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Periodic wave solutions and stability analysis for the KP-BBM equation with abundant novel interaction solutions

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Abstract

This paper aims at investigating periodic wave solutions for the (2+1)-dimensional KP-BBM equation, from its bilinear form, obtained using the Hirota operator. Two major cases were studied from two different ansatzes. The 3D, 2D and density representation illustrating some cases of solutions obtained have been represented from a selection of the appropriate parameters. The modulation instability is employed to discuss the stability of got solutions. That will be extensively used to report many attractive physical phenomena in the fields of acoustics, heat transfer, fluid dynamics, classical mechanics and so on.

Keywords: KP-BBM equation, Hirota bilinear operator method, periodic wave solution, modulation instability

(Some figures may appear in colour only in the online journal)

1. Introduction

Nonlinear phenomena play a very important role in the fields of mathematics, chemistry, biology, and physics sciences. These phenomena are for the most part modeled by nonlinear mathematical equations. For instance, in nonlinearphysics, physical mechanisms of natural phenomena in the field of applied sciences and engineering, especially inplasma physics, elastic media, optical fibers, fluid dynamics, quantum mechanics, chimerical physics, biotechnology,signal processing, solid state physics, shallow water wave theory, are modeled by nonlinear partial differential equations(NLPDEs). However, the quest for the exact explicit solutions of these equations remains a hot topic. Moreover,looking for localized solutions and more specifically the solitary wave solutions [1–9], the lumps-type solutions [10–36], the interaction soliton-soliton, soliton-kink and kink-kink [37, 38], the interactions between solitary wave solutionsand lump solutions [39–41], as well as the periodic wave solutions [42–44] remains a very interesting subject forresearchers. Several mathematical methods are used in the search for these solutions. For example, the Exp-function method [45, 46], the Homotpy perturbation technique [47], the inverse scattering method [48] and so on. For getting the lump solutions and their interactions authors have conjugated sufficient time to search the exact rationalsoliton solutions, for example, the nonlinear evolution equations [14], the Kadomtsev-Petviashvili (KP) equation [15], the reduced pgKP and pgbKP equations [17], the (2+1)-dimensional KdV equation [18], the (2+1)-dimensionalSawada-Kotera equation [34], the (2+1)-dimensional bSK equation [35, 36], the (2 + 1)-dimensional generalized fifthorderKdV equation [37], the (2+1)-dimensional Burger equation [38], the generalized (3+1)-dimensional Shallowwater-like equation [40], and the B-Kadomtsev-Petviashvili equation [46].Various types of work for finding the periodic solitary wave solutions on the interaction between lump and otherkinds of solitary, periodic and kink solitons for the (2+1)-dimensional Breaking Soliton equation [19], lump andinteraction between different types of those on the variable-coefficient Kadomtsev-Petviashvili equation [20], and periodic

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type and periodic cross-kink wave solutions [21], and the (2+1)-dimensional extended Jimbo-Miwa equations [42] are achieved through the Hirota bilinear operator. Baskonus and co-authors constructed a family of wave solutions for some of nonlinear equations such as doublydispersive equation [22], the GilsonPickering model [23], a (2+1)-dimensional coupling system with KdV equation [24], the Lonngren-Wave equation [25], the Ablowitz-Kaup-Newell-Segur equation [26], the Wu-Zhang system [27],the longitudinal wave equation in a magneto-electro-elastic circular rod [28], the generalized double combined Sinh-Cosh-Gordon equation [29], the Sharma-Tasso-Olver equation [30], by utilizing the extended sinh-Gordon equationexpansion method, the modified exp(-φ(ζ))-expansion function method and the improved Bernoulli sub-equation functionmethod. In [31], Yukus and co-authors used the modified exp(-φ(ζ))-expansion function method to constructingsome new analytical solutions with novel structure such as the trigonometric and hyperbolic function solutions. Shafiq, Rashidi, Hammouch and Khan [32] investigated the mixed convective stagnation point flow of Williamson liquid overa vertical stretched plate and convergent solutions for the temperature and velocity were constructed and analyzed.The bilinear method was employed to investigate the rogue wave solutions and the rogue type multiple lump wavesolutions of the (2+1)-dimensional Benjamin-Ono equation by Zhao, He, and Gao [33].

The aim of this study is to construct the invariant solutions of the (2+1)-dimensional Kadomtsev–Petviashvili–Benjamin–Bona–Mahony (KP-BBM) equation of the form

$$u_{xt} + u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxt} + \gamma u_{yy} = 0, \tag{1.1}$$

where α, β and γ the arbitrary constants. Equation (1.1) is formulated using the Kadomtsev–Petviashviii (KP) equation in which reads as [15]

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0, \tag{1.2}$$

and also derived from the standard BBM equation [49] with below form

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{1.3}$$

For equation (1.2), Ma [15] obtained the lump solutions by using the following transformation

$$u = 2(\ln f)_{xx}, \tag{1.4}$$

and also Manakov *et al* [50] acquired subclass of the lump solutions in which involving two free parameters. Besides, Zhao and Ma [51] used the Hirota bilinear form of the KP equation and gained twelve classes of the lump-kinksolutions. Sufficient literature for KP-BBM equation (1.1) is provided here notably Wazwaz [52] constructed the newcompact and noncompact of the KP-BBM and the ZK-BBM equations with the aid of extended tanh method in whichconstitutes the analytical method containing the expansion of tanh series. Saut and Tzvetkov [53] obtained localizedsolitary solutions via global well-posedness structures from the conservation law and generated some soliton solutionsdepending upon the appropriate choice of the arbitrary functions. Continuing, Alam and Akbar [54] proposed thegeneralized (G'/G)-expansion method of Riccati equation and used it to establish the soliton-like solutions with theaid of symbolic computation. Later on, Tang *et al* [55] employed bifurcations of the travelling wave solutions to theKP-BBM equation and obtained the exact solutions. Some the exact lump solution, lump-kink solutions, lum-solitonsolutions, and breather-wave solutions were examined in [56] by means of symbolic computation and the Hirota bilinearmethod. As remarkable work, the Landau-Lifshitz-Gilbert equation had been considered to study the propagationand interaction of magnetic solitons in a ferromagnetic thin film with the interfacial Dzyaloshinskii-Moriya interactio [57] and found the analytical forms for magnetic breathers and the first- to third-order rogue wave solutions [58]. Likewise, Deng and coauthors [59] studied the interaction phenomenon between the lump waves and stripe solitons tothe (2+1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada equation by making use of the Hirota bilinear method. In [60], Dai and co-authors proved the existence of multi-wave solutions and obtianed three-wave solution, periodictwo-solitary-wave solutions, for the (1+2)-D Kadomtsev-Petviashvili equation via the extended three-soliton method.

Consider the Bell polynomial of Kadomtsev–Petviashvili–Benjamin–Bona–Mahony equation as follows

$$\mathbb{P}_{\mathrm{KP-BBM}}(u) := u_{xt} + u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxt} + \gamma u_{yy} = 0. \tag{1.5}$$

Let the Hirota derivatives in terms of the functions f and g can be written as

$$\prod_{i=1}^{3} D_{J_i}^{\beta_i} f.g = \prod_{i=1}^{3}\left(\frac{\partial}{\partial J_i} - \frac{\partial}{\partial {J_i}'}\right)^{\beta_i} f(J)g(J')\Bigg|_{J'=J}, \tag{1.6}$$

where the vectors $J = (J_1, J_2, J_3) = (x, y, t)$, $J' = (J'_1, J'_2, J'_3) = (x', y', t')$ and β_1, β_2, β_3 are the arbitrary nonnegative integers, and its corresponding bilinear formalism equals. It is known that the KP-BBM equation possesses a Hirota bilinear form:

$$\begin{aligned}
\mathbb{B}_{\mathrm{KP-BBM}}(f) &:= (D_x D_t + D_x^2 + \beta D_x^3 D_t \\
&\quad + \gamma D_y^2 + \alpha D_x^2 - 3\beta D_x D_t) f.f \\
&= 2[(1 - 3\beta)(ff_{xt} - f_x f_t) \\
&\quad + (1 + \alpha)(ff_{xx} - f_x^2) + \gamma(ff_{yy} - f_y^2) \\
&\quad + \beta(ff_{xxxt} - 3f_x f_{xxt} + 3f_{xx} f_{xt} - f_t f_{xxx})] = 0.
\end{aligned} \tag{1.7}$$

Use the following transformation between the function f and u:

$$u = 2(\ln f)_{xx}. \tag{1.8}$$

Employing the Bell polynomial theories of soliton equations, we achieve to the following relation as

$$\mathbb{P}_{\mathrm{KP-BBM}}(u) = \left[\frac{\mathbb{B}_{\mathrm{KP-BBM}}(f)}{f^2}\right]_{xx}. \tag{1.9}$$

Theorem 1.1. *f solves (1.8) if and only if $u = 2(\ln f)_{xx}$ demonstrate a solution to the KP-BBM equation (1.5)*

$$\begin{aligned}&(D_xD_t + D_x^2 + \beta D_x^3 D_t + \gamma D_y^2 + \alpha D_x^2\\ &- 3\beta D_x D_t) f f = 2[(1-3\beta)(ff_{xt} - f_x f_t)\\ &+ (1+\alpha)(ff_{xx} - f_x^2) + \gamma(ff_{yy} - f_y^2)\\ &+ \beta(ff_{xxxt} - 3f_x f_{xxt} + 3f_{xx} f_{xt} - f_t f_{xxx})] = 0.\end{aligned} \tag{1.10}$$

Our purpose here is to discover the exact solutions of the KP-BBM equation under consideration the Hirota bilinear method for getting the novel periodic solutions in which can be arisen with twenty one classes. Discussion about the nonlinear KP-BBM equation and the Hirota bilinear method are given. Also, the modulation instability of the KP-BBM equation is offered. In the continuation, we will offer the graphical illustrations of some solutions of the considered model along with the obtained solutions. After that, we will deal with the probe of solutions and we will finish by a conclusion.

