IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **53** (2020) 185202 (16pp) https://doi.org/10.1088/1751-8121/ab7e53

Odd supersymmetrization of elliptic *R*-matrices*

A Levin [1,2], M Olshanetsky [2,3,4] and A Zotov [2,4,5,6]

[1] National Research University Higher School of Economics, Russian Federation, Usacheva str. 6, Moscow, 119048, Russia
[2] Institute for Theoretical and Experimental Physics of NRC 'Kurchatov Institute', B. Cheremushkinskaya str. 25, Moscow, 117218, Russia
[3] Institute for Information Transmission Problems RAS (Kharkevich Institute), Bolshoy Karetny per. 19, Moscow, 127994, Russia
[4] Steklov Mathematical Institute of Russian Academy of Sciences, Gubkina str. 8, Moscow, 119991, Russia
[5] Moscow Institute of Physics and Technology, Inststitutskii per. 9, Dolgoprudny, Moscow region, 141700, Russia

E-mail: alevin2@hse.ru, olshanet@itep.ru and zotov@mi-ras.ru

Received 25 October 2019, revised 22 February 2020
Accepted for publication 10 March 2020
Published 10 April 2020

Abstract

We study a general ansatz for an odd supersymmetric version of the Kronecker elliptic function, which satisfies the genus one Fay identity. The obtained result is used for construction of the odd supersymmetric analogue for the classical and quantum elliptic *R*-matrices. They are shown to satisfy the classical Yang–Baxter equation and the associative Yang–Baxter equation. The quantum Yang–Baxter equation is discussed as well. It acquires additional term in the case of supersymmetric *R*-matrices.

Keywords: elliptic R-matrix, Yang–Baxter equations, supersymmetric elliptic curve

1. Introduction

Kronecker function. In this paper we deal with the Kronecker elliptic function [1] defined on the elliptic curve $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ with the moduli τ. It is fixed by the residue

$$\operatorname*{Res}_{z=0} \phi(\hbar, z) = 1 \tag{1.1}$$

*To the 80th anniversary of Andrei Slavnov.
[6]Author to whom any correspondence should be addressed.

and the quasi-periodic boundary conditions on the lattice $\mathbb{Z}\oplus\tau\mathbb{Z}$:

$$\phi(\hbar,z+1)=\phi(\hbar,z),\quad \phi(\hbar,z+\tau)=\mathrm{e}^{-2\pi\imath\hbar}\phi(\hbar,z). \tag{1.2}$$

Explicit expression is given in terms of the Riemann theta-function

$$\begin{gathered}\phi(\hbar,z;\tau)\equiv\phi(\hbar,z)=\frac{\vartheta'(0)\vartheta(\hbar+z)}{\vartheta(\hbar)\vartheta(z)},\\ \vartheta(z;\tau)\equiv\vartheta(z)=\sum_{k\in\mathbb{Z}}\exp\left(\pi\imath\tau\left(k+\frac{1}{2}\right)^2+2\pi\imath\left(z+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)\right),\quad \vartheta(-z)=-\vartheta(z).\end{gathered} \tag{1.3}$$

The key properties of the function (1.3) are as follows:

- The Kronecker function satisfies the genus one Fay trisecant identity [2]:

$$\phi(\hbar_1,z_{12})\phi(\hbar_2,z_{23})=\phi(\hbar_2,z_{13})\phi(\hbar_1-\hbar_2,z_{12})+\phi(\hbar_2-\hbar_1,z_{23})\phi(\hbar_1,z_{13}), \tag{1.4}$$

  where $z_{ij}=z_i-z_j$;
- The Kronecker function satisfies the heat equation:

$$2\pi\imath\partial_\tau\phi(\hbar,z;\tau)=\partial_z\partial_\hbar\phi(\hbar,z;\tau). \tag{1.5}$$

Using the skew-symmetry property of the Kronecker function

$$\phi(\hbar,z_{12})=-\phi(-\hbar,z_{21}) \tag{1.6}$$

we can rewrite (1.4) in the form

$$\phi(\hbar_1,z_{12})\phi(\hbar_2,z_{23})+\phi(-\hbar_2,z_{31})\phi(\hbar_1-\hbar_2,z_{12})+\phi(\hbar_2-\hbar_1,z_{23})\phi(-\hbar_1,z_{31})=0. \tag{1.7}$$

Yang–Baxter equations. Relations (1.4) and (1.5) play a crucial role in elliptic integrable systems and monodromy preserving equations since they underlie the Lax representations with spectral parameter, classical and quantum R-matrix structures, Sklyanin algebras and the Knizhnik–Zamolodchikov–Bernard equations [3–6]. From algebraic viewpoint the Fay identity (1.7) is the scalar version of the Fomin–Kirillov algebra [7] defined by the associative Yang–Baxter equation

$$R_{12}^{\hbar_1}(z_{12})R_{23}^{\hbar_2}(z_{23})+R_{31}^{-\hbar_2}(z_{31})R_{12}^{\hbar_1-\hbar_2}(z_{12})+R_{23}^{\hbar_2-\hbar_1}(z_{23})R_{31}^{-\hbar_1}(z_{31})=0, \tag{1.8}$$

where notations of the quantum inverse scattering method [6] are used, so that $R_{ab}^{\hbar}(z_{ab})$ (R-matrix) is a matrix valued function of the spectral parameter z_a-z_b and the Planck constant $\hbar$. Put it differently, R_{ab} is an operator in $\mathrm{Mat}(N,\mathbb{C})^{\otimes 3}$ acting non-trivially in the ath and bth tensor components. In particular, equation (1.8) is fulfilled by the properly normalized

Baxter–Belavin elliptic R-matrix [8], which is then treated as a matrix generalization of the Kronecker function (1.3). Applications of (1.8) can be found in [9, 10].

A skew-symmetric and unitary solution of (1.8) satisfies also the quantum Yang–Baxter equation:

$$R^{\hbar}_{12}(z_{12})R^{\hbar}_{13}(z_{13})R^{\hbar}_{23}(z_{23}) = R^{\hbar}_{23}(z_{23})R^{\hbar}_{13}(z_{13})R^{\hbar}_{12}(z_{12}). \tag{1.9}$$

In the classical limit $\hbar \to 0$ it provides the classical Yang–Baxter equation for the classical r-matrix:

$$[r_{12}(z_{12}), r_{13}(z_{13})] + [r_{12}(z_{12}), r_{23}(z_{23})] + [r_{13}(z_{13}), r_{23}(z_{23})] = 0. \tag{1.10}$$

Supersymmetrization. Following [11, 12] (see also [13]) we consider the supersymmetric elliptic curve, which is defined as a quotient of superspace $\mathbb{C}^{1|1}$ (endowed with coordinates z, ζ) by (super)translations

$$\begin{cases} z \to z+1, \\ \zeta \to \zeta, \end{cases} \qquad \begin{cases} z \to z+\tau+2\pi\imath\zeta\omega, \\ \zeta \to \zeta + 2\pi\imath\omega, \end{cases} \tag{1.11}$$

where ζ is a superpartner to the coordinate z, and ω is a superpartner to the moduli of elliptic curve τ. The supersymmetric elliptic curve is equipped with the covariant derivative $D_\zeta = \partial_\zeta + \zeta\partial_z$, $D_\zeta^2 = \partial_z$. In what follows we also use the Grassmann variables μ_i as the superpartners to the parameters $\hbar_i$. Finally, we have the following table of even and odd variables:

$$\begin{array}{rccc} \text{even variables:} & z_k & \tau & \hbar_i \\ \text{odd variables:} & \zeta_k & \omega & \mu_i \end{array} \tag{1.12}$$

The variables ζ_k, μ_i, ω are Grassmann, i.e.

$$\zeta_k^2 = \mu_i^2 = \omega^2 = 0, \quad [\zeta_k, \zeta_l]_+ = [\zeta_k, \mu_i]_+ = [\mu_i, \mu_j]_+ = [\zeta_k, \omega]_+ = [\omega, \mu_i]_+ = 0. \tag{1.13}$$

