

Quantum origin of the Minkowski space

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Abstract

We show that $D = 4$ Minkowski space is an emergent concept related to a class of operators in extended Hilbert space with no positive-definite scalar product. We start with the idea of position-like and momentum-like operators (Plewa 2019 *J. Phys. A: Math. Theor.* **52** 375401), introduced discussing a connection between quantum entanglement and geometry predicted by ER = EPR conjecture. We examine eigenequations of the simplest operators and identify $D = 4$ Minkowski space as to be spanned by normalized eigenvectors corresponding to the zero eigenvalue. Both spacetime dimension and signature of the metric are fixed by the regularization procedure. We generalize the result to the case of more general operators, being analogues to quantum fields. We reproduce the Minkowski space again, however, now in a holographic way, as being identified with the conformal boundary of AdS_5 . We observe an interesting analogy to string theory and, in particular, to AdS/CFT correspondence.

Keywords: ER=EPR, AdS/CFT, extended Hilbert space

1. Introduction

It has been proposed that classical spacetime may not be a fundamental concept, but having a quantum origin. Earlier observations presented in [2, 3] suggest that quantum entanglement could be a quantum notion originating space. More precisely, using Ryu–Takayanagi formula [4] it was shown that connectivity of space is related to quantum entanglement. This suggests a sort of duality between quantum states and spacetime. A realization of such a duality is the ER = EPR conjecture [5–8], stating that there is a link between connected back hole solutions and maximally entangled quantum states. Itself, the conjecture serves as a potential resolution of AMPS firewall paradox [9]. From a broader perspective, ER = EPR provides even wider connection between entangled particles and wormholes. In the strongest form, it predicts that every maximally entangled state can be interpreted geometrically in terms of some (connected)

black holes. Still, quantum entanglement may be ‘not enough’ [7], and additional notions, like quantum complexity [7, 10–13] are required. In particular, quantum complexity seems to be essential in understanding quantum nature of the growth of the Einstein–Rosen bridge [11]. One expects a significant modification of a classical black hole geometry for sufficiently long time scales, caused by quantum processes behind the horizon. All of these facts make black holes quantum objects, revealing interesting connections between quantum mechanics and gravity.

It was also suggested that gravity and quantum mechanics are not separate subjects [14]. Instead, they are strongly correlated and this holds true even far away from the fundamental scale: gravity is needed for consistent description of seemingly non-gravitational systems. To some extent, this is already the case within the AdS/CFT correspondence [15–17]. Strongly coupled states of conformal supersymmetric Yang–Mills theory translate into weakly coupled gravitational solutions in (effectively) five-dimensional spacetime. The main difference is that this holographic description has nothing to do with ‘ordinary’ gravity in asymptotically flat four-dimensional Minkowski space. It is possible, however, that further generalization of various dualities can eventually result in more general gravitational language, covering even larger class of relativistic states. It could happen that finding gravitational imprint of various quantum correlations can make these dualities ‘real’. This may uncover the true quantum nature of gravity.

Following motivations presented above and keeping in mind the ER = EPR hypothesis, we continue the analysis outlined in [1] and examine mathematical formalism based on the idea of operators in the extended Hilbert space. Here the term *extended Hilbert space* refers to a generalization incorporating negative norm states, i.e. the Hilbert space with no positive definite scalar product [18, 19]. As it was shown in [1], one can construct a special type of operators in that space, which play a role of the standard position and momentum. They can be used to reformulate the quantum harmonic oscillator in such a way that there is a link with spacetime. In particular, it was shown that there is a connection between maximally entangled ground state of a two-dimensional oscillator and the geometry of AdS₃.

Rather than discussing the extension of the duality for a larger class of states, we look closer on the operators themselves, considering their eigenequations. We show that solutions to these equations are highly non-trivial and can be interpreted geometrically.

This paper is organized as follows. In section 2 we discuss the eigenequations of the simplest operators, the *position* e and the *momentum* $-i\partial$ as in [1]. We show that there is an infinite set of finite-dimensional spaces spanned by normalized eigenvalues. We identify the space corresponding to either zero position or zero momentum to be special because of four-dimensional orthonormal basis consisting of one negative and three positive-norm states. The space will be then interpreted geometrically in terms of four-dimensional Minkowski spacetime. We identify the Poincaré symmetry group as a subgroup of more general complex Lorentz rotations, being symmetry transformations in the corresponding extended Hilbert space. In section 3 we consider more general operators labeled by continuous labels, $e(x)$ and $-i\partial(x)$. Constructing eigenvectors, we identify an additional constraint equation to get Hermitian operators. We interpret the equation geometrically as specifying a class of pseudo-hyperbolic spaces [20]. We derive AdS₅ again as a special solution to the constraint, identifying Minkowski space with the conformal boundary of anti-de Sitter space. Subsequently, we generalize the results to the case of the most general operators, finding a surprising connection with string theory. We conclude in section 4.

2. The simplest operators

2.1. Operators in the extended Hilbert space

We start with basic definitions introduced¹ in [1].

Definition 1. Let $F_0 \subset \mathcal{H}$ be a subspace of some extended Hilbert space \mathcal{H} , i.e. the Hilbert space with no positive-definite scalar product [18, 19]. We call F_0 the *fiducial subspace* and assume that it contains at least one normalizable state. Let $\{e_i, \partial_i\}_{i=1}^N$ stands for a set of operators in \mathcal{H} , defined so that acting on any $|\phi_a\rangle \in F_0$, one has

$$\langle \phi_b | e_i^{n_1} \dots e_{i_k}^{n_k} | \phi_a \rangle = \langle \phi_b | \partial_i^{n_1} \dots \partial_{i_k}^{n_k} | \phi_a \rangle := 0, \quad (1)$$

$$[e_i, e_j] := 0, \quad [\partial_i, \partial_j] := 0, \quad (2)$$

$$\partial_i e_{j_1} \dots e_{j_n} | \phi_a \rangle := \delta_{ij_1} e_{j_2} \dots e_{j_n} | \phi_a \rangle + \dots + \delta_{ij_n} e_{j_1} \dots e_{j_{n-1}} | \phi_a \rangle, \quad (3)$$

$$e_i \partial_{j_1} \dots \partial_{j_n} | \phi_a \rangle := -\delta_{ij_1} \partial_{j_2} \dots \partial_{j_n} | \phi_a \rangle - \dots - \delta_{ij_n} \partial_{j_1} \dots \partial_{j_{n-1}} | \phi_a \rangle, \quad (4)$$

$$e_i^\dagger := e_i, \quad (5)$$

for $N, i_k, j_k, n_k \in \mathbb{N}$. We call e_i and ∂_j *e-operators*.

We defined *e-operators* postulating how they act on states belonging to the fiducial subspace F_0 . As in [1], we restrict to a single normalizable vector in that space, the so-called *fiducial vector* $|\phi\rangle$ (still the space F_0 itself can be highly non-trivial). Acting with products of either e_i or ∂_i operators results in zero-norm states in the form $e_1|\phi\rangle, e_1e_2|\phi\rangle, \partial_2^2|\phi\rangle$, etc. Despite they are zero norm states, their linear combinations can be normalizable. In particular, a single pair (e_1, ∂_1) of *e-operators* and a single fiducial vector give rise infinitely-dimensional extended Hilbert space. The latter is spanned by positive, negative and zero norm states resulting from products of operators acting on $|\phi\rangle$. Equations (3) and (4) ensure that any product of operators can be expressed solely in terms of either e_i or ∂_j . Notice that according to equation (3), ∂_i looks like ‘derivative’ with respect to e_i . Similarly, ignoring the minus sign in equation (4), e_i resembles ‘derivative’ with respect to ∂_j . This is why we reserved a single name² *e-operators* to both e_i and ∂_j .

Following the notations adopted in [1] we introduce two spaces, $E[F_0]$ and $D[F_0]$, defined as being spanned by combinations of respectively e_i and ∂_i acting on the fiducial vector but not the fiducial vector itself. For instance, $e_1e_2|\phi\rangle, e_i^2|\phi\rangle, \dots \in E[F_0]$, $\partial_1\partial_2|\phi\rangle, \partial_i^2|\phi\rangle, \dots \in D[F_0]$, but $|\phi\rangle \notin E[F_0]$ and $|\phi\rangle \notin D[F_0]$. We call the (full) extended Hilbert space³ $\mathcal{H} = F_0 \oplus E[F_0] \oplus D[F_0]$ the *maximally extended space*, reserving the term *extended space* to $E[F_0] \oplus D[F_0]$. The extended space plays a special role constructing the quantum harmonic oscillator [1].

The important thing is that equation (5) does not automatically imply that e_i are Hermitian in the standard sense, i.e.

$$(u, e_i v) = (e_i u, v), \quad (6)$$

¹ In this paper we adopt some small change of terminology regarding operator names.

² In [1] they are called *external operators*.

³ More specifically, since the definition bases on the choice of the fiducial subspace, one should write $\mathcal{H}[F_0] = F_0 \oplus E[F_0] \oplus D[F_0]$. However, restricting to a single fiducial vector, we intentionally ignore such details.

for $u, v \in \mathcal{H}$. Instead, equation (5) defines a Hermitian conjugate of the object e_i . Perhaps it would be better to rewrite equation (5) as $e_i^* = e_i$. However, because both e_i and ∂_j are operators, following the notation adopted in [1], we will continue to use the standard symbol of Hermitian conjugate. As we shall see, the operators are indeed Hermitian in the corresponding space of eigenstates. Finding that space will be the main subject of this paper.

It turns out that the rule (5) has to be supplemented by the analogous condition for ∂_i

$$\partial_i^\dagger := -\partial_i. \tag{7}$$

Rather than being postulated, this is required by consistency of the algebra generated by commutators of e -operators (see [1] for more details). The scalar product in the maximally extended space is trivially induced by the product in F_0 , and the rules (1)–(5) and (7). For instance, for two vectors $u = (e_1 + \partial_1)|\phi\rangle$ and $v = (e_1 - \partial_1)|\phi\rangle$, one finds

$$(u, v) = \langle \phi | (e_1 + \partial_1)^\dagger (e_1 - \partial_1) | \phi \rangle = \langle \phi | e_1^2 | \phi \rangle - \langle \phi | e_1 \partial_1 | \phi \rangle - \langle \phi | \partial_1 e_1 | \phi \rangle + \langle \phi | \partial_1^2 | \phi \rangle = 0. \tag{8}$$

Making use of equations (2)–(5) and (7), leads to the following commutation relation

$$[\partial_i, e_j] = \delta_{ij} \hat{\eta}, \tag{9}$$

where

$$\begin{aligned} \hat{\eta}|u\rangle &:= 2|u\rangle : & |u\rangle \in F_0, \\ \hat{\eta}|u\rangle &:= |u\rangle : & |u\rangle \in E[F_0] \oplus D[F_0]. \end{aligned} \tag{10}$$

Notice that $\hat{\eta} = id$ in the extended space $E[F_0] \oplus D[F_0]$. Equation (9) simplifies in that space becoming $[\partial_i, e_j] = \delta_{ij}$. The last allows to interpret e_i as analogue of the position operator, while $-i\partial_i$ plays a role of the momentum. From now on we will call them respectively the e -position and e -momentum.

2.2. Position eigenequation

Below we discuss the position eigenequation. Restrict for simplicity to a single pair of operators (e, ∂) ; here we skipped the labels for convenience. Acting on a given normalized fiducial vector $|\phi\rangle$ the operators span infinitely-dimensional maximally extended space \mathcal{H} . The eigenequation reads

$$e|e_\lambda\rangle = \lambda|e_\lambda\rangle. \tag{11}$$

Looking for a solution, one may start with the most general form of the vector in the corresponding maximally extended space:

$$|e_\lambda\rangle = \left(a + \sum_{n=1}^{\infty} b_n e^n + \sum_{n=1}^{\infty} c_n \partial^n \right) |\phi\rangle, \tag{12}$$

where a, b_n, c_n are unknown coefficients to be fixed. Inserting the ansatz (12) into the eigenequation (11), and solving the corresponding recurrence equations, gives

$$|e_\lambda\rangle = aE_\lambda|\phi\rangle, \tag{13}$$

where

$$E_\lambda = 1 + \sum_{n=1}^{\infty} \left(\lambda^{-n} e^n + \frac{(-\lambda)^n}{n!} \partial^n \right), \tag{14}$$

and a stands for the overall normalization coefficients. The recurrence equations fix both b_n and c_n , leaving a as a free parameter. Being irrelevant from perspective of the eigenequation, the coefficient a is expected to be fixed by the normalization condition (if the eigenvector is normalizable). We ignore this for a moment, letting temporarily $a = 1$. The product of two eigenvectors (13), labeled by two eigenvalues λ_i, λ_j , reads

$$\langle e_{\lambda_i} | e_{\lambda_j} \rangle = 1 + \sum_{n=1}^{\infty} \left[\left(\frac{\lambda_i^*}{\lambda_j} \right)^n + \left(\frac{\lambda_j}{\lambda_i^*} \right)^n \right], \tag{15}$$

where ‘*’ stands for complex conjugate. Clearly, λ_i should be real for Hermitian operator. We kept the complex conjugate in equation (15) because we have not proved that e is Hermitian. If so, eigenvectors from different eigenvalues $\lambda_i \neq \lambda_j$ should be orthogonal.

