

A Hamilton–Jacobi formalism for higher order implicit Lagrangians

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Abstract

In this paper, we present a generalization of the Hamilton–Jacobi theory for higher order implicit Lagrangian systems. We propose two different backgrounds to deal with higher order implicit Lagrangian theories: the Ostrogradski approach and the Schmidt transform, which convert a higher order Lagrangian into a first order one. The Ostrogradski approach involves the addition of new independent variables to account for higher order derivatives, whilst the Schmidt transform adds gauge invariant terms to the Lagrangian function. In these two settings, the implicit character of the resulting equations will be treated in two different ways in order to provide a Hamilton–Jacobi equation. On one hand, the implicit Lagrangian system will be realized as a Lagrangian submanifold of a higher order tangent bundle that is generated by a Morse family. On the other hand, we will rely on the existence of an auxiliary section of a certain bundle that allows the construction of local vector fields, even if the differential equations are implicit. We will illustrate some examples of our proposed schemes, and discuss the applicability of the proposal.

Keywords: Hamilton–Jacobi, second order Lagrangian, Schmidt–Legendre transform, Ostrogradsky–Legendre transform, implicit differential equations, Morse family, higher-order Lagrange equations

1. Introduction

In classical mechanics there exist two main approaches to deal with mechanical systems: namely the Hamiltonian and the Lagrangian formalism [1, 4, 41]. Assuming that the configuration space of a physical system is an n -dimensional manifold Q , the Lagrangian formulation of the dynamics is generated by a Lagrangian function L defined on the tangent bundle TQ . Physically, TQ can be considered as the velocity phase space of the dynamics whereas L is given as the difference of kinetic and potential energies. The extremals of the integral action lead to the determination of the Euler–Lagrange equations governing the motion. In a local chart (q^A, \dot{q}^A) on TQ , the Euler–Lagrange equations can be written as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^A} - \frac{\partial L}{\partial q^A} = 0. \quad (1)$$

Note that, the Euler–Lagrange equations conform a system of n second order differential equations.

On the other hand, Hamiltonian realization of a physical system is achieved on the cotangent bundle T^*Q , which is the momentum-phase space of a physical system. T^*Q is equipped with a canonical symplectic (closed, and non-degenerate) two-form Ω_Q [46, 50]. The non-degeneracy of the symplectic two-form establishes that for an exact one-form dH , where H is the Hamiltonian function on T^*Q , there exists a unique vector field X_H on T^*Q . This vector field is the Hamiltonian vector field and it is implicitly defined by the following equation:

$$\iota_{X_H} \Omega = dH. \quad (2)$$

Here, ι is a contraction operator or interior derivative [2, 63]. In most of the cases, the Hamiltonian function corresponds with the total energy of physical system, which is a manifestation of the skew-symmetry of Ω_Q , and it is conserved all along the motion. There is a distinguished coordinate system (q^A, p_A) , known as the Darboux' coordinates, on T^*Q . In this chart, the symplectic two form turns out to be $\Omega_Q = dq^A \wedge dp_A$, and the Hamilton equations (2) can be written as

$$\dot{q}^A = \frac{\partial H}{\partial p_A}(q^A, p_A), \quad \dot{p}_A = -\frac{\partial H}{\partial q^A}(q^A, p_A). \quad (3)$$

Notice that, the Hamilton equations conform a system of $2n$ first order differential equations. Two questions may arise at this point. One is how to find a passage between the Lagrangian and the Hamiltonian formulation of a given dynamical system. The other one is how to solve the Euler–Lagrange or/and the Hamilton equations. These two questions are essential and motivational for the present work. In this introductory part, we use local coordinates and avoid technical details as much as possible in order to exhibit the motivation and the aim of the present paper in a clearer form. Accordingly, we will explain the global definitions and mathematical foundations along the main body of the paper.

1.1. The Legendre transformation

To find a passage between the Lagrangian and the Hamiltonian formalisms of a dynamical system, one needs a one-to-one transformation from the tangent bundle TQ to the cotangent bundle T^*Q . But, unfortunately, there is no canonical way to arrive at such a mapping between these two bundles [1, 4]. Nevertheless, by means of a Lagrangian function L , more concretely by taking the fiber derivative of L , we can define the Legendre transformation

$$\mathbb{F}L : TQ \longrightarrow T^*Q : (q^A, \dot{q}^A) \mapsto \left(q^A, \frac{\partial L}{\partial \dot{q}^A} \right). \quad (4)$$

Here, the momenta is defined to be $p_A = \partial L / \partial \dot{q}^A$ so that it is a function of the positions and the velocities (q^A, \dot{q}^A) . By referring once more to the Lagrangian function, define a real valued function

$$H = p_A \dot{q}^A - L(q^A, \dot{q}^A) \quad (5)$$

depending on the positions, the velocities and the momenta.

Considering the Legendre transformation (4), we see that there are two possibilities. One possibility is the case where all the velocities (\dot{q}^A) can be written as functions of the position and the momenta (q^A, p_A) , which is the case when the Legendre transformation is locally invertible. This case occurs if the Lagrangian function is non-degenerate, that is if the Hessian matrix $[\partial^2 L / \partial \dot{q}^A \partial \dot{q}^B]$ has rank n . Other possibility occurs when some of the velocities cannot be written as functions of (q^A, p_A) . That is, the Legendre transformation is not invertible, so the Hessian condition does not hold. Let us discuss these two cases one by one.

Non-degenerate Lagrangians. If the Lagrangian function is non-degenerate, that is, if the Hessian matrix $[\partial^2 L / \partial \dot{q}^A \partial \dot{q}^B]$ has rank n , then, the implicit function theorem guarantees the existence of a local solution $\dot{q}^A = \dot{q}^A(q, p)$. That is, we can locally write the velocities as functions of the positions and the momenta. In this case, the Legendre transformation (4) becomes invertible so that it turns out to be a local diffeomorphism. By substituting the local inversion $\dot{q}^A = \dot{q}^A(q, p)$ into the function (5), one arrives at a well-defined Hamiltonian function $H = H(q^A, p_A)$ that only depends on the positions and momenta. It is now easy to compute that the Hamilton equation (3) generated by this Hamiltonian function are exactly the same as the Euler–Lagrange equation (1). So that, in this case, Hamiltonian analysis of the Euler–Lagrange equations can be achieved immediately.

Degenerate Lagrangians. If, on the other hand, a Lagrangian function is degenerate, then the Legendre transformation (4) fails to be invertible even locally. In this case, the image $\mathbb{F}L(TQ)$ of the tangent bundle TQ under the Legendre transformation can at most be a (so called as the primary constraint) submanifold of T^*Q [19]. Since some of the velocities in the set (\dot{q}^A) cannot be written as functions of the positions and the momenta, the function H , in (5), must also depend on the velocities. That is, $H = H(q^A, p_A, \dot{q}^A)$. One way to understand such a function is to consider it as a family of functions defined on T^*Q parameterized by the variables (\dot{q}^A) . More terminologically, H can be taken as a Morse family of functions [8, 46]. In this case, the dynamical equations on the cotangent bundle become implicit differential equations

$$\dot{q}^A = \frac{\partial H}{\partial p_A}, \quad \dot{p}_A = -\frac{\partial H}{\partial q^A}, \quad \frac{\partial H}{\partial \dot{q}^A} = 0 \Leftrightarrow p_A - \frac{\partial L}{\partial \dot{q}^A} = 0. \quad (6)$$

A key observation here is to see that the explicit Hamiltonian dynamics in (3) and the implicit Hamiltonian dynamics in (6) are both Lagrangian submanifolds of the iterated tangent bundle TT^*Q . This leads to a pure geometric approach to the Legendre transformation which makes this transformation even applicable in constrained or/and degenerate Lagrangians. This interpretation was first introduced by Tulczyjew, and the passing from the higher order tangent to the higher order cotangent of manifold (and viceversa) is known as the Tulczyjew triple [67]. In the main body of the paper we shall present the theory of Tulczyjew triples in detail. Here, TT^*Q is a symplectic manifold equipped with the symplectic two-form

$$\Omega_Q^T = d\dot{q}^A \wedge dp_A + dq^A \wedge d\dot{p}_A \quad (7)$$

obtained from the complete lift of the canonical symplectic two-form Ω_Q on T^*Q [65].

We wish to make an additional remark here by recalling an alternative way of reaching a well-defined explicit Hamiltonian formulation for degenerate systems. In this method, to have a Hamiltonian realization, one needs to employ the Dirac–Bergmann constraint algorithm [19] or its geometric equivalent, the Gotay–Nester–Hinds algorithm [27–29] to the system.

Implicit Hamiltonian dynamics. A Lagrangian submanifold of a symplectic manifold is determined by two conditions. One is that the dimension of the Lagrangian submanifold must be a half of the dimension of the symplectic manifold, and the second is that the symplectic two-form must be identically zero when it is restricted to the Lagrangian submanifold. Notice that, direct substitutions of the explicit Hamiltonian dynamics in (3) or the implicit Hamiltonian dynamics in (6) into the symplectic two-form (7) result with the vanishing of the symplectic two-form. So that, both of them determine Lagrangian submanifolds of TT^*Q .

In general, we say that a system of differential equations is a Hamiltonian system if it can be recast as a Lagrangian submanifold of a certain symplectic manifold. It is evident that if a Lagrangian submanifold is horizontal, according to the Poincaré lemma [46], there locally exists a Hamiltonian function generating the dynamics. This results with an explicit differential system. The existence of Darboux’ coordinates on symplectic manifolds guarantees that such an explicit Hamiltonian system can be written in form (3). If a Lagrangian submanifold is non-horizontal then it is not possible to find either a Hamiltonian vector field or a Hamiltonian function [48]. Instead, it is possible to find (inevitably not a classical Hamiltonian function but) a generating function for a non-horizontal, Lagrangian submanifold, so that it can locally be written as

$$\dot{q}^A = \frac{\partial F}{\partial p_A}, \quad \dot{p}_A = -\frac{\partial F}{\partial q^A}, \quad \frac{\partial F}{\partial \lambda^\alpha} = 0 \quad (8)$$

for a Morse family of functions F . Notice that, in this local picture, F depends on auxiliary variables (λ^α) as well as on the position and the momenta (q^A, p_A) . The existence of this local realization (8) is a manifestation of the generalized Poincaré lemma (also called as Maslov–Hörmander theorem) [9, 33, 46, 69] under the light of special symplectic structures [35, 60].

1.2. Hamilton–Jacobi theory

In order to find some possible analytical solutions of the Hamilton equations, one may employ the Hamilton–Jacobi (HJ) theory. The HJ theory is rooted in the idea of finding an appropriate canonical transformation [4, 9, 34] that leads the system to equilibrium and pairs of action-angle variables that render the dynamics trivial. This philosophy has brought many interesting results, deriving into integrability theories, reduction, KAM theory, among others [20, 24, 61, 62]. Our interest resides in the geometric interpretation of this theory [1, 46, 49], its formulation, and applications. See for example [11], where a geometric framework for the Hamilton–Jacobi theory was presented and the Hamilton–Jacobi equation was formulated both in the Lagrangian and in the Hamiltonian formalisms of autonomous and non-autonomous mechanics.

Let us describe this geometry in local coordinates. Consider a Hamiltonian system defined by a Hamiltonian vector field X_H on the symplectic manifold (T^*Q, Ω_Q) . The time-independent Hamilton–Jacobi problem is a partial differential equation, whose solution is a function W defined on Q , given by

$$H\left(q^A, \frac{\partial W}{\partial q^A}\right) = \varepsilon. \quad (9)$$

Here, the function H is the Hamiltonian function generating the Hamiltonian dynamics, and ε is a constant real number. Notice that, in (9), the momenta is replaced by the partial derivative of a function W that is $p_A = \partial W / \partial q^A$. We call a solution W of (9) general if it additionally depends on n number of some other variables (q^A) as well, that is $W = W(q^A, \varrho^A)$.

The Hamilton–Jacobi theory finds solutions on the lower dimensional manifold Q and retrieves them on the higher dimensional manifold T^*Q by the existence of a section dW of the cotangent bundle. Here, W is a solution of the Hamilton–Jacobi equation (9). Let us now comment how we relate the solutions of Hamilton–Jacobi equation and the solutions of the Hamilton equations. Consider the differential equation

$$\dot{q}^A = \left. \frac{\partial H}{\partial p_A} \right|_{p_A = \partial W / \partial q^A} \quad (10)$$

on Q defined by taking the first set of the Hamilton equation (3) at $p_A = \partial W / \partial q^A$. Notice that, the system (10) consists of a number n of first order differential equations, that is a half of the number of equations in (3). Accordingly, it is easier to solve this system than solving the Hamilton equations. If W is a solution of the Hamilton–Jacobi problem, then a solution of (10) can be lifted to a solution of the Hamilton equations. More concretely, if $\varphi_t = (\varphi_t^A)$ is a solution of (10) then $(\varphi_t, \partial W / \partial q^A(\varphi_t))$ is a solution of the Hamilton equations.

A Hamilton–Jacobi theory for implicit Hamiltonian dynamics. It is important to remark here that the classical HJ theory only deals with explicit Hamiltonian systems. In [25], we presented a generalization of the classical Hamilton–Jacobi theory that proved its suitability in the case of implicit Hamiltonian dynamics. To elaborate this theory, we started with a (possibly non-horizontal) Lagrangian submanifold S of TT^*Q . This submanifold S projects to a submanifold $T\pi_Q(S)$ of TQ by the mapping $T\pi_Q$. Here, $T\pi_Q$ is the tangent mapping of the cotangent bundle projection $\pi_Q : T^*Q \mapsto Q$. Note that $T\pi_Q(S)$ determines an implicit differential equations on Q . The philosophy of the implicit HJ theory is to retrieve solutions of S , provided the solutions of $T\pi_Q(S)$. In similar fashion as in the classical Hamilton–Jacobi theorem 1, in order to lift the solutions in Q to T^*Q , we are still in need of a closed one-form dW on Q , but two ingredients of the theory are missing. One is that the base manifold, denoted by $C = \tau_{T^*Q}(S)$, is not necessarily the whole T^*Q , but possibly a proper submanifold of it. The second is the nonexistence of a Hamiltonian vector field due to the implicit character of the equations. We present the following diagram to summarize this discussion:

$$\begin{array}{ccc}
 S \hookrightarrow TT^*Q & & \\
 \swarrow \tau_{TT^*Q} & & \searrow T\pi_Q \\
 C \hookrightarrow T^*Q & & TQ \hookrightarrow T\pi_Q(S) \\
 \swarrow \pi_Q & & \searrow \tau_Q \\
 & Q &
 \end{array}
 \quad (11)$$

dW (curved arrow from T^*Q to Q)

where τ_Q is the tangent bundle projection from TQ to Q whereas τ_{TT^*Q} is the tangent bundle projection from TT^*Q to T^*Q .

Our first idea to work with a non-horizontal submanifold is to make use of the generalized Poincaré lemma, which affirms that there exists a Morse function F (a family of generating functions) that generates the dynamics of the implicit system. Recall this local realization exhibited in (8). The Morse function F plays the role of the Hamiltonian in the explicit picture. A Hamilton–Jacobi theory for this system consists in finding a function W on Q satisfying

$$F\left(q^A, \frac{\partial W}{\partial q^A}, \lambda^\alpha\right) = \varepsilon, \quad \left. \frac{\partial F}{\partial \lambda^\alpha} \right|_{p_A = \partial W / \partial q^A} = 0, \quad (12)$$

where ε is a constant.

The second idea is to deal with the implicit character of the system by constructing a local a vector field. For this construction we need to consider an auxiliary section $\sigma : C \cap \text{Im}(\text{d}W) \rightarrow S$, because since S is implicit, there may exist several vectors in S projecting to the same point in C . The role of the section σ is to reduce this unknown number to one. As a result, we arrive at a vector field X_σ that satisfies Hamilton-like equations, and which is suitable for the application of the classical HJ theory. We will show the details in the main body of the paper.

1.3. Higher order systems

In our previous work [25], where the HJ formalism was generalized for implicit systems, we also addressed the problem of constructing HJ theory for degenerate Lagrangian theories. Now, as a complement to our previous work, in the present paper we wish to apply the implicit HJ theory to degenerate higher order Lagrangian systems.

Higher order Euler–Lagrange equations. The Euler–Lagrange equations (1) consist of second order differential equations. It is also possible to formulate differential equations involving higher order derivatives in the realm of the Lagrangian formalism. Let us depict here the second order case and postpone the general case to the upcoming sections. Consider a Lagrangian function L depending on the acceleration as well as the position and the momenta, that is $L = L(q^A, \dot{q}^A, \ddot{q}^A)$. Geometrically, the triplets $(q^A, \dot{q}^A, \ddot{q}^A)$ are the elements of the second order tangent bundle T^2Q . After the variational of the action integral, one arrives at the second order Euler–Lagrange equations

$$\frac{\text{d}^2}{\text{d}t^2} \frac{\partial L}{\partial \ddot{q}^A} - \frac{\text{d}}{\text{d}t} \frac{\partial L}{\partial \dot{q}^A} + \frac{\partial L}{\partial q^A} = 0. \quad (13)$$

Note that, the second order Euler–Lagrange equations in (13) give rise to a system of fourth order differential equations if the term $\partial L / \partial \ddot{q}^A$ depends on \ddot{q}^A . In the literature, higher order systems make appearance in many physical theories involving the mathematical description of relativistic particles with spin, string theories, gravitation, Podolsky electromagnetism, in some problems of fluid mechanics and classical physics, and in numerical models arising from the geometric discretization of first order dynamical systems (see [56, 57] for a long but non-exhaustive list of references).

Ostrogradski momenta. The Hamiltonian formulation of the second order Lagrangian formulation (13) is also possible. This can be, for example, achieved by means of the Ostrogradski momenta [55]

$$p_A^{(0)} = \frac{\partial L}{\partial \dot{q}^A} - \frac{\text{d}}{\text{d}t} \left(\frac{\partial L}{\partial \ddot{q}^A} \right), \quad p_A^{(1)} = \frac{\partial L}{\partial \ddot{q}^A}. \quad (14)$$

See that, in this case, we have two sets of momenta denoted by $p_A^{(0)}$ and $p_A^{(1)}$. In this case, define a function

$$H = p_A^{(0)} \dot{q}^A + p_A^{(1)} \ddot{q}^A - L(q^A, \dot{q}^A, \ddot{q}^A) \quad (15)$$

depending on the positions, the velocities, the accelerations, and the Ostrogradski momenta. A second order Lagrangian function is said to be non-degenerate in the sense of Ostrogradski if the rank of the matrix $[\partial^2 L / \partial \ddot{q}^A \partial \ddot{q}^B]$ is maximal. If the Lagrangian function is non-degenerate, then all the accelerations (\ddot{q}^A) can be written as functions of the first order terms ($q^A, \dot{q}^A, p_A^{(1)}$) by employing the implicit function theorem to the second momenta in (14). After the substitution of these solutions, the function in (15) turns out to be a well defined Hamiltonian function $H = H(q^A, \dot{q}^A, p_A^{(0)}, p_A^{(1)})$ on the iterated cotangent bundle T^*TQ equipped with the local coordinates $(q^A, \dot{q}^A, p_A^{(0)}, p_A^{(1)})$. In this case, the Hamilton equations are computed to be

$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A^{(0)}}, \quad \frac{d\dot{q}^A}{dt} = \frac{\partial H}{\partial p_A^{(1)}}, \quad \frac{dp_A^{(0)}}{dt} = -\frac{\partial H}{\partial q^A}, \quad \frac{dp_A^{(1)}}{dt} = -\frac{\partial H}{\partial \dot{q}^A}. \quad (16)$$

After a direct substitution of the Hamiltonian function $H = H(q^A, \dot{q}^A, p_A^{(0)}, p_A^{(1)})$ into the Hamilton equations, one arrives at the second order Euler–Lagrange equation (13). This establishes a Hamiltonian realization of the higher order Lagrangian formalism. If a Lagrangian function is degenerate in the sense of Ostrogradski, then one cannot solve all the acceleration (\ddot{q}^A) as functions of ($q^A, \dot{q}^A, p_A^{(1)}$) referring to the second momenta in (14). So that, the function H in (15) cannot be written as a function depending solely on $(q^A, \dot{q}^A, p_A^{(0)}, p_A^{(1)})$. The accelerations cannot be accommodated by the first order terms. Hence, there is no explicit Hamiltonian function or no explicit Hamilton equations as given in (16). This results in systems of implicit differential equations defined on T^*TQ . In this paper, we are interested in such kind of higher order degenerate Lagrangian systems.

Schmidt method. There is an alternative way to arrive at the Hamiltonian picture of a higher order Lagrangian system. For this, initially, one needs to recast a Lagrangian function depending on higher order derivatives into a Lagrangian function depending only on the first order derivatives. Then, one applies directly the classical Legendre transformation (4) to this first order formalism. There are two ways of writing a higher order Lagrangian as a first order one. The first method is based on the idea that consecutive time derivatives of the initial coordinates form new coordinates with the introduction of Lagrange multipliers. This eventually coincides with the method of Ostrogradski momenta (14). The second one is the Schmidt method [3, 21, 59] where the acceleration is defined as a new coordinate instead of the velocity. One of the advantages of the Schmidt method is the non-existence of Lagrange multipliers in the reduction procedure. Instead, the Lagrangian function is modified by adding a gauge term such that the associated energy function contains additional terms preserving the equations of motion. Another important feature about the Schmidt method is that it equally deals with degenerate or non-degenerate Lagrangians.