In our recent paper [14] we proposed an odd supersymmetric version of the Kronecker function (1.3). It is of the following form:

$$\begin{aligned} \boldsymbol{\Phi}(\hbar, z_1, z_2; \tau|\, \mu, \zeta_1, \zeta_2; \omega) \equiv \boldsymbol{\Phi}^{\hbar|\,\mu}(z_1, z_2|\, \zeta_1, \zeta_2) &= (\zeta_1 - \zeta_2)\phi(\hbar, z_{12}) \\ &+ \omega\partial_1\phi(\hbar, z_{12}) + 2\pi\imath\zeta_1\zeta_2\omega\partial_\tau\phi(\hbar, z_{12}) + \zeta_1\zeta_2\mu\partial_1\phi(\hbar, z_{12}) \\ &+ \frac{1}{2}(\zeta_1 + \zeta_2)\mu\omega\partial_1^2\phi(\hbar, z_{12}), \end{aligned} \tag{1.14}$$

where $\partial_1\phi(x, y) = \partial_x\phi(x, y)$, $\partial_2\phi(x, y) = \partial_y\phi(x, y)$. And the truncated version is given by

$$\boldsymbol{\Phi}^{\hbar|\,0}(z_1, z_2|\, \zeta_1, \zeta_2) = (\zeta_1 - \zeta_2)\phi(\hbar, z_{12}) + \omega\partial_1\phi(\hbar, z_{12}) + 2\pi\imath\zeta_1\zeta_2\omega\partial_\tau\phi(\hbar, z_{12}). \tag{1.15}$$

The latter is (1.14) without two last terms. It was shown in [14] that both functions (1.14), (1.15) satisfy

- The Fay identity (1.7) written as

$$\mathbf{\Phi}_{12}^{\hbar_1|\,\mu_1}\mathbf{\Phi}_{23}^{\hbar_2|\,\mu_2}+\mathbf{\Phi}_{31}^{-\hbar_2|\,-\mu_2}\mathbf{\Phi}_{12}^{\hbar_1-\hbar_2|\,\mu_1-\mu_2}+\mathbf{\Phi}_{23}^{\hbar_2-\hbar_1|\,\mu_2-\mu_1}\mathbf{\Phi}_{31}^{-\hbar_1|\,-\mu_1}=0, \tag{1.16}$$

where $\mathbf{\Phi}_{ab}^{\hbar|\,\mu}=\mathbf{\Phi}^{\hbar|\,\mu}(z_a,z_b|\,\zeta_a,\zeta_b)$.

- The supersymmetric version of the heat equation

$$(\partial_\omega+2\pi\imath(\zeta_1+\zeta_2)\partial_\tau)\,\mathbf{\Phi}_{12}^{\hbar|\,\mu}=\left(\partial_{\zeta_1}+\zeta_1\partial_{z_1}-\frac{1}{2}\mu\partial_\hbar\right)\partial_\hbar\mathbf{\Phi}_{12}^{\hbar|\,\mu}. \tag{1.17}$$

For the truncated function (1.15) the last term is absent in the rhs of (1.17).

The identity (1.16) was used to construct the odd supersymmetric version of the quantum Baxter–Belavin R-matrix in the fundamental representation of GL_N group. The R-matrix was shown to satisfy the associative Yang–Baxter (1.8) written as

$$\mathbf{R}_{12}^{\hbar_1|\,\mu_1}\mathbf{R}_{23}^{\hbar_2|\,\mu_2}+\mathbf{R}_{31}^{-\hbar_2|\,-\mu_2}\mathbf{R}_{12}^{\hbar_1-\hbar_2|\,\mu_1-\mu_2}+\mathbf{R}_{23}^{\hbar_2-\hbar_1|\,\mu_2-\mu_1}\mathbf{R}_{31}^{-\hbar_1|\,-\mu_1}=0, \tag{1.18}$$

where $\mathbf{R}_{ab}^{\hbar|\,\mu}=\mathbf{R}_{ab}^{\hbar|\,\mu}(z_a,z_b|\,\zeta_a,\zeta_b)$ is defined through (1.14). At the same time the supersymmetric analogue of the classical r-matrix satisfies the classical (super) Yang–Baxter equation:

$$[\mathbf{r}_{12},\mathbf{r}_{13}]_++[\mathbf{r}_{12},\mathbf{r}_{23}]_++[\mathbf{r}_{13},\mathbf{r}_{23}]_+=0,\quad \mathbf{r}_{ab}=\mathbf{r}_{ab}(z_a,z_b|\,\zeta_a,\zeta_b). \tag{1.19}$$

The latter equation was introduced and studied in [9, 15].

Purpose of the paper. In [14] the function (1.14) was derived from the two conditions. The first one is the residue condition

$$\operatorname*{Res}_{z_1=z_2}\mathbf{\Phi}^{\hbar|\,\mu}(z_1,z_2|\,\zeta_1,\zeta_2)=\zeta_1-\zeta_2. \tag{1.20}$$

The second one is the quasi-periodic boundary condition:

$$\begin{gathered}\mathbf{\Phi}^{\hbar|\,\mu}(z_1+1,z_2|\,\zeta_1,\zeta_2)=\mathbf{\Phi}^{\hbar|\,\mu}(z_1,z_2+1|\,\zeta_1,\zeta_2)=\mathbf{\Phi}^{\hbar|\,\mu}(z_1,z_2|\,\zeta_1,\zeta_2),\\ \mathbf{\Phi}^{\hbar|\,\mu}(z_1+\tau+2\pi\imath\zeta_1\omega,z_2|\zeta_1+2\pi\imath\omega,\zeta_2)=\exp\left(-2\pi\imath(\hbar+\mu\zeta_1+\pi\imath\mu\omega)\right)\mathbf{\Phi}^{\hbar|\mu}(z_1,z_2|\,\zeta_1,\zeta_2).\\ \mathbf{\Phi}^{\hbar|\,\mu}(z_1,z_2+\tau+2\pi\imath\zeta_2\omega|\,\zeta_1,\zeta_2+2\pi\imath\omega)=\exp\left(2\pi\imath(\hbar+\mu\zeta_2-\pi\imath\mu\omega)\right)\mathbf{\Phi}^{\hbar|\,\mu}(z_1,z_2|\,\zeta_1,\zeta_2).\end{gathered} \tag{1.21}$$

It is similar to derivation of (1.3) from (1.1) and (1.2).

In this paper we consider a generalization of (1.14), where every term is multiplied by an arbitrary $\mathbb{C}$-valued coefficient $A_1,\ldots,A_5$ (the set of the coefficients is denoted by A):

$$\begin{aligned}\mathbf{\Phi}^{\hbar|\,\mu}(z_1,z_2|\,\zeta_1,\zeta_2|\,A)\;&=A_1(\zeta_1-\zeta_2)\phi(\hbar,z_{12})+A_2\omega\partial_1\phi(\hbar,z_{12})\\&\quad+A_3\zeta_1\zeta_2\omega\partial_\tau\phi(\hbar,z_{12})+A_4\zeta_1\zeta_2\mu\partial_1\phi(\hbar,z_{12})\\&\quad+\frac{A_5}{2}(\zeta_1+\zeta_2)\mu\omega\partial_1^2\phi(\hbar,z_{12}).\end{aligned} \tag{1.22}$$

The function (1.14) is reproduced when $A_1 = A_2 = A_4 = A_5 = 1, A_3 = 2\pi\imath$ and the truncated function (1.15) appears for $A_1 = A_2 = 1$, $A_3 = 2\pi\imath$ and $A_4 = A_5 = 0$. In what follows the conditions (1.20) and (1.21) are not imposed. Otherwise we are left with (1.14) or (1.15) only.

The paper is organized as follows. In next section we study expression (1.22) as an ansatz for the Fay identity (1.16) and supersymmetric version of the heat equation (1.17). Equations (1.16) and (1.17) provide conditions for the coefficients. In particular, we will show that (1.16) holds true if $A_1A_5 = A_2A_4$. As a by product we include two $\mathbb{C}$-valued parameters k, κ into the heat equation (1.17):

$$(\kappa\partial_\omega + 2\pi\imath(\zeta_1 + \zeta_2)\partial_\tau)\,\mathbf{\Phi}^{\hbar|\,\mu}(A) = \left(\partial_{\zeta_1} + \zeta_1\partial_{z_1} - \frac{k}{2}\mu\partial_\hbar\right)\partial_\hbar\mathbf{\Phi}^{\hbar|\,\mu}(A). \tag{1.23}$$

In section 3 we describe construction of elliptic R-matrices based on (1.22). This leads to additional constraint for the coefficients A_k. Besides the classical and associative Yang–Baxter (1.18) and (1.19) we also discuss the quantum Yang–Baxter (1.9). It acquires additional term in the case of supersymmetric R-matrix. A summary of results is given in conclusion.

2. Generalized ansatz for the Kronecker function

Consider the function (1.22) depending on five arbitrary coefficients. As mentioned earlier we do not impose the quasi-periodic boundary condition (1.21). This is due to the following statement, which is verified by direct calculation:

Proposition 2.1. *The function* (1.22) *satisfies the boundary conditions* (1.21) *if*

$$A_1 = A_2 = A_4 = A_5, \quad A_3 = 2\pi\imath A_1. \tag{2.1}$$

For the case when the variable μ is absent, the conditions (1.21) *provide the truncated function* (1.15).