Contrary to what is expected, the product (15) implies that this is not the case unless $|\lambda_i| = |\lambda_j|$. If $|\lambda_i| \neq |\lambda_j|$, the product is divergent and the eigenvectors are not orthogonal. What is more, they are non-normalizable, i.e. $\|e_\lambda\| = \infty$. Despite the latter is common for operators with continuous spectrum (they are mostly non-normalizable), the former means that e is not Hermitian in the whole maximally extended space \mathcal{H} . Still, it could happen the operator is Hermitian if it is restricted to some subspace $\mathcal{E} \subset \mathcal{H}$. Keeping this in mind, we call \mathcal{E} the *restricted Hilbert space* (within \mathcal{H}). If exists, the space is of special importance because it makes e -position a candidate for observable. Since \mathcal{H} is not positive-definite, no one guaranties that \mathcal{E} will be free of negative norm states. In fact, this is in parallel analogy to construction of the harmonic oscillator presented in [1]. Hilbert space of the quantum harmonic oscillator emerges as a natural positive-definite sector in the extended space. Finding that sector, one finds the oscillator. Similarly, we now expect that constructing the restricted Hilbert space \mathcal{E} will get some insight into physical meaning of e -position.

Having said that, we are now ready to start the construction. Since, by definition, e is Hermitian in \mathcal{E} , the eigenvalues should be real, i.e. $\lambda \in \mathbb{R}$ in equation (15). Another observation is that despite the eigenvectors (13) are not normalizable, they can be easily regularized. In order to do so, consider a finite cut-off $\Lambda \gg 1$ and the following regularized version of the operator (14)

$$E_\lambda^\Lambda := 1 + \sum_{n=1}^{\Lambda} \left(\lambda^{-n} e^n + \frac{(-\lambda)^n}{n!} \partial^n \right). \tag{16}$$

Eigenvectors (13) can be now rewritten as $|e_\lambda\rangle = \lim_{\Lambda \rightarrow \infty} E_\lambda^\Lambda |\phi\rangle$. It is worth to underline that vectors constructed with the help of the operator (16) with finite Λ are not the exact position eigenstates. In fact, the reason for introducing the form (16) consists in parameterization of the limit $\Lambda \rightarrow \infty$. Rather than imposing a real cut-off, we will let $\Lambda \rightarrow \infty$ in the end. Note that the operator (16) is defined up to the overall normalization constant and any vector in the form $a \lim_{\Lambda \rightarrow \infty} E_\lambda^\Lambda |\phi\rangle$, with $a \in \mathbb{C}$, is an eigenvector as well. One can formally make a infinitesimally small dividing by infinite length and requiring the resulting state will be finite. This can be done introducing ‘regularized’ eigenvectors in the form

$$|e_\lambda^r\rangle := \lim_{\Lambda \rightarrow \infty} \frac{1}{\sqrt{2\Lambda}} E_\lambda^\Lambda |\phi\rangle. \tag{17}$$

The latter is to be supplemented by a subtle redefinition of the product:

$$\langle e_{\lambda_i}^r | e_{\lambda_j}^r \rangle := \lim_{\Lambda \rightarrow \infty} \left(\frac{1}{2\Lambda} \langle \phi | (E_{\lambda_i}^\Lambda)^\dagger E_{\lambda_j}^\Lambda | \phi \rangle \right). \quad (18)$$

It is straightforward to observe that equations (17) and (18) imply the eigenvectors are unit, i.e. $\langle e_\lambda^r | e_\lambda^r \rangle = 1$. Alternatively, one can define the vectors (17) as $|e_\lambda^r\rangle := \lim_{\Lambda \rightarrow \infty} \frac{1}{\sqrt{2\Lambda_0}} E_\lambda^\Lambda | \phi \rangle$, supplementing the definition with the following version of product

$$\langle e_{\lambda_i}^r | e_{\lambda_j}^r \rangle := \lim_{\Lambda \rightarrow \infty} (\langle e_{\lambda_i}^r | e_{\lambda_j}^r \rangle |_{\Lambda_0 = \Lambda}). \quad (19)$$

For the sake of simplicity we adopt the form (17) of regularized eigenvectors, understanding they are normalized according to the rule (18).

One can check that eigenvectors (17) corresponding to opposite non-zero eigenvalues are orthogonal, i.e. $\langle e_{-\lambda}^r | e_\lambda^r \rangle = 0$, $\lambda \neq 0$. However, this is not the case in general. Fixing a single $|\lambda| = \lambda_0 > 0$, one specifies a two-dimensional space $\mathcal{E}_{\lambda_0} \subset \mathcal{H}$ spanned by eigenvectors of two opposite eigenvalues $\lambda = \pm\lambda_0$. Clearly, \mathcal{E}_{λ_0} is an ordinary two-dimensional Hilbert space and e is Hermitian in that space. The positive real parameter λ_0 is arbitrary, but should be fixed. Once this is done, all other eigenvalues $|\lambda| \neq \lambda_0$ are forbidden. Still, none of the values $\lambda_0 > 0$ is special. One can formally incorporate all of them considering direct sum of the corresponding spaces \mathcal{E}_{λ_0}

$$\mathcal{E}_+ = \bigoplus_{\lambda_0 > 0} \mathcal{E}_{\lambda_0}, \quad (20)$$

and supplementing \mathcal{E}_+ with an additional superselection rule eliminating superpositions of states from different absolute value of the eigenvalues. In what follows, $|\lambda|$ plays a role of the conserved charge. The only allowed superpositions involve eigenvectors corresponding to opposite eigenvalues. This guaranties that the e -position is Hermitian. All superselection sectors in equation (20), despite separate, are equivalent in the sense there is a trivial isomorphism preserving the scalar product. In fact, all the sectors are (ordinary) two-dimensional Hilbert spaces.

The construction above may seem to be unfamiliar from quantum mechanical point of view, since constructing the restricted Hilbert space \mathcal{E} involves additional superselection rules. As we shall see, they have a deeper geometrical meaning.

2.3. Generalization to complex eigenvalues

So far we have restricted ourselves to non-zero eigenvalues. It turns out, however, the case $\lambda = 0$ will be interesting from geometrical point of view. Before discussing the limit $\lambda \rightarrow 0$, it will be convenient ignoring for a moment the fact we construct a space where e is manifestly Hermitian. Instead, we temporarily allow general complex eigenvalues. The reason is that we would like to enlarge the space (20), incorporating eigenvectors from arbitrarily small but complex eigenvalues.

As in the case of equation (17), we start considering a pair of normalized eigenvectors. Letting in equation (15) $\lambda_i = i$, one finds

$$\|e_{\lambda_i}\| = \lim_{\Lambda \rightarrow \infty} (-1)^\Lambda. \quad (21)$$

Again, this is ill-defined. On the other hand, the right-hand side of the equation above can be easily regularized letting Λ to be either even or odd. This leads to a pair of normalized

eigenvectors, defined as follows

$$|e_i^{(\Lambda_1)}\rangle := \lim_{N \rightarrow \infty} E_i^{\Lambda_1(N)} |\phi\rangle, \quad |e_i^{(\Lambda_2)}\rangle := \lim_{N \rightarrow \infty} E_i^{\Lambda_2(N)} |\phi\rangle, \quad (22)$$

where $\Lambda_1 = \Lambda_1(N) = 2N$, $\Lambda_2 = \Lambda_2(N) = 2N - 1$; $N \in \mathbb{N}$. The limit (21) is now well-defined. In particular,

$$\|e_i^{(\Lambda_1)}\| = 1, \quad \|e_i^{(\Lambda_2)}\| = -1. \quad (23)$$

Note that there is one positive and one negative norm eigenvector. Imposing of the cut-off Λ is not regularization in the standard sense, since instead of interpreting the divergent term, we formally allowed all the possible outcomes. Each of them is associated with different normalized vector, making this simple procedure free from ambiguities.

Unfortunately, the product $\langle e_i^{(\Lambda_i)} | e_i^{(\Lambda_j)} \rangle$ is not well-defined due to the presence of double limits:

$$\langle e_i^{(\Lambda_1)} | e_i^{(\Lambda_2)} \rangle = \lim_{N_1 \rightarrow \infty} \lim_{N_2 \rightarrow \infty} \langle \phi | (E_i^{\Lambda_1(N_1)})^\dagger E_i^{\Lambda_2(N_2)} | \phi \rangle. \quad (24)$$

They have to be clarified to get the result. This can be done introducing an order operation $::$, defined as follows

$$:\langle \phi | (E_{\lambda_i}^{\Lambda_i})^\dagger E_{\lambda_j}^{\Lambda_j} | \phi \rangle: \equiv \frac{1}{2} \langle \phi | (E_{\lambda_i}^{\Lambda_i})^\dagger E_{\lambda_j}^{\Lambda_j} | \phi \rangle \Big|_{\Lambda_i \leq \Lambda_j} + \frac{1}{2} \langle \phi | (E_{\lambda_i}^{\Lambda_i})^\dagger E_{\lambda_j}^{\Lambda_j} | \phi \rangle \Big|_{\Lambda_i \geq \Lambda_j}. \quad (25)$$

We can now define

$$:\langle e_{\lambda_i}^{(\Lambda_i)} | e_{\lambda_j}^{(\Lambda_j)} \rangle: = \lim_{N_i \rightarrow \infty} \lim_{N_j \rightarrow \infty} : \langle \phi | (E_{\lambda_i}^{\Lambda_i(N_i)})^\dagger E_{\lambda_j}^{\Lambda_j(N_j)} | \phi \rangle: \quad (26)$$

In other words, taking the limits we impose additional constraints, i.e. $\Lambda_i \leq \Lambda_j$ and $\Lambda_i \geq \Lambda_j$, treating them in a symmetric way. As it is shown in appendix A, the rule (26) transforms the double limit into the following single ones:

$$:\langle e_{\lambda_i}^{(\Lambda_i)} | e_{\lambda_j}^{(\Lambda_j)} \rangle: = \frac{1}{2} \lim_{N_i \rightarrow \infty} \langle \phi | (E_{\lambda_i}^{\Lambda_i(N_i)})^\dagger E_{\lambda_j}^{\Lambda_j(N_i)} | \phi \rangle + \frac{1}{2} \lim_{N_j \rightarrow \infty} \langle \phi | (E_{\lambda_i}^{\Lambda_i(N_j)})^\dagger E_{\lambda_j}^{\Lambda_j(N_j)} | \phi \rangle, \quad (27)$$

and both they are well defined. For vectors (22) the rule (27) gives $:\langle e_i^{(\Lambda_i)} | e_i^{(\Lambda_j)} \rangle: = 0$. In what follows, the two degenerated eigenvectors are orthogonal. Note that the order (25) only matters if $\Lambda_i \neq \Lambda_j$. If they are equal, $:\langle e_{\lambda_i}^{(\Lambda_i)} | e_{\lambda_j}^{(\Lambda_j)} \rangle: = \langle e_{\lambda_i}^{(\Lambda_i)} | e_{\lambda_j}^{(\Lambda_j)} \rangle |_{\Lambda_i = \Lambda_j}$.

Having specified exemplary eigenvectors (22) corresponding to the single eigenvalue $\lambda = i$, one can ask if the procedure can be generalized to the case of general complex eigenvalues. As it is shown in appendix B, this is indeed the case for

$$\lambda = re^{i\varphi}, \quad \varphi = \frac{p\pi}{q}, \quad (28)$$

where $\frac{p}{q} \in (0, 1) \cup (1, 2)$, $p, q \in \mathbb{N}$. The corresponding regularized eigenvectors read

$$|e_\lambda^{(l)}\rangle = \lim_{N \rightarrow \infty} E_\lambda^{\Lambda_{q,l}(N)} |\phi\rangle. \quad (29)$$

The eigenvectors are labeled by additional discrete labels l in such a way that

$$\Lambda_{q,l}(N) = qN + l, \quad l \in \{0, 1, \dots, q - 1\}. \quad (30)$$

In particular, letting $r = 1$, $\varphi = \frac{\pi}{2}$ corresponds to $\lambda = i$, $p = 1$, $q = 2$ and $l \in \{0, 1\}$. These two values define two different eigenvectors, specified by two cut-offs, $\Lambda_{2,0}(N) = 2N$, $\Lambda_{2,1}(N) = 2N + 1$. This reproduces the eigenvectors (22).

