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Dynamics of poles of elliptic solutions to the BKP equation

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Abstract
We derive equations of motion for poles of elliptic solutions to the B-version of the Kadomtsev–Petviashvili equation (BKP). The basic tool is the auxiliary linear problem for the Baker–Akhiezer function. We also discuss integrals of motion for the pole dynamics which follow from the equation of the spectral curve.

Keywords: BKP equation, elliptic solutions, dynamics of poles

1. Introduction

In the seminal paper [1] the motion of poles of singular solutions to the Korteweg–de Vries and Boussinesq equations was investigated. It was discovered that the poles move as particles of the many-body Calogero–Moser system [2–4] with some additional restrictions in the phase space. In [5, 6] it was shown that in the case of the Kadomtsev–Petviashvili (KP) equation this correspondence becomes an isomorphism: the dynamics of poles of rational solutions to the KP equation is given by equations of motion for the Calogero–Moser system with pairwise interaction potential $1/(x_i - x_j)^2$. This remarkable connection was further generalized to elliptic (double periodic) solutions by Krichever in [7]: poles x_i of the elliptic solutions move according to the equations of motion

$$\ddot{x}_i = 4 \sum_{k \neq i} \wp'(x_i - x_k) \tag{1}$$

of Calogero–Moser particles with the elliptic interaction potential $\wp(x_i - x_j)$ ($\wp$ is the Weierstrass $\wp$-function). This many-body system of classical mechanics is known to be integrable. For a review of the models of the Calogero–Moser type (including the models associated with the classical root systems) see [8]. (For further progress, generalizations and related models see, e.g. [9–16].)

This result allows for generalizations in various directions. The extension to the matrix KP equation was discussed in [17]; in this case the poles and matrix residues at the poles move as particles of the spin generalization of the Calogero–Moser model known also as the Gibbons–Hermsen model [9]. Another generalization of the Calogero–Moser many-body systems with elliptic interaction is their relativistic extension known also as the Ruijsenaars–Schneider systems [10, 11] and their versions with spin degrees of freedom [18]. These relativistic systems emerge as dynamics of poles of elliptic solutions to the two-dimensional Toda lattice (see [18]).

We are going to suggest a generalization in yet another direction: our goal is to find out what dynamical system governs the dynamics of poles of elliptic solutions to the B-version of the KP equation. We will see that the result is a new, previously unknown dynamical system with elliptic interaction which does not look like any kind of the Calogero–Moser system.

The method of derivation of equations of motion for the poles of singular solutions to integrable non-linear equations suggested by Krichever consists in substituting the pole ansatz not in the non-linear equation itself but in the auxiliary linear problems for it. We apply this method in this paper.

In section 2 we derive equations of motion for poles of elliptic solutions to the *B*-version of the KP equation (BKP). The BKP equation is the first member of an infinite BKP hierarchy with independent variables ('times') $t_1, t_3, t_5, t_7, \ldots$ [19, 20], see also [21–23]. We set $t_1 = x$. The BKP equation has the form of a system of two partial differential equations for two dependent variables u, w:

$$\begin{cases} 3w' = 10u_{t_3} + 20u''' + 120uu' \\ \\ w_{t_3} - 6u_{t_5} = w''' - 6u''''' - 60uu''' - 60u'u'' + 6uw' - 6wu', \end{cases} \tag{2}$$

where prime means differentiation w.r.t. x. In fact the variable w can be excluded and the equation can be written in terms a single dependent variable $U = \int^x u dx$. Equation (2) are equivalent to the Zakharov–Shabat ('zero curvature') equation $\partial_{t_5} B_3 - \partial_{t_3} B_5 + [B_3, B_5] = 0$ for the differential operators

$$B_3 = \partial_x^3 + 6u\partial_x, \qquad B_5 = \partial_x^5 + 10u\partial_x^3 + 10u'\partial_x^2 + w\partial_x. \tag{3}$$

In its turn, the Zakharov–Shabat equation is the compatibility condition for the auxiliary linear problems

$$\partial_{t_3}\psi = B_3\psi, \qquad \partial_{t_5}\psi = B_5\psi$$

for the Baker–Akhiezer function ψ which depends on a spectral parameter z.

