_i^*$ for all i. The point is usually *strictly minimal* — no other point in the polydisk is a pole of F.

(iii) Near each contributing point $\mathbf{x}^*$ we can solve the equation $H(\mathbf{x}^*) = 0$, without loss of generality $x_d = g(\mathbf{x}')$ with $\mathbf{x}' = (x_1, ..., x_{d-1})$. However, the asymptotic is independent of which coordinate we solve for. Then we define the function $\phi = \log g(\mathbf{x}')$ locally parametrizing the hyper-surface $\{H = 0\}$ in logarithmic coordinates. The Hessian $\mathcal{H}$ of ϕ is an essential part of the asymptotic formula, and we can compute it in terms of the original data as follows. We construct the Hessian matrix $\mathcal{H} := \left(\mathcal{H}_{ij}\right)_{i,j=1}^{d-1}$ with elements $\mathcal{H}_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$. The diagonal and off-diagonal matrix elements can be written explicitly in terms of g:

$$\begin{aligned} \mathcal{H}_{ii} &= -x_i \frac{\partial_i g(\mathbf{x}') + x_i \partial_i^2 g(\mathbf{x}')}{g(\mathbf{x}')} + x_i^2 \left(\frac{\partial_i g(\mathbf{x}')}{g(\mathbf{x}')}\right)^2 \quad \text{diagonal}, \\ \mathcal{H}_{ij} &= -x_i x_j \left(\frac{\partial_i \partial_j g(\mathbf{x}')}{g(\mathbf{x}')} - \frac{\partial_j g(\mathbf{x}') \partial_i g(\mathbf{x}')}{g(\mathbf{x}')^2}\right) \quad \text{off-diagonal}. \end{aligned} \tag{13}$$

Thus we can write the Hessian matrix as

$$\mathcal{H} = \left(\frac{x_i x_j}{g}\left(\frac{\partial_i g \partial_j g}{g} - \partial_i \partial_j g\right) - \frac{x_i \partial_i g}{g} \delta_{ij}\right)_{i,j=1}^{d-1}. \tag{14}$$

The above holds for all values of the variables. We are only interested in the evaluation of the Hessian determinant at each contributing point, and this allows further simplification which we now carry out. The critical equation $H = 0$ implies

$$\mathrm{d}H = \sum_{i=1}^{d-1} \partial_i H \, \mathrm{d}x_i + \partial_\mathrm{d} H \, \mathrm{d}x_\mathrm{d} = \sum_{i=1}^{d-1} \left[\partial_i H + \partial_\mathrm{d} H \frac{\partial x_\mathrm{d}}{\partial x_i}\right] \mathrm{d}x_i = 0, \tag{15}$$

where we omitted the star for the critical points for simplicity. It implies the following relation for any $j = 1, ..., d-1$:

$$\frac{\partial_j H}{\partial_d H} = -\partial_j g. \tag{16}$$

Putting this relation together with (12), we obtain

$$\frac{x_j \partial_j g}{x_d} = -\frac{r_j}{r_d}. \tag{17}$$

Applying identity (17) to the Hessian matrix (14) we obtain the elements of the Hessian matrix evaluated at critical points,

$$\mathcal{H}^*_{ij} = \frac{r_i r_j}{r_d^2} - \frac{x_i^* x_j^* \partial_i \partial_j g}{g} + \frac{r_i}{r_d}\delta_{ij}. \tag{18}$$

The computation of $\frac{x_i^* x_j^* \partial_i \partial_j g}{g}$ depends on the relative positions of the loops i, j and d in the quiver diagram.

(iv) The final step is to derive the asymptotic formula for $a_{\mathbf{r}}$ from the critical points and Hessian determinant. The Cauchy integral formula yields

$$a_{\mathbf{r}} = \frac{1}{(2\pi \mathrm{i})^d} \int_T \mathbf{x}^{-\mathbf{r}} F(\mathbf{x}) \frac{\mathrm{d}\mathbf{x}}{\mathbf{x}}, \tag{19}$$

where the torus T is a product of small circles around the origin in each coordinate and $\frac{\mathrm{d}\mathbf{x}}{\mathbf{x}} = (x_1 \cdots x_d)^{-1} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$. In the asymptotic regime $|\mathbf{r}| \to \infty$, the following *smooth point asymptotic formula* is obtained in [26, Theorem (1.3)]. We write simply $\mathbf{x}$ for $\mathbf{x}^*(\mathbf{r})$.