Hamilton–Jacobi theory for higher order systems. In [17], the authors propose a HJ theory for higher order systems that are explicit. Nonetheless, they do not propose any generalization of the HJ formalism for higher order implicit equations. This observation has been another point of motivation for our present work. As we have pointed out, the degeneracy levels of Lagrangian function bring some complications when we pursue their Hamiltonian formulation. Apart from this, it is important to note that other technicalities appear in the

case of higher order dynamics. Notice that for the first order theory, the Lagrangian function is defined on TQ , whereas the Hamiltonian function is defined on T^*Q . So that both the domain TQ and the range T^*Q of the Legendre transformation have the same dimensions $2n$. But, for example, in the second order case, the Lagrangian function is defined on T^2Q , and its Hamiltonian counterpart is defined on T^*TQ . The dimension of T^2Q is $3n$, whereas the dimension of T^*TQ is $4n$. This issue also becomes relevant when we write a Hamilton–Jacobi theory for higher order Lagrangians, and concerning the implicit character of the equations. Hence, one needs to perform a careful and coordinate-free analysis to have a correct picture. This is one of the reasons why we prefer to use tools of global analysis in the main body of the paper, while we use local expressions along the rest of the text.

1.4. Statement of the problem and contents

After presenting some fundamentals and pointing out what is missing in the literature, we are now ready to state our aim in the present paper.

Goal of the present paper. In this paper, our goal is to construct a Hamilton–Jacobi theory for the higher order degenerate Lagrangians. More concretely, we will generalize the geometry that we have proposed [25] in such a way that it allows us to construct a Hamilton–Jacobi formalism for higher order systems. As we mentioned previously in [25], two main ideas are proposed to deal with the implicit character of the Lagrangian/Hamiltonian formulation: one is the use of Morse families, and other one is to introduce auxiliary sections. In the present work, we generalize both of these two ideas to a higher order degenerate frame. Further, in this paper, we aim to compare the Ostrogradski and Schmidt method in the realm of a HJ theory for both degenerate and non-degenerate higher order Lagrangians.

Organization of the present paper. The following section is reserved for some basics as well as for a brief summary of Hamilton–Jacobi theory of implicit systems of first order. Additionally, Morse families and special symplectic structures, Tulczyjew triple for the first order dynamics are summarized. In section 3, we review the fundamentals of higher order tangent bundles, Tulczyjew triples for higher order frameworks, and implicit higher-order differential equations. In section 4, we construct a Hamilton–Jacobi theory for higher order implicit Lagrangian systems. We explain our two main approaches to work with the implicit character of the arising higher order implicit equations: one is the Lagrangian submanifold method or Morse family approach, and the second is the construction of a local vector field by the existence of an additional section that reduces the number of vectors in the implicit submanifold projecting to a same point of a lower dimensional bundle. For the Morse family approach, we will elaborate a list of subcases considering the Ostrogradski method and the Schmidt transformation, comparing both cases and illustrating their utility in nondegenerate and degenerate cases. Section 5 shows the applications of an implicit Hamilton–Jacobi theory for higher order dynamical systems in the particular case of second order Lagrangians. We will depict such application making use of the Ostrogradski approach and the Schmidt–Legendre transform. For second order Lagrangians, we also introduce the setting of the acceleration bundle for the Schmidt–Legendre transform, in order to deal likewise with degenerate or nondegenerate higher order implicit Lagrangians. Two particular examples are: a deformed elastic cylindrical beam with fixed ends, and the end of a javelin. The Ostrogradski and Schmidt methods will be compared in this same section for nondegenerate cases, as it is the case in which Ostrogradski applies. As more general models, we will depict the second and third order Lagrangians with affine dependence on their highest order terms. Section 6 contains further commentaries on the usefulness and the limitations of the theory that we propose, as well as

some open questions on possible Hamilton–Jacobi realizations of second order degenerate Lagrangian theories especially coming from the gravitation theory.

2. Fundamentals

2.1. Hamiltonian dynamics on the cotangent bundle

Consider an n -dimensional manifold Q and its cotangent bundle T^*Q with the canonical projection π_Q from T^*Q to Q . T^*Q is equipped with a canonical (Liouville) one-form θ_Q defined as

$$\theta_Q(X) = \langle T\pi_Q(X), \tau_{T^*Q}(X) \rangle, \quad (17)$$

where the pairing on the right hand side is the duality between T^*Q and TQ , [1, 4, 41, 46, 69]. Here, $T\pi_Q$ is the tangent mapping of the projection π_Q whereas τ_{T^*Q} is the tangent bundle projection from TT^*Q to T^*Q . To see these mapping more explicitly, we refer to the commutative diagram exhibited in (11). Minus of the exterior derivative of the canonical one form, that is $\Omega_Q = -d\theta_Q$, is the canonical symplectic two-form on T^*Q . A Hamiltonian system on T^*Q is determined by the triple (T^*Q, Ω_Q, H) , where H being a Hamiltonian function. Geometrically, the Hamilton equations are defined by

$$\iota_{X_H}\Omega_Q = dH, \quad (18)$$

where X_H is the Hamiltonian vector field associated with the Hamiltonian function H , and ι_{X_H} is the interior derivative [2, 63].

Local picture. Let (q^A, p_A) be bundle coordinates on T^*Q that are Darboux coordinates for the symplectic form. In these coordinates, the canonical one-form reads $\theta_Q = p_A dq^A$ whereas the canonical symplectic two-form becomes $\Omega_Q = dq^A \wedge dp_A$. In this local picture, the Hamiltonian vector field X_H is written as

$$X_H = \frac{\partial H}{\partial p_A} \frac{\partial}{\partial q^A} - \frac{\partial H}{\partial q^A} \frac{\partial}{\partial p_A} \quad (19)$$

so that the Hamilton equations turn out to be the ones in (3).

2.2. Geometric Hamilton–Jacobi theory

Consider a Hamiltonian system (T^*Q, Ω_Q, H) with the Hamiltonian vector field X_H . Let γ be a one-form on Q , and define a vector field X_H^γ on Q by

$$X_H^\gamma = T\pi \circ X_H \circ \gamma. \quad (20)$$

This definition implies the commutativity of the following diagram.

$$\begin{array}{ccc} T^*Q & \xrightarrow{X_H} & TT^*Q \\ \downarrow \pi_Q & & \downarrow T\pi_Q \\ Q & \xrightarrow{X_H^\gamma} & TQ \end{array} \quad . \quad (21)$$

We state the geometric Hamilton–Jacobi theorem as follows [11, 37].

Theorem 1. For a closed one-form γ on Q , the following conditions are equivalent:

(i) The vector fields X_H and X_H^γ are γ -related, that is

$$T\gamma(X_H^\gamma) = X_H \circ \gamma. \quad (22)$$

(ii) The equation is fulfilled

$$d(H \circ \gamma) = 0.$$

Notice that, since a solution γ is assumed to a closed one-form, by Poincaré lemma there locally exists a function W satisfying $dW = \gamma$. Substitution of this into the condition $d(H \circ \gamma) = 0$ results with the classical formulation of the Hamilton–Jacobi problem (9) where the constant ε appears as a manifestation of the integration.

Geometric HJ theories in the literature. This realization of the Hamilton–Jacobi theory has been devised in various situations, as it is the case of nonholonomic systems [13, 24, 32, 36, 52, 53], geometric mechanics on Lie algebroids [5], almost-Poisson manifolds [38], singular systems [40], Nambu–Poisson framework [44], control theory [7], classical field theories [37, 39, 45], partial differential equations in general [68], the geometric discretization of the Hamilton–Jacobi equation [43, 54], and others [6, 12].

2.3. Lagrangian submanifolds and Morse families

Let (M, Ω) be a symplectic manifold, and S_M be a submanifold of M . We define the symplectic orthogonal complement of TS_M as the set of tangent vectors

$$TS_M^\perp = \{u \in TM \mid \Omega(u, v) = 0, \forall v \in TS_M\}. \quad (23)$$

S_M is called a Lagrangian submanifold of M if $TS_M = TS_M^\perp$. In this case, the dimension of S_M is equal to the half of the dimension of M . For different types of manifolds (Poisson, Nambu–Poisson, etc), the definition of a Lagrangian submanifold has been accommodated to its background. See for example [46]. In the case of mechanical systems, the Lagrangian submanifolds have a physical interpretation as a generalization of the set of possible initial momenta of a given point in the configuration space.

Consider the canonical symplectic manifold (T^*Q, Ω_Q) . As it is well known, the image space of a closed one-form γ on Q is a Lagrangian submanifold of T^*Q . We call such kind of Lagrangian submanifolds as horizontal Lagrangian submanifolds. If a Lagrangian submanifold of T^*Q is non-horizontal, that is, if it is not possible to determine it as the image of a closed one-form, then one possibility is to employ the theory of Morse families in order to express the Lagrangian submanifold in terms of a generating function.

Morse families. Let (R, τ, N) be a fiber bundle. The vertical bundle VR over R is the space of vertical vectors $U \in TR$ satisfying $T\tau(U) = 0$. The conormal bundle of VR is defined by

$$V^0R = \{\alpha \in T^*R : \langle \alpha, U \rangle = 0, \forall U \in VR\}.$$

Let E be a real-valued function on R , then the image of its exterior derivative is a submanifold of T^*R . We say that E is a Morse family (or an energy function) if

$$T_z \text{Im}(dE) + T_z V^0R = T_z T^*R, \quad (24)$$

for all $z \in \text{Im}(dE) \cap V^0R$. A Morse family defined on (R, τ, N) generates a Lagrangian submanifold of the canonical symplectic manifold (T^*N, Ω_N) in the following way [8]:

$$S_N = \{w \in T^*N : T^*\tau(w) = dE(z), \text{ for some } z \in T^*R\}. \quad (25)$$

In this case, we say that S_N is generated by the Morse family E . Note that, in the definition of S_N , there is an intrinsic requirement that $\tau(z) = \pi_N(w)$. The inverse of this statement is also true, that is, any Lagrangian submanifold is generated by a Morse family. This is known as the generalized Poincaré lemma [9, 33, 46, 66, 69]. Here, we are presenting the following diagram in order to summarize this discussion.

$$\begin{array}{ccc} \mathbb{R} & \xleftarrow{E} & R \\ & & \downarrow \tau \\ & & N \\ & & \text{=====} \\ & & N \end{array} \quad \begin{array}{c} S_N \\ \curvearrowright \\ T^*N \\ \downarrow \pi_N \\ N \end{array} \quad . \quad (26)$$

Local picture for Morse families. Assume that N is equipped with local coordinates (x^a) , and consider the bundle local coordinates (x^a, λ^α) on the total space R . In this picture, a function E is called a Morse family if the rank of the matrix

$$\begin{pmatrix} \frac{\partial^2 E}{\partial x^a \partial x^b} & \frac{\partial^2 E}{\partial x^a \partial \lambda^\alpha} \end{pmatrix} \quad (27)$$

is maximal. In such a case, the Lagrangian submanifold (25) generated by E locally looks like

$$S_N = \left\{ \left(x^a, \frac{\partial E}{\partial x^a}(x, \lambda) \right) \in T^*N : \frac{\partial E}{\partial \lambda^\alpha}(x, \lambda) = 0 \right\}. \quad (28)$$

See that the dimension of S_N is half of the dimension of T^*N , and that the canonical symplectic two-form Ω vanishes on S_N .

Special symplectic structures. Let P be a symplectic manifold carrying an exact symplectic two-form $\Omega = d\Theta$. Assume also that, P is the total space of a fibre bundle (P, π, N) . A special symplectic structure is a quintuple $(P, \pi, N, \Theta, \chi)$ where χ is a fiber preserving symplectic diffeomorphism from P to the cotangent bundle T^*N [35, 60]. Here, χ can uniquely be characterized by

$$\langle \chi(p), \pi_* X(p) \rangle = \langle \Theta(p), X(p) \rangle \quad (29)$$

for a vector field X on P , for any point p in P where $\pi(p) = n$. Note that, pairing on the left hand side of (29) is the natural pairing between the cotangent space T_n^*N and the tangent space $T_n N$. Pairing on the right hand side of (29) is the one between the cotangent space T_p^*P and the tangent space $T_p P$. We refer [8, 35, 60] for further discussions on special symplectic structures. Here is a diagram exhibiting the special symplectic structure.

$$\begin{array}{ccc} T^*N & \xleftarrow{\chi} & P \\ \pi_N \downarrow & & \swarrow \pi \\ & & N \end{array} \quad . \quad (30)$$

The two-tuple (P, Ω) is called as underlying symplectic manifold of the special symplectic structure $(P, \pi, N, \Theta, \chi)$.

Let $(P, \pi, N, \Theta, \chi)$ be a special symplectic structure. Assume also that S_P be a Lagrangian submanifold of P . The image $S_N = \chi(S_P)$ of S_P is a Lagrangian submanifold of T^*N . By referring to the generalized Poincaré lemma presented in the previous subsection, we argue that the Lagrangian submanifold S_N can locally be generated by a Morse family E on a fiber bundle (R, τ, N) . Accordingly, we call the Morse family E a generator of both S_N and S_P since they are

the same up to χ . The following diagram summarizes this discussion by equipping a Morse family (26) to a special symplectic structure (30).

$$\begin{array}{ccccc}
\mathbb{R} & \xleftarrow{E} & R & & \\
& & \downarrow \tau & & \\
& & N \equiv N & & \\
& & \uparrow \pi_N & \swarrow \pi & \\
& & T^*N & \xleftarrow{\chi} & P \\
& & \uparrow S_N & & \uparrow S_P
\end{array}
\quad (31)$$

2.4. Tulczyjew triple

In this subsection, we demonstrate that the tangent bundle TT^*Q admits two different special symplectic structures [65–67]. Let us start this analysis by elaborating the symplectic two-form on TT^*Q .

Symplectic structure on TT^*Q . Consider the canonical symplectic manifold T^*Q equipped with the exact symplectic two-form $\Omega_Q = -d\Theta_Q$. Define a derivation i_T taking the symplectic two-form Ω_Q on T^*Q to a one-form on TT^*Q as, for X in TTT^*Q ,

$$i_T \Omega_Q(X) = \Omega_Q(T\tau_{T^*Q}(X), \tau_{TT^*Q}(X)).$$

Here, $T\tau_{T^*Q}$ is the tangent mapping of the bundle projection τ_{T^*Q} whereas τ_{TT^*Q} is the tangent bundle projection TTT^*Q to TT^*Q . Accordingly, we define two one-forms on TT^*Q as $\theta_1 = i_T\Omega_Q$ and $\theta_2 = d_T\theta_Q$ where the derivation d_T is the commutator $[d, i_T]$. Minus of the exterior derivatives of these one-forms results with a symplectic two-form

$$\Omega_Q^T = -d\theta_1 = -d\theta_2 \quad (32)$$

on TT^*Q [65, 67]. In terms of the induced local coordinate chart $(q^A, p_A; \dot{q}^A, \dot{p}_A)$, the potential one-forms are computed to be

$$\theta_1 = i_T \Omega_Q = \dot{p}_A dq^A - \dot{q}^A dp_A, \quad \theta_2 = d_T \Theta_Q = \dot{p}_A dq^A + p_A d\dot{q}^A. \quad (33)$$

Notice that, in this case, the symplectic two-form Ω_Q^T turns out to be the one in (7). Note that, the difference $\theta_1 - \theta_2$ is an exact one-form. Actually, it is the exterior derivative of coupling function $\dot{q}^A p_A$ in the Legendre transformation (4).

Special symplectic structures on TT^*Q . The non-degeneracy of the canonical symplectic structure Ω_Q on T^*Q leads to the existence of the following symplectomorphism

$$\Omega_Q^b : TT^*Q \mapsto T^*T^*Q : X \mapsto \iota_X \Omega_Q : (q^A, p_A, \dot{q}^A, \dot{p}_A) \mapsto (q^A, p_A, -\dot{p}_A, \dot{q}^A). \quad (34)$$

It is a matter of a direct calculation to prove that, the quintuple

$$(TT^*Q, T^*Q, \tau_{T^*Q}, \theta_1, \Omega_O^b) \quad (35)$$

is a special symplectic manifold. Here, θ_1 is the differential one-form defined in (33). There exists a canonical involution on TTQ . A *dualization* of this determines a symplectomorphism Ξ_Q from TT^*Q to the cotangent bundle T^*TQ . In local the coordinates, we obtain

$$\Xi_Q: T^*Q \mapsto T^*TQ: (q^A, p_A, \dot{q}^A, \dot{p}_A) = (q^A, \dot{q}^A, \dot{p}^A, p_A). \quad (36)$$

Then it becomes easy to prove that

$$(TT^*Q, TQ, T\pi_Q, \theta_2, \Xi_Q) \quad (37)$$

is a special symplectic manifold. Here, θ_2 is the differential one-form defined in (33). As a result, we provide two special symplectic structures for the symplectic manifold (TT^*Q, Ω_Q^T) . Tulczyjew triple is a combination of these two special symplectic structures in one commutative diagram as given below [30, 31, 48, 64, 66, 67].

$$\begin{array}{ccccc} T^*TQ & \xleftarrow{\Xi_Q} & TT^*Q & \xrightarrow{\Omega_Q^b} & T^*T^*Q \\ & \searrow \pi_{TQ} & \searrow T\pi_Q & \searrow \tau_{T^*Q} & \searrow \pi_{T^*Q} \\ & & TQ & & T^*Q \end{array} \quad (38)$$

We cite [22, 23, 70] for the Tulczyjew triple in the Lie group framework.

2.5. Hamilton–Jacobi theory for implicit Hamiltonian systems

In [25], we have proposed two methods to construct Hamilton–Jacobi formulations of implicit Hamiltonian systems. The first method consists of a theory which does refer to Morse families. The second is based on the construction of a local vector field defined on the image of a section, but not globally on the phase space.

The method of Morse families. By referring to the special symplectic structure (35) accommodated on the right wing of the Tulczyjew triple, we see that for every Lagrangian submanifold S of TT^*Q , there exists a Morse family F generating S so that

$$S = \left\{ \left(q^A, p_A; \frac{\partial F}{\partial p_A}, -\frac{\partial F}{\partial q^A} \right) \in TT^*Q : \frac{\partial F}{\partial \lambda^a} = 0 \right\} \quad (39)$$

where $F = F(q^A, p_A, \lambda^a)$. Notice that F is defined on a fiber bundle R (equipped with coordinates (q^A, p_A, λ^a)) over the cotangent bundle T^*Q . Here is the diagram summarizes the Morse family.

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{F} & R & & T^*T^*Q \xleftarrow{\Omega_Q^b} T^*T^*Q \\ & & \downarrow \tau & & \uparrow \tau_{T^*Q} \\ & & T^*Q & \xlongequal{\quad} & T^*Q \end{array} \quad (40)$$

where we have employed the right wing of the Tulczyjew triple (38).

We consider the restriction of the momenta (p_A) to the image space of a closed one-form $\gamma = \gamma_A dq^A$ on Q . This reads the following restricted submanifold

$$S|_{\text{Im}(\gamma)} = \left\{ \left(q^A, \gamma_A(q); \frac{\partial F}{\partial p_A} \Big|_{\text{Im}(\gamma)}, -\frac{\partial F}{\partial q^A} \Big|_{\text{Im}(\gamma)} \right) \in TT^*Q : \frac{\partial F}{\partial \lambda^a} \Big|_{\text{Im}(\gamma)} = 0 \right\}. \quad (41)$$

Note that, if the Lagrangian submanifold S is the image of a Hamiltonian vector field X_H , then $S|_{\text{Im}(\gamma)}$ reduces to the image space of the composition $X_H \circ \gamma$. Since the submanifold $S|_{\text{Im}(\gamma)}$ does not depend on the momentum variables, we can project it to a submanifold S^γ of TQ via the tangent mapping $T\pi_Q$ as follows

$$S^\gamma = T\pi_Q \circ S|_{\text{Im}(\gamma)} = \left\{ \left(q^i, \frac{\partial F}{\partial p_A} \Big|_{\text{Im}(\gamma)} \right) \in TQ : \frac{\partial F}{\partial \lambda^a} \Big|_{\text{Im}(\gamma)} = 0 \right\}. \quad (42)$$

The submanifold S^γ defines an implicit differential equation on Q . We state the generalization of the Hamilton–Jacobi theorem 1 as follows [25].

Theorem 2. *The following conditions are equivalent for a closed one-form γ :*

- (i) *The Lagrangian submanifold S in (39) and the submanifold S^γ in (42) are γ -related, that is*

$$T\gamma(S^\gamma) = S|_{\text{Im}(\gamma)}$$

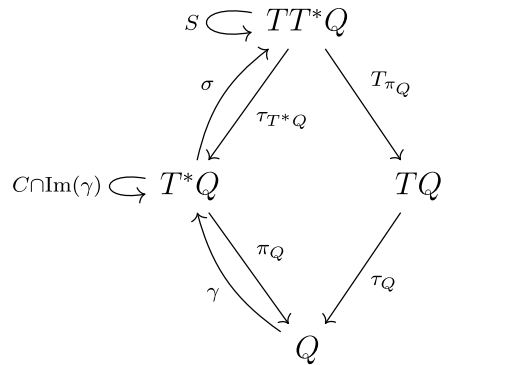
- (ii) *$dF(q, \gamma(q), \lambda) = 0$, where F is the Morse family generating S .*

In a local chart, we can take γ as the exterior derivative a function $W = W(q)$. Hence, the local version of the second condition in theorem 2 turns out to be (12). Notice that, if the Lagrangian submanifold S is horizontal, then the Morse function F becomes independent of the auxiliary variables (λ^a) . In this case, the Morse function F becomes a well defined Hamiltonian function on T^*Q so that the implicit HJ theorem 2 reduces to the classical HJ theorem 1.