The change of dependent variables from u, w to the tau-function $\tau = \tau(x, t_3, t_5, \ldots)$

$$u = \partial_x^2 \log\tau, \qquad w = \frac{10}{3}\,\partial_{t_3}\partial_x \log\tau + \frac{20}{3}\,\partial_x^4 \log\tau + 20(\partial_x^2 \log\tau)^2 \tag{4}$$

makes the first of the equation (2) trivial and the other one turns into the bilinear form [20]

$$\Big(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5\Big)\tau\cdot\tau = 0, \tag{5}$$

where D_i are the Hirota operators. Their action is defined by the rule

$$P(D_1, D_3, D_5, \ldots)\tau\cdot\tau = P(\partial_{y_1}, \partial_{y_3}, \partial_{y_5}, \ldots)\tau(x + y_1, t_3 + y_3, \ldots)\tau(x - y_1, t_3 - y_3, \ldots)\Big|_{y_i = 0}$$

for any polynomial $P(D_1, D_3, D_5, \ldots)$. The Baker–Akhiezer function is known to be expressed through the tau-function according to the formula

$$\psi = A(z) \exp\Big(\sum_{k\geqslant 1,\, k \text{ odd}} t_k z^k\Big) \frac{\tau\Big(t_1 - 2z^{-1}, t_3 - \frac{2}{3}z^{-3}, t_5 - \frac{2}{5}z^{-5}, \ldots\Big)}{\tau(t_1, t_3, t_5, \ldots)}. \tag{6}$$

Here $A(z)$ is the normalization factor.

Our aim is to study double-periodic (elliptic) in the variable $t_1 = x$ solutions of the BKP equation. For such solutions the tau-function is an 'elliptic polynomial' in the variable x:

$$\tau = A e^{cx^2/2} \prod_{i=1}^{N} \sigma(x - x_i) \tag{7}$$

with some constants A, c, where

$$\sigma(x) = \sigma(x|\,\omega, \omega') = x \prod_{s\neq 0} \Big(1 - \frac{x}{s}\Big) e^{\frac{x}{s} + \frac{x^2}{2s^2}}, \quad s = 2\omega m + 2\omega' m' \quad \text{with integer } m, m',$$

is the Weierstrass σ-function with quasi-periods 2ω, $2\omega'$ such that $\mathrm{Im}(\omega'/\omega) > 0$. It is connected with the Weierstrass ζ- and $\wp$-functions by the formulas $\zeta(x) = \sigma'(x)/\sigma(x)$, $\wp(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x)$. The roots x_i are assumed to be all distinct. Correspondingly, the function $u = \partial_x^2 \log \tau$ is an elliptic function with double poles at the points x_i:

$$u = c - \sum_{i=1}^{N} \wp(x - x_i). \tag{8}$$

The poles depend on the times t_3, t_5. We will show that the dependence on the time $t_3 = t$ is described by the equations of motion

$$\ddot{x}_i + 6 \sum_{j\neq i} (\dot{x}_i + \dot{x}_j) \wp'(x_i - x_j) - 72 \sum_{j\neq k\neq i} \wp(x_i - x_j) \wp'(x_i - x_k) = 0. \tag{9}$$

This is the main result of the paper. A characteristic feature of the system (9) is the presence of both two-body and three-body interaction and dependence on the first time derivatives. Note that the latter is also the case for the relativistic Calogero–Moser systems [10, 11] while the former seems to be a novel phenomenon for classical integrable systems.

In section 3 we discuss integrals of motion for the dynamical system (9). It is shown that there is a large set of integrals of motion. In section 4 properties of the spectral curve are studied. Section 5 is devoted to analytic properties of the ψ-function on the spectral curve.

2. Elliptic solutions to the BKP equation and dynamics of poles

According to Krichever's method [7], the basic tool for studying t-dynamics of poles is the auxiliary linear problem $\partial_t \psi = B_3 \psi$ for the function ψ, i.e.

$$\partial_t \psi = \partial_x^3 \psi + 6u \partial_x \psi. \tag{10}$$

Since the coefficient function u is double-periodic, one can find double-Bloch solutions $\psi(x)$, i.e. solutions such that $\psi(x + 2\omega) = b\psi(x)$, $\psi(x + 2\omega') = b'\psi(x)$ with some Bloch multipliers b, b'. Equations (6) and (7) tell us that the Baker–Akhiezer function has simple poles at the points x_i. The pole ansatz for the ψ-function is

$$\psi = \mathrm{e}^{xz+tz^3} \sum_{i=1}^{N} c_i \Phi(x - x_i, \lambda), \tag{11}$$

where the coefficients c_i do not depend on x. Here we use the function

$$\Phi(x, \lambda) = \frac{\sigma(x+\lambda)}{\sigma(\lambda)\sigma(x)}\, \mathrm{e}^{-\zeta(\lambda)x}$$

which has a simple pole at $x = 0$ (ζ is the Weierstrass ζ-function). The expansion of Φ as $x \to 0$ is

$$\Phi(x, \lambda) = \frac{1}{x} + \alpha_1 x + \alpha_2 x^2 + \ldots, \qquad x \to 0,$$

where $\alpha_1 = -\frac{1}{2}\,\wp(\lambda)$, $\alpha_2 = -\frac{1}{6}\,\wp'(\lambda)$. The parameters z and λ are spectral parameters, they are going to be connected by equation of the spectral curve. Using the quasiperiodicity properties of the function Φ,