The construction additionally requires that eigenvectors corresponding to different complex eigenvalues does not belong to a single space (see appendix B for more details). This is a direct analogue of the superselection rule for regularized eigenvectors corresponding to real eigenvalues (17). However, now each space corresponds to a single complex eigenvalue, while different complex λ specify different spaces of regularized eigenvectors. Each space contains one positive, one negative and $q - 2$ zero norm states (if $q \leq 2$ then there is no zero norm vectors). The resulted eigenvectors are degenerated, whereas the spaces they belong to are equivalent in the sense they differ only by the number of zero norm eigenstates.

2.4. The limit of zero eigenvalues and $D = 4$ Minkowski space

We are now ready to discuss the problem of zero eigenvalues. In order to do so, consider $\lambda = \pm\epsilon$ first, where ϵ stands for an infinitesimal real parameter. Define

$$|e_{\pm}\rangle := |e_{\pm\epsilon}^r\rangle, \quad (31)$$

where $|e_{\pm\epsilon}^r\rangle$ are given by equation (17) for $\lambda_0 = \epsilon$. That is, there are two eigenvectors of two opposite eigenvalues $\pm\epsilon$. Both eigenvalues go to zero in the limit $\epsilon \rightarrow 0$. To make the product (18) well-defined we let ϵ to be arbitrary small, but non-zero. In what follows, the limit $\lambda \rightarrow 0$ consists in treating ϵ as infinitesimal parameter, and letting $\epsilon \rightarrow 0$ in the end. This is in parallel analogy to introducing regularization parameter Λ . The only difference is that we first eliminate Λ , letting $\Lambda \rightarrow \infty$ while keeping ϵ as a fixed parameter, and then eliminate ϵ letting $\epsilon \rightarrow 0$. Alternatively, one can keep ϵ as arbitrarily small but fixed all the time, treating the vectors given by equation (31) as eigenvectors corresponding to approximately zero eigenvalue. The smaller the value ϵ , the better the approximation. We ignore such details now, however. Physically it does not matter whether some quantity is exactly zero or is arbitrarily small. From now on we interpret equation (31) as defining a pair of degenerated eigenvectors corresponding to the zero eigenvalue. This value can be found in the limit of either small negative $\lambda = -\epsilon$ or small positive $\lambda = \epsilon$ eigenvalues.

In the next step we generalize the procedure to the case of complex eigenvalues. The idea is to reach $\lambda = 0$ by small complex λ (i.e. $|\lambda| \ll 1$). For instance, let $\lambda = i\epsilon$. Recalling equations (28)–(30), this corresponds to $r = \epsilon$, $\varphi = \pi/2$, leading to two eigenvectors labeled by $l = 0$ and $l = 1$. Define

$$|e\rangle_0 := |e_{i\epsilon}^{(1)}\rangle, \quad |e\rangle_1 := |e_{i\epsilon}^{(0)}\rangle, \quad (32)$$

where $|e_{i\epsilon}^{(l)}\rangle$ are given by equation (29). These are defined as respectively negative and positive norm eigenvectors, i.e. ${}_0\langle e|e\rangle_0 = -1$, ${}_1\langle e|e\rangle_1 = 1$. For small but fixed ϵ the operator e is approximately Hermitian on states (32) in the sense that the deviation $|(e^k, e^l) - (e^k, e^l)| = 2i\epsilon\delta_{kl}$ is small of order ϵ (here e^k, e^l stand for eigenvectors (32)). Letting ϵ arbitrarily close to zero makes the deviation from Hermiticity, as well as the eigenvalue λ , arbitrarily small. This is similar to the case of eigenvectors from real eigenvalues (31). Again, we treat ϵ as arbitrarily close to zero, eliminating it in the limit $\epsilon \rightarrow 0^+$. The vectors (32) turns out to be normalized e -eigenvectors corresponding to the zero eigenvalue. In contrast to the vectors (31),

one gets the value $\lambda = 0$ in the limit at which the imaginary part of the eigenvalue goes to zero.

Despite the vectors (31) and (32) were found using different methods, they are orthogonal. This is due to the prefactor $(2\Lambda)^{-1/2}$ in the definition (17). Hence, the two-dimensional space corresponding to $\lambda_0 = \epsilon$, i.e. the space of eigenstates corresponding to opposite eigenvalues $\lambda = \pm\epsilon$, can be effectively enlarged supplementing with vectors (32). This results in the total number of four unit eigenvectors:

$$|e\rangle_0 := |e^0\rangle, \quad |e\rangle_1 := |e^1\rangle, \quad |e\rangle_2 := |e_+\rangle, \quad |e\rangle_3 := |e_-\rangle. \quad (33)$$

They all correspond to the zero eigenvalue. The first two vectors are the eigenvectors (32), the last two ones are the eigenvectors (31).

At this point one may wonder if the basis (33) can be further modified incorporating eigenvectors from different complex (infinitesimal) eigenvalues. The important thing to note is that there is a superselection rule forbidding superposition of states corresponding to different eigenvalues, even if they are arbitrarily close to each other (see appendix B for more details). Choosing the complex eigenvalue fixes the choice, as in the case of eigenvectors corresponding to real eigenvalues (see equation (20) and the corresponding superselection rule). In particular, instead of the vectors (32), one could consider eigenvectors from different infinitesimally small complex eigenvalues. As was shown in appendix B, different choices correspond to different number of zero norm states, but this do not affect the number of non-zero norm vectors. Since the zero norm states do not contribute to the scalar product, they are meaningless and can be ignored. In what follows, trying to change regularized eigenvectors, one cannot affect the dimension and signature of the space spanned by the basis of degenerated eigenvectors corresponding to the zero eigenvalue. Increasing dimension is forbidden by superselection rules, while the signature remains the same because we get always one negative and three positive norm eigenvectors.

Let \mathcal{E}_0 stands for the restricted Hilbert space spanned by the basis (33) (a subspace of \mathcal{H} in which e is Hermitian). The set of disconnected spaces (20) can be then enlarged by \mathcal{E}_0 :

$$\mathcal{E} = \bigoplus_{\lambda_0 \geq 0} \mathcal{E}_{\lambda_0}. \quad (34)$$

Being restricted to any of the spaces \mathcal{E}_{λ_0} , the operator e is manifestly Hermitian. As in the case of equation (20), writing down equation (34) we keep in mind that there are additional superselection rules forbidding superpositions of states corresponding to different $|\lambda|$.

Among various different spaces in the direct sum (34), the restricted Hilbert space \mathcal{E}_0 is special because of the biggest number of orthogonal eigenvectors and the fact that they are all degenerated. Now we will take a closer look at that space. Note that any complex superposition of states corresponding to the zero eigenvalue is an eigenvector as well. Clearly, this is only possible if $\lambda = 0$. Let

$$|v\rangle = \xi^\mu |e_\mu\rangle \quad (35)$$

be a superposition chosen so that $|\xi^\mu| \ll L$ for $L \gg \epsilon$; $\xi^\mu \in \mathbb{C}$. In what follows, $e|v\rangle = \mathcal{O}(\epsilon)$. The smaller ϵ , the better the approximation. In particular, in the limit $\epsilon \rightarrow 0$, one finds $e|v\rangle \rightarrow 0$. Hence the vector (35) is a zero eigenvector in the same sense as any of the basis eigenstates (33). Again, ϵ is identified with additional regularization parameter, similarly as Λ . Notice that $|e\rangle_0$ is a negative norm vector, while the rest ones, $|e\rangle_i$, $i = 1, 2, 3$, have positive norms. This suggest a connection with Minkowski space. Indeed, the basis (33) is orthonormal in the sense of the scalar product in Minkowski space, $\langle e_\mu | e_\nu \rangle = \langle e_\nu | e_\mu \rangle = \eta_{\mu\nu}$. Here $\mu, \nu \in \{0, \dots, 3\}$

and $\eta_{\mu\nu}$ stands for the standard Minkowski metric. However, the scalar product is symmetric under complex Lorentz rotations, i.e. the symmetry group $O(3, 1, \mathbb{C})$. Since the negative norm vector is present, we cannot impose the standard normalization condition for the superposition (35). For the same reason in the extended Hilbert space complex amplitudes ξ^μ have no probabilistic interpretation. Still, as it was shown in [1], in special circumstances they could play a role analogous to the standard probabilities. We ignore this details now, considering the most general symmetry group $O(3, 1, \mathbb{C})$.

It turns out that coefficients ξ^μ can be identified with coordinates in the Minkowski space. To see this note that the symmetry under $O(3, 1, \mathbb{C})$ rotations means that

$$\eta_{\alpha\beta} = \eta_{\gamma\delta} (\Lambda_\alpha^\gamma)^* \Lambda_\beta^\delta, \quad (36)$$

where $*$ stands for complex conjugate. Consider an infinitesimal transformation $\delta\Lambda_\nu^\mu = \delta_\nu^\mu + \Omega_{\mu\nu}^\mu$, where $\Omega_{\mu\nu} = -\Omega_{\mu\nu}^*$. Extracting real and imaginary parts,

$$\Omega_{\mu\nu} = \omega_{\mu\nu} + i\tilde{\omega}_{\mu\nu}, \quad (37)$$

one rewrites the rotation as

$$\delta\Lambda_\nu^\mu = \delta_\nu^\mu + \omega_\nu^\mu + i\tilde{\omega}_\nu^\mu, \quad (38)$$

where $\omega_{\mu\nu}, \tilde{\omega}_{\mu\nu}$ are antisymmetric tensors. The transformation involves twelve real parameters, six for each of two antisymmetric tensors: $\omega_{\mu\nu}$ and $\tilde{\omega}_{\mu\nu}$. Acting on a complex vector $\xi^\mu = \sigma^\mu + i\theta^\mu$; $\sigma^\mu, \theta^\mu \in \mathbb{R}$, the transformation (38) gives

$$(\xi^\mu)' = \delta\Lambda_\nu^\mu \xi^\nu = (\delta_\nu^\mu + \omega_\nu^\mu) \sigma^\nu - \tilde{\omega}_\nu^\mu \theta^\nu + i [(\delta_\nu^\mu + \omega_\nu^\mu) \theta^\nu + \tilde{\omega}_\nu^\mu \sigma^\nu]. \quad (39)$$

The real $(\sigma^\mu)'$ and imaginary $(\theta^\mu)'$ parts of $(\xi^\mu)'$ transform as follows

$$(\sigma^\mu)' = (\delta_\nu^\mu + \omega_\nu^\mu) \sigma^\nu - \tilde{\omega}_\nu^\mu \theta^\nu, \quad (40)$$

$$(\theta^\mu)' = (\delta_\nu^\mu + \omega_\nu^\mu) \theta^\nu + \tilde{\omega}_\nu^\mu \sigma^\nu. \quad (41)$$

For a fixed $\theta^\mu = \theta_0^\mu = \text{const}$, the real part of equation (39), given by equation (40), is a Poincare transformation

$$(\sigma^\mu)' = (\delta_\nu^\mu + \omega_\nu^\mu) \sigma^\nu + a^\mu. \quad (42)$$

Here $a^\mu = -\tilde{\omega}_\nu^\mu \theta_0^\nu$ plays a role of the translation, while $\delta_\nu^\mu + \omega_\nu^\mu$ is a Lorentz rotation. Similarly, for a fixed $\sigma^\mu = \sigma_0^\mu = \text{const}$, the imaginary part (41) becomes

$$(\theta^\mu)' = (\delta_\nu^\mu + \omega_\nu^\mu) \theta^\nu + b^\mu, \quad (43)$$

where $b^\mu = \tilde{\omega}_\nu^\mu \sigma_0^\nu$. Again, this is the Poincare transformation with the same Lorentz rotation and the translation b^μ . In what follows, imposing one of the two following constraints

$$\theta^\mu = \theta_0^\mu, \quad (44)$$

$$\sigma^\mu = \sigma_0^\mu, \quad (45)$$

i.e. letting $\theta^\mu = \text{const}$ or $\sigma^\mu = \text{const}$, respectively the real or imaginary part of the complex vector ξ^μ transforms under standard four-dimensional Poincare symmetry group. This means that either σ^μ or θ^μ can be identified with coordinates in four-dimensional Minkowski space.

Notice that the constrains (44) and (45) are essential for this identification. The last fact has a natural interpretation in terms of a higher-dimensional space. Suppose we interpret imaginary parts of the vector ξ^μ as coordinates in extra dimensions. To this end, consider a vector $\vec{f} = y^a \vec{f}_a$, $a = 1, \dots, 8$, where $(y^A) := \{\sigma^0, \dots, \sigma^3, \theta^0, \dots, \theta^3\}$ are coordinates in an orthonormal, eight-dimensional basis \vec{f}_a . The latter is constructed in such a way that vector spaces spanned by the two subsets $\{\vec{f}_0, \dots, \vec{f}_3\}$, $\{\vec{f}_4, \dots, \vec{f}_7\}$ are isomorphic (in the sense of preserving the scalar product) to the space spanned by the eigenvectors $\{|e\rangle_0, \dots, |e\rangle_3\}$. In what follows, $\vec{f} \in \mathbb{R}^{6,2}$. Imposing one of the two constraints (44) and (45) results in a four-dimensional hypersurface, the four-dimensional Minkowski space.

