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Interconnections among nonlinear field equations

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Abstract
Galileons are related to other implicitly integrable equations such as the first order nonlinear wave equation and the universal field equation, which is a Lagrangian for the Galileon.

Keywords: Galileons, nonlinear, Lagrangian

1. Introduction

Peter Freund, physicist, novelist, linguist, singer, and raconteur was the epitome of the cultured European, who enlivened many meetings with his personality. In the present crowded world his distinctive voice is much missed.

In this article, I should like to offer a small contribution in the spirit of his activities in physics [1]. There is a class of nonlinear equations which have certain properties in common; they are all integrable, but with solutions which are only implicit. I should like to address the relationship between the maximal Galileon equation, or the homogeneous Monge–Ampère equation, i.e. the Hessian det $|\phi_{\mu\nu}|$ set to zero [2]; and what I have called the universal field equation (UFE), because any function of a solution is also a solution [3]. There is a direct connection between this equation and the nonlinear wave (Monge) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{1}$$

with general implicit solution

$$u = F(ut - x), \tag{2}$$

where F is an arbitrary differentiable function. In two space dimensions this generalizes to

$$u_t + uu_x + vu_y = 0, \qquad v_t + uv_x + vv_y = 0, \tag{3}$$

with general implicit solution

$$u = F(ut - x, vt - y), \qquad v = G(ut - x, vt - y), \tag{4}$$

where F and G are two arbitrary differentiable functions. If u and v are both functions of a single field $\phi(x, y, t)$, then the two equations become one and ϕ satisfies the UFE in three dimensions [3]

$$\det \begin{vmatrix} 0 & \phi_x & \phi_y & \phi_t \\ \phi_x & \phi_{xx} & \phi_{xy} & \phi_{xt} \\ \phi_y & \phi_{xy} & \phi_{yy} & \phi_{yt} \\ \phi_t & \phi_{xt} & \phi_{yt} & \phi_{tt} \end{vmatrix} = 0. \tag{5}$$

The UFE also serves as a Lagrangian for the Galileon. In what follows, the implicit metric is assumed to be Euclidean, for ease of verification, but this is not essential.

Suppose that we have a field ϕ dependent upon n variables x_i and there is a functional relationship among the derivatives of ϕ of the form

$$G\left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \ldots, \frac{\partial\phi}{\partial x_n}\right) = 0. \tag{6}$$

Differentiate this expression successively with respect to x_1, x_2, etc and eliminate the first derivatives, $\frac{\partial G}{\partial \phi_{x_i}}$. This leads to the result that the Hessian is given by $\det\left|\frac{\partial^2\phi}{\partial x_i \partial x_j}\right| = 0$, i.e. the Maximal Galileon equation. On the other hand, introducing the abbreviation $\alpha_i = \frac{\partial\phi}{\partial x_i}$, consider the expression

$$\sum_j \frac{\partial G}{\partial \alpha_j}\frac{\partial^2 G}{\partial \alpha_i \partial x_j} = \sum_{j,k} \frac{\partial G}{\partial \alpha_j}\frac{\partial^2 G}{\partial \alpha_k \partial \alpha_i}\frac{\partial^2 \phi}{\partial x_j \partial x_k}. \tag{7}$$

Perform the sum over j; since $\frac{\partial G}{\partial x_i} = \frac{\partial G}{\partial \alpha_j}\frac{\partial^2\phi}{\partial x_i \partial x_j} = 0$, the right hand side of the equation vanishes. Setting $\frac{\partial G}{\partial \alpha_i} = u_i$, we have the Monge equation generalised to many dimensions

$$\sum_j u_j \frac{\partial u_i}{\partial x_j} = 0. \tag{8}$$

1.1. Some solutions

(1) As has already been remarked, if there is a functional relationship between the first derivatives of a differentiable field ϕ, then that field automatically satisfies the Maximal Galileon equation, though it is not the most general solution. A class of explicit solutions to the Galileon and the UFE is easily constructed. If $\sum_i x_i \frac{\partial\phi}{\partial x_i} = \phi$, i.e. the field ϕ is homogeneous and of degree 1, then ϕ satisfies the Galileon equation; if, on the other hand $\sum_i x_i \frac{\partial\phi}{\partial x_i} = 0$, then ϕ is homogeneous of degree zero, and satisfies the UFE. Furthermore, solutions of this type admit linear superposition, as functions of the same degree have an additive property.