The purpose of the section is to find out if the generalized function (1.22) satisfies the Fay identity (1.16) and the supersymmetric heat equation (1.23).

2.1. Fay identity

The main result of the paragraph is formulated as follows:

Proposition 2.2. *The Fay identity* (1.16) *holds true for the function* (1.22) *if*

$$A_1A_5 = A_2A_4. \tag{2.2}$$

Proof. Verification is straightforward. Using notations for derivatives from (1.14) and the (anti)commutation relations (1.19) let us write down the first term from (1.16):

$$
\begin{aligned}
&\mathbf{\Phi}_{12}^{\hbar_1|\,\mu_1}(z_1,z_2|\,\zeta_1,\zeta_2|\,A)\mathbf{\Phi}_{23}^{\hbar_2|\,\mu_2}(z_2,z_3|\,\zeta_2,\zeta_3|\,A)\\
&\quad= A_1^2(\zeta_1-\zeta_2)(\zeta_2-\zeta_3)\phi(\hbar_1,z_{12})\phi(\hbar_2,z_{23})+A_1A_2(\zeta_1-\zeta_2)\omega\phi(\hbar_1,z_{12})\partial_1\phi(\hbar_2,z_{23})\\
&\qquad+A_1A_3\zeta_1\zeta_2\zeta_3\omega\phi(\hbar_1,z_{12})\partial_\tau\phi(\hbar_2,z_{23})+\frac{1}{2}A_1A_5(\zeta_1-\zeta_2)\\
&\qquad\times(\zeta_2+\zeta_3)\mu_2\omega\phi(\hbar_1,z_{12})\partial_1^2\phi(\hbar_2,z_{23})\\
&\qquad+A_1A_4\zeta_1\zeta_2\zeta_3\mu_2\phi(\hbar_1,z_{12})\partial_1\phi(\hbar_2,z_{23})+A_1A_2\omega(\zeta_2-\zeta_3)\partial_1\phi(\hbar_1,z_{12})\phi(\hbar_2,z_{23})\\
&\qquad+A_2A_4\omega\zeta_2\zeta_3\mu_2\partial_1\phi(\hbar_1,z_{12})\partial_1\phi(\hbar_2,z_{23})+A_1A_3\zeta_1\zeta_2\zeta_3\omega\partial_\tau\phi(\hbar_1,z_{12})\phi(\hbar_2,z_{23})\\
&\qquad+A_1A_4\zeta_1\zeta_2\zeta_3\mu_1\partial_1\phi(\hbar_1,z_{12})\phi(\hbar_2,z_{23})+\frac{1}{2}A_4A_5\zeta_1\zeta_2\mu_1\zeta_3\mu_2\omega\partial_1\phi(\hbar_1,z_{12})\partial_1^2\phi(\hbar_2,z_{23})\\
&\qquad+A_2A_4\zeta_1\zeta_2\mu_1\omega\partial_1\phi(\hbar_1,z_{12})\partial_1\phi(\hbar_2,z_{23})+\frac{1}{2}A_4A_5\zeta_1\zeta_2\zeta_3\mu_1\omega\mu_2\partial_1^2\phi(\hbar_1,z_{12})\partial_1\phi(\hbar_2,z_{23})\\
&\qquad+\frac{1}{2}A_1A_5(\zeta_1+\zeta_2)(\zeta_2-\zeta_3)\mu_1\omega\partial_1^2\phi(\hbar_1,z_{12})\phi(\hbar_2,z_{23}),
\end{aligned}
\tag{2.3}
$$

and similarly for the second and the third terms. Summing them up we should then verify if the coefficients behind any Grassmann monomial equals zero. For example, the coefficient behind $\zeta_1\zeta_2$, $\zeta_2\zeta_3$ and $\zeta_3\zeta_1$ is the lhs of the ordinary Fay identity (1.4) multiplied by A_1^2. It is equal to zero and do not provide any constraints for the coefficients A_k.

The rest of the coefficients behind Grassmann monomials vanish due to identities obtained as some derivatives of (1.4). For example, the coefficients behind $\zeta_3\omega$ and $\zeta_1\zeta_2\zeta_3\mu_1$ vanish due to identity obtained as derivative of (1.4) with respect to $\hbar_1$, and the coefficient behind $\zeta_1\zeta_2\zeta_3\mu_1\mu_2\omega$ vanishes due to the one appearing from (1.4) by the action of $\partial_{\hbar_1}\partial_{\hbar_2}(\partial_{\hbar_1}+\partial_{\hbar_2})$. All of them do not impose any constrains for A_k except the monomials of type $\zeta^2\mu\omega$. They contain the terms proportional to A_2A_4 and A_1A_5. For each of such terms there exists an identity in the form of some derivative of (1.4), which yields the condition (2.2). Namely, for $\zeta_1\zeta_2\omega\mu_1$ one should apply the identity $(\partial_{\hbar_1}^2+2\partial_{\hbar_1}\partial_{\hbar_2})[(1.4)]$, for $\zeta_2\zeta_3\omega\mu_1$ and $\zeta_3\zeta_1\omega\mu_1$—$\partial_{\hbar_1}^2[(1.4)]$, for $\zeta_1\zeta_2\omega\mu_2$ and $\zeta_3\zeta_1\omega\mu_2$—$\partial_{\hbar_2}^2[(1.4)]$. Finally, for the coefficient behind $\zeta_2\zeta_3\omega\mu_2$ one should use $(\partial_{\hbar_2}^2+2\partial_{\hbar_1}\partial_{\hbar_2})[(1.4)]$. ■

2.2. Supersymmetric heat equation

Here we prove the following statement:

Proposition 2.3. *The heat equation* (1.23) *holds true for the function* (1.22) *if*

$$
\kappa A_2=A_1,\quad \kappa A_3=2\pi\imath A_1,\quad A_4=kA_1,\quad \kappa A_5=kA_1. \tag{2.4}
$$

Proof. Let us write down all terms entering the heat equation (1.23):

$$
\kappa\partial_\omega\mathbf{\Phi}_{12}^{\hbar|\,\mu}(A)=\kappa A_2\partial_1\phi(\hbar,z_{12})+\kappa A_2\zeta_1\zeta_2\partial_\tau\phi(\hbar,z_{12})+\frac{\kappa A_5}{2}(\zeta_1+\zeta_2)\mu\partial_1^2\phi(\hbar,z_{12}), \tag{2.5}
$$

$$
2\pi\imath(\zeta_1+\zeta_2)\partial_\tau\mathbf{\Phi}_{12}^{\hbar|\,\mu}(A)=-4\pi\imath A_1\zeta_1\zeta_2\partial_\tau\phi(\hbar,z_{12})+2\pi\imath A_2(\zeta_1+\zeta_2)\omega\partial_\tau\partial_1\phi(\hbar,z_{12}),
\tag{2.6}
$$

$$\partial_{\zeta_1}\partial_\hbar \mathbf{\Phi}_{12}^{\hbar|\,\mu}(A) = A_1\partial_1\phi(\hbar,z_{12}) + A_3\zeta_2\omega\partial_\tau\partial_1\phi(\hbar,z_{12}) + A_4\zeta_2\mu\partial_1^2\phi(\hbar,z_{12}) + \frac{1}{2}A_5\mu\omega\partial_1^3\phi(\hbar,z_{12}), \tag{2.7}$$

$$\zeta_1\partial_{z_1}\partial_\hbar \mathbf{\Phi}_{12}^{\hbar|\,\mu}(A) = A_2\zeta_1\omega\partial_1^2\partial_2\phi(\hbar,z_{12}) - A_1\zeta_1\zeta_2\partial_1\partial_2\phi(\hbar,z_{12}) + \frac{1}{2}A_5\zeta_1\zeta_2\mu\omega\partial_1^3\partial_2\phi(\hbar,z_{12}), \tag{2.8}$$

$$\mu\partial_\hbar^2 \mathbf{\Phi}_{12}^{\hbar|\,\mu}(A) = A_1\mu(\zeta_1-\zeta_2)\partial_1^2\phi(\hbar,z_{12}) + A_2\mu\omega\partial_1^3\phi(\hbar,z_{12}) + A_3\zeta_1\zeta_2\mu\omega\partial_\tau\partial_1^2\phi(\hbar,z_{12}). \tag{2.9}$$

Plugging these expressions into (1.23) we again should verify if the relation holds true for every Grassmann monomial including the trivial one. The vanishing of the coefficient behind the trivial monomial provides relation $A_1 = \kappa A_2$, and for those behind $\zeta_1\mu$ and $\zeta_2\mu$—we get the constraint $A_4 = kA_1 = \kappa A_5$. The rest of the coefficients requires also to use the ordinary heat equation (1.5). ∎

This result contains the previously obtained statements from [14] as particular cases. For example, when $k=0$ and $\kappa=1$ the last term in the rhs of equation (1.23) vanishes, and the function (1.22) turns into (1.15). In the case $k=\kappa=1$ the conditions (2.4) are solved as given in (2.1), and we come back to the function (1.14).