The method of local vector fields. Assuming that the integrability conditions are all satisfied [47], let S be a (possibly non-horizontal) Lagrangian submanifold of TT^*Q , and consider its projection $C = \tau_{T^*Q}(S)$. Due to the non-horizontal character of S , there may exist several vectors in S projecting to the same point in C , so that C does not have to be the whole T^*Q . Define an auxiliary section $\sigma : T^*Q \rightarrow TT^*Q$ satisfying $\sigma(C) \subset S$. The role of the section σ is to reduce this unknown number to one. We are additionally require that the domain of the section σ be the intersection of $\text{Im}(\gamma)$ and C . Here, γ is a closed one-form on Q . As a result, we arrive at a vector field X_σ . Note that X_σ satisfies

$$\iota_{X_\sigma} \Omega_Q = \Theta(\gamma(q)) \quad (43)$$

for an arbitrary one-form Θ defined on $\gamma(q)$. We record here the following diagram in order to visualize the sections more explicitly.



We define a vector field X_σ^γ on the tangent bundle TQ as

$$X_\sigma^\gamma = T\pi \circ X_\sigma \circ \gamma. \quad (44)$$

In local coordinates, the vector field X_σ and its projection X_σ^γ can be written as

$$X_\sigma = \sigma^A(q, \gamma(q)) \frac{\partial}{\partial q^A} + \sigma_A(q, \gamma(q)) \frac{\partial}{\partial p_A}, \quad X_\sigma^\gamma = \sigma^A(q, \gamma(q)) \frac{\partial}{\partial q^A}, \quad (45)$$

respectively. Using a one-form section γ on Q , the tangent lift of the projected vector field X_σ^γ is

$$T\gamma(X_\sigma^\gamma) = \sigma^A \left(\frac{\partial}{\partial q^A} + \frac{\partial \gamma_B}{\partial q^A} \frac{\partial}{\partial p_B} \right). \quad (46)$$

Using (44), we find an expression relating the section σ and the vector fields as follows.

$$\sigma^A(q, \gamma(q)) \frac{\partial \gamma_B}{\partial q^A}(q) = \sigma_B(q, \gamma(q)). \quad (47)$$

We are ready now to state the following theorem [25].

Theorem 3. *Given the conditions above, we say that: the two vector fields X_σ and X_σ^γ are γ -related if and only if (47) is fulfilled.*

3. Higher order dynamical systems

Let us consider differential manifolds and standard tensor bundle calculus. It is assumed throughout the text that all structures and mappings are smooth (C^∞ -class). For very detailed descriptions of fundamentals, we refer to [14] and we shall skip to our notation and brief comments on the essentials.

3.1. Geometry of higher order bundles

Jet bundles. Given a fibration (P, π, N) , consider the dimension of P be p and that of N be n . Consider a section $s : N \rightarrow P$ and let us denote by $\text{Sec}(P)$ the set of all sections on P . We say that two sections $s, s' \in \text{Sec}(P)$ are k -related for $0 \leq k \leq \infty$ in a point $x \in N$ if $s(x) = s'(x)$ and for all functions $f : P \rightarrow \mathbb{R}$, the function $f \circ s - f \circ s' : N \rightarrow \mathbb{R}$ is flat of order k at x , that is, this function and all the derivatives up to order k included are zero at x . The equivalence class determined by the k -relation is called jet of order k for a section $j^k s(x)$ [58]. The set of all k -jets at x is denoted by $J_x^k(P, \pi, N)$. For the union of all of them at any point x , we say $J^k(P, \pi, N)$. More generally, we can define now at a point a mapping from N to P . Consider a function $f : N \rightarrow P$, then the equivalence class determined by the k -equivalence is called the k -jet of f at x . For a representative of the class we use $j^k f(x)$ and the set of k -jets is represented by $J_x^k(P, N)$ and again, the union of these at every x will be represented by $J^k(P, N)$. Notice that the manifold $J^k(P, \pi, N)$ is a submanifold of $J^k(P, N)$ and so the above projections admit restrictions to it.

Both in the case of sections or mappings it is possible to define jets for local sections or mappings. For it, one works with the germs, which are the equivalence classes determined by the relation that two section/mappings are related if they have the same value at every point in the intersection of their domains.

The k -jet manifold of section/mappings can be fibered in different ways, we have

$$\alpha^k : J^k(P, N) \rightarrow N; \quad \alpha^k(j^k(x)) = x, \quad (48)$$

$$\beta^k : J^k(P, N) \rightarrow P \quad \alpha^k(f^k(x)) = x \quad (49)$$

$$\rho_r^k : J^k(P, N) \rightarrow J^r(P, N) \quad \rho_r^k(f^k(x)) = f^r(x), \quad r \leq k. \quad (50)$$

Here the α^k projection is called the source projection and β^k is the target projection. Now, consider $J^1(J^k(P, N), \alpha^k, N)$ be a manifold of 1-jets of $\alpha^k : J^k(P, N) \rightarrow N$. The interest of this manifold is that the fibered manifold $J^{k+1}(P, N)$ can be regularly immersed into $J^1(J^k(P, N), \alpha^k, N)$.

$$\begin{array}{ccccc} J^k P & \xleftarrow{\rho_k^{k+1}} & J^{k+1} P & \xrightarrow{\psi} & J^1(J^k P) \\ & \nwarrow \rho_k^{k+1} \circ u & \uparrow u & \nearrow j^1(\rho_k^{k+1} \circ u) & \\ & & N & & \end{array}$$

such that

$$u(x) = J^1(\rho_k^{k+1} \circ u)(x) \quad (51)$$

for a function $u : N \rightarrow J^{k+1}P$. Note that $J^{k+1}P \subset J^1(J^k P)$. We can set local coordinates for jets, the jet manifold $J^k(P, N)$ has an atlas when it is modeled in the space $\mathbb{R}^p \times \mathbb{R}^n \times J^k(\mathbb{R}^p, \mathbb{R}^n)$, locally we may think $J^k(P, N)$ as $J^k(\mathbb{R}^p, \mathbb{R}^n)$. Locally, $(x_a, \xi^A, \xi_{a(r)}^A)$ with $1 \leq r \leq k$ is the coordinate representation of a point of $J^k(P, N)$.

Higher order tangent bundles. Now, a particular type of jet manifold is the tangent bundle of higher order. Consider Q a configuration space of dimension n , TQ is the tangent bundle and T^*Q is the cotangent bundle or phase space for a dynamical system. The k -order tangent bundle can be identified with k order jets in the following way

$$T^k Q = J_0^k(\mathbb{R} \times Q, \pi_1, \mathbb{R}). \quad (52)$$

So that $T^k Q$ is a submanifold of $J_0^k(\mathbb{R} \times Q, \pi_1, \mathbb{R})$. Here, π_1 is the projection from $\mathbb{R} \times Q$ to the first factor \mathbb{R} . One has the same type of fibrations as for the jets above. In fact, if $r \leq k$, we have the canonical projection $\rho_r^k : T^k Q \rightarrow T^r Q$, given by $\rho_r^k(\sigma^k(0)) = \sigma^r(0)$, and the target projection is $\beta^k : T^k Q \rightarrow Q$, given by $\beta^k(\sigma^k(0)) = \sigma(0)$. One has obviously $\rho_0^k = \beta^k$, where $T^0 Q$ is identified canonically with Q .

To describe the local coordinates in $T^k Q$, let (U, φ) be a local chart in Q , with $\varphi = (\varphi^A)$, $1 \leq A \leq n$, and $\varphi : \mathbb{R} \rightarrow Q$ is a curve in Q such that $\varphi(0) \in U$; by writing $\phi^A = \varphi^A \circ \phi$, the k -jet $\tilde{\phi}^k(0)$ is uniquely represented in $(\beta^k)^{-1}(U) = T^k U$ by

$$(q_{(0)}^A, q_{(1)}^A, q_{(2)}^A, \dots, q_{(k)}^A) := (q^A, \dot{q}^A, \ddot{q}^A, \dots, q^{(k)A}) \quad (53)$$

where

$$q^A = \phi^A(0) \quad ; \quad q_{(i)}^A = \left. \frac{d^{(i)} \phi^A}{dt^{(i)}} \right|_{t=0}$$

in the open set $(\beta^k)^{-1}(U) \subseteq T^k Q$. The local expression of the canonical projections β^k and ρ_r^k are

$$\rho_r^k(q_{(0)}^A, q_{(1)}^A, \dots, q_{(k)}^A) = (q_{(0)}^A, q_{(1)}^A, \dots, q_{(r)}^A), \quad \beta^k(q_{(0)}^A, q_{(1)}^A, \dots, q_{(k)}^A) = (q_{(0)}^A).$$

Hence, local coordinates in the open set $(\beta^k)^{-1}(U) \subseteq T^k Q$ adapted to the ρ_r^k -bundle structure are

$$(q_{(0)}^A, \dots, q_{(r)}^A, q_{(r+1)}^A, \dots, q_{(k)}^A),$$

and a section $s \in \Gamma(\rho_r^k)$ is locally given in this open set by

$$s(q_{(0)}^A, \dots, q_{(r)}^A) = (q_{(0)}^A, \dots, q_{(r)}^A, s_{(r+1)}^A, \dots, s_{(k)}^A),$$

where $s_{(j)}^A$ (with $r+1 \leq j \leq k$) are local functions. This approach is very useful to work on the tangent bundle $TT^{k-1}Q$. Accordingly, we denote the induced coordinates on $TT^{k-1}Q$ as

$$(q_{(\kappa)}^A; \dot{q}_{(\kappa)}^A) = (q_{(0)}^A, q_{(1)}^A, \dots, q_{(k)}^A; \dot{q}_{(0)}^A, \dot{q}_{(1)}^A, \dots, \dot{q}_{(k)}^A) \in TT^{k-1}Q,$$

where κ runs from 0 to $k-1$.

3.2. Tulczyjew triples for higher order bundles

Consider the natural embedding of $T^k Q$ into the iterated tangent bundle $TT^{k-1}Q$ of $T^{k-1}Q$. This is locally given by

$$\iota : T^k Q \mapsto TT^{k-1}Q : (q_{(0)}^A, q_{(1)}^A, \dots, q_{(k)}^A) \mapsto (q_{(0)}^A, q_{(1)}^A, \dots, q_{(k-1)}^A; q_{(1)}^A, \dots, q_{(k-1)}^A, q_{(k)}^A), \quad (54)$$

see [42]. Here, the induced coordinates on $TT^{k-1}Q$ are assumed to be

$$(q_{(\kappa)}^A; \dot{q}_{(\kappa)}^A) = (q_{(0)}^A, q_{(1)}^A, \dots, q_{(k-1)}^A; \dot{q}_{(0)}^A, \dot{q}_{(1)}^A, \dots, \dot{q}_{(k-1)}^A) \in TT^{k-1}Q, \quad (55)$$

where κ runs from 0 to $k-1$. For future reference, let us record here the particular case $s=2$ that is

$$T^2 Q \longrightarrow TTQ : (q_{(0)}^A, q_{(1)}^A, q_{(2)}^A) \longrightarrow (q_{(0)}^A, q_{(1)}^A; q_{(1)}^A, q_{(2)}^A). \quad (56)$$

This embedding will enable us to study the dynamics on the higher order bundles in the framework of Tulczyjew triples.

Recall first the first order Tulczyjew triple presented in section 2.4. By replacing the configuration manifold Q in the classical first order Tulczyjew triple by the $(k-1)$ th order tangent bundle $T^{k-1}Q$, we draw the following generalized Tulczyjew triple [21] proper for the higher order frameworks

$$\begin{array}{ccccc} T^*TT^{k-1}Q & \xleftarrow{\Xi_{T^{k-1}Q}} & TT^*T^{k-1}Q & \xrightarrow{\Omega_{T^{k-1}Q}^b} & T^*T^*T^{k-1}Q \\ & \searrow \pi_{TT^{k-1}Q} & \swarrow T\pi_{T^{k-1}Q} & \searrow \tau_{T^*T^{k-1}Q} & \swarrow \pi_{T^*T^{k-1}Q} \\ & & TT^{k-1}Q & & T^*T^{k-1}Q \end{array} \quad (57)$$

Here, $\pi_{TT^{k-1}Q}$ is the cotangent bundle projection, $T\pi_{T^{k-1}Q}$ is the tangent lift of $\pi_{T^{k-1}Q}$, $\tau_{T^*T^{k-1}Q}$ is the tangent bundle projection, and $\pi_{T^*T^{k-1}Q}$ is the cotangent bundle projection.

Since $T^*T^{k-1}Q$ is a cotangent bundle, the pair $(T^*T^{k-1}Q, \Omega_{T^{k-1}Q})$ is a symplectic manifold with the canonical symplectic two-form $\Omega_{T^{k-1}Q} = d\Theta_{T^{k-1}Q}$. On $T^*T^{k-1}Q$, the Darboux coordinates are

$$(q_{(\kappa)}^A; p_A^{(\kappa)}) = (q_{(0)}^A, q_{(1)}^A, \dots, q_{(k-1)}^A, p_A^{(0)}, p_A^{(1)}, \dots, p_A^{(k-1)}) \in T^*T^{k-1}Q,$$

so we write the canonical two-form as

$$\Omega_{T^{k-1}Q} = \sum_{\kappa=0}^{k-1} dp_A^{(\kappa)} \wedge dq_{(\kappa)}^A. \quad (58)$$

On $TT^*T^{k-1}Q$, introduce the following local coordinate system

$$(q_{(\kappa)}^A, p_A^{(\kappa)}, \dot{q}_{(\kappa)}^A, \dot{p}_A^{(\kappa)}) \in TT^*T^{k-1}Q \quad (59)$$

where κ runs from 0 to $k-1$. The pair $(TT^*T^{k-1}Q, \Omega_{TT^*T^{k-1}Q}^T)$ is a symplectic manifold with lifted symplectic two-form. In terms of the coordinates, $\Omega_{TT^*T^{k-1}Q}^T$ can be written as

$$\Omega_{TT^*T^{k-1}Q}^T = \sum_{\kappa=0}^{k-1} d\dot{p}_A^{(\kappa)} \wedge dq_{(\kappa)}^A + \sum_{\kappa=0}^{k-1} dp_A^{(\kappa)} \wedge d\dot{q}_{(\kappa)}^A. \quad (60)$$

Then, we define the adapted symplectic diffeomorphism $\Xi_{T^{k-1}Q}$ and $\Omega_{T^{k-1}Q}^b$ from the symplectic diffeomorphism Ξ_Q and Ω_Q^b in the first order Tulczyjew triple (57). Accordingly, they are computed as

$$\begin{aligned} \Xi_{T^{k-1}Q}(q_{(\kappa)}, p^{(\kappa)}, \dot{q}_{(\kappa)}, \dot{p}^{(\kappa)}) &= (q_{(\kappa)}, \dot{q}_{(\kappa)}, \dot{p}^{(\kappa)}, p^{(\kappa)}), \\ \Omega_{T^{k-1}Q}^b(q_{(\kappa)}, p^{(\kappa)}, \dot{q}_{(\kappa)}, \dot{p}^{(\kappa)}) &= (q_{(\kappa)}, p^{(\kappa)}, \dot{p}^{(\kappa)}, -\dot{q}_{(\kappa)}), \quad \kappa = 0, \dots, k-1. \end{aligned} \quad (61)$$

We remark here that both the left and the right wings of the higher order Tulczyjew triple are special symplectic structures, and the triple is merging them to enable a Legendre transformation for the singular or/and constrained higher order dynamical systems.

3.3. Explicit higher order differential equations

Consider the bundle projection $\pi_1: \mathbb{R} \times Q \rightarrow \mathbb{R}$ onto the first factor. If $\phi: \mathbb{R} \rightarrow Q$ is a curve in Q , the canonical lifting of ϕ to T^kQ is the curve $j^k\phi: \mathbb{R} \rightarrow T^kQ$. We consider the module of vector fields $\mathfrak{X}(\pi_s')$ along the projection $\pi_s': J^r\pi \rightarrow J^s\pi$. The k th holonomic lift of $X = X_o\partial/\partial t \in \mathfrak{X}(\mathbb{R})$ is given by

$$j^kX = X_o \left(\frac{\partial}{\partial t} + \sum_{i=0}^k q_{(i+1)}^A \frac{\partial}{\partial q_{(i)}^A} \right).$$

Using the identification $J^k\pi \cong \mathbb{R} \times T^kQ$ and denoting by $\pi_2: \mathbb{R} \times Q \rightarrow Q$ the natural projection onto the second factor, and all the induced projections in higher order jet bundles, we have the following diagram.

$$\begin{array}{ccc} \mathbb{R} \times T^{k+1}Q & \xrightarrow{\pi_2^{k+1}} & T^{k+1}Q \\ \downarrow \pi_k^{k+1} & & \downarrow \rho_k^{k+1} \\ \mathbb{R} \times T^kQ & \xrightarrow{\pi_2^k} & T^kQ \end{array}.$$

Definition 1. A curve $\psi: \mathbb{R} \rightarrow T^k Q$ is holonomic of type r , $1 \leq r \leq k$, if $j^{k-r+1}\phi = \rho_{k-r+1}^k \circ \psi$, where $\phi = \beta^k \circ \psi, \phi: \mathbb{R} \rightarrow Q$; that is, the curve ψ is the lifting of a curve in Q up to $T^{k-r+1}Q$.

$$\begin{array}{ccc}
 & & T^k Q \\
 & \nearrow \psi & \downarrow \rho_{k-r+1}^k \\
 \mathbb{R} & \xrightarrow{\rho_{k-r+1}^k \circ \psi} & T^{k-r+1} Q \\
 & \searrow j^{k-r+1}\phi & \downarrow \text{Id} \\
 & & T^{k-r+1} Q \\
 & \searrow \phi = \beta^k \circ \psi & \downarrow \beta^{k-r+1} \\
 & & Q
 \end{array}$$

In particular, a curve ψ is holonomic of type 1 if $j^k\phi = j_0^k\phi$, with $\phi = \beta^k \circ \psi$. Throughout this paper, holonomic curves of type 1 are simply called holonomic.

Definition 2. A vector field $X \in \mathfrak{X}(T^k Q)$ is a semispray of type r , $1 \leq r \leq k$, if every integral curve ψ of X is holonomic of type r .

The local expression of a semispray of type r is

$$X = q_{(1)}^A \frac{\partial}{\partial q_{(0)}^A} + q_{(2)}^A \frac{\partial}{\partial q_{(1)}^A} + \dots + q_{(k-r+1)}^A \frac{\partial}{\partial q_{(k-r)}^A} + F_{k-r+1}^A \frac{\partial}{\partial q_{(k-r+1)}^A} + \dots + F_k^A \frac{\partial}{\partial q_{(k)}^A} \quad (62)$$

where $F_{k-r+1}^A, \dots, F_k^A$ are functions of $q_{(i)}, i = 1, \dots, k$. Observe that semisprays of type 1 in $T^k Q$ are the analogue to holonomic vector fields in first order mechanics. Their local expressions are

$$X = q_{(1)}^A \frac{\partial}{\partial q_{(0)}^A} + q_{(2)}^A \frac{\partial}{\partial q_{(1)}^A} + \dots + q_{(k)}^A \frac{\partial}{\partial q_{(k-1)}^A} + F^A \frac{\partial}{\partial q_{(k)}^A}. \quad (63)$$

If $X \in \mathfrak{X}(T^k Q)$ is a semispray of type r , a curve $\phi: \mathbb{R} \rightarrow Q$ is said to be a *path* or *solution* of X if $j^k\phi$ is an integral curve of X ; that is, $\widetilde{j^k\phi} = X \circ j^k\phi$, where $\widetilde{j^k\phi}$ denotes the canonical lifting of $j^k\phi$ from $T^k Q$ to $TT^k Q$.

3.4. Implicit higher order differential equations

Consider a k th order system

$$\Phi^l(q, \dot{q}, \ddot{q}, \dots, q^{(k)}) = 0, \quad l = 1, \dots, r$$

of differential equations defined by r number equations on a configuration manifold Q . Geometrically, the functions Φ^l define a submanifold S of $T^k Q$. Using the induced coordinates on the higher order tangent bundle, this submanifold is given locally by

$$S = \{\mathbf{q} := (q, \dot{q}, \ddot{q}, \dots, q^{(k)}) \mid \Phi^l(\mathbf{q}) = 0\}. \quad (64)$$

A differentiable curve ϕ on Q whose canonical k -lifting is a curve $\psi = j_0^k\phi$ on $T^k Q$ is a solution of $S \subset T^k Q$ if the lifted curve lies in S . The submanifold S can be understood as a first

order differential equation defined on $TT^{k-1}Q$ as well. To this end we first consider the natural embedding of T^kQ into the iterated tangent bundle $TT^{k-1}Q$ of $T^{k-1}Q$. This is locally described as in (54). For $s = 2$, recall (56). Using the mapping in (54), image $\iota(S)$ of S is a submanifold of $TT^{k-1}Q$. The differential equation S is called explicit if there exists a vector field X on $T^{k-1}Q$ such that $\text{Im}(X)$ is $\iota(S)$. Otherwise, S is called an implicit differential equation.

Looking for a Lagrangian function generating a differential equation is the inverse problem of calculus of variations. See, for example, [51] for a geometric approach to this problem for the case of $s = 1$. For the fourth order explicit systems, in [26], some conditions are proposed for the existence and uniqueness of a Lagrangian function. In this work, we assume that there exists already a Lagrangian function generating the dynamics.