The interpretation above may seem to be artificial because of introduced ad hoc the eight-dimensional basis \vec{f}_a . Still, the construction provides a clear interpretation of the constraints (44) and (45). As we shall see discussing more general e -operators, the Minkowski space emerges in a similar fashion, being solution to the constraint equation in a higher-dimensional space.

We close this section recalling the form of space of eigenvectors (34). The fact it splits into separate subspaces is crucial for geometric interpretation of \mathcal{E}_0 . In particular, this prohibits enlarging the space by additional eigenvectors corresponding to different infinitesimal eigenvalues. The procedure fixes the dimension and signature of the restricted Hilbert space \mathcal{E}_0 , interpreting it geometrically in terms of Minkowski spacetime. This means that regularized eigenvectors corresponding to the zero eigenvalue are more like geometric objects, giving rise the Poincare symmetry in four dimensions.

3. General operators

Below we discuss eigenequation of e -position labeled by continuous variables, written symbolically as $x = \{x^0, \dots, x^{D-1}\}$. In general, generalized e -operators $e_i(x)$, $\partial_j(x)$ are given by the following generalization of definition 1:

$$[e_i(x), e_j(y)] = [\partial_i(x), \partial_j(y)] := 0, \tag{46}$$

$$\langle \varphi | e_{i_1}^{n_1}(x_1) \dots e_{i_k}^{n_k}(x_k) | \phi \rangle = \langle \varphi | \partial_{i_1}^{n_1}(x_1) \dots \partial_{i_k}^{n_k}(x_k) | \phi \rangle := 0, \tag{47}$$

$$e_i^\dagger(x) := e_i(x), \quad \partial_i^\dagger(x) := -\partial_i(x), \tag{48}$$

$$\partial_i(x) e_j(y) | \phi \rangle := \delta_{ij} \delta^D(x - y) | \phi \rangle. \tag{49}$$

Additionally we require

$$\begin{aligned} \partial_i(x) e_{j_1}(y_1) \dots e_{j_n}(y_n) | \phi \rangle &= \delta_{ij_1} \delta^D(x - y_1) e_{j_2}(y_2) \dots e_{j_n}(y_n) | \phi \rangle + \dots \\ &+ \delta_{ij_n} \delta^D(x - y_n) e_{j_1}(y_1) \dots e_{j_{n-1}}(y_{n-1}) | \phi \rangle, \end{aligned} \tag{50}$$

$$\begin{aligned} e_i(x) \partial_{j_1}(y_1) \dots \partial_{j_n}(y_n) | \phi \rangle &= -\delta_{ij_1} \delta^D(x - y_1) \partial_{j_2}(y_2) \dots \partial_{j_n}(y_n) | \phi \rangle + \dots \\ &- \delta_{ij_n} \delta^D(x - y_n) \partial_{j_1}(y_1) \dots \partial_{j_{n-1}}(y_{n-1}) | \phi \rangle. \end{aligned} \tag{51}$$

It can be verified that operators satisfy the following commutation relation

$$[\partial_j(y), e_i(x)] = \delta_{ij} \delta^D(x - y) \hat{\eta}. \tag{52}$$

Linear combinations of $e_i(x)$ and $\partial_j(y)$ leads to creation $\alpha_i^\dagger(x)$ and annihilation $\alpha_i(x)$ operators, satisfying

$$[\alpha_i(x), \alpha_j^\dagger(y)] = \delta_{ij} \delta^D(x - y) \hat{\eta}, \quad (53)$$

$$[\alpha_i(x), \alpha_j(y)] = [\alpha_i^\dagger(x), \alpha_j^\dagger(y)] = 0. \quad (54)$$

In the extended space, i.e. the space spanned by combinations of e -operators acting on the fiducial vector but not the ‘bare’ fiducial vector itself, equation (53) simplifies taking the form of the standard algebra for creation and annihilation operators, i.e.

$$[\alpha_i(x), \alpha_j^\dagger(y)] = \delta_{ij} \delta^D(x - y), \quad (55)$$

whereas the commutation relations (54) remain unchanged. Consider for simplicity operators labeled only by continuous labels, i.e. $e(x)$, $\partial(x)$. For instance, one can identify $e(x) = e_1(x)$, $\partial(x) = \partial_1(x)$, rewriting equation (55) as

$$[\alpha(x), \alpha^\dagger(y)] = \delta^D(x - y). \quad (56)$$

The most general e -operators will be discussed in the end of this paper. The eigenequation of $e(x)$ operator reads

$$e(x)|e(x)\rangle = \lambda(x)|e(x)\rangle. \quad (57)$$

Here we cannot identify $\lambda(x) = x$, because for $D > 1$ this would make no sense from perspective of the eigenequation (λ is a scalar-like object). Instead, we have to assume that eigenvalues are given by a scalar function $\lambda(x) = \lambda(x^0, \dots, x^{D-1})$. We do not fix the dimension D now, letting it to be a free parameter.

The eigenvector $|e(x)\rangle$ is a direct analogue of the standard position eigenstate in position representation $|x\rangle$, i.e.

$$\hat{x}|x\rangle = x|x\rangle. \quad (58)$$

Here

$$\langle x|x'\rangle = \delta^3(x - x'). \quad (59)$$

Being non-normalizable, eigenvectors $|x\rangle$ are orthogonal. Keeping this in mind, we now examine the position $e(x)$. The solution of equation (57) takes the form

$$|e(x)\rangle = E_\lambda(x)|\phi\rangle, \quad (60)$$

where $|\phi\rangle$ is a fiducial vector and $E_\lambda(x)$ stands for the most general combination of (generalized) e -operators:

$$\begin{aligned}
 E_\lambda(x) &= a(x) + \sum_{n=1}^{\infty} \int d^D x_1 \dots d^D x_n b_n(x; x_1, \dots, x_n) e(x_1) \dots e(x_n) \\
 &+ \sum_{n=1}^{\infty} \int d^D x_1 \dots d^D x_n c_n(x; x_1, \dots, x_n) \partial(x_1) \dots \partial(x_n).
 \end{aligned}
 \tag{61}$$

Here $a(x)$, $b_n(x; x_1, \dots, x_n)$, $c_n(x; x_1, \dots, x_n)$ are functional coefficients to be fixed. Supplementing equation (60) with the ansatz (61), and then substituting the resulting vector $|e(x)\rangle$ into the eigenequation (57), leads to recurrence equations for the functional coefficients. Solving them, one finds

$$b_n(x; x_1, \dots, x_n) = \frac{a(x)}{\lambda^n(x)} \prod_{i=1}^n \delta^D(x - x_i),
 \tag{62}$$

$$c_n(x; x_1, \dots, x_n) = a(x) \frac{(-\lambda(x))^n}{n!},
 \tag{63}$$

Here $\lambda(x)$, $a(x)$ are remaining free functional coefficients. Consider the product $\langle e(x')|e(x)\rangle = \langle \phi|E_\lambda(x')^\dagger E_\lambda(x)|\phi\rangle$. Making use of equations (61)–(63) gives

$$\langle e(x')|e(x)\rangle = a^*(x')a(x) \left(1 + \sum_{n=1}^{\infty} \left[\left(\frac{\lambda(x)}{\lambda^*(x')} \right)^n + \left(\frac{\lambda^*(x')}{\lambda(x)} \right)^n \right] \right).
 \tag{64}$$

Notice that letting $a(x) = 1$, $\lambda(x) = \lambda_j$ and $\lambda(x') = \lambda_i$, we restore the form (15). This is expected since with the lack of manifest dependence on continuous variables x, x' , the product should reduce to what has been found for e operator. On the other hand, there are two remaining functional parameters in equation (64): the eigenvalue $\lambda(x)$ and $a(x)$.

As in the case of e operator, we will now search for a restricted Hilbert space in which $e(x)$ is Hermitian. In order to do so, we assume that $\lambda(x)$ is real and non-zero. We will not reconsider the problem of zero eigenvalues. In fact, due to relative similarity between equations (64) and (15), we would get nothing interesting. Instead, we will concentrate on a small region in the neighborhood of two close points X^a and $X^a + dX^a$, where X^a stand for continuous labels of e -operators ($A \in \{0, \dots, D - 1\}$). We ask what guaranties that $e(x)$ is Hermitian in that region.

Assume that $a(x) = \text{const}$, letting $a(x) = 1$ in equation (64), and impose a finite cut-off Λ for $E_\lambda(x)$ in equation (61). The cut-off will be eliminated by taking the limit $\Lambda \rightarrow \infty$ in the end. The regularized product reads

$$\langle e_\lambda(x')|e_\lambda(x)\rangle = \left(1 + \sum_{n=1}^{\Lambda} \left[\left(\frac{\lambda(x)}{\lambda(x')} \right)^n + \left(\frac{\lambda(x')}{\lambda(x)} \right)^n \right] \right),
 \tag{65}$$

where $(x')^A = X^a + dX^a$. We will treat the arguments $x, x' \in \mathbb{R}^D$ as dimensionless parameters. Performing the sum (65) explicitly and expanding the result in dX^a gives

$$\langle e_\lambda(x')|e_\lambda(x)\rangle = 2\Lambda \left(1 + \left(\frac{\Lambda^2}{6} + \frac{\Lambda}{4} + \frac{1}{12} \right) \frac{\partial_{AB}\lambda(x)}{\lambda(x)} dx^A dx^B \right) + \mathcal{O}(dx^3).
 \tag{66}$$

If $\Lambda \gg 1$, one can neglect two terms in the internal bracket, rewriting equation (65) in a bit

simpler form

$$\langle e_\lambda(x') | e_\lambda(x) \rangle = 2\Lambda \left(1 + \Lambda^2 \Theta_{AB}(x) dx^A dx^B \right), \tag{67}$$

where

$$\Theta_{\mu\nu}(x) := \frac{\partial_{\mu\nu} \lambda(x)}{6\lambda(x)}. \tag{68}$$

Here we skipped the terms of order $\mathcal{O}(dx^3)$ ⁴. If $X^a = (x')^A$, i.e. $dX^a = 0$, the product (67) reads

$$\langle e_\lambda(x) | e_\lambda(x) \rangle = 2\Lambda. \tag{69}$$

This is the norm of the eigenvector $|e_\lambda(x)\rangle$; it diverges in the limit $\Lambda \rightarrow \infty$. This is the same divergence we have met discussing e -eigenvectors corresponding to real eigenvalues. As we recall, regularized eigenvectors (17) have been constructed multiplying by infinitesimal prefactor $1/\sqrt{2\Lambda}$. However, since now the eigenvectors are labeled by continuous variables, we will not do so. For instance, for the standard position operator equation (59) gives $\langle x|x \rangle = \delta^3(0)$. This is why, contrary to the analysis presented in the last section, we will not demand the position eigenvectors (60) to be normalized. We only require they are orthogonal.

The last translates into the additional constraint. Recalling the form (67) of the product, the constraint reads

$$\Theta_{AB}(x) dx^A dx^B = -\Lambda^{-2}. \tag{70}$$

The last is non-trivial only if $dX^a \neq 0$ (if $dX^a = 0$ the constraint is a non-issue because the product reduces to infinite norm (69)). Assume for a moment that equation (70) has already been satisfied. One gets

$$\langle e_{\lambda'}(x') | e_\lambda(x) \rangle = \begin{cases} \lim_{\Lambda \rightarrow \infty} 2\Lambda, & \text{if } x = x' \\ 0, & \text{if } x \neq x'. \end{cases} \tag{71}$$

In the limit $\Lambda \rightarrow \infty$, one identifies $\delta(0) = \Lambda/(2\pi)$. Therefore equation (71) can be interpreted as related with Dirac delta, i.e. $\langle e_{\lambda'}(x') | e_\lambda(x) \rangle \propto \delta(\lambda(x) - \lambda(x'))$. The operator $e(x)$ is Hermitian in the resulting restricted Hilbert space and has continuous spectrum. We should also keep in mind that equation (71) makes sense only in a close neighborhood of a fixed point. Hence, equation (70) is a local constraint.