$$\Phi(x + 2\omega, \lambda) = \mathrm{e}^{2(\zeta(\omega)\lambda - \zeta(\lambda)\omega)}\Phi(x, \lambda),$$

$$\Phi(x + 2\omega', \lambda) = \mathrm{e}^{2(\zeta(\omega')\lambda - \zeta(\lambda)\omega')}\Phi(x, \lambda),$$

one can see that ψ given by (11) is indeed a double-Bloch function with Bloch multipliers

$$b = \mathrm{e}^{2(\omega z + \zeta(\omega)\lambda - \zeta(\lambda)\omega)}, \qquad b' = \mathrm{e}^{2(\omega' z + \zeta(\omega')\lambda - \zeta(\lambda)\omega')}.$$

We will often suppress the second argument of Φ writing simply $\Phi(x) = \Phi(x, \lambda)$. We will also need the x-derivatives $\Phi'(x, \lambda) = \partial_x \Phi(x, \lambda)$, $\Phi''(x, \lambda) = \partial_x^2 \Phi(x, \lambda)$, etc.

It is evident from (8) and (10) that the constant c in the pole expansion for the function u can be eliminated by the simple transformation $x \to x - 6ct$, $t \to t$ (or $\partial_x \to \partial_x$, $\partial_t \to \partial_t + 6c\partial_x$ for the vector fields). Because of this we will put $c = 0$ from now on for simplicity.

Substituting (11) into (10) with $u = -\sum_i \wp(x - x_i)$, we get:

$$\sum_i \dot{c}_i \Phi(x - x_i) - \sum_i c_i \dot{x}_i \Phi'(x - x_i) = 3z^2 \sum_i c_i \Phi'(x - x_i) + 3z \sum_i c_i \Phi''(x - x_i) + \sum_i c_i \Phi'''(x - x_i)$$

$$-6z\Big(\sum_k \wp(x - x_k)\Big)\Big(\sum_i c_i \Phi(x - x_i)\Big) - 6\Big(\sum_k \wp(x - x_k)\Big)\Big(\sum_i c_i \Phi'(x - x_i)\Big).$$

It is enough to cancel all poles which are at the points x_i (up to fourth order). It is easy to see that poles of the fourth and third order cancel identically. A direct calculation shows that the conditions of cancellation of second and first order poles have the form

$$c_i \dot{x}_i = -(3z^2 + 6\alpha_1)c_i - 6z \sum_{k \neq i} c_k \Phi(x_i - x_k) - 6 \sum_{k \neq i} c_k \Phi'(x_i - x_k) + 6c_i \sum_{k \neq i} \wp(x_i - x_k), \tag{12}$$

$$\begin{aligned} \dot{c}_i \;\; = \;\; & -6z\alpha_1 c_i - 12\alpha_2 c_i - 6z \sum_{k \neq i} c_k \Phi'(x_i - x_k) - 6zc_i \sum_{k \neq i} \wp(x_i - x_k) \\ & -6 \sum_{k \neq i} c_k \Phi''(x_i - x_k) + 6c_i \sum_{k \neq i} \wp'(x_i - x_k) \end{aligned} \tag{13}$$

which have to be valid for all $i = 1, \ldots, N$. These conditions can be rewritten in the matrix form as linear problems for a vector $\mathbf{c} = (c_1, \ldots, c_N)^T$:

$$\begin{cases} L\mathbf{c} = (3z^2 + 6\alpha_1)\mathbf{c} \\ \\ \dot{\mathbf{c}} = M\mathbf{c}, \end{cases} \tag{14}$$

where

$$L = -\dot{X} - 6zA - 6B + 6D, \tag{15}$$

$$M = -(6z\alpha_1 + 12\alpha_2)I - 6zB - 6zD - 6C + 6D' \tag{16}$$

and the matrices X, A, B, C, D, D', I are given by $X_{ik} = \delta_{ik}x_i$, $I_{ik} = \delta_{ik}$,

$$A_{ik} = (1 - \delta_{ik})\Phi(x_i - x_k),$$

$$B_{ik} = (1 - \delta_{ik})\Phi'(x_i - x_k),$$

$$C_{ik} = (1 - \delta_{ik})\Phi''(x_i - x_k),$$

$$D_{ik} = \delta_{ik}\sum_{j\neq i}\wp(x_i - x_j),$$

$$D'_{ik} = \delta_{ik}\sum_{j\neq i}\wp'(x_i - x_j).$$

The matrices A, B, C are off-diagonal while the matrices D, D' are diagonal. The equation of the spectral curve is $\det\Big(L - (3z^2 + 6\alpha_1)I\Big) = 0$.