2. New periodic-waves solutions for KP-BBM equation

Here, we will compose the periodic-waves solutions of equation (1.1), next three waves hypothesis can be discovered through employing the Hirota operator [47]. The solution can be expressed in the below as:

$$\mathbf{f} = a_1H_1 + a_2H_2 + a_3H_3 + a_4H_4, \tag{2.1}$$

$$\begin{aligned}&H_1 = \exp(\Omega_1 x + \Omega_2 y + \Omega_3 t),\\ &H_2 = \exp(-\Omega_1 x - \Omega_2 y - \Omega_3 t),\\ &H_3 = \cos(\Omega_4 x + \Omega_5 y + \Omega_6 t),\\ &H_4 = \cosh(\Omega_7 x + \Omega_8 y + \Omega_9 t),\end{aligned} \tag{2.2}$$

$$\begin{aligned}&H_5 = \sin(\Omega_4 x + \Omega_5 y + \Omega_6 t),\\ &H_6 = \sinh(\Omega_7 x + \Omega_8 y + \Omega_9 t),\\ &\mathbf{u} = 2\frac{\partial^2}{\partial x^2}\ln(\mathbf{f})\\ &= 2\,\frac{a_1\Omega_1^2H_1 + a_2\Omega_1^2H_2 - a_3H_3\Omega_4^2 + a_4H_4\Omega_7^2}{f}\\ &\quad - 2\,\frac{(a_1\Omega_1H_1 - a_2\Omega_1H_2 - a_3H_5\Omega_4 + a_4H_6\Omega_7)^2}{f^2},\end{aligned} \tag{2.3}$$

where Ω_i, $i = 1, \ldots, 9 a_i$, $j = 1, \ldots, 4$, are free parameters in which are to find later. Plugging (2.3) into equation (1.10) and then collecting the coefficients, we get to the following results:

$$\begin{cases}\omega\,\Omega_4^4 - 6\,\omega\,\Omega_4^2\Omega_7^2 + \omega\,\Omega_7^4 + 4\,\Omega_4\Omega_6 - 3\,\Omega_5^2 - 4\,\Omega_7\Omega_9 + 3\,\Omega_8^2 = 0\\ \omega\,\Omega_1^4 + 6\,\omega\,\Omega_1^2\Omega_7^2 + \omega\,\Omega_7^4 - 4\,\Omega_1\Omega_3 + 3\,\Omega_2^2 - 4\,\Omega_7\Omega_9 + 3\,\Omega_8^2 = 0\\ 2\,\omega\,\Omega_4^3\Omega_7 - 2\,\omega\,\Omega_4\Omega_7^3 + 2\,\Omega_4\Omega_9 - 3\,\Omega_5\Omega_8 + 2\,\Omega_6\Omega_7 = 0,\\ 2\,\omega\,\Omega_1^3\Omega_7 + 2\,\omega\,\Omega_1\Omega_7^3 - 2\,\Omega_1\Omega_9 + 3\,\Omega_2\Omega_8 - 2\,\Omega_3\Omega_7 = 0\\ \omega\,\Omega_1^4 - 6\,\omega\,\Omega_1^2\Omega_4^2 + \omega\,\Omega_4^4 - 4\,\Omega_1\Omega_3 + 3\,\Omega_2^2 + 4\,\Omega_4\Omega_6 - 3\,\Omega_5^2 = 0\\ 2\,\omega\,\Omega_1^3\Omega_4 - 2\,\omega\,\Omega_1\Omega_4^3 - 2\,\Omega_1\Omega_6 + 3\,\Omega_2\Omega_5 - 2\,\Omega_3\Omega_4 = 0\\ 16\omega\Omega_1^4a_1a_2 + 4\omega\Omega_4^4a_3^2 + 4\omega\Omega_7^4a_4^2 - 16\Omega_1\Omega_3a_1a_2 + 12\Omega_2^2a_1a_2 + 4\Omega_4\Omega_6a_3^2 - 3\Omega_5^2a_3^2 - 4\Omega_7\Omega_9a_4^2 + 3\Omega_8^2a_4^2 = 0.\end{cases} \tag{2.4}$$

Solving the equations we get:

Case I:

$$\begin{aligned}&\mathbf{u}_1 = 2\,\frac{a_1\Omega_1^2H_1 + a_4H_4\Omega_7^2}{f} - 2\,\frac{(a_1\Omega_1H_1 + a_4H_6\Omega_7)^2}{f^2},\\ &f = a_1H_1 + a_4H_4,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&H_1 = \mathrm{e}^{\Omega_1 x + \frac{\sqrt{-\beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1)}}{\beta\,\gamma}y - \frac{3\Omega_1(\alpha+1)}{4\,\beta\,\Omega_1^2 - 3\,\beta + 1}t},\\ &H_4 = \cosh\left(\frac{\sqrt{-3\,\beta(\beta\,\Omega_1^2 - 3\,\beta + 1)}}{3\beta}x\right.\\ &\qquad\left. + \frac{3\Omega_7(\alpha+1)}{4\,\beta\,\Omega_1^2 - 3\,\beta + 1}t\right).\end{aligned} \tag{2.6}$$

Also, the existence condition of the solution is that

$$\begin{aligned}&\beta(\beta\,\Omega_1^2 - 3\,\beta + 1) < 0,\\ &\beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1) < 0,\\ &4\,\beta\,\Omega_1^2 - 3\,\beta + 1 \neq 0.\end{aligned}$$

Case II:

$$\begin{aligned}&\mathbf{u}_2 = 2\,\frac{-a_3H_3\Omega_4^2 + a_4H_4\Omega_7^2}{f}\\ &\quad - 2\,\frac{(-a_3H_5\Omega_4 + a_4H_6\Omega_7)^2}{f^2},\quad f = a_3H_3 + a_4H_4,\end{aligned} \tag{2.7}$$

$$H_3 = \cos\left(\sqrt{-\frac{3\beta(\alpha+1)}{\gamma(\beta\,\Omega_7^2 a_3^2 - 4\,\beta\,\Omega_7^2 a_4^2 - 3\,\beta\,a_3^2 + 3\,\beta\,a_4^2 + a_3^2 - a_4^2)}}\,a_4\Omega_7^2 y\right),$$

$$H_4 = \cosh\left(\Omega_7 x - \frac{(\alpha\,a_3^2 - \alpha\,a_4^2 + a_3^2 - a_4^2)\Omega_7}{\beta\,\Omega_7^2 a_3^2 - 4\,\beta\,\Omega_7^2 a_4^2 - 3\,\beta\,a_3^2 + 3\,\beta\,a_4^2 + a_3^2 - a_4^2}t\right).$$

Also, the existence condition of the solution is that

$$\gamma(\beta\,\Omega_7^2 a_3^2 - 4\,\beta\,\Omega_7^2 a_4^2 - 3\,\beta\,a_3^2 + 3\,\beta\,a_4^2 + a_3^2 - a_4^2)\beta(\alpha+1) < 0.$$

Case III:

$$\mathbf{u}_3 = 2\,\frac{a_4 H_4 \Omega_7^2}{f} - 2\,\frac{(a_4 H_6 \Omega_7)^2}{f^2},\ f = a_3 H_3 + a_4 H_4, \quad (2.8)$$

$$H_3 = \cos\left(\frac{\sqrt{\beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1)}}{\beta\,\gamma}y + \Omega_6 t\right),$$
$$H_4 = \cosh\left(\frac{\sqrt{\beta(3\,\beta-1)}}{\beta}x + \frac{\alpha\,a_3^2 - \alpha\,a_4^2 + a_3^2 - a_4^2}{\sqrt{\beta(3\,\beta-1)}\,a_4^2}t\right).$$