4. The Hamilton–Jacobi problem for higher order implicit systems

For regular higher order Lagrangians, the submanifold S in $TT^*T^{k-1}Q$ projects via $T\pi_{T^{k-1}Q}$ on the whole $T^*T^{k-1}Q$. In the singular case, this projection is only a part of $T^*T^{k-1}Q$. If we would like to construct a Hamilton–Jacobi theory in this setting, there must be a way in which we obtain a Lagrangian submanifold of $TT^*T^{k-1}Q$. To find a solution, we need to find a section $\gamma : T^*T^{k-1}Q \rightarrow TT^*T^{k-1}Q$. Nonetheless, starting from an implicit differential equation on $TT^*T^{k-1}Q$, by the projection $T\pi_{T^{k-1}Q}$, we arrive at a submanifold in $TT^{k-1}Q$. Hence, we need to make use of Tulczyjew triple (57) to pass from the Lagrangian to Hamiltonian pictures $TT^{k-1}Q$ and $T^*T^{k-1}Q$ through some morphisms. In this section we develop a geometric Hamilton–Jacobi theory for higher order implicit differential equations using two different approaches.

The first method consists of a theory which does refer to vector fields, that we will refer to as the Morse family method. The second is based on the construction of a local vector field defined on the image of a section, but not defined globally on the phase space. In this case, the definitions above apply locally, and the philosophy of the Hamilton–Jacobi approach matches the explanation above. Let us then start first with the method which is not so related to the usual definitions and that implements as a novelty the use of Lagrangian submanifolds generated by a Morse function. Hence, we start with our so-called Morse family method. Notice also that we will rely on the Ostrogradski approach (59) in this subsection, but there is an alternative, the Schmidt approach that we will introduce in the next section.

4.1. The Morse family method—general approach

Let us start with the first method. We start with a Lagrangian submanifold S of the symplectic manifold $TT^*T^{k-1}Q$ equipped with the symplectic two-form $\Omega_{T^{k-1}Q}^T$ exhibited in (60). If it is a horizontal Lagrangian submanifold then it is possible to find a Hamiltonian vector field on $T^*T^{k-1}Q$ whose image is exactly the submanifold itself [60]. This corresponds to an explicit dynamical system. If the Lagrangian submanifold fails to be horizontal then there is no Hamiltonian vector field generating the Lagrangian submanifold. In this case, dynamical equations governing the dynamical system can only be written in an implicit differential equation form. We now propose a Hamilton–Jacobi formalism valid for both the explicit and implicit systems.

Lagrangian submanifolds of $TT^*T^{k-1}Q$ Consider a Lagrangian submanifold S of $TT^*T^{k-1}Q$. If it is projectable, by projecting S via the mapping $\tau_{T^*T^{k-1}Q}$, we reach a submanifold of $T^*T^{k-1}Q$. On the other hand, if we project S with $T\pi_{T^{k-1}Q}$, we reach a submanifold of $TT^{k-1}Q$, where we have a first order implicit differential equation. Accordingly, we can

iteratively project these resulting bundles: from $T^*T^{k-1}Q$ to Q . Let us summarize this in the following diagram.

$$\begin{array}{ccccc}
 S & \xhookrightarrow{i} & TT^*T^{k-1}Q & & \\
 & \searrow \tau_{T^*T^{k-1}Q} & \searrow T\pi_{T^{k-1}Q} & & \\
 T^*T^{k-1}Q & & TT^{k-1}Q & \xleftarrow{i} & S^\gamma \\
 & \searrow \pi_{T^{k-1}Q} & \searrow \tau_{T^{k-1}Q} & \searrow T\tau_Q^{k-1} & \\
 & \searrow \gamma & T^{k-1}Q & & TQ \\
 & & \downarrow \tau_Q^{k-1} & \nearrow \tau_Q & \uparrow j^1\phi \\
 & & Q & \xleftarrow{\phi} & \mathbb{R}
 \end{array}$$

where $\tau_Q^{k-1} : T^{k-1}Q \rightarrow Q$. Notice that if S is integrable, then $T\pi_{T^{k-1}Q}(S)$ is integrable too. We see this by considering the projection $T\pi_{T^{k-1}Q}(V)$ of an element $V \in S$. Note that, if φ is a curve lying in $T^*T^{k-1}Q$ and it is tangent to $V \in S$, then $\pi_{T^{k-1}Q} \circ \varphi$ is curve on $T^{k-1}Q$ that is tangent to $T\pi_{T^{k-1}Q}(V)$. This shows that the projections of the solutions of S are solutions of $T\pi_{T^{k-1}Q}(S)$. The inverse question is precisely the basis of a Hamilton–Jacobi theory, i.e. if starting from the solutions of $T\pi_{T^{k-1}Q}(S)$ we are able to construct solutions of S , that is to lift the solutions on $TT^{k-1}Q$ to the iterated bundle $TT^*T^{k-1}Q$.

Notice that S may not be projectable, that means that S is only projectable when it is restricted to the image space of a differential one-form γ on $T^{k-1}Q$. We denote the restriction of S to the image space of a one-form γ as follows $S|_{\text{Im}(\gamma)}$. For this procedure, we need to introduce a section $\gamma : T^{k-1}Q \rightarrow T^*T^{k-1}Q$ such that for a solution $\psi : \mathbb{R} \rightarrow T^{k-1}Q$ of $S^\gamma = T\pi_{T^{k-1}Q}(S)$, we have that $\gamma \circ \psi : \mathbb{R} \rightarrow T^*T^{k-1}Q$ is a solution of S . We say that S and S^γ are γ -related. In accordance to the usual Hamilton–Jacobi theory [5, 12, 13, 38], recall (1), we have

$$\begin{array}{ccccc}
 T^*T^{k-1}Q & \xrightarrow{\widetilde{j^{k-1} \circ \gamma \circ \phi}} & TT^*T^{k-1}Q & \xleftarrow{\quad} & S \\
 \uparrow \gamma & \downarrow \pi_{T^{k-1}Q} & \downarrow T\pi_{T^{k-1}Q} & & \\
 T^{k-1}Q & \xrightarrow{\widetilde{j^{k-1} \circ \phi}} & TT^{k-1}Q & \xleftarrow{\quad} & S^\gamma
 \end{array} \quad (65)$$

Morse families. In this case, since S and S^γ are implicit, we do not have a vector field. Nonetheless, as we have discussed previously, for every Lagrangian submanifold S in $TT^*T^{k-1}Q$, there exists a Morse family E defined over a smooth bundle structure $(R, \tau, T^*T^{k-1}Q)$ generating S . Let us recall in a diagram the Lagrangian submanifold that is generated and the Lagrangian submanifold we need for a Hamilton–Jacobi theory:

$$\begin{array}{ccccc}
& & D & & S \\
& & \downarrow & & \downarrow \\
\mathbb{R} & \xleftarrow{E} & R & & T^*T^*T^{k-1}Q \\
& & \downarrow \tau & & \downarrow \Omega_{T^{k-1}Q}^b \\
& & T^*T^{k-1}Q & \xleftarrow{\pi_{T^*T^{k-1}Q}} & TT^*T^{k-1}Q \\
& & \text{=====} & & \nwarrow \tau_{T^*T^{k-1}Q} \\
& & T^*T^{k-1}Q & &
\end{array}
\quad (66)$$

where the triangle is the special symplectic structure presented as the right wing of the Tulczyjew triple (57). Here, D is the image of S under the musical mapping $\Omega_{T^{k-1}Q}^b$ hence a Lagrangian submanifold of $T^*T^*T^{k-1}Q$. Assume the local coordinates $(q_{(\kappa)}^A, p_A^{(\kappa)}, \lambda^\alpha)$ on the fiber bundle R . Here $(q_{(\kappa)}^A, p_A^{(\kappa)})$ is the Darboux' coordinates on $T^*T^{k-1}Q$ since κ runs from 0 to $k-1$. The Lagrangian submanifold S , generated by the Morse family $E = E(q_{(\kappa)}, p^{(\kappa)}, \lambda)$, can be written as

$$D = \left\{ \left(q_{(\kappa)}^A, p_A^{(\kappa)}; \frac{\partial E}{\partial q_{(\kappa)}^A}, \frac{\partial E}{\partial p_A^{(\kappa)}} \right) \in T^*T^*T^{k-1}Q : \frac{\partial E}{\partial \lambda^\alpha} = 0 \right\}. \quad (67)$$

The isomorphic image of D is the Lagrangian submanifold describing the dynamics and computed to be

$$S = \left\{ \left(q_{(\kappa)}^A, p_A^{(\kappa)}; \frac{\partial E}{\partial p_A^{(\kappa)}}, -\frac{\partial E}{\partial q_{(\kappa)}^A} \right) \in TT^*T^{k-1}Q : \frac{\partial E}{\partial \lambda^\alpha} = 0 \right\}. \quad (68)$$

The Lagrangian submanifold S generates the following systems of implicit differential equations

$$\dot{q}_{(\kappa)}^A = \frac{\partial E}{\partial p_A^{(\kappa)}}, \quad \dot{p}_A^{(\kappa)} = -\frac{\partial E}{\partial q_{(\kappa)}^A}, \quad \frac{\partial E}{\partial \lambda^\alpha} = 0. \quad (69)$$

We introduce a closed one-form γ on $T^{k-1}Q$ with local picture

$$\gamma(q_{(\kappa)}) = \gamma_A^{(\kappa)} dq_{(\kappa)}^A,$$

where $\gamma_A^{(\kappa)}$ are real valued functions on $T^{k-1}Q$. See that, $\text{Im}(\gamma)$ is a Lagrangian submanifold of $T^*T^{k-1}Q$, so that there is an inclusion $\iota : \text{Im}(\gamma) \hookrightarrow T^*T^{k-1}Q$. We use the inclusion to pull the bundle $(R, \tau, T^*T^{k-1}Q)$ back over $\text{Im}(\gamma)$. By this, one arrives at a fiber bundle $(\iota^*R, \iota^*\tau, \text{Im}(\gamma))$.

$$\begin{array}{ccc}
\iota^*(R) & \xrightarrow{\varepsilon} & R \\
\downarrow \iota^*\tau & & \downarrow \tau \\
\text{Im}(\gamma) & \xrightarrow{\iota} & T^*T^{k-1}Q
\end{array}
\quad (70)$$

Here, the total space the pull-back bundle is

$$\iota^*(R) = \{(\gamma(q_{(\kappa)}), z) \in \text{Im}(\gamma) \times R : \tau(z) \in \text{Im}(\gamma)\}$$

with ε is the corresponding inclusion. Although restriction of the Morse family on $\iota^*(R)$ should formally be written as $E \circ \varepsilon$, we will abuse notation using E . The submanifold generated by $E = E(q_{(\kappa)}, \gamma^{(\kappa)}, \lambda)$ is given by

$$S|_{\text{Im}(\gamma)} = \left\{ \left(q_{(\kappa)}^A, \gamma_A^{(\kappa)}; \frac{\partial E}{\partial q_{(\kappa)}^A}, -\frac{\partial E}{\partial q_{(\kappa)}^A} \right) \in TT^*T^{k-1}Q : \frac{\partial E}{\partial \lambda^\alpha} = 0 \right\}. \quad (71)$$

Note that, if the Lagrangian submanifold S was explicit, and would be understood as the image of a Hamiltonian vector field X_H , then $S|_{\text{Im}(\gamma)}$ reduces to the image space of the composition $X_H \circ \gamma$.

The submanifold $S|_{\text{Im}(\gamma)}$ exhibited in (71) does not depend on the momentum variables. This enables us to project it to a submanifold S^γ of TQ by the tangent mapping $T\pi_Q$ as follows

$$S^\gamma = T\pi_{T^{k-1}Q} \circ S|_{\text{Im}(\gamma)} = \left\{ \left(q_{(\kappa)}^A, \frac{\partial E}{\partial q_{(\kappa)}^A}(q_{(\kappa)}, \gamma^{(\kappa)}, \lambda) \right) \in TT^{k-1}Q : \frac{\partial E}{\partial \lambda^\alpha} = 0 \right\}. \quad (72)$$

Note that the submanifold S^γ defines an implicit differential equation on $T^{k-1}Q$. We state the generalization of the Hamilton–Jacobi theorem 1 as follows.

Theorem 4 (Higher order implicit HJ theorem). *The following conditions are equivalent for a closed one-form γ that is a solution of the implicit higher order Hamilton–Jacobi problem:*

- (i) *The Lagrangian submanifold S in (67) and the submanifold S^γ in (72) are γ -related, that is*

$$T\gamma(S^\gamma) = S|_{\text{Im}(\gamma)}$$

- (ii) *The Morse family E that generates the submanifold S fulfills the equation*

$$dE(q_{(\kappa)}, \gamma^{(\kappa)}, \lambda) = 0. \quad (73)$$

Proof. The one-form $\gamma = \gamma_A^{(\kappa)} dq_{(\kappa)}^A$ is closed, that is, $\partial \gamma_A^{(\kappa)} / \partial q_{(\hat{\kappa})}^B = \partial \gamma_B^{(\hat{\kappa})} / \partial q_{(\kappa)}^A$. The first assertion in theorem 4 can be written locally as

$$\delta_{\hat{\kappa}}^\kappa \frac{\partial \gamma_A^{(\kappa)}}{\partial q_{(\hat{\kappa})}^A} \frac{\partial E}{\partial p_A^{(\kappa)}} + \frac{\partial E}{\partial q_{(\kappa)}^A} = 0, \quad (74)$$

for $A = 1, \dots, n$, and $\hat{\kappa}, \kappa = 1, \dots, k-1$ and with the condition $\partial E / \partial \lambda^\alpha = 0$. Let us now compute

$$dE(q_{(\kappa)}, \gamma^{(\kappa)}, \lambda) = \frac{\partial E}{\partial q_{(\kappa)}^A} dq_{(\kappa)}^A + \frac{\partial E}{\partial p_A^{(\kappa)}} \gamma_A^{(\kappa, \hat{\kappa})} dq_{(\hat{\kappa})}^A + \frac{\partial E}{\partial \lambda^\alpha} d\lambda^\alpha. \quad (75)$$

Note that, after the substitution of (74) into (75) and by employing the closure of the one-form, we conclude that the exterior derivative of E vanishes when $p^{(\kappa)} = \gamma^{(\kappa)}$. ■

4.1.1. The Morse family method—Ostrogradski momenta. Now, we come to the problem of deciding the total space R of the bundle $T^*T^{k-1}Q$. We are proposing two alternative ways for this. In this case our interest is focused in the Lagrangian submanifolds generated by a

Lagrangian function. Accordingly, consider a Lagrangian function depending on higher order differential terms on the higher order tangent bundle $T^k Q$ of the configuration space Q . If Q is an n -dimensional manifold with a local chart $(q_{(0)}^A)$, then $T^k Q$ is a $(k+1) \times n$ -dimensional manifold with the induced local chart $(q_{(0)}^A, q_{(1)}^A, \dots, q_{(k)}^A)$. Now, consider the Whitney product

$$W = T^k Q \times_{T^{k-1} Q} T^* T^{k-1} Q \quad (76)$$

equipped with the local coordinates

$$(q_{(\kappa)}^A, q_{(k)}^A, p_A^{(\kappa)}) = (q_{(0)}^A, \dots, q_{(k-1)}^A, q_{(k)}^A, p_A^{(0)}, \dots, p_A^{(k-1)}) \quad (77)$$

of the higher order tangent bundle $T^k Q$ and the iterated cotangent bundle $T^* T^{k-1} Q$ fibered over $T^{k-1} Q$. Here, we have assumed the canonical coordinates $(q_{(\kappa)}^A, p_A^{(\kappa)})$ on $T^* T^{k-1} Q$ where κ runs from 0 to $k-1$. Note that, we can realize this Whitney product as the total space of the smooth fiber bundle

$$\tau : T^k Q \times_{T^{k-1} Q} T^* T^{k-1} Q \mapsto T^* T^{k-1} Q : (q_{(\kappa)}^A, q_{(k)}^A, p_A^{(\kappa)}) \mapsto (q_{(\kappa)}^A, p_A^{(\kappa)}), \quad (78)$$

where the base is $T^* T^{k-1} Q$. In this fibration the fibers are given by $(q_{(k)}^A)$ and they are n -dimensional.

For a given higher order Lagrangian $L = L(q_{(0)}^A, \dots, q_{(k)}^A)$, the corresponding energy function E is defined on the Whitney product $T^k Q \times_{TQ} T^* T^{k-1} Q$ and explicitly given by

$$E(q_{(\kappa)}^A, q_{(k)}^A, p_A^{(\kappa)}) = p_A^{(0)} q_{(1)}^A + p_A^{(1)} q_{(2)}^A + \dots + p_A^{(k-1)} q_{(k)}^A - L. \quad (79)$$

It is immediate to see that E is a Morse family and that it generates a Lagrangian submanifold of the cotangent bundle $T^* T^* T^k Q$, as it was mentioned in the theory on Morse families. Diagrammatically, we replace the total space R with the Whitney product in (76) with the projection (78). Hence, the Lagrangian submanifold D in (67) takes the particular form

$$\left(q_{(\kappa)}^A, p_A^{(\kappa)}; -\frac{\partial L}{\partial q_{(0)}^A}, p_A^{(0)} - \frac{\partial L}{\partial q_{(1)}^A}, \dots, p_A^{(k-2)} - \frac{\partial L}{\partial q_{(k-1)}^A}, q_{(1)}^A, \dots, q_{(k)}^A \right) \quad (80)$$

equipped with constraints

$$p_A^{(k-1)} - \frac{\partial L}{\partial q_{(k)}^A} = 0. \quad (81)$$

See that the Lagrangian submanifold exhibited in (80) and (81) is in $T^* T^* T^{k-1} Q$ where $q_{(k)}^A$ are auxiliary variables presenting the implicit character of the system. In this framework, Ostrogradski momenta are given by

$$p^{(\kappa)} = \sum_{j=\kappa}^{k-1} \left(-\frac{d}{dt} \right)^{j-\kappa} \left(\frac{\partial L}{\partial q^{(\kappa+1)}} \right) \quad (82)$$

where κ runs from 0 to $k-1$.

Legendre transformation by means of Tulczyjew triple. Using the right wing of the Tulczyjew' triple (57), and referring directly to the musical isomorphism $\Omega_{T^{k-1} Q}^T$ in (61), we map the Lagrangian submanifold in (80) and (81) to $TT^* T^{k-1} Q$. This reads

$$(q_A^{(\kappa)}, p_A^{(\kappa)}; q_{(1)}^A, \dots, q_{(k)}^A, \frac{\partial L}{\partial q_{(0)}^A}, \frac{\partial L}{\partial q_{(1)}^A} - p_A^{(0)}, \dots, \frac{\partial L}{\partial q_{(k-1)}^A} - p_A^{(k-2)}). \quad (83)$$

The submanifold in (80) and (81) is a Lagrangian submanifold of $TT^*T^{k-1}Q$. The dynamics in this submanifold is represented by a systems of implicit differential equations

$$\begin{aligned} \dot{q}_{(0)}^A &= q_{(1)}^A, \dots, \dot{q}_{(k-1)}^A = q_{(k)}^A, \quad \dot{p}_A^{(0)} = \frac{\partial L}{\partial q_{(0)}^A}, \\ \dot{p}_A^{(1)} &= \frac{\partial L}{\partial q_{(1)}^A} - p_A^{(0)}, \dots, \dot{p}_A^{(k-1)} = \frac{\partial L}{\partial q_{(k-1)}^A} - p_A^{(k-2)}, \quad p_A^{(k-1)} - \frac{\partial L}{\partial q_{(k)}^A} = 0 \end{aligned} \quad (84)$$

equipped with the constraints given in (81). It is immediate now to check that the Lagrangian submanifold (80) and (81), or the system of implicit equations (84) correspond to the higher order Euler–Lagrange equations

$$\frac{\partial L}{\partial q_{(0)}^A} - \frac{d}{dt} \frac{\partial L}{\partial q_{(1)}^A} + \frac{d^2}{dt^2} \frac{\partial L}{\partial q_{(2)}^A} \dots + (-1)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial q_{(k)}^A} = 0. \quad (85)$$

Note that these identifications are independent of the regularity of the Lagrangian functions.