3.1. Spacetime interpretation of the constraint equation and geometry of AdS_5

We will now interpret and solve the constraint (70). We start with an observation that linear combinations of e -operators give rise the standard creation and annihilation operators, leading to the standard algebra (54) of the harmonic oscillator. Physically, this means that variables x of e -operators can be identified with positions in space. Indeed, recalling standard interpretation of quantum fields, each point in space can be viewed as associated with

⁴More specifically, these terms are of order $\mathcal{O}(\Lambda^2 dx^3)$ and are small if $\Lambda^2 dx^3$ is also small. In particular, this would be the case for $|dx| = \mathcal{O}(\Lambda^{-1})$. The regularization parameter Λ is then interpreted as inverse of $|dx|$. Once $|dx| \ll 1$, then $\Lambda \gg 1$.

a quantum harmonic oscillator, while the field itself can be interpreted as infinite number of such oscillators. The vacuum space can be related with oscillators in the ground state.

This simple logic can be further generalized to incorporate time. To be more specific, consider the possibility that the variables $x = x^0, \dots, x^{D-1}$ represent spacetime labels, just like for operators in the Heisenberg picture. However, in light of the algebra (54), this would mean there are oscillations also in temporal direction; something which requires the concept of operator of time. The latter is not a new idea and has already been discussed [21–23]. One of the reasons behind such an approach is that time and space are treated on equal footing, as in the case of general relativity (or more generally, classical theory of gravity). Since on the quantum level time and space are represented by operators, this suggests a way of incorporating quantum gravity effects.

Despite operator of time is a promising idea, we will not go deeply into the subject. Instead, keeping in mind all what we have said, we restrict to say that the labels x of $e(x)$ and $\partial(x)$ can be potentially connected with coordinates in spacetime. The way they are related should be, at least at the most superficial level, determined by solutions to the constraint (70).

Unfortunately, equation (70) cannot say much about the connection. This is because we actually did not discuss any quantum system. The latter would require introducing quantum fields, or performing a simplified analysis similar to the one presented in [1]. Rather than searching for the connection between quantum states and geometry, we ask about the potential geometrical meaning of the constraint itself.

The starting point will be the result found in section 2.4, a class of regularized eigenvectors spanning four-dimensional Minkowski space. As we recall, for $x = \text{const}$ and $a(x) = 1$, eigenvectors (60) are equivalent to the simple e -eigenvectors (13). Since for $\lambda = 0$ the former was interpreted geometrically in terms of Minkowski space, the same is expected for the latter. Therefore, solving the constraint equation (70), one expects to find a solution sharing the same geometrical interpretation. This should hold for sufficiently small eigenvalues. In particular, we expect to get the four-dimensional Minkowski space in the limit $\lambda(x) \rightarrow 0$. We now examine this in details.

In order to do so, assume that $\lambda(x)$ vanishes at a fixed point $x^A = \vec{0}$, i.e. $\lambda(0) = 0$. Obviously the point $x^A = \vec{0}$ is not special, this is only to illustrate the procedure. We also assume the function is slowly varying in a close vicinity of this point. This means that in the Taylor series approximating $\lambda(x)$ close to zero all gradient terms are small. In particular, $|\partial_A \lambda(0)|, |\partial_A \partial_B \lambda(0)| \ll 1$ and $|\partial_A \partial_B \lambda(0)| \ll |\partial_A \lambda(0)|$, etc. For a slowly varying function, equation (70) can be approximated as

$$\Theta_{AB}(0) dx^A dx^B = -\Lambda^{-2}, \tag{72}$$

where

$$\Theta_{AB}(0) = \frac{\partial_{AB} \lambda(0)}{6\lambda(0)}. \tag{73}$$

Since the matrix $\Theta_{AB}(0)$ is symmetric, it can be diagonalized. Let $\Theta_{\mu\nu}^{(0)} = \text{diag}\{c_0, \dots, c_{D-1}\}$ stands for the result of diagonalization procedure. This procedure involves linear transformation of variables X^a . If the original matrix is already diagonal, they remain the same. If not, they will be changed. For simplicity we ignore such details now, assuming that X^a refer to

already diagonal form $\Theta_{\mu\nu}^{(0)}$. The constraint (72) reads

$$\sum_{n=0}^{D-1} c_n (dx^n)^2 = -\Lambda^{-2}. \tag{74}$$

Here all (constant) coefficients c_n are non-zero. In general, diagonalization of $\Theta_{AB}(0)$ may result also in zero diagonal components c_n . We simply assume that $\lambda(x)$ was chosen in such a way that $\Theta_{\mu\nu}^{(0)}$ has D non-zero components. Alternatively, one can reinterpret D as representing the number of non-zero components after diagonalization.

Equation (74) can be further simplified introducing the following new variables

$$\tilde{x}^A := \sqrt{|c_A|} x^A \tag{75}$$

(there is no summation convention here). The constraint (74) takes the form

$$\sum_{n=0}^{D-1} (\pm)_n (d\tilde{x}^n)^2 = -\Lambda^{-2}, \tag{76}$$

where $(\pm)_n := \text{sign}(c_n)$.

We would like to examine if it is possible to get four-dimensional Minkowski space as a solution to the constraint. Obviously, this requires $D \geq 4$. In the simplest possible case $D = 4$ while $\Theta_{AB}^{(0)}$ has a signature $(-, +, +, +)$. Recalling equation (68) one concludes that this will be the case for

$$\lambda(x) = \lambda_0 \sin(\omega x^0) + \lambda_0 \sinh(k_1 x^1) + \lambda_0 \sinh(k_2 x^2) + \lambda_0 \sinh(k_3 x^3), \tag{77}$$

where ω, k_1, k_2, k_3 are assumed to be small to guaranty the function is slowly varying. Letting l_0 to be a dimensional length scale, the function will be slowly varying in the corresponding region if $\omega \ll l_0^{-1}$ and $k_i \ll l_0^{-1}$. The constraint (76) reads

$$\eta_{\mu\nu} dx^\mu dx^\nu = -\Lambda^{-2}. \tag{78}$$

As we recall, Λ stands for a regularization parameter, which goes to infinity in the end. Geometrically, the left-hand side of equation (78) can be identified with the line element in four-dimensional Minkowski space, i.e. $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. In the limit $\Lambda \rightarrow \infty$ the latter goes to zero meaning that we cover not the whole space, but a hypersurface corresponding to the set of events $ds^2 = 0$. This can be identified with trajectories of massless particles. They define a hypersurface in $D = 4$ Minkowski space, but not the whole space.

Fortunately, there is an even larger class of solutions. Notice that the right-hand side of equation (76) is invariant under $SO(n, m)$ rotations, where (n, m) stands for the signature of the matrix $\Theta_{AB}^{(0)}$. As such, equation (76) resembles the constraint defining the so-called pseudo-hyperbolic space [20] as embedding in $\mathbb{R}^{n,m}$. Depending on the signature of $\Theta_{AB}^{(0)}$ this includes hyperbolic, anti-de Sitter, and many other spaces with no clear spacetime interpretation⁵. Another observation is that equation (76) (as well as equation (74)) becomes scale-invariant in the limit $\Lambda \rightarrow \infty$. The last two facts show that if anti-de Sitter space is a solution to the constraint, the solution is special. Below we show that equation (76) indeed support that space and, in particular, AdS₅.

⁵This is because more than one time-like direction and the presence of closed timelike curves which cannot be eliminated in a consistent way.

In order to do so, assume that $\lambda(x)$ is a function of six continuous variables such that the matrix Θ_{AB}^0 has a signature $(-, -, +, +, +, +)$. The constraint (76) reads

$$-(d\tilde{x}^0)^2 + (d\tilde{x}^1)^2 + (d\tilde{x}^2)^2 + (d\tilde{x}^3)^2 + (d\tilde{x}^4)^2 - (d\tilde{x}^5)^2 = -\Lambda^{-2}. \tag{79}$$

Again, we can interpret the constraint (79) as defining a hypersurface $ds^2 = 0$. However, we would not like to do so. Instead, we solve the constraint explicitly, installing a coordinate system on the resulting manifold. Note that the left-hand side of equation (79) is invariant under $SO(4, 2)$ rotations, the isometry group of AdS_5 . Hence, we expect that anti-de Sitter space is a solution to the constraint (79). More precisely, let $\delta L := \Lambda^{-1}$ and

$$\delta x^A := dx^A \tag{80}$$

stand for components of an infinitesimal, normalized vector in $\mathbb{R}^{4,2}$. The normalization is provided by the constraint (79):

$$-(\delta x^0)^2 + (\delta x^1)^2 + \dots + (\delta x^4)^2 - (\delta x^5)^2 = -\delta L^2. \tag{81}$$

The left and the right-hand side of the equation above are built out of infinitesimally small objects of the same order. A convenient way of solving the constraint (81) is installing Poincare coordinate system $(t_\delta, x_{\delta 1}, x_{\delta 2}, x_{\delta 3}, z_\delta)$ [24], defined as

$$\begin{aligned} \delta x^0 &= \frac{-t_\delta^2 + x_{\delta 1}^2 + x_{\delta 2}^2 + x_{\delta 3}^2 + z_\delta^2 + \delta L^2}{2z_\delta}, \\ \delta x^i &= \delta L \frac{x_{\delta i}}{z_\delta}, \quad i = 1, 2, 3, \\ \delta x^4 &= \frac{-t_\delta^2 + x_{\delta 1}^2 + x_{\delta 2}^2 + x_{\delta 3}^2 + z_\delta^2 - \delta L^2}{2z_\delta}, \\ \delta x^5 &= \delta L \frac{t_\delta}{z_\delta}. \end{aligned} \tag{82}$$

Inserting equation (82) into equation (81) one checks the constraint is satisfied. Note that unlike the standard construction of AdS_5 as embedding in $\mathbb{R}^{4,2}$, equation (81) is a constraint on a vector $\delta x^A \in \mathbb{R}^{4,2}$ built out of infinitesimal variables. Solutions (82) lead to the following metric on the resulting manifold

$$ds^2 = \delta L^2 \frac{-dt_\delta^2 + dx_{\delta 1}^2 + dx_{\delta 2}^2 + dx_{\delta 3}^2 + dz_\delta^2}{z_\delta^2}. \tag{83}$$

This is the metric of AdS_5 in Poincare coordinates; δL stands for AdS radius. Formally, the right-hand side of equation (83) is small of order four. This is actually not a problem since the line element is manifestly scale-invariant. Small lengths can be isomorphically map into their arbitrarily large equivalents. In particular, the line element (83) is invariant under scaling transformation $(t_\delta, x_{\delta 1}, x_{\delta 2}, x_{\delta 3}, z_\delta) \rightarrow (\Omega t_\delta, \Omega x_{\delta 1}, \Omega x_{\delta 2}, \Omega x_{\delta 3}, \Omega z_\delta)$, where Ω is arbitrary real parameter. This scale-invariance reflects the scale-invariance of the constraint (81) or, alternatively, the form (72).

Since $\delta L = \Lambda^{-1}$, AdS-radius goes to zero in the limit $\Lambda \rightarrow \infty$, while the scalar of curvature $R = -20\Lambda^2 = -20\delta L^{-2}$ becomes infinite. Fortunately, this is not an issue at the conformal bound, corresponding to $z_\delta = 0$ in Poincare coordinates. To be more precise, taking the limit

$z_\delta \rightarrow 0$ leads to the ‘boundary’ of AdS₅. This turns out to be nothing but the four-dimensional Minkowski space:

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2. \tag{84}$$

The metric above can be easily derived from equation (83), assuming that $\delta L, z_\delta \rightarrow 0$ while keeping $\delta L/z_\delta = \text{const}$. Letting $z_\delta \rightarrow 0$ is a way to compensate the problematic limit $\delta L \rightarrow 0$. The five-dimensional geometry effectively reduces to the four-dimensional one, and the latter is manifestly non-singular. It takes the form of the four-dimensional Minkowski space. However, the space emerges in a rather unexpected way as the boundary of a higher-dimensional spacetime.

At this point it is worth underlying that all what we have said concerns the assumption that $\lambda(x)$ is small and slowly varying. What is more, we restricted ourselves to the local version of the constraint equation, valid in a close neighborhood of a fixed point. Even with these strong assumptions we arrived with a large class of solutions, identifying AdS₅ to be special because of the form of its boundary. However, global solutions to the constraint equation can be qualitatively different. Without specifying the form of the function $\lambda(x)$ we cannot say much about them. The only what we know is that locally they should look like AdS₅ to match the results found for e operator. Quite interestingly, the last fact can be viewed as a holographic analogue of the standard local flatness. The Minkowski space emerges in a holographic manner, being the boundary of the anti-de Sitter space.

We close the analysis with a simple observation regarding the form of the product (65). It is straightforward to observe that the latter is invariant under scaling transformations $\lambda(x) \rightarrow b\lambda(x)$. This means that scale-invariance is a symmetry of the product in its most general form. In the spacial case, restricting to small regions in the parameter space and small $\lambda(x)$, this symmetry is reflected by scale-invariance of AdS₅.