Notice also that the relation (2.2) coming from the Fay identity is valid on the constraints (2.4). So that (2.4) is a sufficient condition for both—the Fay identity (1.16) and the heat equation (1.22).

3. *R*-Matrices and Yang–Baxter equations

In this section we derive the Yang–Baxter equations and find out if they provide restrictions on possible values of the coefficients A_k.

3.1. *Baxter–Belavin's R-matrix*

Let us briefly recall the widely known construction of the elliptic Baxter–Belavin R-matrix [3] in the fundamental representation of $\mathrm{GL}(N,\mathbb{C})$ Lie group. We deal with a special basis in $\mathrm{Mat}(N,\mathbb{C})$ known as the sine-algebra basis. It consists of N^2 matrices

$$T_a = T_{a_1a_2} = \exp\left(\frac{\pi\imath}{N}a_1a_2\right)Q^{a_1}\Lambda^{a_2}, \quad a=(a_1,a_2)\in\mathbb{Z}_N\times\mathbb{Z}_N, \tag{3.1}$$

defined in terms of

$$Q_{kl} = \delta_{kl}\exp\left(\frac{2\pi\imath}{N}k\right), \quad \Lambda_{kl} = \delta_{k-l+1=0 \bmod N}, \quad Q^N = \Lambda^N = 1_N. \tag{3.2}$$

The latter matrices Q,Λ can be regarded as the finite-dimensional representation of the Heisenberg group since

$$\exp\left(\frac{2\pi\imath}{N}a_1a_2\right)Q^{a_1}\Lambda^{a_2} = \Lambda^{a_2}Q^{a_1}, \quad a_1,a_2\in\mathbb{Z}_+. \tag{3.3}$$

The product of pair of basis matrices (3.1) is easily computed from (3.3):

$$T_\alpha T_\beta = \kappa_{\alpha,\beta} T_{\alpha+\beta}, \quad \kappa_{\alpha,\beta} = \exp\left(\frac{\pi\imath}{N}(\beta_1\alpha_2 - \beta_2\alpha_1)\right), \tag{3.4}$$

where $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$.

In accordance with the numeration of basis matrices (3.1) let us define the set of N^2 functions

$$\varphi_a(\hbar + \Omega_a, z) = \exp\left(2\pi\imath \frac{a_2}{N} z\right) \phi(\hbar + \Omega_a, z), \quad \Omega_a = \frac{a_1 + a_2\tau}{N}, \tag{3.5}$$

where $a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$. Then the quantum elliptic R-matrix is defined as follows:

$$R_{12}^\hbar(z) = \sum_\alpha T_\alpha \otimes T_{-\alpha}\, \varphi_a(\hbar + \Omega_a, z). \tag{3.6}$$

It was constructed as solution of the quantum Yang–Baxter equation (1.9). Later it was also shown [8] to satisfy the associative Yang–Baxter equation (1.8).

Remark. Let us remark that in the definition (3.5) the index $a = (a_1, a_2)$ was assumed to be an element of $\mathbb{Z}_N \times \mathbb{Z}_N$. Let us verify that the functions (3.5) are invariant with respect to shifts $a_{1,2} \to a_{1,2} + N$ of indices (discrete variables). Indeed, if $a_1 \to a_1 + N$ then $\Omega_a \to \Omega_a + 1$ and the function is periodic $\varphi_a(\hbar + \Omega_a, z) = \varphi_a(\hbar + \Omega_a + 1, z)$ due to (1.2). For $a_2 \to a_2 + N$ we have $\Omega_a \to \Omega_a + \tau$ and $\varphi_a(\hbar + \Omega_a, z) \to \exp(2\pi\imath z)\varphi_a(\hbar + \Omega_a + \tau, z) = \varphi_a(\hbar + \Omega_a, z)$ again due to (1.2).

3.2. *Supersymmetric basis functions.*

In our previous paper [14] we considered the function (1.14). The following three equivalent definitions for the odd supersymmetric analogues of the basis functions (3.5) were suggested:

(a) The first one is as follows:

$$\begin{aligned}\mathbf{\Phi}_\alpha^{\hbar+\Omega_\alpha|\,\mu}(z_1, z_2|\, z_1, \zeta_2) &= \exp\left(2\pi\imath\frac{\alpha_2}{N}(z_1 - z_2 + \zeta_1\zeta_2)\right) \mathbf{\Phi}^{\hbar+\Omega_\alpha|\,\mu}(z_1, z_2|\, z_1, \zeta_2)\\ &= \left(1 + 2\pi\imath\frac{\alpha_2}{N}\zeta_1\zeta_2\right) \mathbf{\Phi}^{\hbar+\Omega_\alpha|\,\mu}(z_1, z_2|\, z_1, \zeta_2)\\ &= \mathbf{\Phi}^{\hbar+\Omega_\alpha|\,\mu}(z_1, z_2|\, z_1, \zeta_2) + 2\pi\imath\frac{\alpha_2}{N}\zeta_1\zeta_2\omega\partial_1\phi(\hbar + \Omega_\alpha, z_{12})\,;\end{aligned} \tag{3.7}$$

(b) The second is

$$\mathbf{\Phi}_\alpha^{\hbar+\Omega_\alpha|\,\mu}(z_1, z_2|\, z_1, \zeta_2) = \exp\left(2\pi\imath\frac{\alpha_2}{N}(z_1 - z_2)\right) \mathbf{\Phi}^{\hbar+\Omega_\alpha|\,\mu+2\pi\imath\frac{\alpha_2}{N}\omega}(z_1, z_2|\, z_1, \zeta_2), \tag{3.8}$$

(c) and the last one is

$$\mathbf{\Phi}_\alpha^{\hbar+\Omega_\alpha|\,\mu}(z_1, z_2|\, z_1, \zeta_2) = \exp\left(2\pi\imath\frac{\alpha_2}{N}(z_1 - z_2)\right) \tilde{\mathbf{\Phi}}_\alpha^{\hbar+\Omega_\alpha|\,\mu}(z_1, z_2|\, z_1, \zeta_2), \tag{3.9}$$

with

$$\tilde{\mathbf{\Phi}}_{\alpha}^{\hbar+\Omega_\alpha|\,\mu}(z_1,z_2|\,z_1,\zeta_2)=(\zeta_1-\zeta_2)\varphi_\alpha(\hbar+\Omega_\alpha,z_{12})+\omega\partial_1\varphi_\alpha(\hbar+\Omega_\alpha,z_{12})$$
$$+2\pi\imath\zeta_1\zeta_2\omega\frac{d}{d\tau}\varphi_\alpha(\hbar+\Omega_\alpha,z_{12})+\zeta_1\zeta_2\mu\partial_1\varphi_\alpha(\hbar+\Omega_\alpha,z_{12})$$
$$+\frac{1}{2}(\zeta_1+\zeta_2)\mu\omega\partial_1^2\varphi_\alpha(\hbar+\Omega_\alpha,z_{12}), \tag{3.10}$$

where the derivative with respect to τ in the third term of (3.10) includes also partial derivative with respect to the first argument (it contains $\Omega_\alpha(\tau)$), and thus provides the same answer as in (3.8) or (3.9). The set of functions were shown to satisfy the following equations (Fay identities):

$$\mathbf{\Phi}_{\alpha}^{\hbar_1+\Omega_\alpha|\,\mu_1}(z_1,z_2|\,\zeta_1,\zeta_2)\mathbf{\Phi}_{\beta}^{\hbar_2+\Omega_\beta|\,\mu_2}(z_2,z_3|\,\zeta_2,\zeta_3)$$
$$+\ \mathbf{\Phi}_{-\beta}^{-\hbar_2-\Omega_\beta|\,-\mu_2}(z_3,z_1|\,\zeta_3,\zeta_1)\mathbf{\Phi}_{\alpha-\beta}^{\hbar_1-\hbar_2+\Omega_{\alpha-\beta}|\,\mu_1-\mu_2}(z_1,z_2|\,\zeta_1,\zeta_2)$$
$$+\ \mathbf{\Phi}_{\beta-\alpha}^{\hbar_2-\hbar_1+\Omega_{\beta-\alpha}|\,\mu_2-\mu_1}(z_2,z_3|\,\zeta_2,\zeta_3)\mathbf{\Phi}_{-\alpha}^{-\hbar_1-\Omega_\alpha|\,-\mu_1}(z_3,z_1|\,\zeta_3,\zeta_1)=0. \tag{3.11}$$