Also, the existence condition of the solution is that

$$\beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1) > 0,\quad \beta(3\,\beta - 1) > 0.$$

Case IV:

$$\mathbf{u}_4 = 2\,\frac{a_1\Omega_1^2 H_1 - a_3 H_3 \Omega_4^2}{f} - 2\,\frac{(a_1\Omega_1 H_1 - a_3 H_5 \Omega_4)^2}{f^2},$$
$$\mathbf{f} = a_1 H_1 + a_3 H_3, \quad (2.9)$$

$$H_1 = \mathrm{e}^{\Omega_1 x + \frac{\sqrt{-\beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1)}}{\beta\,\gamma}y - \frac{3\Omega_1(\alpha+1)}{4\,\beta\,\Omega_1^2 - 3\,\beta + 1}t},$$
$$H_3 = \cos\left(\frac{\sqrt{3\,\beta(\beta\,\Omega_1^2 - 3\,\beta + 1)}}{3\beta}x + \frac{\sqrt{3\,\beta(\beta\,\Omega_1^2 - 3\,\beta + 1)}(\alpha+1)}{\beta(4\,\beta\,\Omega_1^2 - 3\,\beta + 1)}t\right).$$

Moreover, the existence condition of the solution is that

$$\beta(\beta\,\Omega_1^2 - 3\,\beta + 1) > 0,$$
$$\beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1) < 0.$$

Case V:

$$\mathbf{u}_5 = 2\,\frac{a_4 H_4 \Omega_7^2}{f} - 2\,\frac{(a_4 H_6 \Omega_7)^2}{f^2},\ f = a_3 H_3 + a_4 H_4, \quad (2.10)$$

$$H_3 = \cos(\Omega_6 t),$$
$$H_4 = \cosh\left(\frac{\sqrt{\beta(3\,\beta - 1)}}{\beta}x + \frac{\sqrt{-\beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1)}}{\beta\,\gamma}y\right).$$

Moreover, the existence condition of the solution is that

$$\beta(3\,\beta - 1) > 0,\quad \beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1) < 0.$$

Case VI:

$$\mathbf{u}_5 = 2\,\frac{a_1\Omega_1^2 H_1 + a_2\Omega_1^2 H_2 - a_3 H_3 \Omega_4^2 + a_4\Omega_7^2}{f} - 2\,\frac{(a_1\Omega_1 H_1 - a_2\Omega_1 H_2 - a_3 H_5 \Omega_4)^2}{f^2}, \quad (2.11)$$

$$\mathbf{f} = a_1 H_1 + a_2 H_2 + a_3 H_3 + a_4,$$
$$H_1 = \mathrm{e}^{\Omega_1 x - \frac{\Omega_1(\alpha+1)}{\beta\,\Omega_1^2 - 3\,\beta + 1}t},$$
$$H_2 = \mathrm{e}^{-\Omega_1 x + \frac{\Omega_1(\alpha+1)}{\beta\,\Omega_1^2 - 3\,\beta + 1}t},$$
$$H_3 = \cos\left(\frac{\sqrt{-\beta(3\,\beta - 1)}}{\beta}x + \frac{\sqrt{\beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1)}}{\beta\,\gamma}y + \frac{(\alpha+1)\sqrt{-\beta(3\,\beta - 1)}}{\beta(\beta\,\Omega_1^2 - 3\,\beta + 1)}t\right),\ H_4 = 1.$$

Moreover, the existence condition of the solution is that

$$\beta(3\,\beta - 1) < 0,\quad \beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1) > 0.$$

Case VII:

$$\mathbf{u}_7 = 2\,\frac{a_1\Omega_1^2 H_1 + a_2\Omega_1^2 H_2 + a_4 H_4 \Omega_7^2}{f} - 2\,\frac{(a_1\Omega_1 H_1 - a_2\Omega_1 H_2 + a_4 H_6 \Omega_7)^2}{f^2}, \quad (2.12)$$

$$
\begin{aligned}
\mathbf{f} &= -\frac{3\,\beta\,a_3^2 - 3\,\beta\,a_4^2 - a_3^2 + a_4^2}{4(2\,\beta\,\Omega_1^2 - 3\,\beta + 1)a_2}H_1\\
&\quad + a_2H_2 + a_3H_3 + a_4H_4,\\
H_1 &= \mathrm{e}^{\Omega_1 x + \frac{\sqrt{-3\,\beta\,\gamma(\alpha\,\beta\,\Omega_1^2 + \beta\,\Omega_1^2 - 3\,\alpha\,\beta + \alpha - 3\,\beta + 1)}}{3\beta\,\gamma}y - \frac{2(\alpha+1)}{3\beta\,\Omega_1}t},\\
H_2 &= \mathrm{e}^{-\Omega_1 x - \frac{\sqrt{-3\,\beta\,\gamma(\alpha\,\beta\,\Omega_1^2 + \beta\,\Omega_1^2 - 3\,\alpha\,\beta + \alpha - 3\,\beta + 1)}}{3\beta\,\gamma}y + \frac{2(\alpha+1)}{3\beta\,\Omega_1}t},\\
H_4 &= \cosh\left(\frac{\sqrt{\beta(3\,\beta - 1)}}{\beta}x\right),
\end{aligned}
$$

$$
\begin{aligned}
H_3 = \cos\Bigg(&\frac{\sqrt{\beta\gamma(3\alpha\beta - \alpha + 3\beta - 1)}}{\beta\,\gamma}y\\
&- \frac{2}{3}\,\frac{\sqrt{\beta\gamma(3\alpha\beta - \alpha + 3\beta - 1)}\sqrt{-3\beta\gamma(\alpha\beta\Omega_1^2 + \beta\Omega_1^2 - 3\alpha\beta + \alpha - 3\beta + 1)}}{\beta^2\gamma(\beta\Omega_1^2 - 3\beta + 1)\Omega_1}t\Bigg).
\end{aligned}
$$

Moreover, the existence condition of the solution is that

$$
\begin{aligned}
&(2\,\beta\,\Omega_1^2 - 3\,\beta + 1)a_2 \neq 0,\\
&\beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1) > 0,\\
&\beta\,\gamma(\alpha\,\beta\,\Omega_1^2 + \beta\,\Omega_1^2 - 3\,\alpha\,\beta + \alpha - 3\,\beta + 1) < 0.
\end{aligned}
$$

Case VIII:

$$
\begin{aligned}
\mathbf{u}_8 &= 2\,\frac{a_2\Omega_1^2H_2 + a_4H_4\Omega_7^2}{f}\\
&\quad - 2\,\frac{(-a_2\Omega_1H_2 + a_4H_6\Omega_7)^2}{f^2},\\
\mathbf{f} &= a_2H_2 - a_4H_3 + a_4H_4,
\end{aligned}
\tag{2.13}
$$

$$
H_2 = \mathrm{e}^{-\Omega_1 x + \frac{\sqrt{-(3\,\beta\,\Omega_1^2 - 9\,\beta + 3)(3\,\beta - 1)}(\alpha+1)}{3\sqrt{\beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1)}}y + \frac{2}{3}\frac{\alpha+1}{\beta\,\Omega_1}t},
$$

$$
\begin{aligned}
H_3 &= \cos\Bigg(\frac{\sqrt{\beta\,\gamma(3\,\alpha\,\beta - \alpha + 3\,\beta - 1)}}{\beta\,\gamma}y\\
&\quad + \frac{2}{3}\frac{\sqrt{-(3\,\beta\,\Omega_1^2 - 9\,\beta + 3)(3\,\beta - 1)}(\alpha + 1)}{(\beta\,\Omega_1^2 - 3\,\beta + 1)\Omega_1\beta}t\Bigg),\\
H_4 &= \cosh\left(\frac{\sqrt{\beta(3\,\beta - 1)}}{\beta}x\right).
\end{aligned}
$$

Moreover, the existence condition of the solution is that

$$
\begin{aligned}
&\beta(3\beta - 1) > 0,\\
&(3\,\beta\,\Omega_1^2 - 9\,\beta + 3)(3\,\beta - 1) < 0,\\
&(\beta\,\Omega_1^2 - 3\,\beta + 1)\Omega_1 \neq 0.
\end{aligned}
$$