3.2. Momentum eigenequation

So far we considered e -position, represented either by e or $e(x)$ operators. Now we will discuss the momentum. In the simplest possible case, it reads $\hat{p} = -i\partial$. The corresponding eigenequation

$$\hat{p}|p\rangle = \lambda|p\rangle, \tag{85}$$

has a solution

$$|p\rangle = \hat{P}_\lambda|\phi\rangle, \tag{86}$$

where

$$\hat{P}_\lambda = \exp(i\lambda e) + \sum_{i=1}^{\infty} \frac{\partial^i}{(i\lambda)^i}. \tag{87}$$

The scalar product of two eigenvectors (86) reads

$$\langle p_{\lambda_i} | p_{\lambda_j} \rangle = 1 + \sum_{n=1}^{\infty} \left[\left(\frac{\lambda_i^*}{\lambda_j} \right) + \left(\frac{\lambda_j}{\lambda_i^*} \right) \right]. \tag{88}$$

Notice that the right-hand side of equation (88) is the same as the right-hand side of equation (15), found for e -position. Therefore, the future analysis, including construction of normalized eigenvectors and their geometric interpretation, will be also the

same. The superselection rules guaranty the momentum is conserved, while the whole $D = 4$ Minkowski space corresponds to the zero momentum eigenvalue. Again, both spacetime dimension and signature of the metric are determined by the regularization procedure.

For generalized momentum $-i\partial(x)$ the eigenequation takes the form

$$-i\partial(x)|p(x)\rangle = \lambda(x)|p(x)\rangle. \tag{89}$$

The eigenvector can be found writing down the most general form of the state in the maximally extended space:

$$|p(x)\rangle = \left(a(x) + \sum_{n=1}^{\infty} \int d^D x_1 \dots d^D x_n b_n(x; x_1, \dots, x_n) e(x_1) \dots e(x_n) + \sum_{n=1}^{\infty} \int d^D x_1 \dots d^D x_n c_n(x; x_1, \dots, x_n) \partial(x_1) \dots \partial(x_n) \right) |\phi\rangle. \tag{90}$$

Substituting equation (90) into (89) and solving the resulting equations gives

$$b_n(x; x_1, \dots, x_n) = a(x) \frac{i^n}{n!} \lambda^n(x), \tag{91}$$

$$c_n(x; x_1, \dots, x_n) = a(x) \frac{(-i)^n}{\lambda^n(x)} \prod_{k=1}^n \delta(x - x_k). \tag{92}$$

Here $a(x)$ is a remaining functional parameter. The product of two eigenvectors (90) reads

$$\langle p(x') | p(x) \rangle = a^*(x') a(x) \left(1 + \sum_{n=1}^{\infty} \left[\left(\frac{\lambda(x)}{\lambda^*(x')} \right)^n + \left(\frac{\lambda^*(x')}{\lambda(x)} \right)^n \right] \right). \tag{93}$$

Again, we found the same result as for the extended position (see equation (64)). Therefore, the constraint equation will be also the same. Having found $D = 4$ Minkowski space discussing e -position, the same can be found discussing e -momentum. The space emerges as a conformal boundary of AdS_5 .

3.3. The most general operators

We are ready to generalize the results to the case of the most general e -operators, $e_a(x)$ and $\partial_a(x)$, $a = 1, \dots, d$. The position eigenequation reads

$$e_a(x)|e(x)\rangle = \lambda_a(x)|e(x)\rangle, \tag{94}$$

where

$$|e(x)\rangle = \bigotimes_{a=1}^d |e_a(x)\rangle, \tag{95}$$

and $|e_a(x)\rangle$ stand for eigenvectors (60). There are d copies of such eigenvectors, one for each label a . It is straightforward to observe that the rules (46)–(51) guaranty that eigenvectors $|e_a(x)\rangle$ indexed by different discrete labels are orthogonal.

Notice that the solution (95) is similar to the previously found (60). In fact, the only what changed is that now we have d functional eigenvalues $\lambda_a(x)$, while eigenvectors (95) were built out of ‘simple’ eigenvectors (60). We have a copy of the constraint equation (70) for each discrete label of the e -position.

It is convenient adopting a bit more natural notation, rewriting the eigenequation (94) as

$$\hat{X}^a(x)|X(x)\rangle = X^a(x)|X(x)\rangle. \tag{96}$$

Here we defined

$$\hat{X}^a(x) := e_a(x), \quad X^a(x) := \lambda_a(x), \quad |e(x)\rangle := |X(x)\rangle. \tag{97}$$

Similarly, for the momentum $\hat{P}^a(x) := -i\partial_a(x)$, one has

$$\hat{P}^a(x)|P(x)\rangle = P^a(x)|P(x)\rangle. \tag{98}$$

The operators $\hat{X}^a(x)$, $\hat{P}^a(x)$ are similar to the standard position and momentum, however, do not share the standard algebra. Instead, equation (52) can be rewritten as

$$[\hat{X}^a(x), \hat{P}^b(y)] = i\delta_{ab}\delta^D(x - y)\hat{\eta}. \tag{99}$$

As it was shown in [1], eigenstates of creation and annihilation operators can be constructed in such a way they belong to extended, but not maximally extended space. This means that one can skip the operator $\hat{\eta}$, reproducing the standard algebra of the quantum harmonic oscillator. However, for e -position and e -momentum the situation is qualitatively different. Solving eigenequations one cannot skip the term responsible for the presence of the bare fiducial vector. In consequence, the commutation relation cannot be simplified in the space of eigenvectors. Therefore we cannot expect the standard uncertainty relation for these operators.

Discussing eigenequation of $e(x)$ operator we identified the constraint (70) guarantying that eigenvectors from different eigenvalues are orthogonal. Itself, the constraint was interpreted geometrically as defining spacetime as embedded manifold. Letting $D = 6$ we found AdS₅ as embedding in $\mathbb{R}^{4,2}$. What makes the solution special is that it matches perfectly the result found for ‘simple’ e -operators: the eigenspace corresponding to the zero eigenvalue can be interpreted geometrically as four-dimensional Minkowski space. As we recall, this was generalized to the case of generalized e -operators in the limit of small $\lambda(x)$. Keeping this in mind, it is natural to ask what will happen in the most general case, i.e. for $e_a(x)$ and $-i\partial_a(x)$.

As before, we start with e -position. Due to the symmetry of the scalar product, the same results can be found for the momentum. In case of $e_a(x)$ eigenstates, one finds d copies of the constraint equation (70):

$$\Theta_{AB}^a(x)dX^A dx^B = -\Lambda_a^{-2}, \tag{100}$$

where

$$\Theta_{AB}^a(x) = \frac{\partial_{AB}X^a(x)}{6X^a(x)}, \tag{101}$$

and $a = 1, \dots, d$. Following the construction presented in section 3.1 we will be interested in covering a small region where the eigenvalues are close to zero. The latter is defined by a neighborhood of $X^a = 0$. Suppose that $X^a(0) = 0$. Repeating the steps of the construction

outlined in section 3.1, we assume that each $X^a(x)$ is a slowly varying function, rewriting the constraints (101) as

$$\sum_{n=0}^{D-1} c_n^a (dx^n)^2 = -\Lambda_a^{-2}. \tag{102}$$

Here Λ_a stand for regularization parameters. Without loosing of generality one can assume they are all equal, i.e. $\Lambda_1 = \dots = \Lambda_d = \Lambda$. Again, one should take the limit $\Lambda \rightarrow \infty$ in the end. Introducing new variables

$$\tilde{x}^{a,A} := \sqrt{|c_A^a|} X^A \tag{103}$$

(there is no summation convention here), one rewrites equation (102) as

$$\sum_{n=0}^{D-1} (\pm)_n^{(a)} (d\tilde{x}^{a,n})^2 = -\Lambda^{-2}. \tag{104}$$

Here $(\pm)_n^{(a)} := \text{sign}(c_n^a)$. Suppose we have chosen $X^a(x)$ in such a way that each matrix Θ_{AB}^a has a signature $(-, -, +, +, +, +)$. This would be the case for

$$X^a(x) = \lambda_0^a \sin(\omega^a x^0) + \lambda_0^a \sum_{i=1}^3 \sinh(k_i^a x^i) + \lambda_0^a \sin(k_4^a x^4), \tag{105}$$

where $\lambda_0^a, \omega, k_1^a, \dots, k_4^a$ are additional parameters. As before, we assume that $\omega, k_1^a, \dots, k_4^a$ are small with respect to some length scale l_0 : $\omega^a \ll l_0^{-1}, k_i^a \ll l_0^{-1}, k_4^a \ll l_0^{-1}$. This guaranties $X^a(x)$ are slowly varying.

Following the construction of AdS₅ in Poincare coordinates presented in section 3.1 one concludes that starting with the form (105), one finds d copies of that space, one for each discrete label a . Since each time one gets the same geometric result, one can start with a simpler form of $X^a(x)$:

$$X^a(x) = \lambda_0^a \sin(\omega x^0) + \lambda_0^a \sum_{i=1}^3 \sinh(k_i x^i) + \lambda_0^a \sin(k_4 x^4), \tag{106}$$

where ω, k_1, \dots, k_4 are the same for all discrete labels a . It is straightforward to check that one finds the same constraint for each discrete label a . Geometrically, this results in AdS₅. This is justified within the adopted assumption that eigenvalues are small, i.e.

$$|X^a(x)| \ll 1. \tag{107}$$

We should keep in mind that all we have said concerns the local form of the constraint given by equation (72). Therefore, the resulting geometry is only a local solution. This is somehow similar to local flatness in general relativity. Note that we have already mentioned this analogy discussing eigenvectors of $e(x)$ operator. As previously, general solutions to the constraint (70) may differ significantly from anti-de Sitter or even Minkowski space. Unfortunately, we cannot say much about them without incorporating matter (quantum fields, vibrations of strings, etc.).

4. Conclusions

We have discussed the solutions to the eigenequation of the e -operators, the position and momentum. Starting with the eigenvectors of the e operator, we show that they are non-normalizable, however, can be easily regularized. We observe that regularized eigenvectors span restricted (extended) Hilbert space in the form of a direct sum of separate spaces of finite dimension. This includes two-dimensional spaces spanned by eigenvectors of opposite eigenvalues, and a four-dimensional space composed of degenerated states corresponding to $\lambda = 0$. We also identified superselection rules forbidding combinations of states with different absolute values of the eigenvalues λ . Our construction guaranties that operator the e is Hermitian if it is restricted to any of these spaces. The same result holds for e -momentum. The space corresponding to the zero eigenvalue turns out to be special, because it has the maximal dimension and because the orthonormal basis consists of three positive and one negative eigenvector. As such, the space can be identified with the four-dimensional Minkowski space. The interpretation relies upon the observation that for $\lambda = 0$ any superposition of eigenvectors is also an eigenvector corresponding to the same zero eigenvalue, and amplitudes in the superposition are linked with spacetime coordinates. The difference is that instead of Lorentz symmetry group, we arrive with a more general group of complex Lorentz rotations $O(3, 1, \mathbb{C})$. Rewritten in terms of real parameters θ^μ and σ^ν , the rotations result in non-trivial transformations given by equations (40) and (41). Assuming that half of the directions in the parameter space are ‘frozen’, i.e. either $\sigma^\mu = \sigma_0^\mu = \text{const}$ or $\theta^\mu = \theta_0^\mu = \text{const}$, leads to the standard four-dimensional Poincare group.

Having found eigenvectors of the simplest e -operators, we generalize the result to the case of operators with continuous labels. Considering the position eigenequation, we notice the presence of the additional constraint equation (70) which ensures that the operator is Hermitian. Eigenvalues $\lambda(x)$ turned out to be labeled by the labels of the e -operators, and except equation (70) there was no further constraints on the form of the function $\lambda(x)$. We discuss the constraint equation (70) restricting to the region where the eigenvalue is close to zero. This is supplemented by technical assumption making $\lambda(x)$ slowly varying function. The reason for considering small eigenvalues was dictated by the observation that for the simplest e -operators, the space corresponding to the zero eigenvalue was identified with the four-dimensional Minkowski space. In what follows, taking $\lambda(x)$ to be close to zero, we expect to reproduce the geometric result corresponding to $\lambda = 0$, i.e. $D = 4$ Minkowski space. We show that this is indeed the case and in a class of various possible solutions to the constraint, one can identify the one interpreted geometrically as AdS_5 . The Minkowski space emerges in a holographic way, as the conformal boundary of the resulting anti-de Sitter space.