The equivalence of three above definitions holds true in the case (1.14), i.e. in the case $A_1=A_2=A_4=A_5, A_3=2\pi\imath A_1$. But the definitions are not equivalent for generic coefficients A_k. Consider the set of functions:

$$\mathbf{\Phi}_{\alpha}^{\hbar+\Omega_\alpha|\,\mu}(z_1,z_2|\,\zeta_1,\zeta_2|\,A,B)\ =\mathbf{\Phi}^{\hbar+\Omega_\alpha|\,\mu}(z_1,z_2|\,\zeta_1,\zeta_2|\,A)$$
$$+\ 2\pi\imath B\frac{\alpha_2}{N}\zeta_1\zeta_2\omega\partial_1\varphi_\alpha(\hbar+\Omega_\alpha,z_{12}), \tag{3.12}$$

where $B\in\mathbb{C}$ is an arbitrary coefficient. It is easy to see that the above definitions (3.7) and (3.10) being applied to the function $\mathbf{\Phi}^{\hbar+\Omega_\alpha|\,\mu}(z_1,z_2|\,\zeta_1,\zeta_2|\,A)$ provide

(a) $B=A_1$,
(b) $B=A_4$,
(c) $B=A_3$

respectively. In fact, any variant is possible. Moreover, we may keep the constant B to be arbitrary. It happens due to

Proposition 3.1. *The set of functions* (3.12) *satisfy the identities* (3.11) *if the condition* (2.2) *holds true, so that the second term in the definition* (3.11) *does not provide any new constraints for the coefficients* $A_1,\ldots,A_5,B$.

Proof. The proof is similar to the one for proposition 2.2. In the latter we have already proved the statement for $B=0$ case. Due to the Grassmann monomial $\zeta_1\zeta_2\omega$ the second term from (3.11) (when $B\neq 0$) provides new terms proportional to A_1B only. They are cancelled out with the help of derivatives of the ordinary Fay identity (1.4) with respect to $\hbar_1$ and $\hbar_2$. ■

3.3. Associative and classical Yang–Baxter equations

As was shown in the previous paragraph, the functions $\mathbf{\Phi}_{\alpha}^{\hbar+\Omega_\alpha|\,\mu}(z_1,z_2|\,\zeta_1,\zeta_2|\,A,B)$ (3.12) with arbitrary constant B satisfy the Fay identities (3.11) without any new constraints for the coefficients.

However there is one more restriction for the coefficients. At the end of section 3.1 we remarked that the functions $\varphi_a(\hbar+\Omega_a,z)$ are invariant with respect the shift of discrete variables (indices) $a_{1,2}\to a_{1,2}+N$.

It is truly important by the following reason. In the Yang–Baxter equations we multiply the basis matrices (3.1) through the rule (3.4). This results in the appearance of sums (or differences) of indices in tensor components of the Yang–Baxter equations. Finally, we use the Fay identities, which also contain the sums (or difference) of the indices. If the basis functions were defined for the indices in the range $0\leqslant a_{1,2}\leqslant N-1$ then the sum or difference of two indices could be out of range. Therefore, we need to verify if the functions (3.12) are invariant with respect to the shifts $a_{1,2}\to a_{1,2}+N$.

Because of the property $\varphi_{a_1+N,a_2}(\hbar+1+\Omega_a,z)=\varphi_a(\hbar+\Omega_a,z)$ the shift $a_1\to a_1+N$ keeps the functions (3.12) invariant. The shift $a_2\to a_2+N$ provides non-trivial additional terms:

$$\begin{aligned}\mathbf{\Phi}^{\hbar+\Omega_\alpha+\tau|\,\mu}_{a_1,a_2+N}(z_1,z_2|\,\zeta_1,\zeta_2|\,A,B) &= \mathbf{\Phi}^{\hbar+\Omega_\alpha|\,\mu}_{a_1,a_2}(z_1,z_2|\,\zeta_1,\zeta_2|\,A,B)\\ &\quad+\,2\pi\imath(B-A_3)\frac{a_2}{N}\zeta_1\zeta_2\omega\partial_1\varphi_a(\hbar+\Omega_a,z_{12}).\end{aligned} \tag{3.13}$$

Finally, we conclude that in order to have invariance of the functions (3.12) with respect to the shifts $a_{1,2}\to a_{1,2}+N$ one should impose condition

$$B=A_3. \tag{3.14}$$

Then the definition (3.10) is valid for the basis functions.

The classical and associative Yang–Baxter equations are proved in the same way as in [14]. Namely, introduce the odd supersymmetric analogue of the Baxter–Belavin's R-matrix (3.6):

$$\mathbf{R}^{\hbar|\,\mu}_{12}(z_1,z_2|\,\zeta_1,\zeta_2|\,A)=\sum_\alpha T_\alpha\otimes T_{-\alpha}\mathbf{\Phi}^{\hbar+\Omega_\alpha|\,\mu}_\alpha(z_1,z_2|\,\zeta_1,\zeta_2|\,A,B)|_{B=A_3}. \tag{3.15}$$

This R-matrix satisfies the associative Yang–Baxter equation

$$\mathbf{R}^{\hbar_1|\,\mu_1}_{12}\mathbf{R}^{\hbar_2|\,\mu_2}_{23}+\mathbf{R}^{-\hbar_2|\,-\mu_2}_{31}\mathbf{R}^{\hbar_1-\hbar_2|\,\mu_1-\mu_2}_{12}+\mathbf{R}^{\hbar_2-\hbar_1|\,\mu_2-\mu_1}_{23}\mathbf{R}^{-\hbar_1|\,-\mu_1}_{31}=0 \tag{3.16}$$

with $\mathbf{R}^{\hbar|\,\mu}_{ab}=\mathbf{R}^{\hbar|\,\mu}_{ab}(z_a,z_b|\,\zeta_a,\zeta_b|\,A)$.

Introduce similarly the odd supersymmetric analogue of the classical elliptic r-matrix

$$\mathbf{r}_{12}(z_1,z_2|\,\zeta_1,\zeta_2|\,A)=\sum_{\alpha\neq0}T_\alpha\otimes T_{-\alpha}\mathbf{\Phi}^{\Omega_\alpha|\,0}_\alpha(z_1,z_2|\,\zeta_1,\zeta_2|\,A,B=A_3). \tag{3.17}$$

The function $\mathbf{\Phi}^{\Omega_\alpha|\,0}_\alpha(z_1,z_2|\,\zeta_1,\zeta_2|\,A,B=A_3)$, where μ is replaced by 0 means that $A_4=A_5=0$. The r-matrix satisfies the classical (super) Yang–Baxter equation:

$$[\mathbf{r}_{12},\mathbf{r}_{13}]_++[\mathbf{r}_{12},\mathbf{r}_{23}]_++[\mathbf{r}_{13},\mathbf{r}_{23}]_+=0 \tag{3.18}$$

with $\mathbf{r}_{ab}=\mathbf{r}_{ab}(z_a,z_b|\zeta_a,\zeta_b|A)$.

3.4. Quantum Yang–Baxter equation

Non-supersymmetric case. Let us recall how the quantum Yang–Baxter equation (1.9) arises from the associative one (1.8). It is enough to require the R-matrix to be

(a) Skew-symmetric, i.e.

$$R_{12}^{\hbar}(z) = -R_{21}^{-\hbar}(-z) \tag{3.19}$$

(b) Unitary

$$R_{12}^{\hbar}(z)R_{21}^{\hbar}(-z) = f(h,z)1_N \otimes 1_N, \tag{3.20}$$

where $f(h,z)$ is a normalization function. For the Baxter–Belavin's R-matrix written as in (3.6) the function is as follows:

$$f(h,z) = N^2(\wp(N\hbar) - \wp(z)). \tag{3.21}$$

Indeed, consider equation (1.8) in the particular case $\hbar_2 = \hbar_1/2$, and then make the substitution $\hbar_1 \to 2\hbar$. As a result we get

$$R_{12}^{2\hbar}R_{23}^{\hbar} + R_{31}^{-\hbar}R_{12}^{\hbar} + R_{23}^{-\hbar}R_{31}^{-2\hbar} = 0, \tag{3.22}$$

where the short notations $R_{ab}^{\eta} = R_{ab}^{\eta}(z_a - z_b)$ are used. Multiply the latter equality by $R_{23}^{\hbar}$ from the left:

$$R_{23}^{\hbar}R_{12}^{2\hbar}R_{23}^{\hbar} + R_{23}^{\hbar}R_{31}^{-\hbar}R_{12}^{\hbar} + R_{23}^{\hbar}R_{23}^{-\hbar}R_{31}^{-2\hbar} = 0. \tag{3.23}$$

By applying the skew-symmetry (3.19) to R_{31} and the unitarity (3.20) to the expression $R_{23}^{\hbar}R_{23}^{-\hbar}$ in the third term we get

$$R_{23}^{\hbar}R_{13}^{\hbar}R_{12}^{\hbar} = R_{23}^{\hbar}R_{12}^{2\hbar}R_{23}^{\hbar} + f(\hbar, z_{23})R_{13}^{2\hbar}. \tag{3.24}$$

The latter equality is, in fact, particular case of more general identities, which can be found in [8, 10, 16].