Hamilton–Jacobi equations. Introduce a closed one-form γ on $T^{k-1}Q$ given locally by

$$\gamma = \gamma_A^{(\kappa)} dq_{(\kappa)}^A = \gamma_A^{(0)} dq_{(0)}^A + \gamma_A^{(1)} dq_{(1)}^A + \dots + \gamma_A^{(k-1)} dq_{(k-1)}^A. \quad (86)$$

Now we apply the implicit Hamilton–Jacobi theorem 4 to the first order implicit system given in (84), which is equivalent to the higher order Euler–Lagrange system. More concretely, we are employing the second condition (73) in theorem 4 to the present case. This reads

$$d \left(\gamma_A^{(0)} q_{(1)}^A + \gamma_A^{(1)} q_{(2)}^A + \gamma_A^{(k-1)} q_{(k)}^A - L(q_{(0)}, q_{(1)}, \dots, q_{(k)}) \right) = 0.$$

Accordingly, we compute the following system of equations

$$\begin{aligned} \frac{\partial \gamma_A^{(0)}}{\partial q_{(0)}^B} q_{(1)}^A + \dots + \frac{\partial \gamma_A^{(k-1)}}{\partial q_{(0)}^B} q_{(k)}^A - \frac{\partial L}{\partial q_{(0)}^B} &= 0, \\ \frac{\partial \gamma_A^{(0)}}{\partial q_{(1)}^B} q_{(1)}^A + \gamma_B^{(0)} + \dots + \frac{\partial \gamma_A^{(k-1)}}{\partial q_{(1)}^B} q_{(k)}^A - \frac{\partial L}{\partial q_{(1)}^B} &= 0, \\ &\dots \\ \frac{\partial \gamma_A^{(0)}}{\partial q_{(k-1)}^B} q_{(1)}^A + \frac{\partial \gamma_A^{(1)}}{\partial q_{(k-1)}^B} q_{(1)}^A + \dots + \gamma_B^{(k-2)} + \frac{\partial \gamma_A^{(k-1)}}{\partial q_{(k-1)}^B} q_{(k)}^A - \frac{\partial L}{\partial q_{(k-1)}^B} &= 0 \\ \gamma_A^{(k-1)} - \frac{\partial L}{\partial q_{(k)}^A} &= 0. \end{aligned} \quad (87)$$

Since γ is a closed one-form, then in a local chart, one may take γ as the exterior derivative dW of a real-valued function W on $T^{k-1}Q$. In this case, we integrate the system as

$$\frac{\partial W}{\partial q_{(0)}^A} q_{(1)}^A + \frac{\partial W}{\partial q_{(1)}^A} q_{(2)}^A + \dots + \frac{\partial W}{\partial q_{(k-1)}^A} q_{(k)}^A - L(q_{(0)}, q_{(1)}, \dots, q_{(k)}) = 0, \quad \frac{\partial W}{\partial q_{(k-1)}^A} - \frac{\partial L}{\partial q_{(k)}^A} = 0. \quad (88)$$

Hamilton–Jacobi equations for nondegenerate cases. Note that, we may solve the Lagrange multipliers $q_{(k)}^A$ from the definition of conjugate momenta $p_A^{(k-1)}$ using the constraint (81) if the matrix $\left[\partial^2 L / \partial q_{(k)}^A \partial q_{(k)}^B\right]$ is nondegenerate. In this case, the solution has the form

$$q_{(k)}^A = \Sigma^A \left(q_{(0)}, q_{(1)}, \dots, q_{(k-1)}, p^{(1)} \right).$$

Further, in a local chart, one may take γ as the exterior derivative dW of a real-valued function W on $T^{k-1}Q$. In this case the requirement that E is constant on the image of dW results in a Hamilton–Jacobi equation in form

$$\frac{\partial W}{\partial q_{(0)}^A} q_{(1)}^A + \frac{\partial W}{\partial q_{(1)}^A} q_{(2)}^A + \dots + \frac{\partial W}{\partial q_{(k-1)}^A} \Sigma^A - L(q_{(0)}, q_{(1)}, \dots, q_{(k-1)}, \Sigma) = 0. \quad (89)$$

4.1.2. The Morse family method—the Schmidt–Legendre transformation for second order systems. We start by recalling some basics on the acceleration bundle and refer the reader to [21] for further details.

Acceleration bundle. Consider the set $K_q(Q)$ of smooth curves passing through $q \in Q$ whose first derivatives vanish at q , that is

$$K_q(Q) = \{ \gamma \in C_q(Q) : D(f \circ \gamma)(0) = 0, \quad \forall f : Q \mapsto \mathbb{R} \}. \quad (90)$$

Define an equivalence relation on $K_q(Q)$ by saying that two curves γ and γ' are equivalent if the second derivatives of γ and γ' are equal at the point q , that is if

$$\gamma(0) = \gamma'(0) = q, \quad D^2(f \circ \gamma)(0) = D^2(f \circ \gamma')(0), \quad \forall f : Q \mapsto \mathbb{R}$$

for all real valued functions f on Q . An equivalence class is denoted by $\alpha\gamma(0)$. The set of all of these equivalence classes is called acceleration space A_qQ at $q \in Q$. If Q is an n -dimensional manifold then union of all acceleration spaces

$$AQ = \bigsqcup_{q \in Q} A_qQ.$$

is a $2n$ -dimensional manifold called as the acceleration bundle of Q . The induced local coordinates on AQ are defined to be

$$(q_{(0)}^A, a_{(0)}^A) : AQ \longrightarrow \mathbb{R}^{2n} : \alpha\gamma(0) \longrightarrow (q_{(0)}^A \circ \gamma(0), D^2(q_{(0)}^A \circ \gamma)(0)). \quad (91)$$

We note that, the third order tangent bundle T^3Q and the tangent bundle of the acceleration bundle are isomorphic. If the induced coordinates assumed on the tangent bundle TAQ are $(q_{(0)}^A, a_{(0)}^A; q_{(1)}^A, a_{(1)}^A)$, then the isomorphism S locally takes the form

$$S : TAQ \rightarrow T^3Q : (q_{(0)}^A, a_{(0)}^A; q_{(1)}^A, a_{(1)}^A) \rightarrow (q_{(0)}^A, q_{(1)}^A; a_{(0)}^A, a_{(1)}^A). \quad (92)$$

Gauge invariance of the Lagrangian formalism and Schmidt method. Consider a second order Lagrangian function

$$L = L(q_{(0)}, q_{(1)}, q_{(2)}) \quad (93)$$

on T^2Q . The gauge invariance of the second order Euler–Lagrange equations implies that the equations of motion generated by L and $L + (d/dt)F$ are the same for any smooth function F on T^2Q . When we consider F , we come up with a third order Lagrangian

$$\begin{aligned}\hat{L}(q_{(0)}, q_{(1)}, q_{(2)}, q_{(3)}) &= L(q_{(0)}, q_{(1)}, q_{(2)}) + \frac{d}{dt}F(q_{(0)}, q_{(1)}, q_{(2)}) \\ &= L(q_{(0)}, q_{(1)}, q_{(2)}) + \frac{\partial F}{\partial q_{(0)}^A} q_{(1)}^A + \frac{\partial F}{\partial q_{(1)}^A} q_{(2)}^A + \frac{\partial F}{\partial q_{(2)}^A} q_{(3)}^A\end{aligned}\quad (94)$$

defined in T^3Q with local coordinates $(q_{(0)}, q_{(1)}, q_{(2)}, q_{(3)})$. By recalling the isomorphism in (92), we pull back the Lagrangian \hat{L} to the tangent bundle TAQ , so it results in a first order Lagrangian function

$$L_2 : TAQ \mapsto \mathbb{R} : (q_{(0)}^A, a_{(0)}^A; q_{(1)}^A, a_{(1)}^A) \mapsto L(q_{(0)}, q_{(1)}, a_{(0)}) + \frac{\partial F}{\partial q_{(0)}^A} q_{(1)}^A + \frac{\partial F}{\partial q_{(1)}^A} a_{(0)}^A + \frac{\partial F}{\partial a_{(0)}^A} a_{(1)}^A \quad (95)$$

defined on the first order tangent bundle TAQ . The Euler–Lagrange equations generated by L_2 are computed to be

$$\frac{\partial L_2}{\partial a_{(0)}^A} - \frac{d}{dt} \frac{\partial L_2}{\partial q_{(1)}^A} = 0, \quad \frac{\partial L_2}{\partial a_{(1)}^A} - \frac{d}{dt} \frac{\partial L_2}{\partial a_{(0)}^A} = 0. \quad (96)$$

The second set of equations in (96) can be rewritten as

$$\left(\frac{\partial L}{\partial a_{(0)}^A} + \frac{\partial F}{\partial q_{(1)}^A} \right) + \frac{\partial^2 F}{\partial q_{(1)}^A \partial a_{(0)}^B} (a_{(0)}^B - q_{(2)}^B) = 0. \quad (97)$$

Assume that the second order Lagrangian function L is a nondegenerate, that is the rank of the Hessian matrix $[\partial^2 L / \partial a_{(0)}^A \partial a_{(0)}^B]$ is maximal, and assume also that the auxiliary function F satisfies

$$\frac{\partial L}{\partial a_{(0)}^A} + \frac{\partial F}{\partial q_{(1)}^A} = 0. \quad (98)$$

In this case, the non-degeneracy of the matrix $[\partial^2 L / \partial a_{(0)}^A \partial a_{(0)}^B]$ implies the non-degeneracy of the matrix $[\partial^2 F / \partial a_{(0)}^A \partial q_{(1)}^B]$. Given this, the equations (97) reduce to the set of constraints $a_{(0)}^B - q_{(2)}^B = 0$. In this case, the first set in (96) results in the same Euler–Lagrange equations generated by L in (93).

Morse family generating the Lagrangian submanifold. Assuming the dual coordinates $(q_{(0)}^A, a_{(0)}^A, p_A^{(0)}, \pi_A^{(0)})$ on the cotangent bundle T^*AQ , define the following Morse family

$$\begin{aligned}E(q_{(0)}, a_{(0)}, p^{(0)}, \pi^{(0)}, q_{(1)}, a_{(1)}) &= p_A^{(0)} q_{(1)}^A + \pi_A^{(0)} a_{(1)}^A - L_2(q_{(0)}, a_{(0)}; q_{(1)}, a_{(1)}) \\ &= p_A^{(0)} q_{(1)}^A + \pi_A^{(0)} a_{(1)}^A - L(q_{(0)}, q_{(1)}, a_{(0)}) - \frac{\partial F}{\partial q_{(0)}^A} q_{(1)}^A - \frac{\partial F}{\partial q_{(1)}^A} a_{(0)}^A - \frac{\partial F}{\partial a_{(0)}^A} a_{(1)}^A\end{aligned}\quad (99)$$

on the Whitney sum $TAQ \times_{AQ} T^*AQ$ over the base manifold T^*AQ . The conjugate momenta are defined by the equations

$$0 = \frac{\partial E}{\partial q_{(1)}^A} = p_A^{(0)} - \frac{\partial L_2}{\partial q_{(1)}^A}, \quad 0 = \frac{\partial E}{\partial a_{(1)}^A} = \pi_A^{(0)} - \frac{\partial L_2}{\partial a_{(1)}^A} = \pi_A^{(0)} - \frac{\partial F}{\partial a_{(0)}^A}. \quad (100)$$

If we substitute the momenta $\pi_A^{(0)}$ in the definition of the Morse family (99) which makes the family free of $(a_{(1)}^A)$, it results in

$$E(q_{(0)}, a_{(0)}, p^{(0)}, \pi^{(0)}, q_{(1)}) = p_A^{(0)} q_{(1)}^A - L(q_{(0)}, q_{(1)}, a_{(0)}) - \frac{\partial F}{\partial q_{(0)}^A} q_{(1)}^A - \frac{\partial F}{\partial q_{(1)}^A} a_{(0)}^A \quad (101)$$

defined on the Whitney sum $T^*AQ \times_Q TAQ$. A further reduction on the Morse family is possible. For this, recall the assumption that the matrix $[\partial^2 F / \partial a_{(0)}^A \partial q_{(1)}^B]$ is nondegenerate. So that we can, at least locally, solve $q_{(1)}^A$ in terms of the momenta from the second equation $\pi_A^{(0)} = \partial F(q_{(0)}, q_{(1)}, q_{(2)}) / \partial a_{(0)}^A$ in (100). Let us write this solution as

$$q_{(1)}^A = z^A(q_{(0)}, a_{(0)}, \pi^{(0)}). \quad (102)$$

This results with a well-defined Hamiltonian function

$$H(q_{(0)}, a_{(0)}; p^{(0)}, \pi^{(0)}) = p_A^{(0)} z^A(q_{(0)}, z, a_{(0)}) - L(q_{(0)}, z, a_{(0)}) - \frac{\partial F}{\partial q_{(0)}^A} z^A - \frac{\partial F}{\partial q_{(1)}^A} a_{(0)}^A \quad (103)$$

on T^*AQ .

Hamilton–Jacobi theory in the acceleration bundle framework. Now, we are ready to write the Hamilton–Jacobi theory for second order nondegenerate Lagrangian functions. For this, assume a real valued function W defined on the acceleration bundle AQ and a Hamiltonian vector field X_H on T^*AQ associated to the Hamiltonian function H in (103). We can define a vector field X_H^γ on the acceleration bundle AQ

$$X_H^\gamma = T\pi_{AQ} \circ X_H \circ \gamma \quad (104)$$

according to the commutativity of the diagram

$$\begin{array}{ccc} T^*AQ & \xrightarrow{X_H} & TT^*AQ \\ \downarrow \pi_{AQ} & & \downarrow T\pi_{AQ} \\ AQ & \xrightarrow{X_H^\gamma} & TAQ \end{array} \quad \gamma = dW \quad (105)$$

Theorem 5 (HJ theorem in the acceleration bundle). Let $\gamma = dW$ be a closed one-form on AQ , we say that γ is a solution of the Hamilton–Jacobi problem in the acceleration bundle if the following two equivalent conditions are satisfied

- (i) The vector fields X_H and X_H^γ are γ -related
- (ii) $d(H \circ \gamma) = 0$.

We can rewrite the second condition as an equation the function $W = W(q_{(0)}, a_{(0)})$ satisfying the partial differential equation

$$\frac{\partial W}{\partial q_{(0)}^A} z^A(q_{(0)}, a_{(0)}, \frac{\partial W}{\partial q_{(0)}}) - L(q_{(0)}, z, a_{(0)}) - \frac{\partial F}{\partial q_{(0)}^A} z^A(q_{(0)}, a_{(0)}, \frac{\partial W}{\partial q_{(0)}}) - \frac{\partial F}{\partial q_{(1)}^A} a_{(0)}^A = E, \quad (106)$$

where E is a constant.

Let us write the second condition explicitly for a particular case. Determine the auxiliary function $F(a_{(0)}, q_{(1)}) = -\delta_{AB} a_0^A q_{(1)}^B$ in (95). In the light of condition (98), the Lagrangian is quadratic with respect to second order time derivatives. More concretely, we see that $\partial L / \partial a_{(0)}^A = \delta_{AB} a_{(0)}^B$. In this case, the Hamiltonian function (103) reduces to

$$H(q_{(0)}, a_{(0)}; p^{(0)}, \pi^{(0)}) = \delta^{AB} p_A^{(0)} \pi_B^{(0)} - L(q_{(0)}, \pi^{(0)}, a_{(0)}) + \delta_{AB} a_{(0)}^A a_{(0)}^B. \quad (107)$$

In this case, for a closed one-form

$$\gamma = \frac{\partial W}{\partial q_{(0)}^A} dq_{(0)}^A + \frac{\partial W}{\partial a_{(0)}^A} da_{(0)}^A,$$

the second condition in theorem 5 provides the following Hamilton–Jacobi equation for the nondegenerate second order Lagrangian function L

$$\delta^{AB} \frac{\partial W}{\partial q_{(0)}^A} \frac{\partial W}{\partial q_{(0)}^B} - L\left(q_{(0)}, \frac{\partial W}{\partial a_{(0)}^A}, a_{(0)}\right) + \delta_{AB} a_{(0)}^A a_{(0)}^B = E. \quad (108)$$

4.1.3. Comparisons of HJ formalisms for nondegenerate cases. Let us consider again the auxiliary function $F = F(q_{(0)}, q_{(1)}, q_{(2)})$ on the second order tangent bundle T^2Q and let us write T^2Q locally as a product space $AQ \times_Q TQ$. Here, the function will have a form $F = F(q_{(0)}, q_{(1)}, a_{(0)})$. In this case, the cotangent bundle of T^*T^2Q can be identified with the product space $T^*AQ \times T^*TQ$. The image of the exterior derivative dF determines a Lagrangian submanifold of $T^*AQ \times T^*TQ$ hence a symplectic diffeomorphism between T^*AQ and T^*TQ . Explicitly, the symplectic diffeomorphism is computed to be

$$T^*AQ \rightarrow T^*TQ : \left(q_{(0)}^A, a_{(0)}^A; p_A^{(0)}, \pi_A^{(0)}\right) \rightarrow \left(q_{(0)}^A, z^A\left(q_{(0)}, a_{(0)}, \pi^{(0)}\right), p_A^{(0)} - \frac{\partial F}{\partial q_{(0)}^A}\left(q_{(0)}, z\left(q_{(0)}, a_{(0)}, \pi^{(0)}\right), a_{(0)}\right), -\frac{\partial F}{\partial z^A}\right). \quad (109)$$

This symplectic diffeomorphism establishes the link between the Morse families (79) (when $k = 1$) and (101). To see this directly, let us now pull back the Morse family E given in (79) by the mapping (109). We compute the result as follows

$$\begin{aligned} & p^{(0)} q_{(1)} + p^{(1)} q_{(2)} - L(q_{(0)}, q_{(1)}, q_{(2)}) \\ &= \left(p_q - \frac{\partial F}{\partial q}(q, z(q, a, p_a), a)\right) z - \frac{\partial F}{\partial z} a - L(q, z, a) \\ &= p_q z - L(q, z, a) - \frac{\partial F}{\partial q} z - \frac{\partial F}{\partial z} a \end{aligned} \quad (110)$$

which is exactly the Hamiltonian function in (103). Here, we have employed the identification $a_{(0)}^A = q_{(2)}^A$. The following examples compare the two methods we have exhibited so far.

Example. Let us consider a pure quadratic one-dimensional Lagrangian

$$L = \frac{1}{2} \mu q_{(2)}^2. \quad (111)$$

If we first apply the Ostragradski method, the momentum $p^{(1)}$ is computed to be $\mu q_{(2)}$. The Hamilton–Jacobi equation (89) for this system is

$$\frac{\partial W}{\partial q_{(0)}} q_{(1)} + \frac{1}{2} \mu \left(\frac{\partial W}{\partial q_{(0)}} \right)^2 = c. \quad (112)$$

Let us now apply the Schmidt method presented in section 4.1.2 to the Lagrangian (111). Condition (98) integrates the function F as

$$F = -\mu a q_{(1)} + g(q_{(0)}) \quad (113)$$

where g is an arbitrary function which can be chosen as zero without loss of any generality. This enables us to use the Hamilton–Jacobi equation in (106), which is exactly

$$-\frac{\partial W}{\partial q_{(0)}} \frac{\partial W}{\partial a_{(0)}} + \frac{1}{2} \mu a_{(0)}^2 = c \quad (114)$$

and which can be solved assuming that $\nabla_a W$ does not equal to zero, and rewrite the Hamilton–Jacobi problem in the form

$$\frac{\partial W}{\partial q_{(0)}} = \frac{\frac{1}{2} \mu a_{(0)}^2 - c}{\frac{\partial W}{\partial a_{(0)}}} = c_2 \quad (115)$$

where c_2 is a constant. Its solution reads:

$$W(q_{(0)}, a_{(0)}) = c_2 q_{(0)} + \frac{1}{6c_2} \mu a_{(0)}^3 - \frac{c}{c_2} a_{(0)}. \quad (116)$$

4.1.4. The Morse family method—the Schmidt method for the third order Lagrangians. Let us start with a third order Lagrangian function $L(q_{(0)}, q_{(1)}, q_{(2)}, q_{(3)})$ defined on T^3Q . Recalling the local diffeomorphism in (92), we pull back the Lagrangian function L to the tangent bundle TAQ of the acceleration bundle. By this, we arrive at a first order Lagrangian function $L = L(q_{(0)}, a_{(0)}; q_{(1)}, a_{(1)})$. Now, we define a manifold M with local coordinates m , its tangent bundle TM with coordinates $(m_{(0)}^A, m_{(1)}^B)$ and the first order Lagrangian function

$$L_3 = L(q_{(0)}, a_{(0)}; q_{(1)}, a_{(1)}) + \frac{\partial F}{\partial q_{(0)}^A} q_{(1)}^A + \frac{\partial F}{\partial q_{(1)}^A} a_{(0)}^A + \frac{\partial F}{\partial a_{(0)}^A} a_{(1)}^A + \frac{\partial F}{\partial m_{(0)}^A} m_{(1)}^A \quad (117)$$

on the tangent bundle $T(AQ \times M)$ equipped with local coordinates

$$(q_{(0)}^A, a_{(0)}^A; q_{(1)}^A, a_{(1)}^A; m_{(0)}^A, m_{(1)}^A).$$

Here, the auxiliary function F depends on $(q_{(0)}, q_{(1)}, a_{(0)}, m_{(0)})$. The Euler Lagrange equations generated by the Lagrangian L_3 are equal to the Euler–Lagrange equations generated by the third order Lagrangian function L if the requirement

$$\det[\partial^2 F / \partial q_{(1)}^A \partial m_{(0)}^B] \neq 0 \quad (118)$$

is assumed [21].