Case IX:

$$
\begin{aligned}
\mathbf{u}_9 &= 2\,\frac{\left(\frac{a_4^2}{4a_2}H_1 + a_2H_2 + a_4H_4\right)\Omega_1^2}{f}\\
&\quad - 2\,\frac{\left(\frac{a_4^2}{4a_2}\Omega_1H_1 - a_2\Omega_1H_2 + a_4H_6\Omega_1\right)^2}{f^2},\\
f &= \frac{a_4^2}{4a_2}H_1 + a_2H_2 + a_4H_4,
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
H_1 &= \mathrm{e}^{\Omega_1 x + \Omega_8 y - \frac{4\,\beta\,\Omega_1^3\Omega_9 + 2\,\alpha\,\Omega_1^2 - 3\,\beta\,\Omega_1\Omega_9 + 2\,\gamma\,\Omega_8^2 + 2\,\Omega_1^2 + \Omega_1\Omega_9}{\Omega_1(4\,\beta\,\Omega_1^2 - 3\,\beta + 1)}t},\\
H_2 &= \mathrm{e}^{-\Omega_1 x - \Omega_8 y + \frac{4\,\beta\,\Omega_1^3\Omega_9 + 2\,\alpha\,\Omega_1^2 - 3\,\beta\,\Omega_1\Omega_9 + 2\,\gamma\,\Omega_8^2 + 2\,\Omega_1^2 + \Omega_1\Omega_9}{\Omega_1(4\,\beta\,\Omega_1^2 - 3\,\beta + 1)}t},
\end{aligned}
$$

$$
\begin{aligned}
&H_4 = \cosh(\Omega_1x + \Omega_8y + \Omega_9t),\\
&H_6 = \sinh(\Omega_1x + \Omega_8y + \Omega_9t),\\
&\Omega_1(4\,\beta\,\Omega_1^2 - 3\,\beta + 1) \neq 0.
\end{aligned}
$$

Case X:

$$
\begin{aligned}
\mathbf{u}_{10} &= 2\,\frac{a_2\Omega_1^2H_2 - a_3H_3\Omega_4^2 + a_4H_4\Omega_7^2}{f}\\
&\quad - 2\,\frac{(-a_2\Omega_1H_2 - a_3H_5\Omega_4 + a_4H_6\Omega_7)^2}{f^2},\\
f &= a_2H_2 + a_3H_3 + a_4H_4,
\end{aligned}
\tag{2.15}
$$

$$
H_2 = \mathrm{e}^{-\Omega_1 x - \frac{\Omega_1(6\,\alpha\,\beta\,\Omega_1^2 + \beta\,\gamma\,\Omega_8^2 + 6\,\beta\,\Omega_1^2 - 6\,\alpha\,\beta + 2\,\alpha - 6\,\beta + 2)}{3\Omega_8\gamma\,\beta\,\Omega_4}y - \frac{-36\,\beta^2\Omega_1^6(\alpha+1)^2 + \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4}{18\beta(4\,\beta\,\Omega_1^2 - 3\,\beta + 1)\gamma\,\Omega_1\Omega_8^2(3\,\beta\,\Omega_1^2 - 3\,\beta + 1)}t},
$$

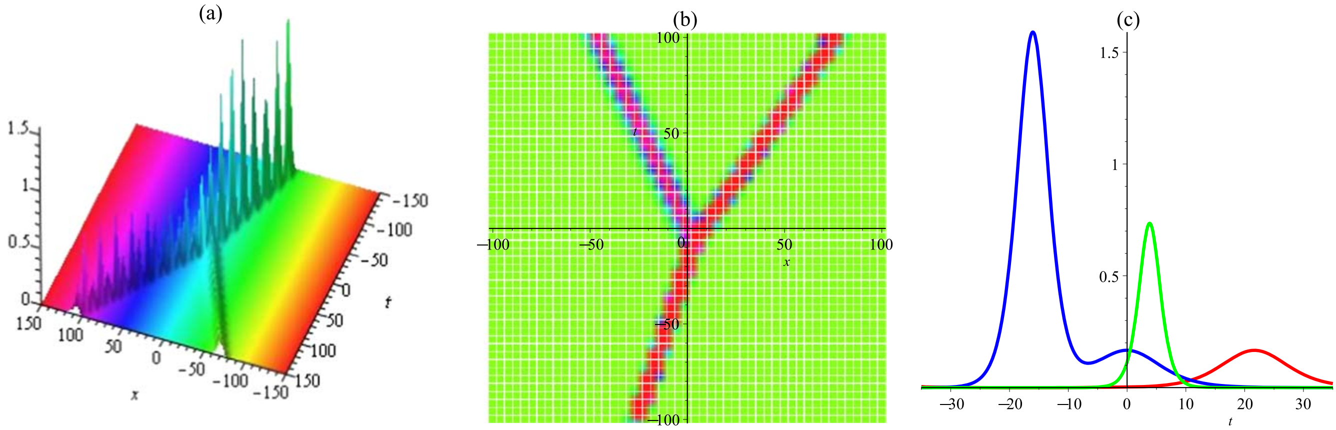

Figure 1. Diagram of periodic-waves (2.5) using values $a_1 = .5$, $a_4 = 1$, $\Omega_1 = 1.5$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $y = -4$, and (a) 3D plot, (b) density plot, and (c) 2D plot with (red $x = -10$, blue $x = 0$, and green $x = 10$).

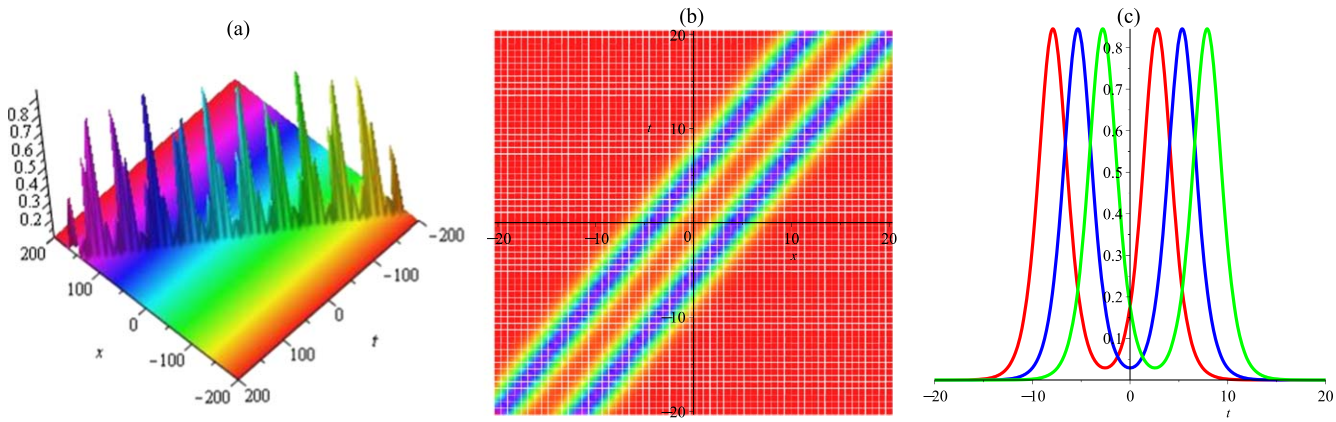

Figure 2. Diagram of periodic-waves (2.8) using values $a_3 = 2.5$, $a_4 = 1$, $\Omega_7 = 1.3$, $\alpha = -2$, $\beta = 1$, $\gamma = 1$, $y = -4$, and (a) 3D plot, (b) density plot, and (c) 2D plot with (red $x = -2$, blue $x = 0$, and green $x = 2$).

$$\begin{aligned}\Xi_1 &= -6\,\Omega_1^4\beta(\alpha+1)\\ &\times(11\,\beta\,\gamma\,\Omega_8^2 - 12\,\alpha\,\beta + 4\,\alpha - 12\,\beta + 4),\\ \Xi_2 &= \gamma\,\Omega_1^2\Omega_8^2\beta(26\,\beta\,\gamma\,\Omega_8^2 + 147\,\alpha\,\beta - 49\,\alpha + 147\,\beta),\end{aligned}$$

$$\begin{aligned}\Xi_3 &= -\Omega_1^2(36\,\alpha^2\beta^2 + 49\,\beta\,\gamma\,\Omega_8^2 - 24\,\alpha^2\beta\\ &+ 72\,\alpha\,\beta^2 + 4\,\alpha^2 - 48\,\alpha\,\beta + 36\,\beta^2 + 8\,\alpha - 24\,\beta + 4),\\ \Xi_4 &= -9\,\gamma\,\Omega_8^2(3\,\beta - 1)(\beta\gamma\,\Omega_8^2 + 3\,\alpha\,\beta - \alpha + 3\,\beta - 1),\end{aligned}$$

$$H_3 = \cos\left(\frac{\sqrt{-\beta(3\beta\Omega_1^2 - 3\beta + 1)}}{\beta}x + \frac{\Omega_8\Omega_4}{\Omega_7}y - \frac{(-3\alpha\beta\Omega_1^2 + \beta\gamma\Omega_8^2 - 3\beta\Omega_1^2 + 3\alpha\beta - \alpha + 3\beta - 1)}{3\sqrt{\beta(3\beta\Omega_1^2 - 3\beta + 1)}\,(4\beta\Omega_1^2 - 3\beta + 1)}t\right),$$

$$H_4 = \cosh\left(\frac{\sqrt{-\beta(3\,\beta\,\Omega_1^2 - 3\,\beta + 1)}}{\beta}x + \Omega_8 y + \frac{-3\alpha\beta\Omega_1^2 + \beta\gamma\Omega_8^2 - 3\beta\Omega_1^2 + 3\alpha\beta - \alpha + 3\beta - 1}{3\sqrt{-\beta(3\beta\Omega_1^2 - 3\beta + 1)}\,(4\beta\Omega_1^2 - 3\beta + 1)}t\right).$$