The linear system (14) is overdetermined. Differentiating the first equation in (14) with respect to t, we see that the compatibility condition of the linear problems (14) is

$$\Big(\dot{L} + [L, M]\Big)\mathbf{c} = 0. \tag{17}$$

One can prove the following matrix identity (see the appendix):

$$\dot{L} + [L, M] = -12D'\Big(L - (3z^2 + 6\alpha_1)I\Big) - \ddot{X} + 12D'(6D - \dot{X}) + 6\dot{D} - 6D''', \tag{18}$$

where $D'''_{ik} = \delta_{ik}\sum_{j\neq i}\wp'''(x_i - x_j)$. It then follows that the compatibility condition (17) is equivalent to vanishing of all elements of the diagonal matrix

$$-\ddot{X} + 12D'(6D - \dot{X}) + 6\dot{D} - 6D'''.$$

This gives equations of motion for the poles x_i. Writing the diagonal elements explicitly, we get:

$$\ddot{x}_i + 6\sum_{j\neq i}(\dot{x}_i + \dot{x}_j)\wp'(x_i - x_j) - 72\sum_{j\neq i}\sum_{k\neq i}\wp(x_i - x_j)\wp'(x_i - x_k) + 6\sum_{j\neq i}\wp'''(x_i - x_j) = 0.$$

Taking into account the identity $\wp'''(x) = 12\wp(x)\wp'(x)$, we obtain the equations of motion (9):

$$\ddot{x}_i + 6\sum_{j\neq i}(\dot{x}_i + \dot{x}_j)\wp'(x_i - x_j) - 72\sum_{j\neq k\neq i}\wp(x_i - x_j)\wp'(x_i - x_k) = 0. \tag{19}$$

The rational limit (when $\wp(x) \to 1/x^2$) reads

$$\ddot{x}_i - 12\sum_{j\neq i} \frac{\dot{x}_i + \dot{x}_j}{(x_i - x_j)^3} + 144 \sum_{j\neq k\neq i} \frac{1}{(x_i - x_j)^2(x_i - x_k)^3} = 0. \tag{20}$$

3. Integrals of motion

The Lax representation of equation (9) is missing. Instead of it, we have the matrix relation

$$\dot{L} + [L, M] = -12D'\Big(L - (3z^2 + 6\alpha_1)I\Big) \tag{21}$$

equivalent to the equations of motion. This is a sort of the Manakov's triple representation [24]. This relation means that in contrast to the KP case, where we have the Lax equation for the Lax matrix of the elliptic Calogero–Moser system, eigenvalues of our 'Lax matrix' L are not conserved and the evolution $L \to L(t)$ is not isospectral. Nevertheless, the equation of the spectral curve, $\det\Big(L - (3z^2 + 6\alpha_1)I\Big) = 0$, is an integral of motion. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\det\Big(L - (3z^2 + 6\alpha_1)I\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}\log\Big(L - (3z^2 + 6\alpha_1)I\Big)$$

$$= \operatorname{tr}\Big[\dot{L}\Big(L - (3z^2 + 6\alpha_1)I\Big)^{-1}\Big] = -12\operatorname{tr}D' = 0,$$

where we have used relation (21) and the fact that $\operatorname{tr} D' = \sum_{i\neq j} \wp'(x_i - x_j) = 0$ ($\wp'$ is an odd function). We recall that $\alpha_1 = -\frac{1}{2}\wp(\lambda)$. The expression

$$R(z, \lambda) = \det\Big(3(z^2 - \wp(\lambda))I - L\Big)$$

is a polynomial in z of degree $2N$. Its coefficients are integrals of motion (some of them may be trivial).

The matrix $L = L(z, \lambda)$, which has essential singularities at $\lambda = 0$, can be represented in the form $L = G\tilde{L}G^{-1}$, where $\tilde{L}$ does not have essential singularities and G is the diagonal matrix $G_{ij} = \delta_{ij}\mathrm{e}^{-\zeta(\lambda)x_i}$. Therefore,

$$R(z, \lambda) = \sum_{k=0}^{2N} R_k(\lambda)z^k,$$

where the coefficients $R_k(\lambda)$ are elliptic functions of λ with poles at $\lambda = 0$.

Let us give some examples. At $N = 2$ we have

$$\det\Big(3(z^2 - \wp(\lambda))I - L\Big) = 9z^4 + 3z^2\Big(\dot{x}_1 + \dot{x}_2 - 18\wp(\lambda)\Big) - 36z\wp'(\lambda) - 3\wp(\lambda)(\dot{x}_1 + \dot{x}_2)$$

$$+ \dot{x}_1\dot{x}_2 - 6(\dot{x}_1 + \dot{x}_2)\wp(x_1 - x_2) - 27\wp^2(\lambda) + 9g_2,$$

where g_2 is the coefficient in the expansion of the $\wp$-function near $x = 0$: $\wp(x) = x^{-2} + \frac{1}{20}g_2x^2 + \frac{1}{28}g_3x^4 + O(x^6)$. Therefore, in this case we have two integrals of motion: $I_1 = \dot{x}_1 + \dot{x}_2$, $I_2 = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + 6(\dot{x}_1 + \dot{x}_2)\wp(x_1 - x_2)$.