The smooth point formula states that if $G(\mathbf{x}) \neq 0$ then

$$a_{\mathbf{r}} \sim (2\pi)^{-(d-1)/2} (\det \mathcal{H}(\mathbf{x}))^{-1/2} \frac{G(\mathbf{x})}{-x_d \partial H / \partial x_d(\mathbf{x})} r_d^{-(d-1)/2} \mathbf{x}^{-\mathbf{r}}, \tag{20}$$

where $\mathbf{x}^{-\mathbf{r}} := \prod_{j=1}^d x_j^{-r_j}$. The expansion is uniform in the direction $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ provided this direction is bounded away from the coordinate axes.

We want to apply the above procedure in our case of interest. For any connected quiver with generating function (8), we have functions $H(\mathbf{x})$ and $G(\mathbf{x})$ as in equation (9). For $\mathbf{x}$ sufficiently close to the origin, G and H are holomorphic and H does not vanish at the origin.

Our next observation is the rediscovery of a folklore result in graph theory [8], that $H(\mathbf{x})$ in equation (9) can be expanded graphically in terms of the loops in the quiver,

$$H = \sum_{k=0}^{d} \sum_{l_1 \sqcup l_2 \sqcup \ldots \sqcup l_k} (-1)^k l_1 l_2 \cdots l_k, \tag{21}$$

where l_1, l_2, ..., l_k are loops which meet each node of the quiver diagram at most once, d is the total number of the loops of the quiver diagram, and loops in the second sum are disjoint. Notice that each loop variable in the above expansion (21) is the product of x_i edge variables in the determinant formula (9), around each loop of the quiver diagram.

To summarize, given a quiver diagram, one can easily find the function H and solve the critical equations to obtain the critical points. Then, by computing the Hessian determinant

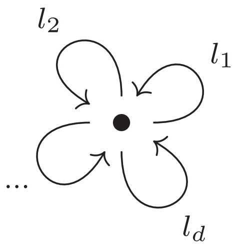

Figure 1. Generalized Clover Quiver.

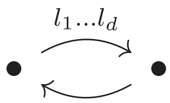

Figure 2. d-Kronecker Quiver.

evaluated at a critical point and inserting these results into equation (20), one can obtain the asymptotic for any quiver diagram. However, owing to the multidimensional nature of the problem, some of the computations, such as solving the critical equations and computing the Hessian determinant, require symbolic mathematical software. On the other hand, as we will present in this paper, alternatively, one can try to find an explicit analytic form of the asymptotic formula for some infinite classes of quivers, with the hope of finding a general analytic result for larger classes of quivers.

4. Some infinite classes of quivers

Having introduced and discussed a general procedure of the asymptotic methods for quiver diagrams, in this section, we implement these methods in two infinite classes of examples and obtain explicit analytic results for the entropy and phase structure of these quiver gauge theories.

4.1. Generalized clover quivers

As the first class of examples, we consider a generating function of the following form:

$$F(\mathbf{x}) = \frac{G(\mathbf{x})}{H(\mathbf{x})} = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{x}^{\mathbf{r}} = \prod_{i=1}^{\infty} \left(1 - \sum_{j=1}^{d} x_j^i\right)^{-1}, \tag{22}$$

with $H(\mathbf{x}) = 1 - \sum_{j=1}^{d} x_j$ and $G(\mathbf{x}) = \prod_{k \geqslant 2} H_k(\mathbf{x}) = \prod_{i=2}^{\infty} (1 - \sum_{j=1}^{d} x_j^i)^{-1}$. This is the *Generalized Clover Quiver* class, see figure 1. It is interesting to notice that the d-Kronecker quivers, consisting of d loops, shown in figure 2, have also the same generating function.

It is easy to observe that for direction $(r_1, \ldots, r_d)$ the unique critical point is

$$\mathbf{x}^* = \left(\frac{r_1}{R}, \frac{r_2}{R}, ..., \frac{r_d}{R}\right),$$

with $R = \sum_{i=1}^{d} r_i$. This point is strictly minimal. To see this, we first show that there are no more zeros of H_1 in the closed polydisk defined by $\mathbf{x}^*$. This is clear because for every such point $\mathbf{x}$ we must have $\sum_{i=1}^{d} x_i = 1$ while $|x_i| \leqslant x_i^*$, and this can only happen when all $x_i = x_i^*$.

We then handle possible poles arising from the other factors H_k for $k \geqslant 2$. To do this, note that $0 < x_i^* < 1$ for all i. Thus if $|x_i| \leqslant x_i^*$ for all i and $k \geqslant 2$ we have

$$\left|1 - \sum_{i=1}^{d} x_i^k\right| \geqslant 1 - \sum_{i=1}^{d} |x_i|^k \geqslant 1 - \sum_{i=1}^{d} |x_i| \geqslant 1 - \sum_{i=1}^{d} x_i^* = 0.$$

Note that the second inequality is strict unless all $x_i = 0$, in which case the third inequality is strict. Thus we can choose a polydisk with radii f_i slightly more than x_i^*, such that for all $k \geqslant 2$, $\sum_i f_i^k < 1$ and so each factor in the product defining G is analytic. Thus $F = G/H$ has the desired form as a quotient of analytic functions in an appropriate polydisk, and $\mathbf{x}^*$ is a strictly minimal critical point for the given direction.