Case XI:

$$\begin{aligned}\mathbf{u}_{11} &= 2\,\frac{a_1\Omega_1^2H_1 + a_4H_4\Omega_7^2}{f} - 2\,\frac{(a_1\Omega_1H_1 + a_4H_6\Omega_7)^2}{f^2},\\ f &= a_1H_1 + \frac{\sqrt{5(2\beta\Omega_1^2 + 3\beta - 1)(4\beta\Omega_1^2 - 3\beta + 1)}}{2\beta\Omega_1^2 + 3\beta - 1}H_3 + a_4H_4,\end{aligned} \tag{2.16}$$

$$\begin{aligned}H_2 &= e^{\Omega_1x + \Omega_2y + \Omega_3t},\\ H_3 &= \cos(\Omega_5y + \Omega_6t),\\ H_4 &= \cosh(\Omega_7x + \Omega_8y + \Omega_9t),\end{aligned}$$

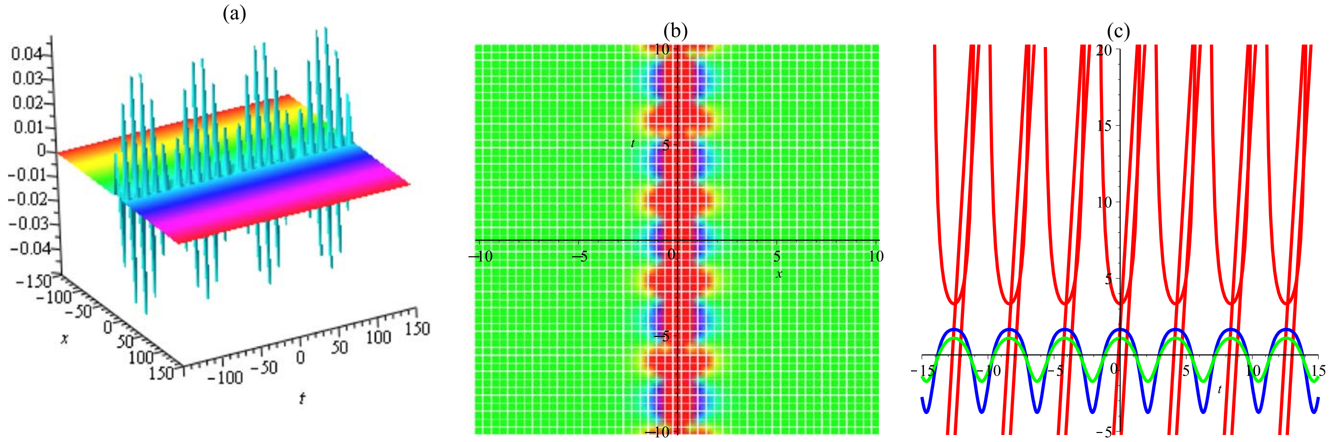

Figure 3. Diagram of periodic-waves (2.8) using values $a_3 = 1.1$, $a_4 = 1$, $\Omega_7 = 1.3$, $\alpha = -1$, $\beta = -2$, $\gamma = -1$, $y = -4$, and (a) 3D plot, (b) density plot, and (c) 2D plot with (red $x = -1.3$, blue $x = 0$, and green $x = 1.3$).

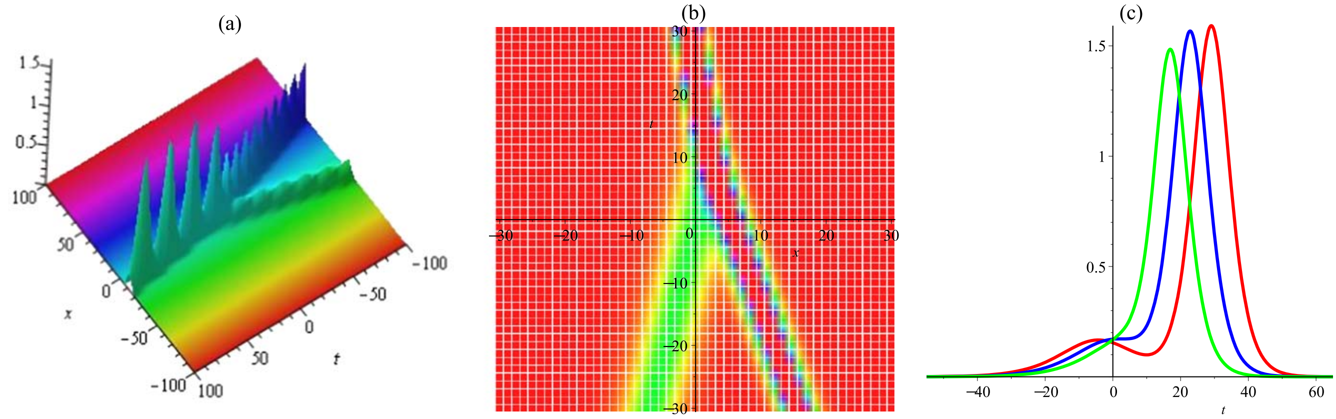

Figure 4. Diagram of periodic-waves (2.9) using values $a_1 = 0.5$, $a_3 = 1$, $\Omega_1 = 1.5$, $\alpha = -2$, $\beta = 2$, $\gamma = 1$, $y = -4$, and (a) 3D plot, (b) density plot, and (c) 2D plot with (red $x = -1$, blue $x = 0$, and green $x = 1$).

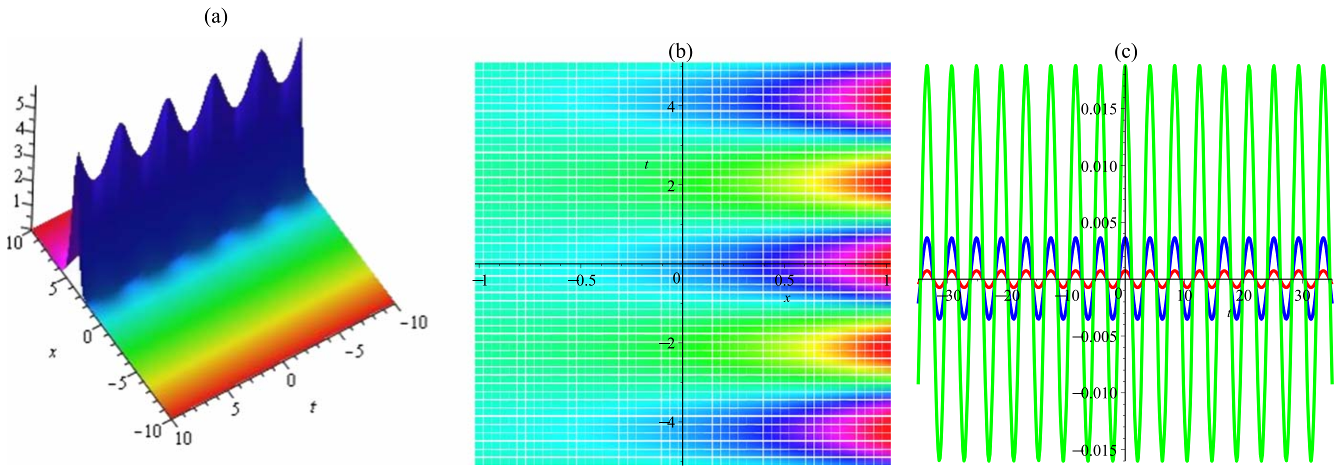

Figure 5. Diagram of periodic-waves (2.10) using values $a_3 = 0.2\gamma$, $a_4 = 1$, $\Omega_6 = 1.5$, $\alpha = -2$, $\beta = 2$, $\gamma = 1$, $y = -4$, and (a) 3D plot, (b) density plot, and (c) 2D plot with (red $x = -1$, blue $x = 0$, and green $x = 1$).