At $N=3$ a non-trivial calculation leads to the following result:

$$\det\Big(3(z^2-\wp(\lambda))I-L\Big)=27z^6+9\Big(I_1-45\wp(\lambda)\Big)z^4-540\wp'(\lambda)z^3$$

$$+\left[\frac{3}{2}I_1^2-3I_2-54\wp(\lambda)I_1-1215\wp^2(\lambda)+243g_2\right]z^2-36\wp'(\lambda)\Big(I_1+9\wp(\lambda)\Big)z$$

$$+I_3-I_1I_2+\frac{1}{6}I_1^3+3\wp(\lambda)\Big(I_2-\frac{1}{2}I_1^2\Big)-27\wp^2(\lambda)I_1+9g_2I_1-135\wp^3(\lambda)-27g_2\wp(\lambda)+216g_3,$$

where

$$\begin{aligned}
I_1 &= \dot x_1+\dot x_2+\dot x_3,\\
I_2 &= \tfrac{1}{2}\Big(\dot x_1^2+\dot x_2^2+\dot x_3^2\Big)+6\dot x_1\Big(\wp(x_{12})+\wp(x_{13})\Big)+6\dot x_2\Big(\wp(x_{21})+\wp(x_{23})\Big)\\
&\quad +6\dot x_3\Big(\wp(x_{31})+\wp(x_{32})\Big)-36\Big(\wp(x_{12})\wp(x_{13})+\wp(x_{12})\wp(x_{23})+\wp(x_{13})\wp(x_{23})\Big),\\
I_3 &= \tfrac{1}{3}\Big(\dot x_1^3+\dot x_2^3+\dot x_3^3\Big)+6\dot x_1^2\Big(\wp(x_{12})+\wp(x_{13})\Big)+6\dot x_2^2\Big(\wp(x_{21})+\wp(x_{23})\Big)\\
&\quad +6\dot x_3^2\Big(\wp(x_{31})+\wp(x_{32})\Big)+12\dot x_1\dot x_2\wp(x_{12})+12\dot x_1\dot x_3\wp(x_{13})+12\dot x_2\dot x_3\wp(x_{23})\\
&\quad -864\wp(x_{12})\wp(x_{13})\wp(x_{23})
\end{aligned}\tag{22}$$

are integrals of motion (here $x_{ik}\equiv x_i-x_k$).

In general, we can prove that the following quantities are integrals of motion:

$$\begin{aligned}
I_1 &= \sum_i \dot x_i,\\
I_2 &= \frac{1}{2}\sum_i \dot x_i^2+6\sum_{i\neq j}\dot x_i\wp(x_{ij})-18\sum_{i\neq j\neq k}\wp(x_{ij})\wp(x_{ik}).
\end{aligned}\tag{23}$$

In the expression for I_2 the last sum is taken over all triples of distinct numbers i,j,k from 1 to N. The conservation of I_1 means that the center of masses moves uniformly, i.e. $\sum_i \ddot x_i=0$.

For the prove that $\dot I_1=0$ we write, using equations of motion (9) and permuting the summation indices,

$$\begin{aligned}
\dot I_1 &= \sum_i \ddot x_i = 72\sum_{i\neq j\neq k}\wp(x_{ij})\wp'(x_{ik})\\
&= 12\sum_{i\neq j\neq k}\Big(\wp(x_{ij})\wp'(x_{ik})+\wp'(x_{ij})\wp(x_{ik})\\
&\quad +\wp(x_{ji})\wp'(x_{jk})+\wp'(x_{ji})\wp(x_{jk})
\end{aligned}$$

$$+\wp(x_{ki})\wp'(x_{kj})+\wp'(x_{ki})\wp(x_{kj})\Big)$$

$$=12\sum_{i\neq j\neq k}\Big[\partial_{x_i}\Big(\wp(x_{ij})\wp(x_{ik})\Big)+\partial_{x_j}\Big(\wp(x_{ji})\wp(x_{jk})\Big)+\partial_{x_k}\Big(\wp(x_{ki})\wp(x_{kj})\Big)\Big]=0,$$

where we have used the identity

$$\partial_{x_i}\Big(\wp(x_{ij})\wp(x_{ik})\Big)+\partial_{x_j}\Big(\wp(x_{ji})\wp(x_{jk})\Big)+\partial_{x_k}\Big(\wp(x_{ki})\wp(x_{kj})\Big)=0. \tag{24}$$

It is in fact equivalent to the well known identity

$$\begin{vmatrix}1 & \wp(x_{ij}) & \wp'(x_{ij})\\ 1 & \wp(x_{jk}) & \wp'(x_{jk})\\ 1 & \wp(x_{ki}) & \wp'(x_{ki})\end{vmatrix}=0$$

and can be proved by expanding near the possible poles at $x_i=x_j$ and $x_i=x_k$.