To compute the Hessian, we observe that $g = x_d = 1 - \sum_{i=1}^{d-1} x_i$ is linear and so the second partial derivatives are zero. Thus from (18) we have

$$\mathcal{H}_{ij} = \frac{r_i r_j}{r_d^2} + \frac{r_i}{r_d}\delta_{ij}.$$

By changing to the variables r_i/r_d the determinant of this matrix is easily computed (see appendix) to be

$$\det \mathcal{H} = r_d^{-(1+d)} R \prod_{i=1}^{d} r_i. \tag{23}$$

By implementing the above explicit formula for the determinant of the Hessian matrix, in equation (20), it is straightforward to obtain the asymptotic result. By using equation (20), for the asymptotics of $a_{\mathbf{r}}$ of the generating function (22), in the 'central region', as $R \to \infty$ and r_i/r_j (for each $i, j = 1$ to d) bounded away from zero, we obtain the following explicit asymptotic formula,

$$a_{\mathbf{r}} \sim \frac{G(\mathbf{x}^*)}{(2\pi)^{(d-1)/2}} R^{R+\frac{1}{2}} \prod_{i=1}^{d} r_i^{-r_i - \frac{1}{2}}. \tag{24}$$

The entropy of the generalized clover quiver is obtained in the following Shannon form:

$$S_{\mathbf{r}} = \log G(\mathbf{x}^*) + \left(R + \frac{1}{2}\right) \log R - \sum_i \left(r_i + \frac{1}{2}\right) \log r_i. \tag{25}$$

- **Univariate case**

In this part we characterize a special type of quiver whose asymptotics cannot be studied with the methods above. This is the one-variable case of the generalized clover quiver and is called the *Jordan Quiver* (see figure 3). In fact, the generating function of this type of quiver is the generating function of the (integer) partitions and derivation of its asymptotics is a classical problem in analytic combinatorics. The reason that the asymptotic method of this paper does not apply is that the relevant singularities in the one-variable case occur at all possible roots of unity, $1 - x^i = 0$, and each factor in the product contributes to the asymptotic, while for example in the two-variable case the exponential order of the contribution of $1 - x - y = 0$ is higher than that of $1 - x^2 - y^2 = 0$, etc.

All the oriented cycle graphs with no multiplicity (multiple edges) and no loop of length one, can be reduced to a quiver consisting of a single node and single loop. This loop variable is a product of edge variables for the cycle graph. The simplest example of this class is the two node graph with two oppositely directed edges. The asymptotics for this class of quivers follows the asymptotics of partitions.

- **Bivariate case**

In the bivariate case ($d = 2$), for Bi-Clover Quiver in figure 4, the generating function is

$$F(x,y) = \frac{G(x,y)}{H(x,y)} = \prod_{i=1}^{\infty}(1 - x^i - y^i)^{-1} = \sum_{r,s} a_{rs}x^r y^s$$

where $G(x,y) = \prod_{i=2}^{\infty}(1 - x^i - y^i)^{-1}$, $H(x,y) = (1 - x - y)$.

The asymptotics for a_{rs} as $r + s \to \infty$ and $r/s, s/r$ are bounded away from zero, can be obtained as a special case of the computations above. Let $\lambda = r/(r + s) \in (0, 1)$. This yields the first order asymptotic

$$a_{rs} \sim \frac{G(\lambda, 1-\lambda)}{\sqrt{2\pi}} \frac{(r+s)^{(r+s)}}{r^r s^s} \sqrt{\frac{r+s}{rs}}.$$

This is uniform in λ as long as it stays in a compact subinterval of $(0, 1)$ (alternatively, the slope r/s lies in a compact interval of $(0,\infty)$—note that $r/s = \lambda/(1-\lambda)$). In particular for the main diagonal $r = s$, corresponding to $\lambda = 1/2$, we obtain

$$a_{nn} \sim \frac{G(1/2, 1/2)}{\sqrt{\pi n}} 4^n.$$

The exact value of G at the critical point is not completely explicit, being given by an infinite product. It is a positive real number greater than 1, since each factor satisfies those same conditions. The minimum value of $G(\lambda)$ over all λ occurs when $\lambda = 1/2$ and equals the reciprocal of the *Pochhammer symbol* $(1/2; 1/2)_\infty$. This has the approximate numerical value 3.46275. The value of $G(\lambda)$ approaches ∞ as $\lambda \to 0$ or $\lambda \to 1$.