$$\Omega_2 = -\frac{(3\alpha\beta^2\Omega_1^4 + 3\beta^2\Omega_1^4 - 12\alpha\beta^2\Omega_1^2 + 4\alpha\beta\Omega_1^2 - 12\,\beta^2\Omega_1^2 + 9\alpha\beta^2 + 4\beta\Omega_1^2 - 6\alpha\beta + 9\beta^2 + \alpha - 6\beta + 1)\Omega_1}{\Omega_7(7\beta\Omega_1^2 - 3\beta + 1)\beta\gamma\,\Omega_8},$$

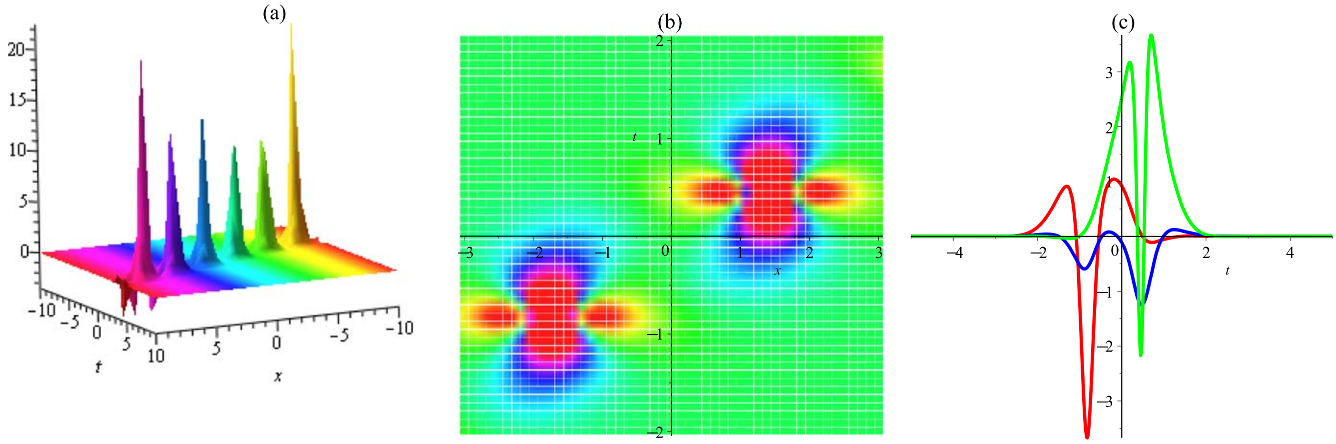

Figure 6. Diagram of periodic-waves (2.11) using values $a_2 = 0.5$, $a_3 = 1.5$, $a_4 = 1$, $\Omega_1 = 1.5$, $\alpha = 1$, $\beta = \frac{1}{4}$, $\gamma = -1$, $y = -4$, and (a) 3D plot, (b) density plot, and (c) 2D plot with (red $x = -1$, blue $x = 0$, and green $x = 1$).

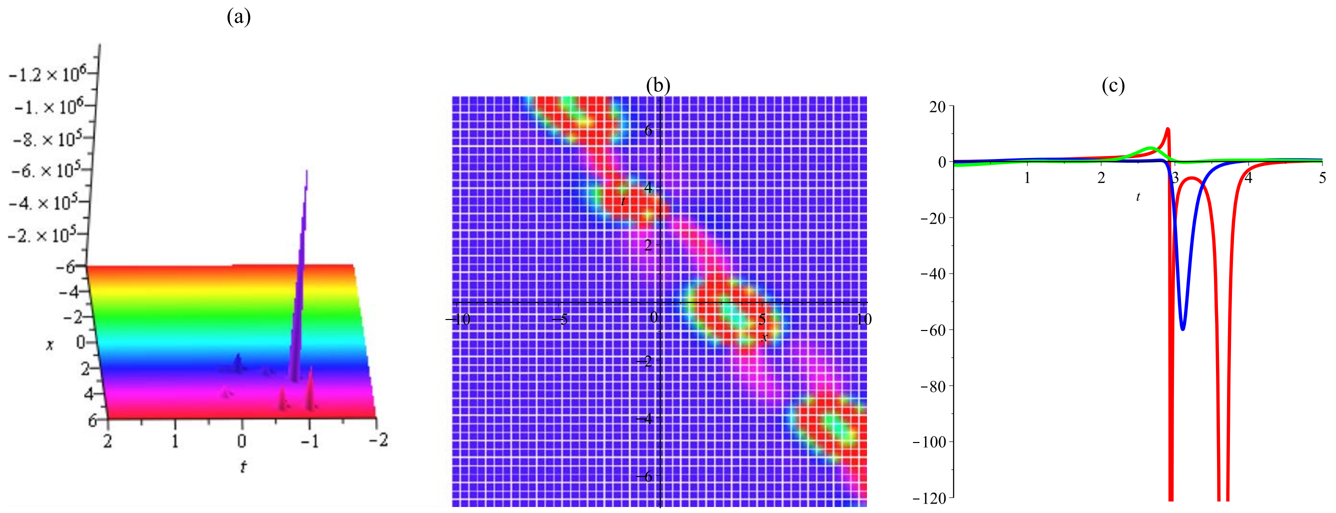

Figure 7. Diagram of periodic-waves (2.12) using values $a_2 = 0.5$, $a_3 = 1.5$, $a_4 = 1$, $\Omega_1 = 1.5$, $\alpha = -2$, $\beta = \frac{1}{4}$, $\gamma = 1$, $y = -4$, and (a) 3D plot, (b) density plot, and (c) 2D plot with (red $x = -1$, blue $x = 0$, and green $x = 1$).

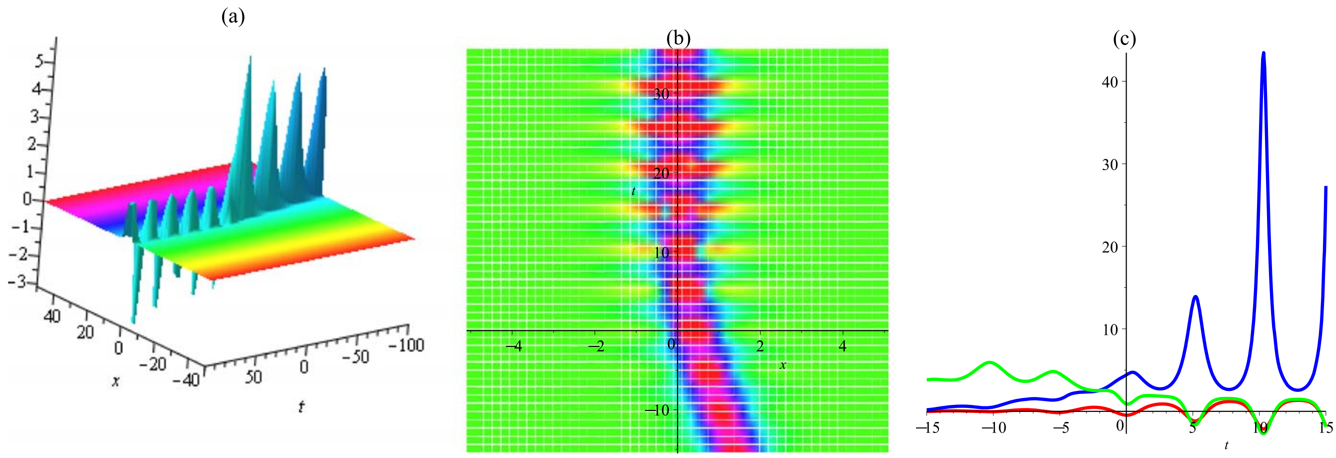

Figure 8. Diagram of periodic-waves (2.13) using values $a_2 = 0.5$, $a_4 = 1.4$, $\Omega_1 = 1.5$, $\alpha = -2$, $\beta = 2$, $\gamma = -1$, $y = -4$, and (a) 3D plot, (b) density plot, and (c) 2D plot with (red $x = -1$, blue $x = 0$, and green $x = 1$).