Next, consider equation (1.8) with indices 2 and 3 being interchanged. The latter means that we conjugate (1.8) by the permutation operator P_{23} and redefine the variables as $z_2 \leftrightarrow z_3$:

$$R_{13}^{\hbar_1}R_{32}^{\hbar_2} + R_{21}^{-\hbar_2}R_{13}^{\hbar_1-\hbar_2} + R_{32}^{\hbar_2-\hbar_1}R_{21}^{-\hbar_1} = 0. \tag{3.25}$$

Substitute again $\hbar_2 = \hbar_1/2$ and denote $\hbar_1 := 2\hbar$:

$$R_{13}^{2\hbar}R_{32}^{\hbar} + R_{21}^{-\hbar}R_{13}^{\hbar} + R_{32}^{-\hbar}R_{21}^{-2\hbar} = 0. \tag{3.26}$$

Then, multiply the equality (3.26) by $R_{23}^{\hbar}$ from the right:

$$R_{13}^{2\hbar}R_{32}^{\hbar}R_{23}^{\hbar} + R_{21}^{-\hbar}R_{13}^{\hbar}R_{23}^{\hbar} + R_{32}^{-\hbar}R_{21}^{-2\hbar}R_{23}^{\hbar} = 0. \tag{3.27}$$

Using the skew-symmetry and unitarity we get

$$R_{12}^{\hbar}R_{13}^{\hbar}R_{23}^{\hbar} = f(\hbar, z_{23})R_{13}^{2\hbar} + R_{23}^{\hbar}R_{12}^{2\hbar}R_{23}^{\hbar}. \tag{3.28}$$

Finally, the quantum Yang–Baxter equation (1.9) follows from comparing (3.24) and (3.28).

Supersymmetric case. Let us make the calculations similar to those from the previous paragraph for the odd supersymmetric R-matrix (3.15).

First, notice that the skew-symmetry property (3.19) turns in the supersymmetric case into the symmetry property due to R-matrix oddness:

$$\mathbf{R}_{ab}^{\hbar|\,\mu} = \mathbf{R}_{ba}^{-\hbar|\,-\mu}, \tag{3.29}$$

where $\mathbf{R}_{ab}^{\hbar|\,\mu} = \mathbf{R}_{ab}^{\hbar|\,\mu}(z_a, z_b|\,\zeta_a, \zeta_b|\,A)$.

Next, let us evaluate the analogue of the unitarity property (3.20):

$$\mathbf{R}_{12}^{\hbar|\,\mu}\mathbf{R}_{21}^{\hbar|\,\mu} = \sum_{\alpha,\beta} T_\alpha T_{-\beta} \otimes T_{-\alpha} T_\beta \mathbf{\Phi}_{\alpha;\,12}^{\hbar+\Omega_\alpha|\,\mu}\mathbf{\Phi}_{\beta;\,21}^{\hbar+\Omega_\beta|\,\mu} \tag{3.30}$$

where we assume $\mathbf{\Phi}_{\alpha;\,ij}^{\hbar+\Omega_\alpha|\,\mu} = \mathbf{\Phi}_{\alpha}^{\hbar+\Omega_\alpha|\,\mu}(z_i, z_j|\,\zeta_i, \zeta_j|\,A, B)\big|_{B=A_3}$. The expression in the sum can be calculated explicitly using the definition (3.12) and (1.22). Most of the terms vanish due to (1.13). The non-zero terms are as follows:

$$\mathbf{\Phi}_{\alpha;\,12}^{\hbar+\Omega_\alpha|\,\mu}\mathbf{\Phi}_{\beta;\,21}^{\hbar+\Omega_\beta|\,\mu} = \left[A_1A_2(\zeta_1-\zeta_2)\omega\partial_\hbar + A_1A_5\zeta_1\zeta_2\mu\omega\partial_\hbar^2\right]\varphi_\alpha(\hbar+\Omega_\alpha, z_{12})\varphi_\beta(\hbar+\Omega_\beta, z_{21}). \tag{3.31}$$

Notice that the derivation of the latter answer used $A_1A_5 = A_2A_4$ as in (2.2). Plugging (3.31) into (3.30) we see that its rhs is represented as action of the differential operator from the quadratic brackets on the expression $R_{12}^{\hbar}(z_{12})R_{21}^{\hbar}(z_{21})$, which is equal to the ordinary unitarity relation, i.e.

$$\mathbf{R}_{12}^{\hbar|\,\mu}\mathbf{R}_{21}^{\hbar|\,\mu} = \left[A_1A_2(\zeta_1-\zeta_2)\omega\partial_\hbar + A_1A_5\zeta_1\zeta_2\mu\omega\partial_\hbar^2\right]R_{12}^{\hbar}(z_{12})R_{21}^{\hbar}(z_{21}). \tag{3.32}$$

Finally, using (3.21) we get the following statement for the analogue of unitarity property:

Proposition 3.2. *The analogue of the unitarity property* (3.20) *and* (3.21) *for the odd R-matrix* (3.15) *is of the form:*

$$\mathbf{R}_{12}^{\hbar|\,\mu}\mathbf{R}_{21}^{\hbar|\,\mu} = \left(A_1A_2(\zeta_1-\zeta_2)\omega N^3\wp'(N\hbar) + A_1A_5\zeta_1\zeta_2\mu\omega N^4\wp''(N\hbar)\right)1_N\otimes 1_N. \tag{3.33}$$

Notice also that

$$\mathbf{R}_{12}^{\hbar|\,\mu}\mathbf{R}_{21}^{\hbar|\,\mu} = -\mathbf{R}_{21}^{\hbar|\,\mu}\mathbf{R}_{12}^{\hbar|\,\mu}. \tag{3.34}$$

Having the properties (3.29) and (3.33) we can make the calculations similar to the previous paragraph. The main statement of the paragraph is as follows.

Proposition 3.3. *Consider the supersymmetric elliptic* $\mathrm{GL}(N,\mathbb{C})$ *odd R-matrix* (3.15). *It satisfies two equations of the Yang–Baxter type with additional terms. The first one is*

$$\mathbf{R}_{12}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{23}^{\hbar|\,\mu} = \mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{12}^{\hbar|\,\mu} + 2\mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{32}^{\hbar|\,\mu}\mathbf{R}_{13}^{2\hbar|\,2\mu} \tag{3.35}$$

or, using it is represented as (3.33)

$$\begin{aligned}\mathbf{R}_{12}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{23}^{\hbar|\,\mu} = \mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{12}^{\hbar|\,\mu} &+ 2\left(A_1A_2(\zeta_2-\zeta_3)\omega N^3\wp'(N\hbar)\right.\\ &\left.+\ A_1A_5\zeta_2\zeta_3\mu\omega N^4\wp''(N\hbar)\right)\mathbf{R}_{13}^{2\hbar|\,2\mu}.\end{aligned} \tag{3.36}$$

And the second is

$$\mathbf{R}_{12}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{23}^{\hbar|\,\mu} = -\mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{12}^{\hbar|\,\mu} - 2\mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{12}^{2\hbar|\,2\mu}\mathbf{R}_{23}^{\hbar|\,\mu}. \tag{3.37}$$

Proof. Consider the associative Yang–Baxter equation (3.16) for $\hbar_2 = \hbar_1/2$, $\mu_2 = \mu_1/2$, and then denote $\hbar_1 := 2\hbar$ and $\mu_1 := 2\mu$:

$$\mathbf{R}_{12}^{2\hbar|\,2\mu}\mathbf{R}_{23}^{\hbar|\,\mu} + \mathbf{R}_{31}^{-\hbar|\,-\mu}\mathbf{R}_{12}^{\hbar|\,\mu} + \mathbf{R}_{23}^{-\hbar|\,-\mu}\mathbf{R}_{31}^{-2\hbar|\,-2\mu} = 0, \tag{3.38}$$

which is a direct analogue of (3.23). Multiplying it by $\mathbf{R}_{23}^{\hbar|\,\mu}$ from the left and using (3.29) we obtain

$$\mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{12}^{\hbar|\,\mu} = -\mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{12}^{2\hbar|\,2\mu}\mathbf{R}_{23}^{\hbar|\,\mu} - \mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{32}^{\hbar|\,\mu}\mathbf{R}_{13}^{2\hbar|\,2\mu}. \tag{3.39}$$