Phase structure

We start with the simplest example of the class, which is the bivariate clover quiver. Following the discussion in the paragraph after equation (9) in section 2, the phase transition line in this example is $1 - x_1 - x_2 = 0$ and the phase diagram is depicted in figure 5. In the case $x_1 = x_2$, we obtain the critical coupling $\beta^* = \log 2$. In the unrefined case of the generalized clover quiver, we have $\beta^* = \log d$. Similarly the phase diagram of the other examples in this class is a hyper-plane obtained from $H(x_1, ..., x_d) = 1 - \sum_{j=1}^{d} x_j = 0$. Using equation (25), up to leading order, the entropy and couplings on the critical hypersurface are obtained as

$$\begin{aligned} S^*_{\mathbf{r}} &= R\log R - \sum_{i=1}^{d} r_i \log r_i = \sum_{i=1}^{d} r_i \cdot \beta^*_i, \\ \frac{\partial S^*_{\mathbf{r}}}{\partial r_i} &= \log R - \log r_i = -\log x^*_i = \beta^*_i. \end{aligned} \tag{26}$$

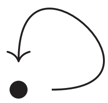

Figure 3. Jordan Quiver.

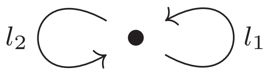

Figure 4. Bi-Clover Quiver.

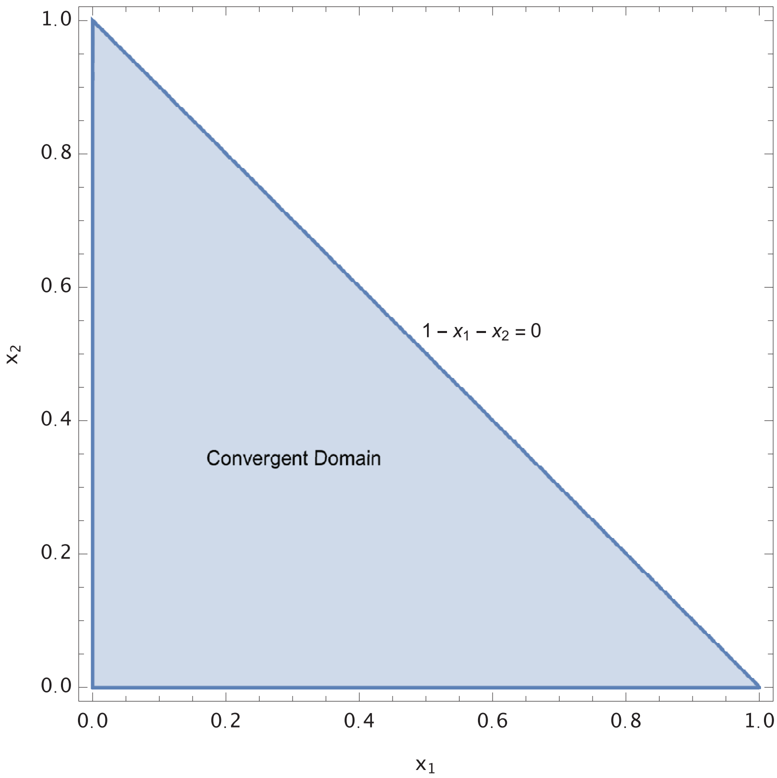

Figure 5. Phase Diagram of Bi-Clover Quiver.

Notice that critical points obtained as above are the same as the solutions of critical equation (12) in the case of generalized clover quivers. The basic physical example in the class of generalized clover quivers is the $\mathbb{C}^3$ quiver gauge theory with three loops in the quiver.

4.1.1. Oriented cycle quivers with multiplicity. An oriented cycle quiver with multiplicity is an oriented cycle graph with multiple edges between any two adjacent nodes and no loop of length one. The generating functions of these quivers reduce to those of the generalized clover quiver where the number of the loops in the clover quiver is determined by the number of the cycles in the cycle graph. In the following, we present some important examples of oriented cycle quivers with multiplicity.