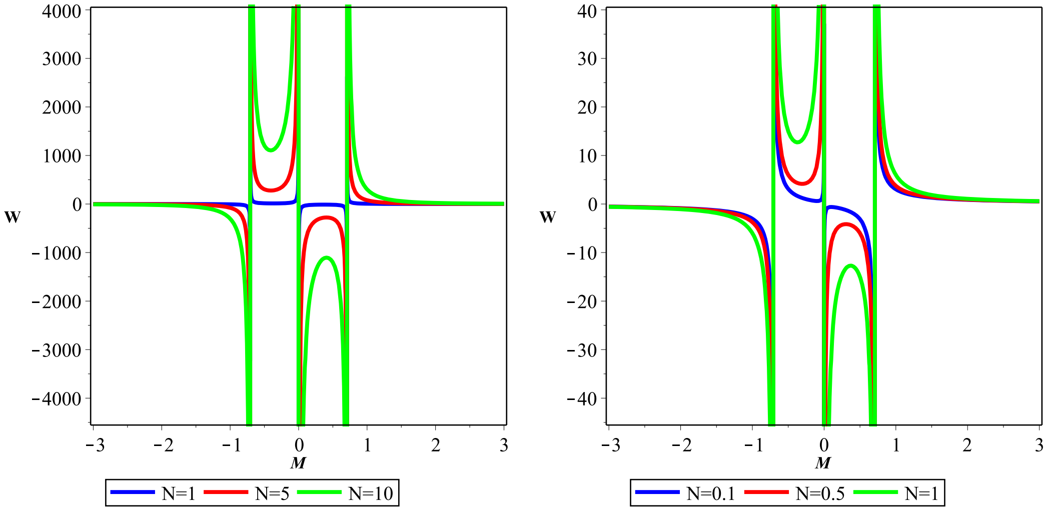

Figure 9. The dispersion relation between frequency $W(M, N)$ and wave number M under the various values $\alpha = 1$, $\beta = 2$, $\gamma = 3$, $q = 1$.

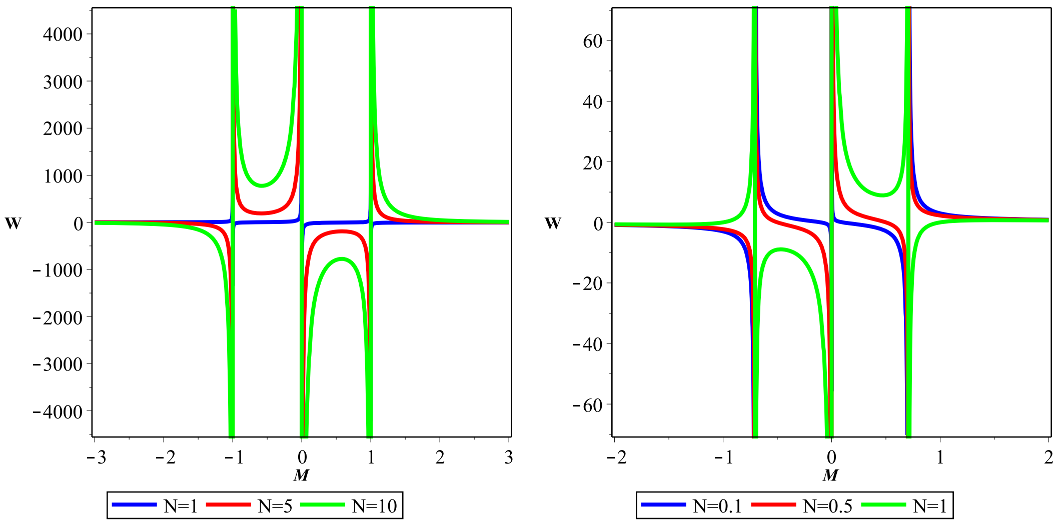

Figure 10. The dispersion relation between frequency $W(M, N)$ and wave number M under the various values (left) $\alpha = -3$, $\beta = 1$, $\gamma = 3$, $q = 1$ (right) $\alpha = 1$, $\beta = 2$, $\gamma = -3$, $q = 1$.

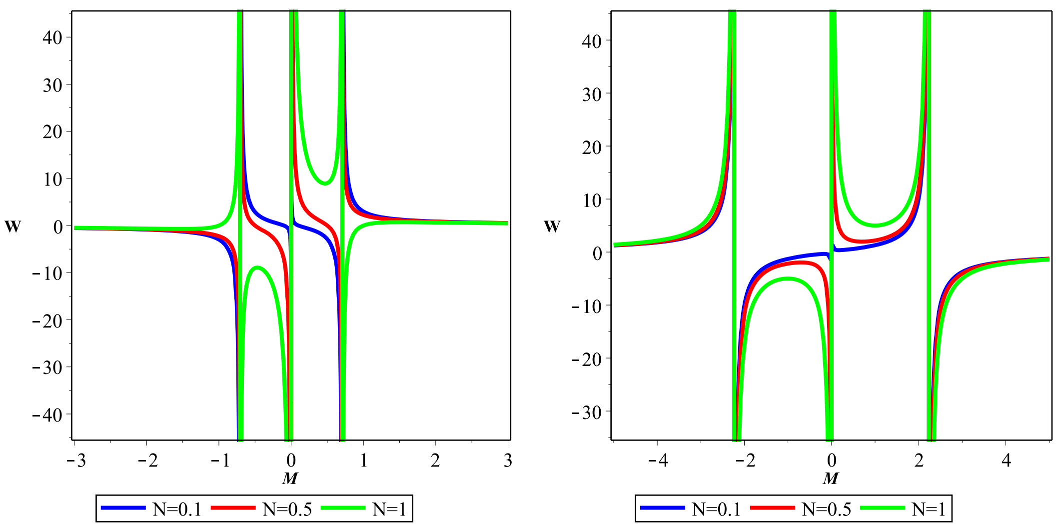

Figure 11. The dispersion relation between frequency $W(M, N)$ and wave number M under the various values (left) $\alpha = -1$, $\beta = 2$, $\gamma = -3$, $q = -1$ (right) $\alpha = 1$, $\beta = 0.2$, $\gamma = -3$, $q = -1$.

$$\Omega_3 = -\frac{2(\alpha+1)(\beta\,\Omega_1^2 - 3\beta + 1)}{3\Omega_1\beta(7\beta\Omega_1^2 - 3\,\beta + 1)},$$
$$\Omega_5 = 3\frac{\Xi_1(\alpha+1)\Omega_7\Omega_1}{(7\beta\Omega_1^2 - 3\beta + 1)\gamma\Omega_8},$$
$$\Omega_6 = 2\frac{\Xi_1(\alpha+1)}{\Omega_1\beta(7\beta\Omega_1^2 - 3\beta + 1)},$$

$$\Omega_7 = \frac{\sqrt{-\beta(3\beta\Omega_1^2 - 3\beta + 1)}}{\beta},$$
$$\Omega_8 = \frac{\sqrt{2\beta\gamma(3\alpha\beta\Omega_1^2 + 3\,\beta\,\Omega_1^2 - 3\alpha\beta + \alpha - 3\beta + 1)}}{\beta\gamma},$$
$$\Omega_9 = \frac{3\,\alpha\,\beta\,\Omega_1^2 + 3\,\beta\,\Omega_1^2 - 3\,\alpha\,\beta + \alpha - 3\,\beta + 1}{3\Omega_7\beta(4\,\beta\,\Omega_1^2 - 3\,\beta + 1)}.$$

$$\Omega_7 = \frac{\sqrt{-\beta(3\,\beta\,\Omega_1^2 - 3\beta + 1)}}{\beta},$$
$$\Omega_8 = \frac{\sqrt{3\,\gamma(7\beta\Omega_1^2 - 3\beta + 1)(3\alpha\beta\Omega_1^2 + 3\beta\Omega_1^2 - 3\alpha\beta + \alpha - 3\beta + 1)}\,\Omega_1}{\gamma(7\beta\Omega_1^2 - 3\beta + 1)},$$

$$\Omega_9 = -2\frac{3\alpha\beta\Omega_1^2 + 3\beta\Omega_1^2 - 3\alpha\beta + \alpha - 3\beta + 1}{\beta\Omega_7(7\beta\Omega_1^2 - 3\beta + 1)},$$
$$\Xi[1] = \frac{1}{3}\sqrt{-18\beta^2\Omega_1^4 - 9\beta^2\Omega_1^2 + 3\beta\Omega_1^2 + 27\beta^2 - 18\beta + 3}.$$