For the proof that $\dot{I}_2=0$ we write:

$$\dot{I}_2=\sum_i\dot{x}_i\ddot{x}_i+6\sum_{i\neq j}\ddot{x}_i\wp(x_{ij})+6\sum_{i\neq j}\dot{x}_i(\dot{x}_i-\dot{x}_j)\wp'(x_{ij})-36\sum_{i\neq j\neq k}(\dot{x}_i-\dot{x}_j)\wp'(x_{ij})\wp(x_{ik}).$$

Substituting the equations of motion, we have:

$$\dot{I}_2=-6\sum_{i\neq j}\dot{x}_i(\dot{x}_i+\dot{x}_j)\wp'(x_{ij})+72\sum_{i\neq j\neq k}\dot{x}_i\wp(x_{ij})\wp'(x_{ik})$$

$$-36\sum_{i\neq j}\sum_{k\neq i}(\dot{x}_i+\dot{x}_k)\wp(x_{ij})\wp'(x_{ik})+432\sum_{i\neq l}\sum_{j\neq k\neq i}\wp(x_{ij})\wp'(x_{ik})\wp(x_{il})$$

$$+6\sum_{i\neq j}\dot{x}_i(\dot{x}_i-\dot{x}_j)\wp'(x_{ij})-36\sum_{i\neq j\neq k}(\dot{x}_i-\dot{x}_j)\wp'(x_{ij})\wp(x_{ik}).$$

The terms containing velocities cancel automatically (taking into account that $\wp'(x_{ij})=-\wp'(x_{ji})$) and we are left with

$$\dot{I}_2=432\sum_{i\neq l}\sum_{j\neq k\neq i}\wp(x_{ij})\wp'(x_{ik})\wp(x_{il})$$

$$=36\sum_{i\neq j\neq k\neq l}\Big[\wp'(x_{ij})\wp(x_{ik})\wp(x_{il})+\wp(x_{ij})\wp'(x_{ik})\wp(x_{il})+\wp(x_{ij})\wp(x_{ik})\wp'(x_{il})$$

$$+\wp'(x_{ji})\wp(x_{jk})\wp(x_{jl})+\wp(x_{ji})\wp'(x_{jk})\wp(x_{jl})+\wp(x_{ji})\wp(x_{jk})\wp'(x_{jl})$$

$$+\wp'(x_{ki})\wp(x_{kj})\wp(x_{kl})+\wp(x_{ki})\wp'(x_{kj})\wp(x_{kl})+\wp(x_{ki})\wp(x_{kj})\wp'(x_{kl})$$

$$+\wp'(x_{li})\wp(x_{lj})\wp(x_{lk})+\wp(x_{li})\wp'(x_{lj})\wp(x_{lk})+\wp(x_{li})\wp(x_{lj})\wp'(x_{lk})\Big]$$

$$+72\sum_{i\neq j\neq k}\Big[\wp(x_{ij})\wp'(x_{ik})\wp(x_{ik})+\wp'(x_{ik})\wp^2(x_{ij})-\wp'(x_{ik})\wp(x_{kj})\wp(x_{ik})-\wp'(x_{ik})\wp^2(x_{kj})$$

$$+\wp(x_{ik})\wp'(x_{ij})\wp(x_{ij})+\wp'(x_{ij})\wp^2(x_{ik})-\wp'(x_{ij})\wp(x_{kj})\wp(x_{ij})-\wp'(x_{ij})\wp^2(x_{kj})$$

$$+\wp(x_{ij})\wp'(x_{jk})\wp(x_{jk})+\wp'(x_{jk})\wp^2(x_{ij})-\wp'(x_{jk})\wp(x_{ki})\wp(x_{jk})-\wp'(x_{jk})\wp^2(x_{ki})\Big]$$

$$=36\sum_{i\neq j\neq k\neq l}\Big[\partial_{x_i}\Big(\wp(x_{ij})\wp(x_{ik})\wp(x_{il})\Big)+\partial_{x_j}\Big(\wp(x_{ji})\wp(x_{jk})\wp(x_{jl})\Big)$$

$$+\partial_{x_k}\Big(\wp(x_{ki})\wp(x_{kj})\wp(x_{kl})\Big)+\partial_{x_l}\Big(\wp(x_{li})\wp(x_{lj})\wp(x_{lk})\Big)\Big]$$

$$+72\sum_{i\neq j\neq k}\Big(\wp(x_{ij})+\wp(x_{jk})+\wp(x_{ki})\Big)\Big[\partial_{x_i}\Big(\wp(x_{ij})\wp(x_{ik})\Big)+\partial_{x_j}\Big(\wp(x_{ji})\wp(x_{jk})\Big)+\partial_{x_k}\Big(\wp(x_{ki})\wp(x_{kj})\Big)\Big],$$