- **conifold $\mathcal{C}$**

  The conifold quiver is an oriented cycle graph with multiplicity, and with two nodes and two couples of parallel edges between the nodes. This is a special case of the generalized clover quiver with four variables. The determinant of the adjacency matrix of the conifold is

$$H = 1 - l_1 - l_2 - l_3 - l_4, \tag{27}$$

  where l_1, l_2, l_3, and l_4 are product of edge variables in the conifold quiver, see [24, section 2]. By direct computation, the critical points and Hessian determinant with the choice of $l_d = l_4$ are

$$l_i^* = \frac{r_i}{R}, \quad \det \mathcal{H} = r_4^{-5} R \prod_{i=1}^{4} r_i \tag{28}$$

  where $R = \sum_{i=1}^{4} r_i$. The asymptotic and the dominant terms in entropy can be obtained from the result for the generalized clover quiver,

$$a_{\mathbf{r}} \sim \frac{G(\mathbf{l}^*)}{(2\pi)^{3/2}} R^{R+\frac{1}{2}} \prod_{i=1}^{4} r_i^{-r_i-\frac{1}{2}}, \quad S_{\mathbf{r}}^* = -\sum_{i=1}^{4} r_i \log r_i + R \log R, \tag{29}$$

  where $G(\mathbf{l}^*) = \prod_{i=2}^{\infty} (1 - l_1^{*i} - l_2^{*i} - l_3^{*i} - l_4^{*i})^{-1}$.

- **Hirzebruch F_0 and del Pezzo dP_0** ($\mathbb{C}^3/\mathbb{Z}_3$)

  As other examples of this class we can mention Hirzebruch F_0 and del Pezzo dP_0, see figure 6 (middle) and figure 1 in [13]. The generating function of Hirzebruch F_0 is the 16 loop variable case of the generalized clover quiver and del Pezzo dP_0 is the generalized clover quiver with 27 loop variables. Their asymptotics can be obtained as special cases of equation (24).

4.2. Affine $\mathbb{C}^3/\hat{A}_n$ orbifold quivers

The next infinite class of examples consists of the affine $\mathbb{C}^3/\hat{A}_n$ orbifold quiver theories, see figure 6. The first observation is that by equations (9) and (21), the denominator H for this quiver can be written in terms of the elementary symmetric functions $e_j(x_1, ..., x_n)$,

$$H(x_1, ..., x_n, x_c) = -x_c + \sum_{j=0}^{n} (-1)^j e_j(x_1, ..., x_n) = -x_c + \prod_{j=1}^{n} (1 - x_j), \tag{30}$$

where $x_1, ..., x_n$ are the loops of length one at the nodes $1, ..., n$, and x_c is the central loop in figure 6. Thus the generating function of this quiver can be written as

$$F(\mathbf{x}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{x}^{\mathbf{r}} = \prod_{i=1}^{\infty} \Big(-x_c^i + \sum_{j=0}^{n} (-1)^j e_j(x_1^i, ..., x_n^i) \Big)^{-1}. \tag{31}$$

First, we choose g function to be the central loop, i.e. $x_d = x_c$. We denote the other loops in the quiver by x_i for $i = 1, ..., n$. We have as above

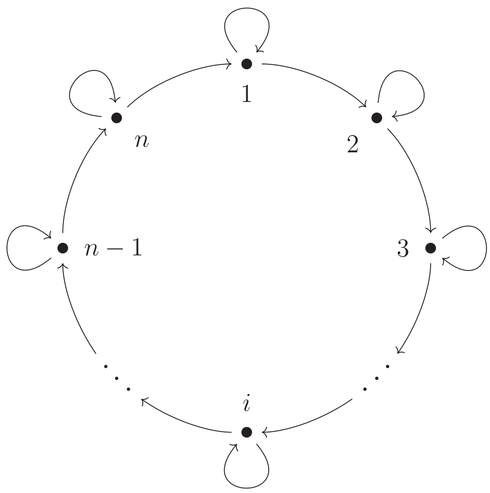

Figure 6. Affine $\mathbb{C}^3/\hat{A}_n$ Orbifold Quivers.

$$G(x_1,\ldots,x_n,x_c) = \prod_{k\geqslant 2} H_k(\mathbf{x}) := \prod_{i=2}^{\infty}\left[-x_c^i + \sum_{j=0}^{n}(-1)^j e_j(x_1^i,\ldots,x_n^i)\right].$$

Suppose that r_i/r_c and r_c/r_i are fixed and bounded away from zero as $r_i, r_c \to \infty$. We claim that the unique critical point and Hessian determinant evaluated at this point are

$$x_i^* = \frac{r_i}{r_c + r_i}, \quad x_c^* = \frac{r_c^n}{\prod_{i=1}^n (r_c + r_i)}, \quad \det \mathcal{H} = r_c^{-2n}\prod_{i=1}^{n} r_i(r_i + r_c). \tag{32}$$

The proof that this point satisfies $H = 0$ follows from the following identity for elementary symmetric function $e_i(r_1,\ldots,r_n)$,

$$\prod_{i=1}^{n}(r_c + r_i) = \sum_{i=0}^{n} r_c^{n-i} e_i(r_1,\ldots,r_n). \tag{33}$$