Finally, we discuss the eigenequation for the most general e -operators, labeled both by discrete and continuous labels. In particular, for the position operator $e_a(x)$, we found d constraint equations, where d stands for the number of discrete labels of the operator. We show that solutions to the constraints in the region of small eigenvalues also supports AdS_5 . We notice an interesting symmetry between e -position and e -momentum, consisting in the structure of the scalar product. Taking a closer look at the position eigenequation (94) and rewriting it in the form (96), we find an analogy to string theory: eigenvalues $X^a(x)$ resemble world-volume coordinates. The latter can be seen as a starting point for string theory. Indeed, on a classical level this provides motivation for writing down the standard p-brane action. However, specifying the classical field $X^a(x)$, one gets automatically the corresponding quantum state $|X^a(x)\rangle$ in the maximally extended space.

The main result presented in this paper is the geometric interpretation of solutions to the eigenequation of e -operators. The interpretation relies upon the number and sign of the norm of regularized eigenvectors corresponding to the zero eigenvalue. In the simplest case this gives rise to the four-dimensional Minkowski space. For more general operators the space emerges as the boundary of AdS_5 . It is also worth underlying that for simple operators, e and ∂ , there is no direct link between classical coordinates and their quantum equivalents. The whole Minkowski space corresponds to a single position or momentum eigenvalue, while superselection rules guaranty these quantities are conserved. In the case of general e -operators the connection becomes slightly more transparent. Still, similarly as in [1], spacetime turns out to be a secondary concept, associated with quantum states.

Compared to the discussion presented in [1–3], the quantum entanglement seems not to be essential for the emergence of spacetime. In what follows, one can ask how it is possible we extracted classical geometry without incorporating quantum correlations. The answer is that these correlations are actually present, being hidden in the adopted assumptions. More specifically, the interpretation of the continuous labels of e -operators bases on the algebra (53) and its connection with the quantum harmonic oscillator. We have also utilized the fact that quantum fields can be interpreted as infinite number of oscillators associated with points in space. This holds true both for excited states and the ground state. The latter is particularly interesting since the vacuum state of relativistic QFT is maximally entangled [25]. Assuming that empty space is filled with quantum fields in the lowest energy state, we get a link with entanglement. In particular, the vacuum anti-de Sitter can be seen as represented by infinite number of oscillators in the ground state.

Talking about the geometry of anti-de Sitter space, it is worth mentioning that in string theory the latter appears when discussing gravitational field of D-branes. In the *closed string description*, they correspond to spacetime geometry in which close strings propagate. In particular, in the case of type IIB supergravity the spacetime metric sourced by a system of N branes, in the strong gravity region, reduces to $\text{AdS}_5 \times S^5$ [16, 26–28]. On the other hand, D-branes themselves are objects where endpoints of open strings are attached. Consequently, in the *open string description*, D-branes can be viewed as sort of hypersurfaces. Their excitations are open strings living on the brane, and closed strings propagating in the bulk [16, 17]. The dual description of D-branes is essential for AdS/CFT conjecture, leading to a link between supersymmetric conformal Yang-Mills theory at the boundary and gravity in asymptotically anti-de Sitter space in the bulk.

Notice that we have met a brane-like analogy two times in our paper. Firstly, discussing the symmetry group of complex rotations $O(3, 1, \mathbb{C})$, we show they lead to the standard Poincare symmetry in four dimensions if the initial symmetry is partially broken. This is interpreted as defining a four-dimensional hypersurface in higher dimensional space, something which resembles a brane. Second, for the most general operators the position eigenvalues take the form of a worldvolume in the sense that there is a map $X^a(x)$, i.e. $X^a \mapsto X^a$. These functions are not fixed by the construction, however, solving the constraint (72) we assume they are small and slowly varying. For a brane this would mean that we are in the region where perturbations are small. Quite interestingly, recalling standard results of the AdS/CFT correspondence, this corresponds to the geometry dual to strongly-coupled Yang-Mills plasma close to thermal equilibrium [29, 30]. The main difference is that in our case the AdS-radius goes to zero, making spacetime curvature in the bulk divergent. Therefore, trying to interpret our resulting geometry as being sourced by a brane, we should keep in mind we are in the region where AdS radius is small. A connection to D-branes in string theory consist in the fact that eigenvalues of the most general e -operators support worldvolume interpretation. Moreover, as in case of D-brane

system, we identified the geometry of anti-de Sitter space. The latter is a special solution to the constraint, reflecting its scale-invariance.

It would be interesting to ask if such an interpretation is more than a formal analogy to string theory. In fact, the formalism of the e -operators in the extended Hilbert space was already used in [1] while searching for the connection between the ground state of a two-dimensional oscillator and AdS₃. Again, this is in full consistency with the holographic principle. All of these facts suggest that Hilbert space with no positive-definite scalar product can be a useful concept combining quantum mechanics and gravity.

The construction presented in this paper can be further generalized in two different ways. The first one is to examine an explicit identification of the position eigenvalues $X^a(x)$ with coordinates on the worldvolume of a brane. That is, one can consider the possibility of letting arbitrary functions $X^a(x)$ to be solutions of the classical equation of motion for a p-brane. Alternatively, one could formally introduce quantum fields, expressing them in terms of the e -operators. The background geometry will be provided by the solution to the constraint, however, one expects a further backreaction induced by quantum correlations. This, in fact, could be the first step towards identifying a more general class of gravitational dualities.

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Appendix A. Regularization

We consider the product $\langle e_{\lambda_i} | e_{\lambda_j} \rangle = \langle \phi | (E_{\lambda_i})^\dagger E_{\lambda_j} | \phi \rangle$ of two eigenvectors (13), labeled by two eigenvalues λ_i and λ_j . Denote $\langle \phi | \dots | \phi \rangle := \langle \dots \rangle$, skipping the fiducial vector for convenience. The product reads

$$\langle e_{\lambda_i} | e_{\lambda_j} \rangle = \langle E_{\lambda_i}^\dagger E_{\lambda_j} \rangle = \left\langle \left(1 + \sum_{k=1}^{\infty} \frac{e^k}{(\lambda_i^*)^k} + \frac{(\lambda_i^*)^k}{k!} \partial^k \right) \left(1 + \sum_{n=1}^{\infty} \frac{e^n}{\lambda_j^n} + \frac{(-\lambda_j)^n}{n!} \partial^n \right) \right\rangle,$$

where in the last equality we substituted the exact form of the operator E_λ (14), making use equations (5) and (7). Multiplying terms on the right-hand side of equation (A1) gives

$$\begin{aligned} \langle E_{\lambda_i}^\dagger E_{\lambda_j} \rangle &= 1 + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{(-\lambda_j)^n \langle e^k \partial^n \rangle}{(\lambda_i^*)^k n!} + \frac{(\lambda_i^*)^k \langle \partial^k e^n \rangle}{\lambda_j^n k!} \right) \\ &= 1 + \sum_{n=1}^{\infty} \left(\left(\frac{\lambda_j}{\lambda_i^*} \right)^n + \left(\frac{\lambda_i^*}{\lambda_j} \right)^n \right). \end{aligned} \tag{A1}$$

To get the first line, we used equation (1), i.e. $\langle e^n \rangle = \langle \partial^n \rangle = 0$ (for $n > 0$). To get the second, we utilized equations (3) and (4), finding

$$\langle e^k \partial^n \rangle = (-1)^n \delta_{nk} n!, \quad \langle \partial^n e^k \rangle = \delta_{nk} k! \tag{A2}$$

We are now ready to consider the product (25):

$$:(E_{\lambda_i}^{\Lambda_i})^\dagger E_{\lambda_j}^{\Lambda_j}: = \frac{1}{2} \langle (E_{\lambda_i}^{\Lambda_i})^\dagger E_{\lambda_j}^{\Lambda_j} \rangle \Big|_{\Lambda_i \leq \Lambda_j} + \frac{1}{2} \langle (E_{\lambda_i}^{\Lambda_i})^\dagger E_{\lambda_j}^{\Lambda_j} \rangle \Big|_{\Lambda_i \geq \Lambda_j}.$$

For the first term, one finds

$$\langle (E_{\lambda_i}^{\Lambda_i})^\dagger E_{\lambda_j}^{\Lambda_j} \rangle \Big|_{\Lambda_i \leq \Lambda_j} = 1 + \sum_{k=1}^{\Lambda_i} \sum_{n=1}^{\Lambda_j} \left(\frac{(-\lambda_j)^n \langle e^k \partial^n \rangle}{(\lambda_i^*)^k n!} + \frac{(\lambda_i^*)^k \langle \partial^k e^n \rangle}{\lambda_j^n k!} \right) \quad (\text{A3})$$

$$= 1 + \sum_{n=1}^{\Lambda_i} \left(\left(\frac{\lambda_j}{\lambda_i^*} \right)^n + \left(\frac{\lambda_i^*}{\lambda_j} \right)^n \right). \quad (\text{A4})$$

In the last line we utilized the fact that, according to equation (A2), the only non-zero contribution to the sum comes from expectation values involving equal number of e -operators e and ∂ . In what follows, $k = n$. This also means that for $\Lambda_i \leq \Lambda_j$ the summation $\sum_{n=1}^{\Lambda_j} (\dots)$ can be terminated at $\Lambda_j = \Lambda_i$. A similar analysis shows that

$$\langle (E_{\lambda_i}^{\Lambda_i})^\dagger E_{\lambda_j}^{\Lambda_j} \rangle \Big|_{\Lambda_i \geq \Lambda_j} = 1 + \sum_{n=1}^{\Lambda_j} \left(\left(\frac{\lambda_j}{\lambda_i^*} \right)^n + \left(\frac{\lambda_i^*}{\lambda_j} \right)^n \right). \quad (\text{A5})$$

Substituting expectation values (A4) and (A5) into (A3) gives

$$\begin{aligned} \langle (E_{\lambda_i}^{\Lambda_i})^\dagger E_{\lambda_j}^{\Lambda_j} \rangle &= 1 + \frac{1}{2} \sum_{n=1}^{\Lambda_i} \left(\left(\frac{\lambda_j}{\lambda_i^*} \right)^n + \left(\frac{\lambda_i^*}{\lambda_j} \right)^n \right) + \frac{1}{2} \sum_{n=1}^{\Lambda_j} \left(\left(\frac{\lambda_j}{\lambda_i^*} \right)^n + \left(\frac{\lambda_i^*}{\lambda_j} \right)^n \right) \\ &= \frac{1}{2} \langle (E_{\lambda_i}^{\Lambda_i})^\dagger E_{\lambda_j}^{\Lambda_i} \rangle + \frac{1}{2} \langle (E_{\lambda_i}^{\Lambda_j})^\dagger E_{\lambda_j}^{\Lambda_j} \rangle. \end{aligned} \quad (\text{A6})$$

Taking into account equation (A6), the product (26) reads

$$\begin{aligned} \langle e_{\lambda_i}^{(\Lambda_i)} | e_{\lambda_j}^{(\Lambda_j)} \rangle &= 1 + \frac{1}{2} \lim_{N_i \rightarrow \infty} \sum_{n=1}^{\Lambda_i(N_i)} \left(\left(\frac{\lambda_j}{\lambda_i^*} \right)^n + \left(\frac{\lambda_i^*}{\lambda_j} \right)^n \right) \\ &\quad + \frac{1}{2} \lim_{N_j \rightarrow \infty} \sum_{n=1}^{\Lambda_j(N_j)} \left(\left(\frac{\lambda_j}{\lambda_i^*} \right)^n + \left(\frac{\lambda_i^*}{\lambda_j} \right)^n \right). \end{aligned} \quad (\text{A7})$$

In particular, for the vectors (22), one has⁶ $\lambda_i = \lambda_j = i$, $\Lambda_i = 2N_i$, $\Lambda_j = 2N_j - 1$, and

$$\begin{aligned} \langle e_i^{(\Lambda_1)} | e_i^{(\Lambda_2)} \rangle &= 1 + \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^{2N} 2(-1)^n + \frac{1}{2} \lim_{N \rightarrow \infty} \sum_{n=1}^{2N-1} 2(-1)^n \\ &= 1 + \lim_{N \rightarrow \infty} \left[\frac{1}{2} (-1 + (-1)^{2N}) + \frac{1}{2} (-1 + (-1)^{2N-1}) \right] = 0. \end{aligned} \quad (\text{A8})$$

⁶Do not confuse the imaginary number i with indexes of N_i , Λ_i .