Case XII:

$$\mathbf{u}_{12} = 2\,\frac{a_2\Omega_1^2H_2 - a_3H_3\Omega_4^2 + a_4H_4\Omega_7^2}{f} - 2\,\frac{(-a_2\Omega_1H_2 - a_3H_5\Omega_4 + a_4H_6\Omega_7)^2}{f^2},$$
$$f = a_2H_2 + a_3H_3 + a_4H_4, \tag{2.17}$$

$$H_2 = e^{\Omega_1x + \Omega_2y + \Omega_3t},$$
$$H_3 = \cos(\Omega_4x + \Omega_5y + \Omega_6t),$$
$$H_4 = \cosh(\Omega_7x + \Omega_8y + \Omega_9t),$$

$$\Omega_2 = \frac{4}{3}\frac{\Omega_1(3\alpha\beta\Omega_1^2 + 3\beta\Omega_1^2 - 3\,\alpha\,\beta + \alpha - 3\beta + 1)}{\Omega_8\gamma\Omega_7\beta},$$
$$\Omega_3 = \frac{(28\beta\Omega_1^2 - 27\,\beta + 9)(\alpha+1)}{18\Omega_1(4\beta\Omega_1^2 - 3\,\beta + 1)\beta},$$

$$\Omega_4 = \frac{\sqrt{\beta(3\beta\Omega_1^2 - 3\beta + 1)}}{\beta},$$
$$\Omega_6 = -\frac{3\alpha\beta\Omega_1^2 + 3\beta\Omega_1^2 - 3\,\alpha\,\beta + \alpha - 3\,\beta + 1}{3\Omega_4\beta(4\beta\Omega_1^2 - 3\beta + 1)},$$

$$\Omega_5 = 2\,\frac{9\,\alpha\beta^2\Omega_1^4 + 9\,\beta^2\Omega_1^4 - 18\alpha\beta^2\Omega_1^2 + 6\alpha\beta\Omega_1^2 - 18\beta^2\Omega_1^2 + 9\alpha\beta^2 + 6\,\beta\Omega_1^2 - 6\alpha\beta + 9\beta^2 + \alpha - 6\beta + 1}{\gamma\beta^2\Omega_4\Omega_7\Omega_8},$$

We obtained forty sets of solutions as mentioned above, we neglect to bring those category of solutions. The three-dimensional dynamic graphs of the wave and corresponding density plots, and two-dimensional plots were successfully depicted in figures 1–8 with the help of the Maple. We can see that the exponential function, the cosine function, and the hyperbolic cosine function react with each other and move forward. Due to analyze the dynamics properties briefly, we would like to discuss the evolution characteristic. By selecting the suitable values of parameters, the analytical treatment of periodic wave solution is presented in 1 and 3 including 3D plot, density plot, and 2D plot, when three spaces arise at spaces $x = -10$, $x = 0$, and $x = 10$. In 1, in origin two lines interact together. But in 3, the periodic wave only move in the direction of the negative t-axis to positive t-axis in the (x, t)-plane. And also, by choosing the given values for 1 and 3, respectively

$$a_1 = 0.5,\ \ a_4 = 1,$$
$$\Omega_1 = 1.5,\ \ \alpha = 1,$$
$$\beta = 2,\ \ \gamma = -1,\ \ y = -4,$$
$$a_3 = 1.1,\ \ a_4 = 1,$$
$$\Omega_7 = 1.3,\ \ \alpha = -1,$$
$$\beta = -2,\ \ \gamma = -1,\ y = -4,$$

it is worth mentioning that this periodic wave solutions have the following features as

$$\lim_{x\longrightarrow\pm\infty} u(x, t) = 0, \qquad \lim_{t\longrightarrow\pm\infty} u(x, t) = 0. \tag{2.18}$$

By increasing x and t to large sufficient value, then the value of $u(x, t)$ will vanish. By considering $a_3 = 2.5$, $a_4 = 1$, $\Omega_7 = 1.3$, $\alpha = -2$, $\beta = 1$, $\gamma = 1$, $y = -4$, the solution $u_2(x, y, t)$ given by (2.8) expresses the move two parallel (x, t)-periodic breathers as line $x = t$. By plugging $a_1 = 0.5$, $a_3 = 1$, $\Omega_1 = 1.5$, $\alpha = -2$, $\beta = 2$, $\gamma = 1$, $y = -4$, in 4, we can see that in origin two lines interact together. In 5, the x-periodic breather is presented. In 6 b, by inserting $a_2 = 0.5$, $a_3 = 1.5$, $a_4 = 1$, $\Omega_1 = 1.5$, $\alpha = 1$, $\beta = \frac{1}{4}$, $\gamma = -1$, $y = -4$, two lump solutions there are in direction of (x,t)-periodic breathers. Finally in 7 and 8 the periodic breather move in directions of (x, t)-periodic breathers and t-periodic breathers, respectively.

3. Stability analysis of KP-BBM equation

In this section, the concept of linear stability analysis will be applied to study the stability analysis for the giving equation (1.5). The perturbed solution of the KP-BBM equation given by

$$u(x, y, t) = q + \lambda U(x, y, t), \tag{3.1}$$

in above the relation constant q is a steady state solution of equation (3.1). Substituting (3.1) into equation, one can obtain

$$\begin{aligned} &2\left(\frac{\partial^2}{\partial x^2}U(x, y, t)\right)U(x, y, t)\alpha\,\lambda \\ &+ 2\left(\frac{\partial}{\partial x}U(x, y, t)\right)^2\alpha\,\lambda + 2\left(\frac{\partial^2}{\partial x^2}U(x, y, t)\right)\alpha\, q \\ &+ \left(\frac{\partial^4}{\partial x^3\partial t}U(x, y, t)\right)\beta \\ &+ \gamma\,\frac{\partial^2}{\partial y^2}U(x, y, t) + \frac{\partial^2}{\partial x\partial t}U(x, y, t) \\ &+ \frac{\partial^2}{\partial x^2}U(x, y, t) = 0, \end{aligned} \tag{3.2}$$

by linerization equation (3.2), we get

$$\begin{aligned} &2\left(\frac{\partial^2}{\partial x^2}U(x, y, t)\right)\alpha\, q \\ &+ \left(\frac{\partial^4}{\partial x^3\partial t}U(x, y, t)\right)\beta \\ &+ \gamma\,\frac{\partial^2}{\partial y^2}U(x, y, t) + \frac{\partial^2}{\partial x\partial t}U(x, y, t) \\ &+ \frac{\partial^2}{\partial x^2}U(x, y, t) = 0. \end{aligned} \tag{3.3}$$

Theorem 3.1. *Presume that the solution of equation (3.3) has the bellow form*

$$U(x, y, z, t) = \rho_1\, e^{i(Mx+Ny+Wt)}, \tag{3.4}$$

where M, N *is the normalized wave numbers, by inserting (3.4) into equation (3.3), then by solving for* W, *we can get the following relation*

$$W(M, N) = \frac{2\,M^2\alpha\, q + N^2\gamma + M^2}{M(M^2\beta - 1)}. \tag{3.5}$$

Proof. By substituting the relation (3.4) in the linear PDE (3.3), we get

$$\begin{aligned} &2\left(\frac{\partial^2}{\partial x^2}U(x, y, t)\right)\alpha\, q + \left(\frac{\partial^4}{\partial x^3\partial t}U(x, y, t)\right)\beta \\ &+ \gamma\,\frac{\partial^2}{\partial y^2}U(x, y, t) + \frac{\partial^2}{\partial x\partial t}U(x, y, t) + \frac{\partial^2}{\partial x^2}U(x, y, t) \\ &= e^{i(Mx+Ny+Wt)}\rho_1 \\ &\times (M^3W\beta - 2\,M^2\alpha\, q - N^2\gamma - M^2 - MW) = 0. \end{aligned} \tag{3.6}$$

By solving and simplifying we can find the value of $W(M, N, P)$ as form

$$W(M, N, P) = \frac{2\,M^2\alpha\, q + N^2\gamma + M^2}{M(M^2\beta - 1)}. \tag{3.7}$$

Therefore, we get to the required solution. Then the proof of the theorem is complete. □

The relations for the propagation in equation (3.5) is investigated. The sign of $W(M, N)$ proposes either the solution will become larger or decay in a given period of time. When the sign of $W(M, N)$ is negative for all value of M, then any superposition of solutions of the form $e^{i(Mx+Ny+Wt)}$ will come to decay and the steady state is stable. In other hand, if the $W(M, N)$ is positive for some values of M, then with time some components of a superposition will become bigger rapidly, the steady state is unstable. If the maximum $W(M, N)$ of the is exactly 0, the dispersion is called marginally stable.

4. Conclusion

In this paper, the periodic wave solutions of the KP-BBM equation have been constructed. From the bilinear form of this equation, two test functions or ansatzes have been chosen. At first, we chose the ansatz as a combination of the exponential, cosine and hyperbolic cosine functions. The 3D, 2D, and density graphs illustrating some solutions were represented, showing different periodic waveforms As a consequence, some new solutions, which include the new multi wave, breather wave, periodic, cross-kink wave solutions were catched. Through of Maple, the evolution phenomenon of these waves is seen in figures 1–11, respectively. Mainly, by choosing specific parameter constraints all cases the two-dimension, and three-dimension in solitons can be captured from the multi wave and periodic wave solutions. The obtained solutions are extended with numerical simulation to analyze graphically, which results into multi wave, breather wave, periodic, cross-kink wave solutions. Moreover, we studied the linear stability analysis of the KP-BBM equation in the previous section. That will be extensively used to report many attractive physical phenomena in the fields of acoustics, heat transfer, fluid dynamics, classical mechanics and so on.

Conflict of Interest

The authors declare that they have no conflict of interest.

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