For the proof of the other critical equations, $\mathbf{r} \times \nabla_{\log} H = 0$, first observe that

$$\partial_j H = \partial_j g, \quad \partial_d H = -1, \tag{34}$$

and denoting the g function of the quiver with n surrounding loops by g_n, we observe that

$$\partial_j g_n = -g_{n-1}. \tag{35}$$

Using the above observation we can prove that the ansatz for x_i^* and x_c^* in equation (32), satisfy

$$r_c x_i^* \partial_i H = r_i x_c^* \partial_d H. \tag{36}$$

The formula for the Hessian determinant in equation (32) follows from the observation that the Hessian matrix evaluated at the critical point $\mathbf{x}^*$ is diagonal, since

$$\begin{aligned} \frac{x_i x_j}{g^2}\partial_i g \partial_j g &= r_i r_j r_c^{-2}, \\ \frac{x_i x_j}{g}\partial_i \partial_j g &= r_i r_j r_c^{-2}, \end{aligned} \tag{37}$$

and moreover, because of the choice $g = x_c$ we have $\partial_i^2 g = 0$, and thus by inserting the critical points in equation (32), the Hessian matrix, equation (14) becomes

$$\mathcal{H}_{ij} = \delta_{ij}\left(\frac{r_i^2}{r_c^2} + \frac{r_i}{r_c}\right), \tag{38}$$

and therefore the result follows.

We now show that the critical point $\mathbf{x}^*$ is strictly minimal and $F = G/H$ has the appropriate form. As in the generalized clover example, this amounts to showing that there are no other zeros of any H_k on the closed polydisk defined by $\mathbf{x}^*$. First note that for each H_k, if there is a zero $\mathbf{x}$ then it is a pole of $1/H_k$, and since $1/H_k$ has nonnegative coefficients, [22, section (2.1)], the coordinatewise modulus $\mathbf{x}^* = (|x_1|, \ldots, |x_n|, |x_c|)$ is also a pole, so that $|\mathbf{x}|$ is a zero of H_k.

We first consider the case $k \geqslant 2$. Suppose that $0 \leqslant x_i \leqslant x_i^*$ for all i, that $0 \leqslant x_c \leqslant x_c^*$. Then if $-x_c^k + \prod_{i=1}^n (1 - x_i^k) = 0$, then using the fact that $-x_c^* + \prod_{i=1}^n (1 - x_i^*) = 0$ we obtain

$$x_c^* - x_c^k = \prod_{i=1}^n (1 - x_i^*) - \prod_{i=1}^n (1 - x_i^k).$$

However $x_c^* - x_c^k > 0$ because $0 < x_c \leqslant x_c^* < 1$, whereas

$$\prod_{i=1}^n (1 - x_i^*) \leqslant \prod_{i=1}^n (1 - x_i^k),$$

yielding a contradiction. Finally when $k = 1$ a similar argument holds: there is no other solution ($\mathbf{x} \neq \mathbf{x}^*$) of $H_1(\mathbf{x}) = 0$ inside the polydisk. To see this, assume to the contrary that $-x_c + \prod_{i=1}^n (1 - x_i) = 0$ and $-x_c^* + \prod_{i=1}^n (1 - x_i^*) = 0$. This leads to two possibilities:

- $x_c - x_c^* = 0$, which yields $\prod_{i=1}^n (1 - x_i^*) = \prod_{i=1}^n (1 - x_i)$ and thus the contradiction $\mathbf{x} = \mathbf{x}^*$;
- $x_c - x_c^* < 0$, which implies $\prod_{i=1}^n (1 - x_i^*) - \prod_{i=1}^n (1 - x_i) > 0$, again yielding a contradiction because $x_i \leqslant x_i^*$ for all i.

Hence the asymptotic approximation and the entropy of the general case can be obtained from equation (20),

$$a_{\mathbf{r}} \sim \frac{G(\mathbf{x}^*)}{(2\pi)^{n/2}} r_c^{-n(r_c+\frac{1}{2})} \prod_{i=1}^n r_i^{-r_i-\frac{1}{2}} (r_i + r_c)^{(r_i+r_c+\frac{1}{2})}, \tag{39}$$

$$S_{\mathbf{r}} = \log G(\mathbf{x}^*) + \sum_{i=1}^n \left((-r_i - \frac{1}{2}) \log r_i + (r_i + r_c + \frac{1}{2}) \log(r_i + r_c) \right) - n(r_c + \frac{1}{2}) \log r_c,$$

where $G(\mathbf{x}^*) = \prod_{i=2}^\infty H(x_1^{*i}, x_2^{*i}, ..., x_n^{*i}, x_c^{*i})^{-1}$.