Appendix B. General complex eigenvalues

Below we discuss normalization of eigenvectors of e operator corresponding to general complex eigenvalues $\lambda = re^{i\varphi}$, where $r > 0$ and $\varphi \in (0, \pi) \cup (\pi, 2\pi)$. Here we excluded two possibilities, $\varphi = 0$ and $\varphi = \pi$, since both they correspond to real eigenvalues discussed in section 2.2 (they require a different procedure). Define

$$|e_\lambda^{(\Lambda)}\rangle := \lim_{N \rightarrow \infty} E_\lambda^{\Lambda(N)} |\phi\rangle. \tag{B1}$$

Let $L_\lambda[\Lambda] := \langle (E_\lambda^\Lambda)^\dagger E_\lambda^\Lambda \rangle$. In the limit $N \rightarrow \infty$ this is norm of the vector (B1). For $\lambda = re^{i\varphi}$, one finds

$$L_\lambda[\Lambda] = 1 + 2 \sum_{n=1}^{\Lambda} \cos(2n\varphi) = 1 + 2 \sin^{-1}(\varphi) \sin(\Lambda\varphi) \cos((\Lambda + 1)\varphi). \tag{B2}$$

Clearly, the latter is ill-defined in the limit $\Lambda \rightarrow \infty$. The idea is to replace the cut-off Λ by a map $\Lambda(N)$ such that the product (B2) will be well defined in the limit $N \rightarrow \infty$. More precisely, we promote the cut-off Λ to a one-to-one function

$$\Lambda \mapsto \Lambda(N). \tag{B3}$$

We expect that its form should be specified by the regularization procedure. We now discuss general conditions guarantying this to be the case. First, we require that at least from some fixed point $N_0 \gg 1$, $\Lambda(N)$ is increasing function of N . Otherwise, the replacement (B3) would make no sense. Second, since we are interested in the limit $N \rightarrow \infty$, looking for a candidate for $\Lambda(N)$ it is sufficient to consider an asymptotic form valid for $N \gg 1$. One can take the following ansatz

$$\Lambda(N) = \sum_{k=0}^{N_{\max}} a_k N^k, \tag{B4}$$

where N_{\max} stands for a cut-off, while a_k are real coefficients chosen so that the function (B4) is increasing. Notice that the original form (B2) is a C^∞ function and, in particular,

$$\left| \lim_{\Lambda \rightarrow \infty} \frac{d^k}{d\Lambda^k} L_\lambda[\Lambda] \right| < \infty \tag{B5}$$

for $k \in \mathbb{N}$. In what follows, if the regularized function $L_\lambda[\Lambda(N)]$ has a well-defined limit, we require it goes smoothly in the limit, i.e.

$$\left| \lim_{N \rightarrow \infty} \frac{d^k}{dN^k} L_\lambda[\Lambda(N)] \right| < \infty. \tag{B6}$$

Recalling the form (B2) of the norm, one concludes that the only way of how equation (B6) can be satisfied is restricting to a class of linear functions $\Lambda(N) = qN + l$, where $q > 0$. Otherwise, the derivatives will become divergent as $N \rightarrow \infty$. Coefficients a, b can be fixed requiring the function $L_\lambda(N)$ has a well-defined limit $N \rightarrow \infty$. It turns out, this will be the case for

$$\varphi = \frac{p}{q}\pi, \quad p, q \in \mathbb{N}, \tag{B7}$$

$$\Lambda_{q,l}(N) = qN + l. \tag{B8}$$

We now look closer the conditions (B7) and (B8). Before doing so, however, we have to make some comments. The first observation is that in equation (B7) we should additionally assume that $p \neq 0$ and $p/q \neq 1/2$. This is to eliminate the possibility $\lambda \in \mathbb{R}$, making the right-hand side of equation (B2) divergent. Without losing of generality, one can also assume that

$$p < 2q, \tag{B9}$$

as a consequence of the fact that $\varphi < 2\pi$ (for complex numbers $re^{i\varphi}$ one has $\varphi \in [0, 2\pi)$). In equation (B8) l stands for additional regularization parameter chosen so that

$$l = 0, \dots, q - 1. \tag{B10}$$

We accept this temporarily without explanation; this will be clear in a moment. Note that for a given pair (p, q) we have maximally $q - 1$ cut-offs $\Lambda_{q,l}(N)$. The sense of introducing the forms (B7) and (B8) consists in the fact the norm (B2) behaves well in the limit $N \rightarrow \infty$ due to the following periodicity:

$$\begin{aligned} L_\lambda[\Lambda_{q,l}(N)] &= 1 + 2 \sin^{-1} \left(\frac{p}{q} \pi \right) \sin \left(\frac{l p}{q} \pi + N p \pi \right) \cos \left((l + 1) \frac{p}{q} \pi + N p \pi \right) \\ &= 1 + 2 \sin^{-1} \left(\frac{p}{q} \pi \right) \sin \left(\frac{l p}{q} \pi \right) \cos \left((l + 1) \frac{p}{q} \pi \right). \end{aligned} \tag{B11}$$

By choosing the angle φ to be rational number $p\pi/q$ and taking the cut-off $\Lambda_{q,k}(N) = qN + l$ we simply guaranty $L_\lambda[\Lambda_{q,l}(N)]$ does not depend on N and, in consequence, behaves well in the limit $N \rightarrow \infty$. It is now clear what is the origin of the constraint (B10): any $\Lambda_{q,l}$ corresponding to $l \geq q$ is equivalent to $\Lambda_{q,l}$ corresponding to $l \leq q - 1$. For instance, according to equation (B10) for $q = 3$, one gets only three functions: $\Lambda_{3,0} = 3N$, $\Lambda_{3,1} = 3N + 1$, $\Lambda_{3,2} = 3N + 2$. Now, suppose we ignore the upper bound, allowing more of them. For the fourth function one gets the cut-off $\Lambda_{3,3} = 3N + 3 = 3(N + 1) = 3\tilde{N}$, where $\tilde{N} := N + 1$. However, this is trivially equivalent to $\Lambda_{3,0}$.

Having said that, we rewrite the initial form of eigenvectors (B1) as

$$|e_\lambda^{(l)}\rangle = \lim_{N \rightarrow \infty} E_\lambda^{\Lambda_{q,l}(N)} |\phi\rangle, \tag{B12}$$

where $\lambda = re^{ip\pi/q}$. It is easily to check that the vectors (22) are the special case of (B12), corresponding to $r = 1$, $(p, q) = (1, 2)$ and $l = 0, 1$.

So far we restricted to normalization conditions and said nothing about the product of two different vectors. Using the rule (26), one finds

$$\begin{aligned} \langle e_{\lambda_i}^{(l_i)} | e_{\lambda_j}^{(l_j)} \rangle &= 1 + \sin^{-1} \left(\frac{\varphi_i + \varphi_j}{2} \right) \lim_{N_i \rightarrow \infty} \sin \left(\frac{\varphi_i + \varphi_j}{2} \Lambda_i(N_i) \right) \\ &\quad \times \cos \left(\frac{\varphi_i + \varphi_j}{2} (\Lambda_i(N_i) + 1) \right) + \sin^{-1} \left(\frac{\varphi_i + \varphi_j}{2} \right) \\ &\quad \times \lim_{N_j \rightarrow \infty} \sin \left(\frac{\varphi_i + \varphi_j}{2} \Lambda_j(N_j) \right) \cos \left(\frac{\varphi_i + \varphi_j}{2} (\Lambda_j(N_j) + 1) \right), \end{aligned} \tag{B13}$$

where $\varphi_{ij} = p_{ij}\pi/q_{ij}$, $\Lambda_{ij} = q_{ij}N_{ij} + l_{ij}$. The product (B13) was obtained under the assumption that $r_i = r_j$ (here $\lambda_i = r_i e^{ip_i\pi/q_i}$). If $r_i \neq r_j$ then $|\langle e_{\lambda_i}^{(l_i)} | e_{\lambda_j}^{(l_j)} \rangle| = \infty$. This can be easily read off from the original form of the product (15). In what follows, eigenvectors labeled by eigenvalues of different r cannot belong to the same (extended) Hilbert space because their product diverges. As we shall see in a moment, there are even stronger constraints on the eigenvalues.

Before we go any further, it is worth mentioning that eigenvectors from different eigenvalues λ_i, λ_j are orthogonal unless the eigenvalues are mutually conjugated, i.e. $\lambda_i = \lambda_j^*$. Indeed,

$$\langle e_{\lambda_j} | e_{\lambda_i} \rangle = \lambda_i \langle e_{\lambda_j} | e_{\lambda_i} \rangle, \tag{B14}$$

$$\langle e_{\lambda_j} | e_{\lambda_i} \rangle = (\langle e_{\lambda_i} | e_{\lambda_j}^\dagger \rangle)^* = \lambda_j^* \langle e_{\lambda_j} | e_{\lambda_i} \rangle. \tag{B15}$$

Combining equations (B14) and (B15), one concludes that either $\lambda_i = \lambda_j^*$ or $\langle e_{\lambda_j} | e_{\lambda_i} \rangle = 0$. This is a direct analogue of the fact that for a Hermitian operator eigenvectors from different eigenvalues are orthogonal. Here we should keep in mind that allowing general complex eigenvectors we *do not* expect that e is Hermitian in the corresponding space. Still, there is a mentioned consistency constraint.

Looking closer at the product (B13) one concludes that i) the product is, in general, not well-defined in the limit $N_m \rightarrow \infty$, ii) vectors corresponding to two different eigenvalues $\lambda_i \neq \lambda_j^*$ are not orthogonal even if the product is well-defined. The former can be circumvented modifying equation (B7) by making p_m two times bigger, however, the latter would be still problematic. The same holds for eigenvectors corresponding to $\lambda_i = \lambda_j^*$: the product is ill-defined. For instance, for $\lambda_i = r e^{i\varphi}$ ($r > 0$) and $\lambda_j = e^{-i\varphi}$ the product diverges. Hence, the only way to construct the space of complex normalized eigenvectors is restricting to a fixed eigenvalue. This leads to a superselection rule, forbidding superpositions of states corresponding to different complex eigenvalues. Note that this rule is stronger than the corresponding rule for the restricted Hilbert space (20). In the latter case we allowed superpositions of states corresponding to the same absolute value $|\lambda|$. Now we require the eigenvalues must be equal. In consequence, all the superselection sectors are built out of degenerated states.

Letting in (B13) $\lambda_i = \lambda_j = \lambda$, one finds

$$\langle e_{\lambda}^{(l_i)} | e_{\lambda}^{(l_j)} \rangle = \sin^{-1} \left(\frac{p\pi}{q} \right) \cos \left(\frac{(l_i - l_j)p\pi}{q} \right) \sin \left(\frac{(1 + l_i + l_j)p\pi}{q} \right). \tag{B16}$$

Now the product (B16) is well-defined and, in particular, does not depend on N_i (which eventually goes to infinity). Notice that eigenvectors from a fixed eigenvalue λ_1 do not belong to the superselection sector corresponding to the eigenvalue λ_2 , etc. The last is in parallel analogy to construction of regularized eigenvectors corresponding to real eigenvalues. The main difference is that each space corresponds to a single eigenvalue.

Constructing the spaces of regularized eigenvectors we need to fix the eigenvalue $\lambda = r e^{ip\pi/q}$ first, and then find the corresponding eigenvectors. Since they are degenerated, any linear combination corresponding to the same eigenvalue $\lambda = r e^{ip\pi/q}$ is also a well-defined eigenvector (with the same eigenvalue). The number of positive, negative and zero norm eigenvectors is given by the signature of the quadratic form:

$$Q_{kn} = \langle e_{\lambda}^{(k)} | e_{\lambda}^{(n)} \rangle. \tag{B17}$$

It turns out, the form above has a signature $\{-1, 1, \vec{0}_{q-2}\}$, where $\vec{0}_{q-2}$ stands for a $q - 2$ dimensional vector composed of zeros, $\vec{0}_{q-2} = \{0_{(1)}, \dots, 0_{(q-2)}\}$. If $q \leq 2$, then there is no zero eigenvectors. For instance, for $\lambda = e^{i\pi/4}$, one identifies $p = 1$, $q = 4$. Recalling equation (B8) and (B10), one finds the following cut-offs

$$\Lambda_{4,0} = 4N, \quad \Lambda_{4,1} = 4N + 1, \quad \Lambda_{4,2} = 4N + 2, \quad \Lambda_{4,3} = 4N + 3. \quad (\text{B18})$$

The corresponding regularized eigenvectors are given by equation (B12). This leads to

$$(Q_{kn}) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix}. \quad (\text{B19})$$

The matrix (B19) has the following eigenvalues $\{-2, 2, 0, 0\}$. Therefore, linear combinations of degenerated eigenvectors $|e_{\lambda}^{(l=0, \dots, 3)}\rangle$, where $\lambda = e^{i\pi/4}$ result in one positive, one negative and two zero norm vectors (the signature is $(-1, 1, 0, 0)$). In general, depending on the value of complex λ , the number of zero norm eigenvectors could be bigger or smaller, however, we get always two normalizable eigenvectors.

Because different spaces of regularized eigenvectors differs only by the number of zero norm vectors, they are all equivalent (in the sense that zero norm eigenvectors do not contribute to the scalar product). Again, this is in parallel analogy to the case of real eigenvalues. As we recall, in this case the spaces are labeled by positive parameter $\lambda_0 > 0$, determining the pair of opposite eigenvalues $\pm\lambda_0$. In the complex case the states are degenerated and both positive and negative norm eigenvectors are present.

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