(2) Suppose $\phi = \frac{xy}{t}$. Then the function G in (6) is

$$G = \phi_t + \phi_x \phi_y, \tag{9}$$

and ϕ satisfies the Galileon as it is of weight one. Also $\frac{\partial \phi}{\partial t} = -\frac{xy}{t^2}$ is of weight zero, and hence this function satisfies the UFE. Differentiate G with respect to x, and also y and set to zero

$$\phi_{xt} + \phi_y \phi_{xx} + \phi_x \phi_{xy} = 0, \tag{10}$$

$$\phi_{yt} + \phi_x \phi_{yy} + \phi_y \phi_{xy} = 0. \tag{11}$$

These are the Monge equations, for $u = \phi_x$, $v = \phi_y$. This is in agreement with the previous analysis of the Monge equation since the previous construction

$$G_{\phi_x}(G_{\phi_x})_x + G_{\phi_y}(G_{\phi_x})_y + G_{\phi_t}(G_{\phi_x})_t = \phi_y \phi_{yx} + \phi_x \phi_{yy} + \phi_{yt} = 0, \tag{12}$$

with $G_{\phi_x} = \frac{y}{t}$, $G_{\phi_y} = \frac{x}{t}$, $G_{\phi_t} = 1$ again yields the same equations.

A particularly attractive solution of this type is to take $\phi = \sqrt{\sum_j (x_j)^2}$.

Setting $G = (\phi_t)^2 + (\phi_x)^2 + (\phi_y)^2 = 0$, we have the generalized Monge equations for (ϕ_t, ϕ_x, ϕ_y)

$$\phi_t \phi_{tt} + \phi_x \phi_{tx} + \phi_y \phi_{ty} = 0, \tag{13}$$

$$\phi_t \phi_{tx} + \phi_x \phi_{xx} + \phi_y \phi_{xy} = 0, \tag{14}$$

$$\phi_t \phi_{ty} + \phi_x \phi_{xy} + \phi_y \phi_{yy} = 0. \tag{15}$$

In fact, in this case, by elimination of the first derivatives, ϕ also satisfies the Maximal Galileon equation in three dimensions. It also satisfies the second member of the Galileon hierarchy $\mathcal{E}_2 = 0$ for three variables, denoting the Galileon equation defined by the sum of determinants of $n \times n$ matrices by $\mathcal{E}_n = \det|\phi_{\mu\nu}| = 0$; in the case of three variables

$$\mathcal{E}_2 = \det|\phi_{x,y}| + \det|\phi_{y,t}| + \det|\phi_{t,x}|, \tag{16}$$

or, more explicitly

$$\mathcal{E}_2 = \frac{\partial^2 \phi}{\partial {x_\mu}^2} \frac{\partial^2 \phi}{\partial {x_\nu}^2} - \left(\frac{\partial^2 \phi}{\partial x_\mu \partial x_\nu} \right)^2 \tag{17}$$

$$= (\phi_{xx}\phi_{yy} - (\phi_{xy})^2) + (\phi_{yy}\phi_{tt} - (\phi_{yt})^2) + (\phi_{tt}\phi_{xx} - (\phi_{tx})^2) = 0. \tag{18}$$

The Maximal Galileon is the one where the number of variables coincides with n, the dimensions of the space-time, so there is only a single determinant. In general, solutions of the Galileon $\mathcal{E}_n = \det|\phi_{\mu\nu}| = 0$ are satisfied by the simultaneous solutions of the constraints

$$\sum_\mu (\partial_\mu \phi)^2 = 0, \qquad \sum_\mu \partial_{\mu\mu} \phi = 0, \tag{19}$$

where derivatives $\frac{\partial}{\partial x_\mu}$ are abbreviated as ∂_μ, etc.

Consider equation (18). The first term vanishes because of the first constraint. The second may be written as

$$(\partial_{\mu\nu}\phi)^2 = \frac{1}{2}\partial_{\mu\mu}(\phi_\nu)^2 - \phi_\nu \phi_{\mu\mu\nu}. \tag{20}$$

The first term vanishes by virtue of the first constraint, the second is zero from the second constraint. Hence $\mathcal{E}_2 = 0$ Now, recursively, since $\mathcal{E}_n\,(\partial_\mu\phi)^2$ is the Lagrangian for $\mathcal{E}_{n+1}$, if $\mathcal{E}_n$ vanishes, so does $\mathcal{E}_{n+1}$, hence the constraints above are sufficient to provide a solution for all Galileons.. They also serve to solve a class of UFE's. The UFE corresponding to $\mathcal{E}_2 = 0$ is

$$(\partial_\mu\phi)^2\,\partial_{\nu\nu}\phi - \partial_\mu\phi\partial_\nu\phi\partial_{\mu\nu}\phi = 0. \tag{21}$$

The first term clearly vanishes, by virtue of the constraints. The second is

$$\partial_{\mu\nu}\phi\partial_\mu\phi\partial_\nu\phi = \frac{1}{2}(\partial_{\mu\mu}\phi)(\partial_\nu\phi)^2 - \partial_\mu((\partial_\nu\phi)^2)\partial_{\mu\mu}\phi) = 0. \tag{22}$$

This vanishes because $\sum_\nu(\phi_\nu)^2 = 0$.