We consider the conjugate momenta on $T^*(AQ \times M)$ determined locally by $(p_A^{(0)}, \pi_A^{(0)}, \mu_A^{(0)})$ and the energy function associated with L_3 is

$$\begin{aligned}
E &= p_A^{(0)} q_{(1)}^A + \pi_A^{(0)} a_{(1)}^A + \mu_A^{(0)} m_{(1)}^A - L_3 \\
&= p_A^{(0)} q_{(1)}^A + \pi_A^{(0)} a_{(1)}^A + \mu_A^{(0)} m_{(1)}^A - L - \frac{\partial F}{\partial q_{(0)}^A} q_{(1)}^A - \frac{\partial F}{\partial q_{(1)}^A} a_{(0)}^A \\
&\quad - \frac{\partial F}{\partial a_{(0)}^A} a_{(1)}^A - \frac{\partial F}{\partial m_{(0)}^A} m_{(1)}^A.
\end{aligned} \tag{119}$$

Notice that this energy function is a Morse family on the Whitney sum $T(AQ \times M) \times T^*(AQ \times M)$ and, in accordance with the following diagram.

$$\begin{array}{ccccc}
TT^*(AQ \times M) & \xleftarrow{\Omega_{(AQ \times M)}^\sharp} & T^*T^*(AQ \times M) & & T(AQ \times M) \times T^*(AQ \times M), \\
& \searrow \tau_{T^*(AQ \times M)} & \swarrow \pi_{T^*(AQ \times M)} & & \downarrow \pi_2 \\
& & T^*(AQ \times M) & \xlongequal{\quad} & T^*(AQ \times M)
\end{array} \tag{120}$$

The family E generates a Lagrangian submanifold D of $T^*T^*(AQ \times M)$, and using the musical isomorphism $\Omega_{(AQ \times M)}^\sharp$, we map this Lagrangian submanifold to a Lagrangian submanifold S of $TT^*(AQ \times M)$, that is a symplectic manifold equipped with the lifted symplectic two-form $\Omega_{(AQ \times M)}^T$. This Lagrangian submanifold exactly determines the third order Euler–Lagrange equations generated by the Lagrangian $L = L(q_{(0)}, q_{(1)}, q_{(2)}, q_{(3)})$.

Let us now apply the implicit Hamilton–Jacobi theorem to this case. Assume a closed one-form γ on $AQ \times M$ given locally by

$$\gamma = \gamma_A dq_{(0)}^A + \alpha_A da_{(0)}^A + \beta_A dm_{(0)}^A. \tag{121}$$

The restriction of the Lagrangian submanifold S to the image space of γ will be denoted by $S|_{\text{Im}(\gamma)}$. Then project $S|_{\text{Im}(\gamma)}$ to the tangent bundle $T(AQ \times M)$ by means of the tangent mapping $T\pi_{AQ \times M}$. This results in a (possibly non horizontal) submanifold $S^\gamma = T_{\pi_{AQ \times M}}(S|_{\text{Im}})$ of $T(AQ \times M)$. Let us depict these in the following diagram

$$\begin{array}{ccccc}
T^*(AQ \times M) & \xrightarrow{\quad} & TT^*(AQ \times M) & \xleftarrow{\quad} & S|_{\text{Im}(\gamma)} \\
\downarrow \pi_{AQ \times M} & \nearrow \gamma = dW & \downarrow T\pi_{AQ \times M} & & \\
AQ \times M & \xrightarrow{\quad} & T(AQ \times M) & \xleftarrow{\quad} & S^\gamma
\end{array} \tag{122}$$

Theorem 6 (HJ theorem for implicit third order Lagrangians in the acceleration space). A solution of the implicit Hamilton–Jacobi problem for third order Lagrangians in the acceleration space is a closed one-form γ that fulfills the two following equivalent relations:

- (i) The Lagrangian submanifold $S|_{\text{Im}(\gamma)}$ and the submanifold S^γ are γ -related, that is $T\gamma(S^\gamma) = S|_{\text{Im}(\gamma)}$.
- (ii) $d(E(q_{(0)}^A, a_{(0)}^A, m_{(0)}^A; \gamma_A, \alpha_A, \beta_A; q_{(1)}^A, a_{(1)}^A, m_{(1)}^A)) = 0$.

The second condition reads the implicit HJ equation

$$E = \gamma_A q_{(1)}^A + \alpha_A a_{(1)}^A + \beta_A m_{(1)}^A - L_3 = c,$$

where c being a constant. Taking the exterior derivative of this equation, we arrive at the following local picture of the Hamilton–Jacobi equation

$$\begin{aligned}
 \frac{\partial \gamma_A}{\partial q_{(0)}^B} q_{(1)}^A + \frac{\partial \alpha_A}{\partial q_{(0)}^B} a_{(1)}^A + \frac{\partial \beta_A}{\partial q_{(0)}^B} m_{(1)}^A - \frac{\partial L_3}{\partial q_{(0)}^B} &= 0 \\
 \frac{\partial \gamma_A}{\partial a_{(0)}^B} q_{(1)}^A + \frac{\partial \alpha_A}{\partial a_{(0)}^B} a_{(1)}^A + \frac{\partial \beta_A}{\partial a_{(0)}^B} m_{(1)}^A - \frac{\partial L_3}{\partial a_{(0)}^B} &= 0 \\
 \frac{\partial \gamma_A}{\partial m_{(0)}^B} q_{(1)}^A + \frac{\partial \alpha_A}{\partial m_{(0)}^B} a_{(1)}^A + \frac{\partial \beta_A}{\partial m_{(0)}^B} m_{(1)}^A - \frac{\partial L_3}{\partial m_{(0)}^B} &= 0 \\
 \gamma_A - \frac{\partial L_3}{\partial q_{(1)}^A} &= 0, \\
 \alpha_A - \frac{\partial L_3}{\partial a_{(1)}^A} &= 0, \\
 \beta_A - \frac{\partial L_3}{\partial m_{(1)}^A} &= 0,
 \end{aligned} \tag{123}$$

where L_3 is the Lagrangian function in (117). As a particular case, we consider that the auxiliary function is taken to be $F = \delta_{AB} q_{(1)}^A m_{(0)}^B$. In this case the Lagrangian function L_3 reduces to

$$L_3(q_{(0)}, a_{(0)}; q_{(1)}, a_{(1)}; m_{(0)}, m_{(1)}) = L(q_{(0)}, a_{(0)}, q_{(1)}, a_{(1)}) + \delta_{AB} q_{(1)}^A m_{(0)}^B. \tag{124}$$

In this case, the last equation in system (123) provides the definition of the Lagrange multiplier as $q_{(1)}^A = \delta^{AC} \beta_C$. So that the substitution of the Lagrangian (124) into (123), we the following reduced Hamilton–Jacobi equations

$$\begin{aligned}
 \delta^{AC} \gamma_C \frac{\partial \gamma_A}{\partial q_{(0)}^B} + \frac{\partial \alpha_A}{\partial q_{(0)}^B} a_{(1)}^A + \frac{\partial \beta_A}{\partial q_{(0)}^B} m_{(1)}^A - \frac{\partial L}{\partial q_{(0)}^B} \Big|_{q_{(1)}^A = \delta^{AB} \beta_B} &= 0 \\
 \delta^{AC} \gamma_C \frac{\partial \gamma_A}{\partial a_{(0)}^B} + \frac{\partial \alpha_A}{\partial a_{(0)}^B} a_{(1)}^A + \frac{\partial \beta_A}{\partial a_{(0)}^B} m_{(1)}^A - \frac{\partial L}{\partial a_{(0)}^B} \Big|_{q_{(1)}^A = \delta^{AB} \beta_B} &= 0 \\
 \delta^{AC} \gamma_C \frac{\partial \gamma_A}{\partial m_{(0)}^B} + \frac{\partial \alpha_A}{\partial m_{(0)}^B} a_{(1)}^A + \frac{\partial \beta_A}{\partial m_{(0)}^B} m_{(1)}^A - \frac{\partial L}{\partial m_{(0)}^B} \Big|_{q_{(1)}^A = \delta^{AB} \beta_B} &= 0 \\
 \gamma_A - \frac{\partial L}{\partial q_{(1)}^A} \Big|_{q_{(1)}^A = \delta^{AB} \beta_B} - \delta_{AB} m_{(1)}^B &= 0, \\
 \alpha_A - \frac{\partial L}{\partial a_{(1)}^A} \Big|_{q_{(1)}^A = \delta^{AB} \beta_B} &= 0,
 \end{aligned} \tag{125}$$

Hamilton–Jacobi theory for degenerate second order Lagrangians Notice that up to now, the non-degeneracy condition has not been assumed. This implies that we can apply this framework in both degenerate and nondegenerate third order Lagrangian systems. It is also interesting to note that we can further study the second order Lagrangian systems in the present framework. Let us study this particular case. In the definition of L_3 given in (117), we choose $L = L(q_{(0)}, a_{(0)}, q_{(1)})$, and consider an auxiliary function $F = F(q_{(0)}, q_{(1)}, m_{(0)})$. So that, we have a Lagrangian function

$$L_{2\text{-deg}}(q_{(0)}, a_{(0)}; q_{(1)}, a_{(1)}; m_{(0)}, m_{(1)}) = L(q_{(0)}, q_{(1)}, a_{(0)}) + \frac{\partial F}{\partial q_{(0)}^A} q_{(1)}^A + \frac{\partial F}{\partial q_{(1)}^A} a_{(0)}^A + \frac{\partial F}{\partial m_{(0)}^A} m_{(1)}^A \quad (126)$$

defined on the tangent bundle $T(AQ \times M)$. In this case, the energy function (119) is reduced to

$$E = p_A^{(0)} q_{(1)}^A + \pi_A^{(0)} a_{(1)}^A + \mu_A^{(0)} m_{(1)}^A - L - \frac{\partial F}{\partial q_{(0)}^A} q_{(1)}^A - \frac{\partial F}{\partial q_{(1)}^A} a_{(0)}^A - \frac{\partial F}{\partial m_{(0)}^A} m_{(1)}^A. \quad (127)$$

This Morse family generates a nonhorizontal Lagrangian submanifold of $TT^*(AQ \times M)$. So that defines an implicit Hamiltonian system. We substitute the Lagrangian $L_{2\text{-deg}}$ into the Hamilton–Jacobi equation (123). This gives the following Hamilton–Jacobi equation for a second order degenerate Lagrangian L . The fifth equation gives us that $\alpha_A = 0$. Under the light of the closure of the differential form γ , this reads $\gamma_A = \gamma_A(q_{(0)}, m_{(0)})$ and that $\beta_A = \beta_A(q_{(0)}, m_{(0)})$ so we have

$$\begin{aligned} \frac{\partial \gamma_A}{\partial q_{(0)}^B} q_{(1)}^A + \frac{\partial \beta_A}{\partial q_{(0)}^B} m_{(1)}^A &= \frac{\partial L}{\partial q_{(0)}^B} + \frac{\partial^2 F}{\partial q_{(0)}^B \partial q_{(0)}^A} q_{(1)}^A + \frac{\partial^2 F}{\partial q_{(0)}^B \partial q_{(1)}^A} a_{(0)}^A + \frac{\partial^2 F}{\partial q_{(0)}^B \partial m_{(0)}^A} m_{(1)}^A, \\ \frac{\partial L}{\partial a_{(0)}^A} + \frac{\partial F}{\partial q_{(1)}^A} &= 0 \\ \frac{\partial \gamma_A}{\partial m_{(0)}^B} q_{(1)}^A + \frac{\partial \beta_A}{\partial m_{(0)}^B} m_{(1)}^A &= \frac{\partial^2 F}{\partial m_{(0)}^B \partial q_{(0)}^A} q_{(1)}^A + \frac{\partial^2 F}{\partial m_{(0)}^B \partial q_{(1)}^A} a_{(0)}^A + \frac{\partial^2 F}{\partial m_{(0)}^B \partial m_{(0)}^A} m_{(1)}^A \\ \gamma_B &= \frac{\partial L}{\partial q_{(1)}^B} + \frac{\partial^2 F}{\partial q_{(1)}^B \partial q_{(0)}^A} q_{(1)}^A + \frac{\partial F}{\partial q_{(0)}^B} + \frac{\partial^2 F}{\partial q_{(1)}^B \partial q_{(1)}^A} a_{(0)}^A + \frac{\partial^2 F}{\partial q_{(1)}^B \partial m_{(0)}^A} m_{(1)}^A \\ \beta_A &= \frac{\partial F}{\partial m_{(0)}^A}. \end{aligned} \quad (128)$$

Let us study the Hamilton–Jacobi equation (128) for the particular choice of $F = \delta_{AB} q_{(1)}^A m_{(0)}^B$. As in the third order case, the last line of the system implies that $q_{(1)}^A = \delta^{AC} \beta_C$. Eventually we have

$$\begin{aligned} \frac{\partial \delta^{AC} \beta_C \gamma_A}{\partial q_{(0)}^B} + \frac{\partial \beta_A}{\partial q_{(0)}^B} m_{(1)}^A &= \frac{\partial L}{\partial q_{(0)}^B} \\ \frac{\partial L}{\partial a_{(0)}^A} + \delta_{AB} m_{(0)}^B &= 0 \\ \delta^{AC} \beta_C \frac{\partial \gamma_A}{\partial m_{(0)}^B} + \frac{\partial \beta_A}{\partial m_{(0)}^B} m_{(1)}^A &= \delta_{BC} a_{(0)}^C \\ \gamma_B &= \frac{\partial L}{\partial q_{(1)}^B} + \delta_{AB} m_{(1)}^A. \end{aligned} \quad (129)$$

4.2. Local vector field method

The second procedure to deal with an implicit higher-order implicit Lagrangian is based on the construction of a local vector field describing the dynamics. Consider an additional section $\sigma : T^*T^{k-1}Q \rightarrow TT^*T^{k-1}Q$ in the same previous picture.

$$\begin{array}{ccccc}
S & \xhookrightarrow{i} & TT^*T^{k-1}Q & & \\
& \searrow \sigma & \downarrow \tau_{T^*T^{k-1}Q} & \searrow T\pi_{T^{k-1}Q} & \\
C \cap \text{Im}(\gamma) & \xrightarrow{\gamma} & T^*T^{k-1}Q & & TT^{k-1}Q \xleftarrow{i} S^\gamma \\
& \searrow \gamma & \downarrow \pi_{T^{k-1}Q} & \searrow \tau_{T^{k-1}Q} & \downarrow T\tau_Q^{k-1} \\
& & T^{k-1}Q & & TQ \\
& & \downarrow \tau_Q^{k-1} & \nearrow \tau_Q & \uparrow j^1\phi \\
& & Q & \xleftarrow{\phi} & \mathbb{R}
\end{array}$$

where $\tau_Q^{k-1} : T^{k-1}Q \rightarrow Q$.

A remark Recall that E is implicit, so there are several vectors in E projecting to the same point. The role of σ is to reduce the unknown number to one. We require that the domain of the section is included in the intersection of $\text{Im}(\gamma)$ and C . Since for implicit systems C may not be the whole $T^*T^{k-1}Q$, as a result we arrive at a vector field X_σ that will satisfy a Hamilton equation of type

$$\iota_{X_\sigma} \Omega_{T^{k-1}Q} = \Theta(\gamma(q)) \quad (130)$$

for a covector Θ defined at a point $\gamma(q)$.

The construction of these local vector field using σ would imply the following diagram

$$\begin{array}{ccc}
T^*T^{k-1}Q & \xrightarrow{X_\sigma} & TT^*T^{k-1}Q \xleftarrow{i} S \\
\downarrow \pi_{T^{k-1}Q} & & \downarrow T\pi_{T^{k-1}Q} \\
T^{k-1}Q & \xrightarrow{X_\sigma^\gamma} & TT^{k-1}Q \xleftarrow{i} S^\gamma
\end{array} \quad (131)$$

Explicitly, the locally constructed vector fields $X_\sigma \in \mathfrak{X}(T^*T^{k-1}Q)$ and $X_\sigma^\gamma \in \mathfrak{X}(T^{k-1}Q)$ in coordinates would read:

$$X_\sigma = \sigma_{(\kappa)}^A(q_{(\kappa)}, \gamma^{(\kappa)}) \frac{\partial}{\partial q_{(\kappa)}^A} + \sigma_A^{(\kappa)}(q_{(\kappa)}, \gamma^{(\kappa)}) \frac{\partial}{\partial p_A^{(\kappa)}}, \quad X_\sigma^\gamma = \sigma_{(\kappa)}^A(q_{(\kappa)}, \gamma^{(\kappa)}) \frac{\partial}{\partial q_{(\kappa)}^A}. \quad (132)$$

If we use the one-form $\gamma : T^{k-1}Q \rightarrow T^*T^{k-1}Q$ and define the projected vector field

$$X_\sigma^\gamma = T_{\pi_{T^{k-1}Q}} \circ X_\sigma \circ \gamma, \quad (133)$$

we have the following theorem.

Theorem 7 (Implicit HJ theorem with an auxiliary section). *The one-form γ will be a solution of an implicit higher order Hamilton–Jacobi problem if it satisfies the following relation*

$$\sigma_{(\kappa)}^A(q_{(\kappa)}^A, \gamma_A^{(\kappa)}(q_{(\kappa)})) \frac{\partial \gamma_A^{(\kappa)}}{\partial q_{(\kappa)}^A} = \sigma_A^{(\kappa)}(q_{(\kappa)}^A, \gamma_A^{(\kappa)}(q_{(\kappa)})), \quad (134)$$

when σ is an auxiliary section $\sigma : T^*T^{k-1}Q \rightarrow TT^*T^{k-1}Q$. It is fulfilled that $\sigma^{-1}(S) = C$. Recall that since S is an implicit submanifold, it does not necessarily project on the whole $T^*T^{k-1}Q$, but in a submanifold C of it.

Proof. It is straightforward using that

$$T_\gamma(X_\sigma^\gamma) = X_\sigma \circ \gamma \quad (135)$$

and the expressions of X_σ^γ and X_σ in coordinates as in (132). ■

5. Applications

5.1. A (homogeneous) deformed elastic cylindrical beam with fixed ends

Let Q be a one-dimensional manifold with coordinate $q_{(0)}$, and introduce the second order Lagrangian

$$L(q_{(0)}, q_{(1)}, q_{(2)}) = \frac{1}{2} \mu q_{(2)}^2 + \rho q_{(0)} \quad (136)$$

in terms of a local coordinate system $(q_{(0)}, q_{(1)}, q_{(2)})$ on T^2Q .

The Morse family method—Ostrogradski momenta. We will first apply the Ostrogradski method. In this method, the corresponding energy function is computed to be

$$E(q_{(0)}, q_{(1)}, q_{(2)}, p^{(0)}, p^{(1)}) = p^{(0)} q_{(1)} + p^{(1)} q_{(2)} - \frac{1}{2} \mu q_{(2)}^2 - \rho q_{(0)}, \quad (137)$$

where $q_{(2)} \in \mathbb{R}$ is the fiber component and $(q_{(0)}, q_{(1)}, p^{(0)}, p^{(1)})$ are the canonical coordinates on T^*TQ . Here, $q_{(2)}$ is a Lagrange multiplier. The Morse family E generates a Lagrangian submanifold S of TT^*TQ , that corresponds with

$$S = \{(q_{(0)}, q_{(1)}, p^{(0)}, \mu q_{(2)}; q_{(1)}, q_{(2)}, \rho, -p^{(0)}) \in TT^*TQ : q_{(2)} \in \mathbb{R}\}.$$

This Lagrangian submanifold defines the following differential equation

$$\ddot{q}_{(0)} = -\frac{\rho}{\mu}, \quad (138)$$

which is exactly the fourth order Euler–Lagrange equation generated by the Lagrangian function L . The projection of S onto the cotangent bundle T^*TQ results in the submanifold

$$C = \{(q_{(0)}, q_{(1)}; p^{(0)}, p^{(1)}) \in T^*TQ : p^{(1)} = \mu q_{(2)} \in \mathbb{R}\}.$$

Let us now consider a closed one-form $\gamma = \gamma^{(0)} dq_{(0)} + \gamma^{(1)} dq_{(1)}$ and write the Hamilton–Jacobi equations.

$$\begin{cases} q_{(1)} \frac{\partial \gamma^{(0)}}{\partial q_{(0)}} + q_{(2)} \frac{\partial \gamma^{(1)}}{\partial q_{(0)}} - \rho = 0 \\ \gamma^{(0)} + q_{(1)} \frac{\partial \gamma^{(0)}}{\partial q_{(1)}} + q_{(2)} \frac{\partial \gamma^{(1)}}{\partial q_{(1)}} = 0 \\ \gamma^{(1)} - \mu q_{(2)} = 0. \end{cases} \quad (139)$$

If we substitute the last equation into the Morse family (137) equal to constant, and we assume that $\gamma = dW$ for some real valued function W on TQ , we arrive at that

$$q_{(1)} \frac{\partial W}{\partial q_{(0)}} + \frac{1}{2\mu} \left(\frac{\partial W}{\partial q_{(1)}} \right)^2 - \rho q_{(0)} = 0. \quad (140)$$

Note that, in this case, solving the Hamilton–Jacobi equation is much more difficult than solving (138).

The Morse family method—the Schmidt method. Let us now propose the Schmidt method section 4.1.2. In this case, we have a two-dimensional acceleration bundle AQ with coordinates $(q_{(0)}, a_{(0)})$. Its tangent bundle TAQ is four-dimensional with coordinates $(q_{(0)}, a_{(0)}; q_{(1)}, a_{(1)})$. We pull back the Lagrangian in (136) to TAQ by means of the isomorphism (92) which reads that

$$L = \frac{1}{2} \mu a_{(0)}^2 + \rho q_{(0)}.$$

The compatibility condition (98) and the non-degeneracy of the Lagrangian suggests the auxiliary function $F = -\mu a_{(0)} q_{(1)}$. So, the extended Lagrangian (95) turns out to be

$$L_2 = \rho q_{(0)} - \frac{1}{2} \mu a_{(0)}^2 - \mu a_{(1)} q_{(1)}.$$

The dual coordinates on the cotangent bundle T^*AQ is given by $(q_{(0)}, a_{(0)}; p^{(0)}, \pi^{(0)})$. The conjugate momenta is computed to be $\pi^{(0)} = -\mu q_{(1)}$. According to (103), this results with the following Hamiltonian function

$$H = \frac{1}{2} \mu a_{(0)}^2 - \frac{1}{\mu} \pi^{(0)} p^{(0)} - \rho q_{(0)}.$$

To arrive at the Hamilton–Jacobi equation, assume a closed one-form $\gamma = \gamma dq_{(0)} + \alpha da_{(0)}$ defined on the acceleration bundle AQ , that is $\frac{\partial \gamma}{\partial a_{(0)}} = \frac{\partial \alpha}{\partial q_{(0)}}$. Recalling the Hamilton–Jacobi theorem asserts that the restriction of H on γ is constant (139). Taking exterior derivative of this, we have the following set of equations

$$\frac{\partial \alpha}{\partial a_{(0)}} \alpha + \frac{\partial \alpha}{\partial a_{(0)}} \gamma = \mu^2 a_{(0)}, \quad (141)$$

$$\frac{\partial \gamma}{\partial q_{(0)}} \alpha + \frac{\partial \alpha}{\partial q_{(0)}} \gamma = -2\rho. \quad (142)$$

Note that, this Hamilton–Jacobi problem reduces to the example studied in section 4.1.3 if $\rho = 0$. In this case, we solve the system as $\gamma = c_2$ and $\alpha = \frac{1}{c_2} \mu^2 a_{(0)} + c$, where c and c_2 are constants.