4.2.1. Phase structure. We start with the simplest example of the class, which is affine $\mathbb{C}^3/\mathbb{Z}_2$, see figure 7. As we discussed in the paragraph after equation (9) in section 2, the phase transition hyper-surface in this example is $1 - x_1 - x_2 - x_3 + x_1 x_2 = 0$ and the phase diagram is depicted in figure 8. In the general case, the phase diagram of this class is a hyper-surface obtained from $H(x_1, ..., x_n, x_c) = 0$ which implies $x_c = \sum_{j=0}^n (-1)^j e_j(x_1, ..., x_n)$. From equation (39) and equation (32), the critical relation on the phase transition hyper-surface, up to leading order, for the affine $\mathbb{C}^3/\hat{A}_n$, yields

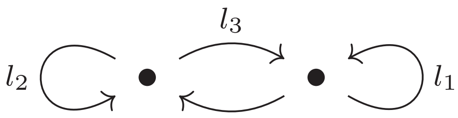

Figure 7. Affine $\mathbb{C}^3/\mathbb{Z}_2$ Quiver.

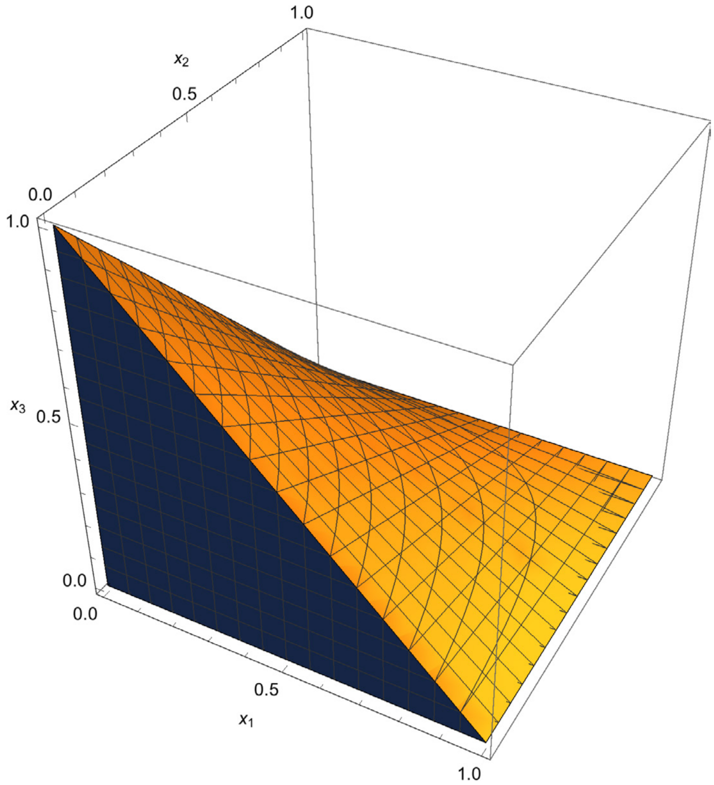

Figure 8. Phase Diagram of $\mathbb{C}^3/\mathbb{Z}_2$. The filled volume which is the convergent domain is separated from the divergent domain by the phase transition hyper-surface.

$$\begin{aligned} S^*_{\mathbf{r}} &= \sum_{i=1}^{n}\Big(-r_i\log r_i + (r_i+r_c)\log(r_i+r_c)\Big) - nr_c\log r_c = \sum_{i=1}^{n} r_i\cdot\beta^*_i + r_c\cdot\beta^*_c, \\ \frac{\partial S^*_{\mathbf{r}}}{\partial r_i} &= \log r_i - \log(r_i+r_c) = -\log x^*_i = \beta^*_i,\ \text{for } i=1,2,\ldots,n, \\ \frac{\partial S^*_{\mathbf{r}}}{\partial r_c} &= n\log r_c - \sum_{i=1}^{n}\log(r_i+r_c) = -\log x^*_c = \beta^*_c. \end{aligned} \tag{40}$$

Notice that critical points obtained as above are the same as the solutions of critical equations equation (12), presented in equation (32).

5. Conclusions and forthcoming research

In this work we adapted recent results in the multivariate asymptotic analysis of generating functions to study the asymptotic counting of operators in the chiral ring of free $\mathcal{N}=1$

quiver gauge theories. We obtained an explicit asymptotic formula for two infinite classes of examples. A next step is to consider other infinite classes of examples, e.g. $L^{a,b,c}$, $Y^{p,q}$ Sasaki–Einstein spaces and three-dimensional ADE-type orbifolds such as $\mathbb{C}^3/\mathbb{Z}_n$. A general formula for the asymptotics for general quivers, expressed in terms of the weighted adjacency matrix, would be an interesting goal.