2. Inclusion of a potential

Consider the Lagrangian

$$\mathcal{L} = (\phi_{\alpha\alpha} + V'(\phi))\left(\frac{1}{2}(\phi_\beta)^2 + V(\phi)\right).$$

The equation of motion is

$$\phi_{\alpha\alpha}\phi_{\beta\beta} - \phi_{\alpha\beta}\phi_{\alpha\beta} - V''(\frac{1}{2}(\phi_\alpha)^2 + V) - V'(\phi_{\alpha\alpha} + V') = 0. \tag{23}$$

If $V(\phi) = 0$, the equation of motion is the pure Galileon

$$\mathcal{E}_2 = \phi_{\alpha\alpha}\phi_{\beta\beta} - \phi_{\alpha\beta}\phi_{\alpha\beta} = 0.$$

What happens under the condition that both factors in the Lagrangian are zero? i.e.

$$\phi_{\alpha\alpha} + V'(\phi) = 0 \tag{24}$$

and

$$\frac{1}{2}(\phi_\beta)^2 + V(\phi) = 0. \tag{25}$$

The answer is the pure Galileon $\mathcal{E}_2 = 0$. This is also the result of eliminating $V(\phi)$ from the two constraints

$$\phi_{\alpha\beta}\phi_\alpha - \phi_{\alpha\alpha}\phi_\beta = 0. \tag{26}$$

Differentiate with respect to x_β and perform the implied sum, again giving the pure Galileon. If, on the other hand, the equation (26) is multiplied by ϕ_β and summed over β, the UFE

$$\phi_{\alpha\beta}\phi_\alpha\phi_\beta - \phi_{\alpha\alpha}(\phi_\beta)^2 = 0 \tag{27}$$

results; thus both Galileon and UFE equations are a consequence of the equation of motion under the two constraints. Thus the introduction of a function of ϕ under these conditions does not affect the answer. It may seem that strictly speaking, $V(\phi)$ is not a true potential; the relative signs in (24) and (25) are wrong for (25) to be a Lagrangian for (24).

In the case of the sine Gordon equation, $V(\phi) = \cos(\phi) - 1$ the single soliton is a solution to $\mathcal{E}_2 = 0$, but this is not surprising as this Galileon equation admits solutions dependent upon $x - vt$, for any v.

3. Another example: the UFE

Take as Lagrangian

$$\mathcal{L}_{\mathrm{UFE}} = \sqrt{\phi_\mu^2 + 2V}. \tag{28}$$

Then the equation of motion (multiplied by $(\phi_\mu^2 + 2V)^{\frac{3}{2}}$) is

$$\partial_\mu \frac{\phi_\mu}{\sqrt{\phi_\mu^2 + 2V}} - \frac{V'}{\sqrt{\phi_\mu^2 + 2V}} \tag{29}$$

$$= \left(\phi_{\mu\mu}\phi_\nu^2 - \phi_\mu\phi_{\mu\nu}\phi_\nu\right) - 2V'(\phi_\mu^2 + 2V) = 0. \tag{30}$$

Clearly if the same conditions (24) and (25) are imposed the equation is just the UFE without a potential. This argument is perhaps suspect as in this case the Lagrangian is singular.

4. Surprise or not?

In some respects this seems a surprising result. However one might ask the converse question: suppose (24) and (25) are taken as equations to be satisfied simultaneously. What is then the result of eliminating $V(\phi)$ and $V'(\phi)$ from these two equations? That such an equation in terms of the derivatives of ϕ should exist is to be expected; what is surprising perhaps is that there are at least two such equations in this case: the UFE and the Galileon. Furthermore, they are of no more than second order in derivatives. This is not a general situation; for example, if one seeks to eliminate V from the pair

$$(\phi_{\alpha\alpha}) + V'(\phi)) = 0 \tag{31}$$

and

$$\left(\frac{1}{2}(\phi_\beta)^2 - V(\phi)\right) = 0, \tag{32}$$

then third derivatives in ϕ are the inevitable consequence of looking for a Galileon equivalent. A second order analogue of the UFE does exist, however; it is simply

$$\left(\phi_{\mu\mu}\phi_\nu^2 + \phi_\mu\phi_{\mu\nu}\phi_\nu\right) = 0. \tag{33}$$

5. Conclusions

The Galileon and the UFE hierarchies are natural nonlinear extensions of the Klein–Gordon equation and are thus good candidates for the search for partially integrable nonlinear equations. It has been demonstrated that there is a close connection with the generalised nonlinear wave equation. The Galileon has possible cosmological relevance to the issue of dark matter, but the jury is still out [4], though it seems increasingly likely that there is no explanation along these lines. Solutions to the Monge equation generically develop singularities, a feature inherited by the Galileon and the UFE, which does not bode well for applications. However, the study of equations with implicit solutions is greatly enriched by these examples, and the subject may well contain further surprises and attractive results. Further background to this article may be found on the ArXiv in an unpublished but compendious article by Thomas Curtright and myself [5], which contains numerous further references.

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