5.2. One dimensional version of the end of a javelin

Let us consider the following Lagrangian on T^2Q of the one-dimensional manifold Q equipped with $(q_{(0)}, q_{(1)}, q_{(2)})$ given by

$$L(q_{(0)}, q_{(1)}, q_{(2)}) = \frac{1}{2} q_{(1)}^2 - \frac{1}{2} q_{(2)}^2. \quad (143)$$

The Morse family method—Ostrogradski momenta. The associated energy function is given by

$$E(q_{(0)}, q_{(1)}, q_{(2)}, p^{(0)}, p^{(1)}) = p^{(0)}q_{(1)} + p^{(1)}q_{(2)} + \frac{1}{2}q_{(2)}^2 - \frac{1}{2}q_{(1)}^2.$$

Here, $q_{(2)} \in \mathbb{R}$ is the fiber component and $(q_{(0)}, q_{(1)}, p^{(0)}, p^{(1)})$ are the canonical coordinates on T^*TQ . The Morse function E generates the Lagrangian submanifold of TT^*TQ given by

$$S = \{(q_{(0)}, q_{(1)}, p^{(0)}, -q_{(2)}; q_{(1)}, q_{(2)}, 0, p^{(0)} - q_{(1)}) \in TT^*TQ : q_{(2)} \in \mathbb{R}\}.$$

This Lagrangian submanifold defines the equations

$$\ddot{q}_{(0)} + \ddot{q}_{(1)} = c,$$

where c is a constant. The projection of S onto the cotangent bundle T^*TQ is a three dimensional manifold

$$C = \{(q_{(0)}, q_{(1)}; p^{(0)}, p^{(1)}) \in T^*TQ : p^{(1)} = -q_{(2)} \in \mathbb{R}\}.$$

for a fixed $q_{(2)}$. For a closed one-form $\gamma^{(0)}dq_{(0)} + \gamma^{(1)}dq_{(1)}$, the Hamilton–Jacobi equation according to theorem (139) turns out to be

$$\begin{cases} q_{(1)} \frac{\partial \gamma^{(0)}}{\partial q_{(0)}} + q_{(2)} \frac{\partial \gamma^{(1)}}{\partial q_{(0)}} = 0 \\ \gamma^{(0)} + q_{(1)} \frac{\partial \gamma^{(0)}}{\partial q_{(1)}} + q_{(2)} \frac{\partial \gamma^{(1)}}{\partial q_{(1)}} - q_{(1)} = 0 \\ \gamma^{(1)} + q_{(2)} = 0. \end{cases} \quad (144)$$

We can solve $q_{(2)}$ from the last equation and if we substitute it in the equation $E = C$, (where C is a constant) under the image of $\gamma = dW$ for some real valued function W on TQ , we arrive at

$$\frac{\partial W}{\partial q_{(0)}} q_{(1)} - \frac{1}{2} \left(\frac{\partial W}{\partial q_{(1)}} \right)^2 - \frac{1}{2} q_{(1)}^2 = 0. \quad (145)$$

There is a solution [18]

$$W(q_{(0)}, q_{(1)}) = Aq_{(0)} + \sqrt{2} \int \sqrt{Aq_{(1)} - \frac{1}{2}q_{(1)}^2 - B} dq_{(1)}$$

which results with a one-form γ solving the system (144) in form

$$\gamma = Adq_{(0)} + \sqrt{2} \sqrt{\left(Aq_{(1)} - \frac{1}{2}q_{(1)}^2 - B \right)} dq_{(1)}.$$

The Morse family method—the Schmidt method. As an alternative realization of the Hamilton–Jacobi problem, we can use the Schmidt method in section 4.1.2. As in the previous subsection, we assume that acceleration bundle AQ is two-dimensional with local coordinates $(q_{(0)}, a_{(0)})$, and TAQ is a four-dimensional manifold with $(q_{(0)}, a_{(0)}; q_{(1)}, a_{(1)})$. We pull back the Lagrangian L in (143) by means of the isomorphism (92) and arrive at that

$$L = \frac{1}{2}q_{(1)}^2 - \frac{1}{2}a_{(0)}^2. \quad (146)$$

In this case, the auxiliary function is taken to be $F = a_{(0)}q_{(1)}$. Note that, F satisfies the compatibility condition in (98). So, the first order Lagrangian function (95) is computed to be

$$L_2 = \frac{1}{2}q_{(1)}^2 + \frac{1}{2}a_{(0)}^2 + q_{(1)}a_{(1)}.$$

The coordinates on the cotangent bundle T^*AQ are $(q_{(0)}, a_{(0)}; p^{(0)}, \pi^{(0)})$ and the conjugate momenta is computed to be $\pi^{(0)} = q_{(1)}$. This results in the following Hamiltonian function

$$H = \pi^{(0)}p^{(0)} - \frac{1}{2}(\pi^{(0)})^2 - \frac{1}{2}a_{(0)}^2$$

The Hamilton–Jacobi theorem in the acceleration bundle (5) asserts that the restriction of H on a closed one-form dW is constant, say c . See that this can be written as

$$\frac{\partial W}{\partial a_{(0)}} \frac{\partial W}{\partial q_{(0)}} - \left(\frac{\partial W}{\partial a_{(0)}} \right)^2 - \frac{1}{2}a_{(0)}^2 = c.$$

A solution of this equation can easily be computed to be

$$W = \frac{1}{\sqrt{2}} \ln \left(a_{(0)} + \sqrt{a_{(0)}^2 + 2c} \right) + \frac{1}{2\sqrt{2}} a_{(0)} \sqrt{a_{(0)}^2 + 2c}.$$

5.3. A simple degenerate model

Now we consider Q as a three dimensional manifold with coordinates (x, y, z) and consider the following degenerate second order Lagrangian

$$L = \frac{1}{2}(\ddot{x} + \ddot{y})^2. \quad (147)$$

The Morse family method—Ostrogradski momenta. On the cotangent bundle T^*TQ , we introduce the momenta $(p_x, p_y, p_z; p_{\dot{x}}, p_{\dot{y}}, p_{\dot{z}})$ and the energy function

$$E = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} + p_{\dot{x}} \ddot{x} + p_{\dot{y}} \ddot{y} + p_{\dot{z}} \ddot{z} - \frac{1}{2}(\ddot{x} + \ddot{y})^2. \quad (148)$$

Assume a function W depending on $(x, y, z; \dot{x}, \dot{y}, \dot{z})$, then the Hamilton–Jacobi problem (87) reads

$$\begin{aligned} & \frac{\partial^2 W}{\partial x \partial x} \dot{x} + \frac{\partial^2 W}{\partial x \partial y} \dot{y} + \frac{\partial^2 W}{\partial x \partial z} \dot{z} + \frac{\partial^2 W}{\partial x \partial \dot{x}} \ddot{x} + \frac{\partial^2 W}{\partial x \partial \dot{y}} \ddot{y} + \frac{\partial^2 W}{\partial x \partial \dot{z}} \ddot{z} = 0 \\ & \frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial \dot{x} \partial x} \dot{x} + \frac{\partial^2 W}{\partial \dot{x} \partial y} \dot{y} + \frac{\partial^2 W}{\partial \dot{x} \partial z} \dot{z} + \frac{\partial^2 W}{\partial \dot{x} \partial \dot{x}} \ddot{x} + \frac{\partial^2 W}{\partial \dot{x} \partial \dot{y}} \ddot{y} + \frac{\partial^2 W}{\partial \dot{x} \partial \dot{z}} \ddot{z} = 0 \\ & \frac{\partial^2 W}{\partial y \partial x} \dot{x} + \frac{\partial^2 W}{\partial y \partial y} \dot{y} + \frac{\partial^2 W}{\partial y \partial z} \dot{z} + \frac{\partial^2 W}{\partial y \partial \dot{x}} \ddot{x} + \frac{\partial^2 W}{\partial y \partial \dot{y}} \ddot{y} + \frac{\partial^2 W}{\partial y \partial \dot{z}} \ddot{z} = 0 \\ & \frac{\partial W}{\partial y} + \frac{\partial^2 W}{\partial \dot{y} \partial x} \dot{x} + \frac{\partial^2 W}{\partial \dot{y} \partial y} \dot{y} + \frac{\partial^2 W}{\partial \dot{y} \partial z} \dot{z} + \frac{\partial^2 W}{\partial \dot{y} \partial \dot{x}} \ddot{x} + \frac{\partial^2 W}{\partial \dot{y} \partial \dot{y}} \ddot{y} + \frac{\partial^2 W}{\partial \dot{y} \partial \dot{z}} \ddot{z} = 0 \\ & \frac{\partial^2 W}{\partial z \partial x} \dot{x} + \frac{\partial^2 W}{\partial z \partial y} \dot{y} + \frac{\partial^2 W}{\partial z \partial z} \dot{z} + \frac{\partial^2 W}{\partial z \partial \dot{x}} \ddot{x} + \frac{\partial^2 W}{\partial z \partial \dot{y}} \ddot{y} + \frac{\partial^2 W}{\partial z \partial \dot{z}} \ddot{z} = 0 \\ & \frac{\partial W}{\partial z} + \frac{\partial^2 W}{\partial \dot{z} \partial x} \dot{x} + \frac{\partial^2 W}{\partial \dot{z} \partial y} \dot{y} + \frac{\partial^2 W}{\partial \dot{z} \partial z} \dot{z} + \frac{\partial^2 W}{\partial \dot{z} \partial \dot{x}} \ddot{x} + \frac{\partial^2 W}{\partial \dot{z} \partial \dot{y}} \ddot{y} + \frac{\partial^2 W}{\partial \dot{z} \partial \dot{z}} \ddot{z} = 0. \end{aligned} \quad (149)$$

Although this system looks cumbersome, the set of constraints in (87) is simply computed as

$$\frac{\partial W}{\partial \dot{x}} = \frac{\partial W}{\partial \dot{y}} = \ddot{x} + \ddot{y}, \quad \frac{\partial W}{\partial \dot{z}} = 0, \quad (150)$$

what reduces this huge system to a more reasonable one. For example, the independence of W to \dot{z} from the last line of the system gives the independence of W to \dot{z} . So that we have actually four number of equations. Then the first two constraints lead to the following reduced system of equations

$$\begin{aligned} \frac{\partial^2 W}{\partial x \partial x} \dot{x} + \frac{\partial^2 W}{\partial x \partial y} \dot{y} + \frac{\partial^2 W}{\partial x \partial \dot{x}} \frac{\partial W}{\partial \dot{x}} &= 0 \\ \frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial \dot{x} \partial x} \dot{x} + \frac{\partial^2 W}{\partial \dot{x} \partial y} \dot{y} + \frac{\partial^2 W}{\partial \dot{x} \partial \dot{x}} \frac{\partial W}{\partial \dot{x}} &= 0 \\ \frac{\partial^2 W}{\partial y \partial x} \dot{x} + \frac{\partial^2 W}{\partial y \partial y} \dot{y} + \frac{\partial^2 W}{\partial y \partial \dot{x}} \frac{\partial W}{\partial \dot{x}} &= 0 \\ \frac{\partial W}{\partial y} + \frac{\partial^2 W}{\partial \dot{y} \partial x} \dot{x} + \frac{\partial^2 W}{\partial \dot{y} \partial y} \dot{y} + \frac{\partial^2 W}{\partial \dot{y} \partial \dot{x}} \frac{\partial W}{\partial \dot{x}} &= 0. \end{aligned} \quad (151)$$

Notice that a solution of this can easily be noticed as

$$W = a\dot{x} + b\dot{y}. \quad (152)$$

5.4. Second order Lagrangian systems with affine dependence on the acceleration

In this subsection we are employing the theoretical parts presented in the previous section to the particular case of second order Lagrangian theories with affine dependence on the acceleration. To this end, we first define the following generic Lagrangian function

$$L(q_{(0)}, q_{(1)}, q_{(2)}) = f_A(q_{(0)}, q_{(1)})q_{(2)}^A + g(q_{(0)}, q_{(1)}) \quad (153)$$

on the second order tangent bundle T^2Q where f_A and g are functions depending only on the position and the velocity.

The Morse family method—Ostrogradski momenta. Let us start with the first approach by introducing the Ostrogradski momenta $(p_A^{(0)}, p_A^{(1)})$ as the fiber coordinates of T^*TQ . Then the energy function take the form

$$E = p_A^{(0)} q_{(1)}^A + p_A^{(1)} q_{(2)}^A - L = p_A^{(0)} q_{(1)}^A + p_A^{(1)} q_{(2)}^A - f_A(q_{(0)}, q_{(1)})q_{(2)}^A - g(q_{(0)}, q_{(1)}). \quad (154)$$

Now, let us introduce an exact one-form

$$\gamma = dW(q_{(0)}, q_{(1)}) = \frac{\partial W}{\partial q_{(0)}^A} dq_{(0)}^A + \frac{\partial W}{\partial q_{(1)}^A} dq_{(1)}^A \quad (155)$$

as given (86) and study the system of Hamilton–Jacobi equations (87). In this case we have that

$$\begin{aligned}
\frac{\partial^2 W}{\partial q_{(0)}^B \partial q_{(0)}^A} q_{(1)}^A + \frac{\partial^2 W}{\partial q_{(0)}^B \partial q_{(1)}^A} q_{(2)}^A &= \frac{\partial f_A}{\partial q_{(0)}^B} q_{(2)}^A + \frac{\partial g}{\partial q_{(0)}^B} \\
\frac{\partial W}{\partial q_{(0)}^B} + \frac{\partial^2 W}{\partial q_{(1)}^B \partial q_{(0)}^A} q_{(1)}^A + \frac{\partial^2 W}{\partial q_{(1)}^B \partial q_{(1)}^A} q_{(2)}^A &= \frac{\partial f_A}{\partial q_{(1)}^B} q_{(2)}^A + \frac{\partial g}{\partial q_{(1)}^B} \\
\frac{\partial W}{\partial q_{(1)}^A} &= f_A(q_{(0)}, q_{(1)}).
\end{aligned} \tag{156}$$

Consider now the third equation in the system (156). Taking the partial derivative of this with respect to $q_{(1)}^B$ result with the following equality

$$\frac{\partial^2 W}{\partial q_{(1)}^A \partial q_{(1)}^B} = \frac{\partial f_A}{\partial q_{(1)}^B}(q_{(0)}, q_{(1)}). \tag{157}$$

Notice that the left hand side is symmetric with respect to the indices A and B whereas this is not generally true for an arbitrary functions f_A . This is the first restriction in the theory. Even though there are numerous physical systems satisfying this symmetry criteria in the literature. There are also interesting physical models involving affine terms violating this symmetry. We provide two example important examples for such kind of systems in the conclusions section by pointing out some possible future works.

We can further investigate more on the integrability of the Hamilton–Jacobi equations. To this end, we substitute the last line of the system (156) into the first two equations. This reads

$$\begin{aligned}
\frac{\partial^2 W}{\partial q_{(0)}^B \partial q_{(0)}^A} q_{(1)}^A &= \frac{\partial g}{\partial q_{(0)}^B} \\
\frac{\partial W}{\partial q_{(0)}^B} &= \frac{\partial g}{\partial q_{(1)}^B} - \frac{\partial f_A}{\partial q_{(0)}^B} q_{(1)}^A.
\end{aligned} \tag{158}$$

Taking the partial derivative of the second line with respect to $\partial q_{(0)}^A$, multiplying by $q_{(1)}^A$, we arrive at the following differential equation

$$\frac{\partial g}{\partial q_{(0)}^B} - \frac{\partial^2 g}{\partial q_{(1)}^B \partial q_{(0)}^A} q_{(1)}^A + \frac{\partial^2 f_C}{\partial q_{(0)}^B \partial q_{(0)}^A} q_{(1)}^C q_{(1)}^A = 0. \tag{159}$$

This is an integrability criterion for the Hamilton–Jacobi problem for second order Lagrangian formalisms that are affine in acceleration. Assuming that this holds, the Hamilton–Jacobi problem can be written in a relatively easy form

$$\begin{aligned}
\frac{\partial W}{\partial q_{(0)}^B} &= \frac{\partial g}{\partial q_{(1)}^B} - \frac{\partial f_A}{\partial q_{(0)}^B} q_{(1)}^A, \\
\frac{\partial W}{\partial q_{(1)}^A} &= f_A(q_{(0)}, q_{(1)}).
\end{aligned} \tag{160}$$

The Morse family method—the Schmidt method. In this case, we shall start with the Lagrangian function (153) once more, but in this case we investigate the associated Hamilton–Jacobi problem by means of the Schmidt Legendre transformation in the framework of the acceleration bundle. By choosing the auxiliary function $F = \delta_{AB} q_{(1)}^A m_{(0)}^B$ we write the equivalent Lagrangian function exhibited in (126) as follows

$$L_{2\text{-deg}} = f_A(q_{(0)}, q_{(1)})a_{(0)}^A + g(q_{(0)}, q_{(1)}) + \delta_{AB}q_{(1)}^A m_{(1)}^B + \delta_{AB}a_{(0)}^A m_{(0)}^B \quad (161)$$

which depends on the base components $(q_{(0)}, a_{(0)}, m_{(0)})$ along with the velocities $(q_{(1)}, a_{(1)}, m_{(1)})$. In this case, the energy function (127) turns out to be

$$E = p_A^{(0)} q_{(1)}^A + \pi_A^{(0)} a_{(1)}^A + \mu_A^{(0)} m_{(1)}^A - f_A(q_{(0)}, q_{(1)})a_{(0)}^A - g(q_{(0)}, q_{(1)}) - \delta_{AB}q_{(1)}^A m_{(1)}^B - \delta_{AB}a_{(0)}^A m_{(0)}^B$$

Assuming an exact one-form

$$\gamma = dW(q_{(0)}, a_{(0)}, m_{(0)}) = \frac{\partial W}{\partial q_{(0)}^A} dq_{(0)}^A + \frac{\partial W}{\partial a_{(0)}^A} da_{(0)}^A + \frac{\partial W}{\partial m_{(0)}^A} dm_{(0)}^A \quad (162)$$

the Hamilton–Jacobi equation (129) turns out to be

$$\begin{aligned} \frac{\partial^2 W}{\partial q_{(0)}^A \partial q_{(0)}^B} q_{(1)}^A + \frac{\partial^2 W}{\partial a_{(0)}^A \partial q_{(0)}^B} a_{(1)}^A + \frac{\partial^2 W}{\partial m_{(0)}^A \partial q_{(0)}^B} m_{(1)}^A &= \frac{\partial f_A}{\partial q_{(0)}^B} a_{(0)}^A + \frac{\partial g}{\partial q_{(0)}^B} \\ \frac{\partial^2 W}{\partial q_{(0)}^A \partial a_{(0)}^B} q_{(1)}^A + \frac{\partial^2 W}{\partial a_{(0)}^A \partial a_{(0)}^B} a_{(1)}^A + \frac{\partial^2 W}{\partial m_{(0)}^A \partial a_{(0)}^B} m_{(1)}^A &= f_B + \delta_{AB} m_{(0)}^A \\ \frac{\partial^2 W}{\partial q_{(0)}^A \partial m_{(0)}^B} q_{(1)}^A + \frac{\partial^2 W}{\partial a_{(0)}^A \partial m_{(0)}^B} a_{(1)}^A + \frac{\partial^2 W}{\partial m_{(0)}^A \partial m_{(0)}^B} m_{(1)}^A &= \delta_{AB} a_{(0)}^A \\ \frac{\partial W}{\partial q_{(0)}^A} &= \frac{\partial f_B}{\partial q_{(1)}^A} a_{(0)}^B + \frac{\partial g}{\partial q_{(1)}^A} + \delta_{AB} m_{(1)}^B \\ \frac{\partial W}{\partial a_{(0)}^A} &= 0 \\ \frac{\partial W}{\partial m_{(0)}^A} &= \delta_{AB} q_{(1)}^B. \end{aligned} \quad (163)$$

From the fifth line we see that W does not depend on $a_{(0)}$. So that, the second line determines the identity $f_B = -\delta_{AB} m_{(0)}^A$. From the fourth and sixth equations, we substitute the Lagrange multipliers $m_{(1)}^B$ and $q_{(1)}^B$, in to the rest of the equations and we arrive at the following reduced system

$$\begin{aligned} \delta^{AC} \frac{\partial^2 W}{\partial q_{(0)}^A \partial q_{(0)}^B} \frac{\partial W}{\partial m_{(0)}^C} + \delta^{AC} \frac{\partial^2 W}{\partial m_{(0)}^A \partial q_{(0)}^B} \left(\frac{\partial W}{\partial q_{(0)}^C} - \frac{\partial f_D}{\partial q_{(1)}^C} a_{(0)}^D - \frac{\partial g}{\partial q_{(1)}^C} \right) &= \frac{\partial f_A}{\partial q_{(0)}^B} a_{(0)}^A + \frac{\partial g}{\partial q_{(0)}^B} \\ \delta^{AC} \frac{\partial^2 W}{\partial q_{(0)}^A \partial m_{(0)}^B} \frac{\partial W}{\partial m_{(0)}^C} + \delta^{AC} \frac{\partial^2 W}{\partial m_{(0)}^A \partial m_{(0)}^B} \left(\frac{\partial W}{\partial q_{(0)}^C} - \frac{\partial f_D}{\partial q_{(1)}^C} a_{(0)}^D - \frac{\partial g}{\partial q_{(1)}^C} \right) &= \delta_{AB} a_{(0)}^A. \end{aligned} \quad (164)$$

In this case, we have arrived at a relatively complicated PDE system comparing with the Ostrogradski method. Indeed, the choice of one of the two methods is important for resolving the equations.