In this work, we only considered the first (leading) asymptotic term in the asymptotic series. The current results in analytic combinatorics for the asymptotic counting of multivariate generating functions [28] allow for the computation of sub-leading contributions in the asymptotics. It would be interesting to apply these results to obtain the higher order asymptotics in quiver gauge theories.

In this work, the matter content of the gauge theory is restricted to matter in bifundamental representations. One can also consider fundamental matter and quiver gauge theories with flavours. The generating functions for counting gauge invariant operators in quivers with flavours are obtained in [22]. Thus it should be straightforward to generalize the method of this work to quivers with flavours and obtain the asymptotic counting of chiral operators in these theories.

In this paper, we considered the zero super-potential limit and exploited the availability of general formula for the generating functions of the multi-trace chiral operators. However, as we mentioned, the $W \neq 0$ sector of the quiver gauge theories and the counting of the chiral operators have been studied vastly. Generating function technology has been introduced and applied successfully to very general classes of quivers [2, 9, 12]. The multivariate asymptotic analysis employed here would be useful in obtaining results for asymptotics of counting in general quiver gauge theories.

The counting of states for the bi-clover quiver (section 4) is the 2-matrix counting which has been recently discussed in the context of small black holes in AdS/CFT [5, 6]. We expect that the consideration of higher order asymptotics will be relevant to this discussion. It would also be interesting to explore the more general quiver asymptotics in the context of gauge/gravity duality.

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Appendix. Hessian determinant of generalized clover quiver

Set $n = d - 1$, and consider the $n \times n$ matrix $\mathfrak{H}$,

$$\mathfrak{H}_{ij} = x_i x_j + x_i \delta_{ij}, \tag{A.1}$$

which is a rescaled version of $\mathcal{H}$. The indices i, j take values from 1 to n. We have

$$\det \mathfrak{H} = \epsilon_{i_1\cdots i_n}\mathfrak{H}_{1i_1}\mathfrak{H}_{2i_2}\cdots\mathfrak{H}_{ni_n}. \tag{A.2}$$

Define the matrix $\mathfrak{X} := \mathrm{diag}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n})$, or equivalently $\mathfrak{X}_{ij} = x_i\delta_{ij}$. Using the expression (A.1) in (A.2), we encounter terms where all the $\mathfrak{H}$ factors contribute the second term, linear in x. The contribution of these terms in $\det \mathfrak{H}$ is

$$\epsilon_{i_1\cdots i_n}x_1\delta_{1i_1}x_2\delta_{2i_2}\cdots x_n\delta_{ni_n} = \det \mathfrak{X}. \tag{A.3}$$

Next there are terms where we pick, from one of the $\mathfrak{H}$'s in (A.2) the term quadratic in x's, and from the rest the linear term. Taking the distinguished $\mathfrak{H}$ to be the first one, we find the contribution of such a term in $\det \mathfrak{H}$, as

$$\begin{aligned}&\epsilon_{i_1\cdots i_n}x_1x_{i_1}\ x_2\delta_{2i_2}\ x_3\delta_{3i_3}\ \cdots\ x_n\delta_{ni_n}\\ &= (\det \mathfrak{X})x_{i_1}\epsilon_{i_1,2,\cdots,n}\\ &= (\det \mathfrak{X})x_1.\end{aligned} \tag{A.4}$$

Summing over all the possible choices of the $\mathfrak{H}$ factor contributing the quadratic term in x's, we find their contributions to $\det \mathfrak{H}$, as

$$(\det \mathfrak{X})(\mathrm{tr}\mathfrak{X}). \tag{A.5}$$

We also have to consider terms where we pick quadratic terms from two or more $\mathfrak{H}$'s. An example of the case with two such quadratic terms, picked from the first two $\mathfrak{H}$ factors in (A.2), is

$$\epsilon_{i_1\cdots i_n}(x_1x_{i_1})(x_2x_{i_2})(x_3\delta_{3i_3})(x_4\delta_{4i_4})\ldots(x_n\delta_{ni_n}). \tag{A.6}$$

This vanishes due to the symmetry of the x's and the antisymmetry of the ϵ under relabelling of $i_1 \leftrightarrow i_2$. This symmetry argument holds whenever we pick two or more quadratic terms. We therefore conclude that

$$\det \mathfrak{H} = (\det \mathfrak{X})(1 + \mathrm{tr}\mathfrak{X}). \tag{A.7}$$

It is now straightforward to obtain equation (23).

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