where we permuted the summation indices and separated the terms with $l=j$ and $l=k$. The last line vanishes because of identity (24). The rest also vanishes due to the identity

$$\partial_{x_i}\Big(\wp(x_{ij})\wp(x_{ik})\wp(x_{il})\Big)+\partial_{x_j}\Big(\wp(x_{ji})\wp(x_{jk})\wp(x_{jl})\Big)$$
$$+\partial_{x_k}\Big(\wp(x_{ki})\wp(x_{kj})\wp(x_{kl})\Big)+\partial_{x_l}\Big(\wp(x_{li})\wp(x_{lj})\wp(x_{lk})\Big)=0. \qquad (25)$$

The proof of this identity is standard. The left hand side is an elliptic function of x_i. Expanding it near the possible poles at $x_i=x_j,\ x_i=x_k,\ x_i=x_l$ one can see that it is regular, so it is a constant independent of x_i. By symmetry, this constant does not depend also on x_j, x_k and x_l. To see that this constant is actually zero, one can put $x_i=x$, $x_j=2x$, $x_k=3x$, $x_l=4x$.

Another integral of motion for any N is

$$J=\det_{1\leqslant i,j\leqslant N}\Big[\delta_{ij}\dot{x}_i-6\delta_{ij}\sum_{k\neq i}\wp(x_{ik})-6(1-\delta_{ij})\wp(x_{ij})\Big]. \qquad (26)$$

The conservation of J follows from the fact that $J=\lim\limits_{\lambda\to 0}R(\lambda^{-1},\lambda)$. Indeed, since

$$\Phi'(x,\lambda)=\Phi(x,\lambda)\Big(\zeta(x+\lambda)-\zeta(x)-\zeta(\lambda)\Big)$$

and

$$\tilde{\Phi}(x,\lambda)=\mathrm{e}^{\zeta(\lambda)x}\Phi(x,\lambda)=\lambda^{-1}+\zeta(x)+\frac{1}{2}\,\frac{\sigma''(x)}{\sigma(x)}\,\lambda+O(\lambda^2),$$

we have

$$\tilde{L}(z,\lambda)=(z-\lambda^{-1})Y(z,\lambda)+\dot{X}-6D-6Q+O(\lambda),$$

where Q is the matrix with matrix elements $Q_{ij}=(1-\delta_{ij})\wp(x_{ij})$ and $Y(z,\lambda)$ is a matrix which is regular at $z=\lambda^{-1}$. Therefore, $R(\lambda^{-1},\lambda)=\det(\dot{X}-6D-6Q)+O(\lambda)$.

4. The spectral curve

The equation of the spectral curve is

$$R(z,\lambda) = \det\Big(3(z^2 - \wp(\lambda))I - L(z,\lambda)\Big) = 0. \tag{27}$$

It is easy to see that $L(-z,-\lambda) = L^T(z,\lambda)$, so the spectral curve admits the involution $\iota : (z,\lambda) \to (-z,-\lambda)$. We have $R(z,\lambda) = \sum_{k=0}^{2N} R_k(\lambda) z^k$, where $R_k(\lambda)$ are elliptic functions of λ such that $R_k(-\lambda) = (-1)^k R_k(\lambda)$. The functions $R_k(\lambda)$ can be represented as linear combinations of $\wp$-function and its derivatives. Coefficients of this expansion are integrals of motion (see examples for $N = 2$ and $N = 3$ in the previous section). Fixing values of these integrals, we obtain via the equation $R(z,\lambda) = 0$ the algebraic curve Γ which is a $2N$-sheet covering of the initial elliptic curve $\mathcal{E}$ realized as a factor of the complex plane with respect to the lattice generated by 2ω, $2\omega'$.

In a neighborhood of $\lambda = 0$ the matrix $\tilde{L}$ can be written as

$$\tilde{L} = -6\lambda^{-1}(z - \lambda^{-1})(E - I) - 6(z - \lambda^{-1})S + O(1),$$

where E is the rank 1 matrix with matrix elements $E_{ij} = 1$ for all $i,j = 1,\ldots,N$ and S is the antisymmetric matrix with matrix elements $S_{ij} = \zeta(x_i - x_j)$, $i \neq j$, $S_{ii} = 0$.