5.5. Third order Lagrangian systems with affine dependence on the third order derivative term

In this subsection, in order to exhibit the application area of the theoretical framework we have proposed, we shall investigate possible Hamilton–Jacobi realization of some class of the third

order singular Lagrangian systems involving affine dependence to the third order derivative terms in the following form

$$L(q_{(0)}, q_{(1)}, q_{(2)}) = f_A(q_{(1)}, q_{(2)})q_{(3)}^A + g(q_{(0)}, q_{(1)}) \quad (165)$$

on the third order tangent bundle T^3Q .

The Morse family method—Ostrogradski momenta. The energy function generating the dynamics of the Lagrangian (165) is

$$E = p_A^{(0)} q_{(1)}^A + p_A^{(1)} q_{(2)}^A + p_A^{(2)} q_{(3)}^A - f_A(q_{(1)}, q_{(2)})q_{(3)}^A - g(q_{(0)}, q_{(1)}), \quad (166)$$

where $(p^{(0)}, p^{(1)}, p^{(2)})$ are the conjugate momenta defining the fiber coordinates of the cotangent bundle T^*T^2Q . Consider an exact one-form on T^2Q which is in coordinates given by

$$\gamma = dW(q_{(0)}, q_{(1)}, q_{(2)}) = \frac{\partial W}{\partial q_{(0)}^A} dq_{(0)}^A + \frac{\partial W}{\partial q_{(1)}^A} dq_{(1)}^A + \frac{\partial W}{\partial q_{(2)}^A} dq_{(2)}^A \quad (167)$$

following (86). We write the system of Hamilton–Jacobi equations (87) as follows

$$\begin{aligned} \frac{\partial^2 W}{\partial q_{(0)}^A \partial q_{(0)}^B} q_{(1)}^A + \frac{\partial^2 W}{\partial q_{(1)}^A \partial q_{(0)}^B} q_{(2)}^A + \frac{\partial^2 W}{\partial q_{(2)}^A \partial q_{(0)}^B} q_{(3)}^A &= \frac{\partial g}{\partial q_{(0)}^B} \\ \frac{\partial W}{\partial q_{(0)}^B} + \frac{\partial^2 W}{\partial q_{(0)}^A \partial q_{(1)}^B} q_{(1)}^A + \frac{\partial^2 W}{\partial q_{(1)}^A \partial q_{(1)}^B} q_{(2)}^A + \frac{\partial^2 W}{\partial q_{(2)}^A \partial q_{(1)}^B} q_{(3)}^A &= \frac{\partial f_A}{\partial q_{(1)}^B} q_{(3)}^A + \frac{\partial g}{\partial q_{(1)}^B} \\ \frac{\partial W}{\partial q_{(1)}^B} + \frac{\partial^2 W}{\partial q_{(0)}^A \partial q_{(2)}^B} q_{(1)}^A + \frac{\partial^2 W}{\partial q_{(1)}^A \partial q_{(2)}^B} q_{(2)}^A + \frac{\partial^2 W}{\partial q_{(2)}^A \partial q_{(2)}^B} q_{(3)}^A &= \frac{\partial f_A}{\partial q_{(2)}^B} q_{(3)}^A \\ \frac{\partial W}{\partial q_{(2)}^B} &= f_B. \end{aligned} \quad (168)$$

Let us try to simplify this system. See that the last line reads that $\partial W / \partial q_{(2)}^B$ is independent of $q_{(0)}$, and leads to the observation that $\partial f_B / \partial q_{(2)}^A$ must be symmetric with respect to the indices A and B . Substitution of the last identity in (168) into the second and third lines we arrive at a fairly more simple system

$$\begin{aligned} \frac{\partial W}{\partial q_{(0)}^B} &= \frac{\partial g}{\partial q_{(1)}^B} + \frac{\partial^2 f_B}{\partial q_{(1)}^A \partial q_{(1)}^C} q_{(2)}^A q_{(2)}^C, \\ \frac{\partial W}{\partial q_{(1)}^B} &= -\frac{\partial f_B}{\partial q_{(1)}^C} q_{(2)}^C, \\ \frac{\partial W}{\partial q_{(2)}^B} &= f_B. \end{aligned} \quad (169)$$

See that, this system is coupled with the first line of the system (168). So that substitution of (169) into the first line of (168) must be identically satisfied. This compatibility condition reads

$$\frac{\partial^2 g}{\partial q_{(0)}^A \partial q_{(1)}^B} q_{(1)}^A + \frac{\partial^3 f_B}{\partial q_{(0)}^A \partial q_{(1)}^D \partial q_{(1)}^C} q_{(2)}^D q_{(2)}^C q_{(1)}^A - \frac{\partial^2 f_B}{\partial q_{(1)}^C \partial q_{(1)}^A} q_{(2)}^C q_{(2)}^A = \frac{\partial g}{\partial q_{(0)}^B}. \quad (170)$$

It is also important to know that one only needs to perform direct integration to find W after the functions f_A and g are determined. But to do this, f_A and g can not be arbitrarily chosen since they have to satisfy the integrability conditions arising from the system.

6. Conclusions and comments

In this paper, we propose a Hamilton–Jacobi theory for higher order Lagrangians. Our theory works well for all non-degenerate and (a large class) of degenerate Lagrangians. The implicit character of singular systems has been studied in two different ways: the first one consisted on making use of Morse families that play the role of the Hamiltonian function, and that give rise to Lagrangian submanifolds as in equivalence to the image of γ , i.e. the solution of a Hamilton–Jacobi problem. The second method consists on the local construction of a vector field associated to the implicit equations and defined on a proper domain compatible with the implicit character. The higher order derivatives are studied through both the Ostrogradski–Legendre and Schmidt–Legendre transformations. In the particular case of second order Lagrangians, we have employed the acceleration bundle picture.

As a future endeavour, we would like to generalize this formalism in such a way that we could be able to work with all degenerate higher order Lagrangian systems, e.g. singular higher order Lagrangians coming from the gravitational theory. Concerning this topic, we will be specifically interested in two examples. These examples cannot be studied in the geometry exhibited in the present framework because of the skew-symmetric Chern–Simons term.

- (1) One is the chiral oscillator in two dimensions, which accounts for mirror symmetry, and in the non-relativistic case, we have an oscillator with a Chern–Simons term (independent of the metric). It reads:

$$L = -\lambda \epsilon_{AB} \dot{q}_{(1)}^A \dot{q}_{(2)}^B + \frac{m}{2} \delta_{AB} \dot{q}_{(1)}^A \dot{q}_{(1)}^B \quad (171)$$

where λ and m are nonvanishing constants [16]. Here, ϵ_{AB} is a skew-symmetric tensor with $\epsilon_{12} = 1$. The Lagrangian (171) is quasi-invariant under the Galilean transformations.

- (2) The second example is the Clément Lagrangian, which is a second order degenerate Lagrangian function [15]. It is defined on the second order tangent bundle T^2Q where Q is a semi-Riemannian manifold equipped with the Minkowskian metric $\theta = [\theta_{AB}]$ with $(+, -, -)$. The Clément Lagrangian is given by

$$L = -\frac{m}{2} \zeta \theta_{AB} \dot{q}_{(1)}^A \dot{q}_{(1)}^B - \frac{2m\Lambda}{\zeta} + \frac{\zeta^2}{2\mu m} \epsilon_{ABC} \dot{q}_{(0)}^A \dot{q}_{(1)}^B \dot{q}_{(2)}^C \quad (172)$$

where $\zeta = \zeta(t)$ is a function that allows arbitrary reparametrizations of the variable t , whereas Λ and $1/2m$ are the cosmological and Einstein gravitational constants, respectively. Here, ϵ_{ABC} is a skewsymmetric three tensor determining the triple product, so this Lagrangian falls into the category of Lagrangians depending linearly on the acceleration [16]. For the Hamiltonian analysis of this singular theory, we cite [10].

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References

- [1] Abraham R and Marsden J E 1978 *Foundations of Mechanics* 2nd edn (Reading, MA: Addison-Wesley)
- [2] Abraham R, Marsden J E and Ratiu T 1988 *Manifolds, Tensor Analysis, and Applications (Applied Mathematical Sciences vol 75)* 2nd edn (New York: Springer)
- [3] Andrzejewski K, Gonera J, Machalski P and Maś lanka P 2010 Modified Hamiltonian formalism for higher-derivative theories *Phys. Rev. D* **82** 045008
- [4] Arnol'd V I 1989 *Mathematical Methods of Classical Mechanics* 2nd edn (Berlin: Springer)
- [5] Balseiro P, Marrero J C, Martín de Diego D and Padrón E 2010 A unified framework for mechanics: Hamilton–Jacobi equation and applications *Nonlinearity* **23** 1887–918
- [6] Barbero-Liñán M, de León M and Martín de Diego D 2013 Lagrangian submanifolds and Hamilton–Jacobi equation *Mon. hefte für Math.* **171** 269–90
- [7] Barbero-Liñán M, de León M, Martín de Diego D, Marrero J C and Muñoz-Lecanda M C 2012 Kinematic reduction and the Hamilton–Jacobi equation *J. Geom. Mech.* **4** 207–37
- [8] Benenti S 2011 *Hamiltonian Structures and Generating Families* (Berlin: Springer)
- [9] Benenti S and Tulczyjew W M 1980 The geometrical meaning and globalization of the Hamilton–Jacobi method *Differential Geometrical Methods in Mathematical Physics* (Berlin: Springer) pp 9–21
- [10] Çağatay Uçgun F, Esen O and Gümrall H 2018 Reductions of topologically massive gravity I: Hamiltonian analysis of second order degenerate Lagrangians *J. Math. Phys.* **59** 013510
- [11] Cariñena J F, Gràcia X, Marmo G, Martínez E, Muñoz-Lecanda M C and Román-Roy N 2006 Geometric Hamilton–Jacobi theory *Int. J. Geom. Methods Mod. Phys.* **3** 1417–58
- [12] Cariñena J F, Gràcia X, Marmo G, Martínez E, Muñoz-Lecanda M C and Román-Roy N 2009 Hamilton–Jacobi theory and the evolution operator *Mathematical Physics and Field Theory Julio Abad, in Memoriam* ed M Asorey et al (Prensas Universitarias de Zaragoza) pp 177–86
- [13] Cariñena J F, Gràcia X, Marmo G, Martínez E, Muñoz-Lecanda M C and Román-Roy N 2010 Nonholonomic geometric Hamilton–Jacobi theory for nonholonomic dynamical systems *Int. J. Geom. Methods Mod. Phys.* **7** 431–54
- [14] Choquet-Bruhat Y, de Witt-Morette C and Dillard-Bleick M 1982 *Analysis, Manifolds and Physics* (Amsterdam: North-Holland)
- [15] Clément G 1994 Particle-like solutions to topologically massive gravity *Class. Quantum Grav.* **11** L115
- [16] Cruz M, Gómez-Cortés R, Molgado A and Rojas E 2016 Hamiltonian analysis for linearly acceleration-dependent Lagrangians *J. Math. Phys.* **57** 062903
- [17] Colombo L, de León M, Prieto-Martínez P D and Román-Roy N 2014 Geometric Hamilton–Jacobi theory for higher order autonomous systems *J. Phys. A: Math. Theor.* **47** 235203
- [18] Constantelos G C 1984 On the Hamilton–Jacobi theory with derivatives of higher order *Nuovo Cimento B (11)* **84** 91–101
- [19] Dirac P A M 1967 *Lectures in Quantum Mechanics (Belfer Graduate School of Science)* (New York: Yeshiva University)
- [20] Dominici D, Gomis J, Longhi G and Pons J M 1984 Hamilton–Jacobi theory for constrained systems *J. Math. Phys.* **25** 2439–60
- [21] Esen O and Guha P 2018 On the geometry of the Schmidt–Legendre transformation *J. Geom. Mech.* **10** 251–91
- [22] Esen O and Gümrall H 2014 Tulczyjew triplet for lie groups I: trivializations and reductions *J. Lie Theory* **24** 1115–60
- [23] Esen O and Gümrall H 2017 Tulczyjew triplet for lie groups II: dynamics *J. Lie Theory* **27** 329–56

- [24] Esen O, Jiménez V M, de León M and Sardón C 2018 Reduction of a Hamilton–Jacobi equation for nonholonomic systems *SIGMA* (arxiv:[1810.04962](#)) (submitted)
- [25] Esen O, de León M and Sardón C 2018 A Hamilton–Jacobi theory for implicit differential systems *J. Math. Phys.* **59** 022902
- [26] Fels M 1996 The inverse problem of the calculus of variations for scalar fourth-order ordinary differential equations *Trans. Am. Math. Soc.* **348** 5007–29
- [27] Gotay M J and Nester J M 1980 Generalized constraint algorithm and special presymplectic manifolds *Geometric Methods in Modern Physics (Lecture Notes in Math.* vol 775) (Berlin: Springer) pp 78–104 (*Proc. NSF-CBMS Conf., Univ. Lowell, Lowell, Mass., 1979*)
- [28] Gotay M J and Nester J M 1984 Apartheid in the Dirac theory of constraints *J. Phys. A: Math. Gen.* **A 17** 3063–6
- [29] Gotay M J, Nester J M and Hinds G 1978 Presymplectic manifolds and the Dirac–Bergmann theory of constraints *J. Math. Phys.* **19** 2388–99
- [30] Grabowski J and Grabowska K 2013 Tulczyjew triples: from statics to field theory *J. Geom. Mech.* **5** 445–72
- [31] Ibort A, de León M, Marmo G and Martín de Diego D 1996 Nonholonomic constrained systems as implicit differential equations *Rend. del Semin. Mat. di Torino* **54** 295
- [32] Iglesias-Ponte D, de León M and Martín de Diego D 2008 Towards a Hamilton–Jacobi theory for nonholonomic mechanical systems *J. Phys. A: Math. Theor.* **41** 015205
- [33] Janeczko S 2000 On implicit Lagrangian differential systems *Ann. Polonici Mathematici* **74** 133–41
- [34] José J V and Saletan E J 1998 *Classical Dynamics: a Contemporary Approach* (Cambridge: Cambridge University Press)
- [35] Lawruk B, Sniatycki J and Tulczyjew W M 1975 Special symplectic spaces *J. Differ. Equ.* **17** 477–97
- [36] Leok M, Ohsawa T and Sosa D 2012 Hamilton–Jacobi theory for degenerate Lagrangian systems with holonomic and nonholonomic constraints *J. Math. Phys.* **53** 072905
- [37] de León M, Marrero J C and Martín de Diego D 2009 A geometric Hamilton–Jacobi theory for classical field theories *Variations, Geometry and Physics* (New York: Nova Sci Publ.) pp 129–40
- [38] de León M, Marrero J C and Martín de Diego D 2010 Linear almost Poisson structures and Hamilton–Jacobi equation applications to nonholonomic mechanics *J. Geom. Mech.* **2** 159–98
- [39] de León M, Martín de Diego D, Marrero J C, Salgado M and Vilariño S 2010 Hamilton–Jacobi theory in k -symplectic field theories *Int. J. Geom. Methods Mod. Phys.* **7** 1491–507
- [40] de León M, Martín de Diego D and Vaquero M 2012 A Hamilton–Jacobi theory for singular Lagrangian systems in the Skinner and Rusk setting *Int. J. Geom. Methods Mod. Phys.* **9** 1250074
- [41] de León M and Rodrigues P R 2011 *Methods of Differential Geometry in Analytical Mechanics* vol 158 (Amsterdam: Elsevier)
- [42] de León M and Rodrigues P R 1985 *Generalized Classical Mechanics and Field Theory (North-Holland Math. Studies* vol 112) (Amsterdam: Elsevier)
- [43] de León M and Sardón C 2017 Geometry of the discrete Hamilton–Jacobi equation. Applications in optimal control *Rep. Math. Phys.* **81** 39–63
- [44] de León M and Sardón C 2017 Geometric Hamilton–Jacobi theory on Nambu–Poisson manifolds *J. Math. Phys.* **58** 033508
- [45] de León M and Vilariño S 2014 Hamilton–Jacobi theory in k -cosymplectic field theories *Int. J. Geom. Methods Mod. Phys.* **11** 1450007
- [46] Libermann P and Marle C M 1987 *Symplectic Geometry and Analytical Dynamics* (Dordrecht: D. Reidel Publishing Company)
- [47] Mendella G, Marmot G and Tulczyjew W M 1995 Integrability of implicit differential equations *J. Phys. A: Math. Gen.* **28** 149
- [48] Marmo G, Mendella G and Tulczyjew W M 1997 Constrained Hamiltonian systems as implicit differential equations *J. Phys. A: Math. Gen.* **30** 277–93
- [49] Marmo G, Morandi G and Mukunda N 1990 A geometrical approach to the Hamilton–Jacobi form of dynamics and its generalizations *Riv. Nuovo Cimento* **13** 1–74
- [50] Marsden J E and Ratiu T S 2013 *Introduction to Mechanics and Symmetry: a Basic Exposition of Classical Mechanical Systems* vol 17 (Berlin: Springer)
- [51] Morandi G, Ferrario C, Vecchio G L, Marmo G and Rubano C 1990 The inverse problem in the calculus of variations and the geometry of the tangent bundle *Phys. Rep.* **188** 147–284
- [52] Ohsawa T and Bloch A M 2011 Nonholomic Hamilton–Jacobi equation and integrability *J. Geom. Mech.* **1** 461–81

- [53] Ohsawa T, Bloch A M and Leok M 2011 Nonholonomic Hamilton–Jacobi theory via Chaplygin Hamiltonization *J. Geom. Phys.* **61** 1263–91
- [54] Ohsawa T, Bloch A M and Leok M 2011 Discrete Hamilton–Jacobi theory *SIAM J. Control Optim.* **49** 1829–56
- [55] Ostrogradski M 1850 Mmoire sur les quations diffrentielles relatives au problme des isoprimitives *Mem. Ac. St. Petersburg.* **6** 385
- [56] Prieto-Martínez P D and Román-Roy N 2011 Lagrangian-Hamiltonian unified formalism for autonomous higher order dynamical systems *J. Phys. A: Math. Theor.* **44** 385203
- [57] Prieto-Martínez P D and Román-Roy N 2012 Unified formalism for higher order non-autonomous dynamical systems *J. Math. Phys.* **53** 032901
- [58] Saunders D J 1989 *The Geometry of Jet Bundles* (London Mathematical Society, Lecture Notes Series vol 142) (Cambridge: Cambridge University Press)
- [59] Schmidt H J 1994 Stability and Hamiltonian formulation of higher derivative theories *Phys. Rev. D* **49** 6354
- [60] Śniatycki J and Tulczyjew W M 1972 Generating forms of Lagrangian submanifolds *Indiana Univ. Math.* **22** 267–75
- [61] Sorrentino A 2017 Action-minimizing methods in Hamiltonian dynamics (MN-50) *An Introduction to Aubry–Mather Theory* (Princeton Scholarship online)
- [62] Su X, Wang L and Yan J 2016 Weak KAM theory for Hamilton–Jacobi equations depending on unknown functions *Discrete Continuous Dyn. Syst.* **36** 6487–522
- [63] Şuhubi E 2013 *Exterior Analysis: Using Applications of Differential Forms* (Amsterdam: Elsevier)
- [64] Tulczyjew W M 1977 A symplectic formulation of particle dynamics *Differential Geometrical Methods in Mathematical Physics* (Berlin: Springer) pp 457–63
- [65] Tulczyjew W M 1977 The Legendre transformation *Ann. de l’IHP Phys. Thorique* **27** 101–14
- [66] Tulczyjew W M and Urbanski P 1996 *Homogeneous Lagrangian Systems, Gravitation, Electromagnetism and Geometric Structures* (Bologna, BO: Pitagora Editrice) pp 91–136
- [67] Tulczyjew W M and Urbanski P 1999 A slow and careful Legendre transformation for singular Lagrangians *Acta Phys. Pol. B* **30** 2909–78
- [68] Vitagliano L 2011 Hamilton–Jacobi diffieties *J. Geom. Phys.* **61** 1932–49
- [69] Weinstein A 1977 Lectures on symplectic manifolds *CBMS Regional Conf. Series in Mathematics* vol 9 p 48
- [70] Zajac M and Grabowska K 2016 The Tulczyjew triple in mechanics on Lie groups *J. Geom. Mech.* **8** 413–35