Similarly to (3.26), consider the equation (3.38) with indices 2 and 3 being interchanged

$$\mathbf{R}_{13}^{2\hbar|\,2\mu}\mathbf{R}_{32}^{\hbar|\,\mu} + \mathbf{R}_{21}^{-\hbar|\,-\mu}\mathbf{R}_{13}^{\hbar|\,\mu} + \mathbf{R}_{32}^{-\hbar|\,-\mu}\mathbf{R}_{21}^{-2\hbar|\,-2\mu} = 0. \tag{3.40}$$

Multiplying it by $\mathbf{R}_{23}^{\hbar|\,\mu}$ from the right and using (3.29) we obtain:

$$\begin{aligned}\mathbf{R}_{12}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{23}^{\hbar|\,\mu} &= -\mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{12}^{2\hbar|\,2\mu}\mathbf{R}_{23}^{\hbar|\,\mu} - \mathbf{R}_{13}^{2\hbar|\,2\mu}\mathbf{R}_{32}^{\hbar|\,\mu}\mathbf{R}_{23}^{\hbar|\,\mu} \\ &\overset{(3.32),(3.34)}{=} -\mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{12}^{2\hbar|\,2\mu}\mathbf{R}_{23}^{\hbar|\,\mu} + \mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{32}^{\hbar|\,\mu}\mathbf{R}_{13}^{2\hbar|\,2\mu}.\end{aligned} \tag{3.41}$$

In contrast to the ordinary case the rhs of (3.39) and (3.41) are not equal to each other because of the property (3.34). Subtracting (3.41) from (3.39) we get (3.35).

Alternatively, we can sum up the equations (3.39) and (3.41). Then the last terms in the rhs are cancelled out, and we get (3.37). ■

Let us comment on the linear R-matrix term, which is the last one in the rhs of (3.36). The necessity of this term becomes obvious in the scalar ($N = 1$) case. In this case we should have $\Phi_{12}^{\hbar|\,\mu}\Phi_{13}^{\hbar|\,\mu}\Phi_{23}^{\hbar|\,\mu} = -\Phi_{23}^{\hbar|\,\mu}\Phi_{13}^{\hbar|\,\mu}\Phi_{12}^{\hbar|\,\mu}$ with the sign minus due to the odd parity of the permutation relating both sides. The equation with the minus sign is also easily follows in the scalar case from (3.35) since $(\Phi_{23}^{\hbar|\,\mu})^2 = 0$. In (3.36) the sign behind the cubic term is plus but it is compensated with the linear term.

Consider also a special case of (3.36) for $A_4 = A_5 = 0$, i.e. when the variable μ is absent. Though (3.36) does not contain A_4 we should require $A_4 = 0$ since the derivation of (3.33) as well as the Fay identity (2.2) used the condition $A_1A_5 = A_2A_4$. Then (3.36) turns into

$$\mathbf{R}_{12}^{\hbar|\,0}\mathbf{R}_{13}^{\hbar|\,0}\mathbf{R}_{23}^{\hbar|\,0} = \mathbf{R}_{23}^{\hbar|\,0}\mathbf{R}_{13}^{\hbar|\,0}\mathbf{R}_{12}^{\hbar|\,0} + 2A_1A_2(\zeta_2 - \zeta_3)\omega N^3\wp'(N\hbar)\mathbf{R}_{13}^{2\hbar|\,0}. \tag{3.42}$$

The second term in the rhs of (3.42) is proportional to $\wp'(N\hbar)$. The function $\wp'(X)$ is double-periodic, $\wp'(X) = -\wp'(-x)$ and has a pole of third order at $x = 0$. Therefore, it has three zeros at non-trivial half-periods $1/2, \tau/2, (\tau+1)/2$. At the same time the R-matrix $\mathbf{R}_{13}^{2\hbar|\,0}$ has poles at $\hbar = \hbar_0 \in \left\{\pm\frac{1}{2N}, \pm\frac{\tau}{2N}, \pm\frac{\tau+1}{2N}\right\}$. Therefore, as a result of the substitution $\hbar = \hbar_0$ only one summand survives in the sum over α in (3.15). Then, according to (3.1) and (3.5) the second term in the rhs of (3.42) is proportional to a constant matrix $T_{(\pm1,0)} \otimes 1_N \otimes T_{(\pm1,0)}$ or $T_{(0,\pm1)} \otimes 1_N \otimes T_{(0,\pm1)}$ or $T_{(\pm1,\pm1)} \otimes 1_N \otimes T_{(\pm1,\pm1)}$ depending on the choice of $\hbar_0$.

4. Conclusion

Let us summarize the obtained results:

- We studied ansatz for the odd supersymmetric Kronecker function in the form

$$\begin{aligned}\mathbf{\Phi}(\hbar, z_1, z_2; \tau |\, \mu, \zeta_1, \zeta_2; \omega |\, A) \equiv \mathbf{\Phi}_{12}^{\hbar|\,\mu} = [&A_1(\zeta_1 - \zeta_2) + A_2\omega\partial_\hbar + 2\pi\imath A_3\zeta_1\zeta_2\omega\partial_\tau \\ &+ A_4\zeta_1\zeta_2\mu\partial_\hbar + \frac{A_5}{2}(\zeta_1+\zeta_2)\mu\omega\partial_\hbar^2\Big]\,\phi(\hbar, z_1 - z_2) \quad (4.1)\end{aligned}$$

  and showed that it satisfies the Fay identity

$$\mathbf{\Phi}_{12}^{\hbar_1|\,\mu_1}\mathbf{\Phi}_{23}^{\hbar_2|\,\mu_2} + \mathbf{\Phi}_{31}^{-\hbar_2|\,-\mu_2}\mathbf{\Phi}_{12}^{\hbar_1-\hbar_2|\,\mu_1-\mu_2} + \mathbf{\Phi}_{23}^{\hbar_2-\hbar_1|\,\mu_2-\mu_1}\mathbf{\Phi}_{31}^{-\hbar_1|\,-\mu_1} = 0 \quad (4.2)$$

  if $A_1A_5 = A_2A_4$.
- We considered the supersymmetric version of the heat equation in the form

$$(\kappa\partial_\omega + 2\pi\imath(\zeta_1+\zeta_2)\partial_\tau)\,\mathbf{\Phi}_{12}^{\hbar|\,\mu} = \left(\partial_{\zeta_1} + \zeta_1\partial_{z_1} - \frac{k}{2}\mu\partial_\hbar\right)\partial_\hbar\mathbf{\Phi}_{12}^{\hbar|\,\mu}. \quad (4.3)$$

  and showed that it holds true if the following conditions valid:

$$\kappa A_2 = A_1, \quad \kappa A_3 = 2\pi\imath A_1, \quad A_4 = kA_1, \quad \kappa A_5 = kA_1. \quad (4.4)$$

- We defined the set of functions

$$\mathbf{\Phi}_\alpha^{\hbar+\Omega_\alpha|\,\mu}(z_1, z_2|\,\zeta_1,\zeta_2|\,A, B) \equiv = \mathbf{\Phi}_{\alpha;\,12}^{\hbar|\,\mu} = \mathbf{\Phi}_{12}^{\hbar|\,\mu} + 2\pi\imath B\frac{\alpha_2}{N}\zeta_1\zeta_2\omega\partial_1\varphi_\alpha(\hbar+\Omega_\alpha, z_{12}) \quad (4.5)$$

  and proved that they satisfy the Fay identities

$$\begin{aligned}&\mathbf{\Phi}_{\alpha;\,12}^{\hbar_1+\Omega_\alpha|\,\mu_1}\mathbf{\Phi}_{\beta;\,23}^{\hbar_2+\Omega_\beta|\,\mu_2} + \mathbf{\Phi}_{-\beta;\,31}^{-\hbar_2-\Omega_\beta|\,-\mu_2}\mathbf{\Phi}_{\alpha-\beta;\,12}^{\hbar_1-\hbar_2+\Omega_{\alpha-\beta}|\,\mu_1-\mu_2} \\ &\quad + \mathbf{\Phi}_{\beta-\alpha;\,23}^{\hbar_2-\hbar_1+\Omega_{\beta-\alpha}|\,\mu_2-\mu_1}|\mathbf{\Phi}_{-\alpha;\,31}^{-\hbar_1-\Omega_\alpha|\,-\mu_1} = 0 \quad (4.6)\end{aligned}$$

  for arbitrary constant B.
- The constant B is fixed to be $B = A_3$ by requirement for the functions (4.5) to be invariant with respect to the shifts $a_{1,2} \to a_{1,2} + N$ of indices.
- Using (4.5), (4.1) the odd elliptic R-matrix is represented in the form:

$$\begin{aligned}\mathbf{R}_{12}(\hbar, z_1, z_2; \tau|\,\mu, \zeta_1, \zeta_2; \omega|\,A) \equiv \mathbf{R}_{12}^{\hbar|\,\mu} = \Big[&A_1(\zeta_1-\zeta_2) + A_2\omega\partial_\hbar + 2\pi\imath A_3\zeta_1\zeta_2\omega\frac{d}{d\tau} \\ &+ A_4\zeta_1\zeta_2\mu\partial_\hbar + \frac{A_5}{2}(\zeta_1+\zeta_2)\mu\omega\partial_\hbar^2\Big]\,R_{12}^\hbar, \quad (4.7)\end{aligned}$$

  where $R_{12}^\hbar$ is the Baxter–Belavin elliptic R-matrix (3.6). It satisfies the associative Yang–Baxter equation

$$\mathbf{R}_{12}^{\hbar_1|\,\mu_1}\mathbf{R}_{23}^{\hbar_2|\,\mu_2} + \mathbf{R}_{31}^{-\hbar_2|\,-\mu_2}\mathbf{R}_{12}^{\hbar_1-\hbar_2|\,\mu_1-\mu_2} + \mathbf{R}_{23}^{\hbar_2-\hbar_1|\,\mu_2-\mu_1}\mathbf{R}_{31}^{-\hbar_1|\,-\mu_1} = 0. \quad (4.8)$$

the symmetry property (3.29) and the unitarity property (3.33). The odd supersymmetric version of the classical elliptic r-matrix (3.17) satisfies the classical (super) Yang–Baxter equation. Notice also that from (1.1) and (4.7) it follows that

$$\operatorname*{Res}_{z=0} \mathbf{R}_{12}^{\hbar_1|\,\mu_1} = (\zeta_1 - \zeta_2) A_1 N P_{12}, \tag{4.9}$$

where P_{12} permutation operator.

- The cubic relations for the supersymmetric extension of the Baxter–Belavin's R-matrix have form of the quantum Yang–Baxter equation with additional term:

$$\begin{aligned} \mathbf{R}_{12}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{23}^{\hbar|\,\mu} = \mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{12}^{\hbar|\,\mu} + 2\,\big(A_1A_2(\zeta_2-\zeta_3)\omega N^3\wp'(N\hbar) \\ + A_1A_5\zeta_2\zeta_3\mu\omega N^4\wp''(N\hbar)\big)\,\mathbf{R}_{13}^{2\hbar|\,2\mu} \end{aligned} \tag{4.10}$$

  and

$$\mathbf{R}_{12}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{23}^{\hbar|\,\mu} = -\mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{13}^{\hbar|\,\mu}\mathbf{R}_{12}^{\hbar|\,\mu} - 2\mathbf{R}_{23}^{\hbar|\,\mu}\mathbf{R}_{12}^{2\hbar|\,2\mu}\mathbf{R}_{23}^{\hbar|\,\mu}. \tag{4.11}$$

- Possible applications of the obtained results include construction of the Knizhnik–Zamolodchikov–Bernard (KZB) equations on supersymmetric elliptic curves. It is the subject of our next paper [17]. The KZ(B) equations can be treated as quantization of the monodromy preserving equations. We should mention the article [18], where the classical rational Schlesinger system on $\mathbb{C}P^{1|1}$ was introduced in the context of studies of the Frobenius (super) manifolds. The definition of the odd connection is similar to the rational limit of the one used in our paper.

The obtained results allow us to describe the quantum version of the Schlesinger system (or the (q)KZ(B) equations) on supersymmetric elliptic curves and the corresponding deformations of the quantum Painlevé equations. It is then also possible to evaluate the integrable deformations of the quantum (spin) Calogero–Ruijsenaars type models and the models of interacting integrable tops. Using the associative Yang–Baxter equation the latter integrable systems were shown to be described by the so-called R-matrix-valued Lax pairs, which lead to integrable long-range spin chains [10, 19]. The constant coefficients studied in this paper also enter to the deformed potentials. The deformations coming from the Grassmann variables for these type integrable models will be studied elsewhere. At the same time a direct usage of the odd R-matrix (4.7) to the quantum inverse scattering method and construction of the spin chains with local interaction is problematic by two reasons. First, due to the additional terms in (4.10) and (4.11), and secondly, due to the R-matrix (4.7) is an odd operator, and therefore, it is not invertible. More promising are the deformations (via the Grassmann variables) of the ordinary elliptic R-matrices. We are going to study these deformations in our future publications.

Acknowledgments

The work was supported in part by RFBR Grants 18-02-01081 (AL and MO), 18-01-00273 (AZ) and by joint RFBR project 19-51-18006 Bolg$_a$ (MO). The work of AL was partially supported by Laboratory of Mirror Symmetry NRU HSE, RF Government Grant, ag. 14.641.31.0001. The research of AZ was also supported in part by the Young Russian Mathematics award.

References

[1] Weil A 1976 *Elliptic Functions According to Eisenstein and Kronecker* (Berlin: Springer)
Mumford D 1983 *Tata Lectures on Theta I, II* (Boston: Birkhäuser)
[2] Fay J D 1973 *Theta Functions on Riemann Surfaces* (*Lecture Notes in Mathematics* vol 352) (Berlin: Springer)
[3] Baxter R J 1972 *Ann. Phys.* **70** 193–228
Belavin A A 1981 *Nucl. Phys.* B **180** 189–200
Belavin A A and Drinfeld V G 1982 *Funct. Anal. Appl.* **16** 159–80
[4] Felder G 1994 *Proc. of the ICM* (Boston: Birkhäuser) 94 pp 1247–55
Levin A M, Olshanetsky M A and Zotov A V 2014 *Russ. Math. Surv.* **69** 35–118
[5] Krichever I M 1980 *Funct. Anal. Appl.* **14** 282–90
Sklyanin E K 1982 *Funct. Anal. Appl.* **16** 263–70
Sklyanin E K 1983 *Funct. Anal. Appl.* **17** 273–84
Buchstaber V M, Felder G and Veselov A V 1994 *Duke Math.* J. **76** 885–911
[6] Sklyanin E K, Takhtadzhyan L A and Faddeev L D 1979 *Theor. Math. Phys.* **40** 688–706
Takhtadzhyan L A and Faddeev L D 1979 *Russ. Math. Surv.* **34** 11–68
[7] Fomin S and Kirillov A N 1999 *Advances in Geometry* (*Progress in Mathematics* vol 172) pp 147–82
[8] Polishchuk A 2002 *Adv. Math.* **168** 56–95
[9] Kirillov A N 2016 SIGMA **12** 002
[10] Levin A, Olshanetsky M and Zotov A 2014 *J. High Energy Phys.* **10** 109
Levin A M, Olshanetsky M A and Zotov A V 2015 *Theor. Math. Phys.* **184** 924–39
Zotov A V 2018 *Theor. Math. Phys.* **197** 1755–70
[11] Levin A M 1987 *Funct. Anal. Appl.* **21** 243–4
Levin A M 1988 *Funct. Anal. Appl.* **22** 60–1
[12] Rabin J M and Freund P G O 1988 *Commun. Math. Phys.* **114** 131–45
Rabin J M 1995 *J. Geom. Phys.* **15** 252–80
[13] D'Hoker E and Phong D H 1989 *Commun. Math. Phys.* **125** 469–513
Rogers A 2007 *Supermanifolds Theory and Applications* (Singapore: World Scientific)
Witten E 2019 *Pure Appl. Math.* Q. **15** 57–211
[14] Levin A, Olshanetsky M and Zotov A 2019 arXiv:1910.01814 [math-ph]
[15] Kulish P P and Sklyanin E K 1982 *J. Sov. Math.* **19** 1596–620
Khoroshkin S M and Tolstoy V N 1991 *Commun. Math. Phys.* **141** 599–617
[16] Levin A, Olshanetsky M and Zotov A 2016 *J. Phys. A: Math. Theor.* **49** 14003
Levin A, Olshanetsky M and Zotov A 2016 *J. Phys. A: Math. Theor.* **49** 395202
Zotov A V 2016 *Theor. Math. Phys.* **189** 1554–62
[17] Levin A, Olshanetsky M and Zotov A *Knizhnik–Zamolodchikov Equations on Supersymmetric Elliptic Curves* (unpublished)
[18] Manin Y I and Merkulov S A 1997 *Topol. Methods Nonlinear Anal.* vol 9 107–61
[19] Sechin I and Zotov A 2018 *Phys. Lett.* B **781** 1–7
Grekov A and Zotov A 2018 *J. Phys.* A **51** 315202