Therefore, near $\lambda = 0$ the function $R(z,\lambda)$ can be represented in the form

$$R(z,\lambda) = \det\Big(3(z^2 - \lambda^{-2})I + 6\lambda^{-1}(z - \lambda^{-1})(E - I) + 6(z - \lambda^{-1})S + O(1)\Big)$$

$$= \det\Big(3(z - \lambda^{-1})^2 I + 6\lambda^{-1}(z - \lambda^{-1})E + 6(z - \lambda^{-1})S + O(1)\Big)$$

$$= 3^N (z - \lambda^{-1})^{2N} \det\left(I + \frac{2}{z\lambda - 1}E + \frac{2\lambda}{z\lambda - 1}S + O(\lambda^2)\right).$$

Using the fact that $\det\Big(A + \varepsilon B\Big) = \det A\Big(1 + \varepsilon\,\mathrm{tr}\,(A^{-1}B)\Big) + O(\varepsilon^2)$ for any two matrices A, B and the relation $(I - \alpha E)^{-1} = I + \frac{\alpha}{1 - N\alpha}E$, we find

$$\det\left(I + \frac{2}{z\lambda - 1}E + \frac{2\lambda}{z\lambda - 1}S + O(\lambda^2)\right)$$

$$= \det\left(I + \frac{2}{z\lambda - 1}E + O(\lambda^2)\right)\left(1 + \frac{2\lambda}{z\lambda - 1}\,\mathrm{tr}\left(S - \frac{2}{z\lambda + 2N - 1}ES\right) + O(\lambda^2)\right).$$

But for any antisymmetric matrix S $\mathrm{tr}\,S = \mathrm{tr}\,(ES) = 0$, so we are left with

$$R(z,\lambda) = 3^N (z - \lambda^{-1})^{2N} \det\left(I + \frac{2}{z\lambda - 1}E + O(\lambda^2)\right).$$

The matrix E has eigenvalue 0 with multiplicity $N - 1$ and another eigenvalue equal to N. Therefore, we can write $R(z,\lambda)$ in the form

$$R(z,\lambda) = 3^N\Big(z + (2N-1)\lambda^{-1} - f_{2N}(\lambda)\Big)\Big(z - \lambda^{-1} - f_1(\lambda)\Big)\prod_{i=2}^{2N-1}\Big(z - \lambda^{-1} - f_i(\lambda)\Big), \tag{28}$$

where f_i are regular functions of λ at $\lambda = 0$. The involution ι implies that f_{2N} and f_1 are odd functions: $f_{2N}(-\lambda) = -f_{2N}(\lambda)$, $f_1(-\lambda) = -f_1(\lambda)$ and the other sheets can be numbered in such a way that $f_i(-\lambda) = -f_{2N+1-i}(\lambda)$, $i = 2, 3, \ldots, N$. This means that the function z has simple poles on all sheets at the points P_j ($j = 1, \ldots, 2N$) located above $\lambda = 0$. Its expansion in the local parameter λ on the sheets near these points is given by the multipliers in the right hand side of (28). So we have the following expansions of the function z near the 'points at infinity' P_j:

$$\begin{aligned} &z = \lambda^{-1} + f_j(\lambda) \quad \text{near } P_j, \quad j = 1, \ldots, 2N-1, \\ &z = -(2N-1)\lambda^{-1} + f_{2N}(\lambda) \quad \text{near } P_{2N}. \end{aligned} \tag{29}$$

Similarly to the spectral curve of the elliptic Calogero–Moser model [7], one of the sheets is distinguished, as it can be seen from (28). We call it the upper sheet. There is also another distinguished sheet, where the point P_1 is located (and where the corresponding function f_1 is odd). We call it the lower sheet for brevity. The points P_1, P_{2N} are two fixed points of the involution ι.

Let us find genus g of the spectral curve Γ. Applying the Riemann–Hurwitz formula to the covering $\Gamma \to \mathcal{E}$, we have $2g - 2 = \nu$, where ν is the number of ramification points of the covering. The ramification points are zeros on Γ of the function $\partial R/\partial z$. Differentiating equation (28) with respect to z, we can see that the function $\partial R/\partial z$ has simple poles at the points P_j ($j = 1, \ldots, 2N-1$) on all sheets except the upper one, where it has a pole of order $2N-1$. The number of poles of any meromorphic function is equal to the number of zeros. Therefore, $\nu = 2(2N-1)$ and so $g = 2N$.

The spectral curve Γ is not smooth because in general position the genus of the curve which is a $2N$-sheet covering of an elliptic curve is $g = N(2N-1) + 1$.

5. Analytic properties of the ψ-function on the spectral curve

Let P be a point of the curve Γ, i.e. $P = (z, \lambda)$, where z and λ are connected by the equation $R(z, \lambda) = 0$. The coefficients c_i in the pole ansatz for the function ψ, after normalization, are functions on the curve Γ: $c_i = c_i(t, P)$. Let us normalize them by the condition $c_1(0,P) = 1$. In fact the non-normalized components $c_i(0,P)$ are equal to $\Delta_i(0, P)$, where $\Delta_i(0, P)$ are suitable minors of the matrix $3(z^2 - \wp(\lambda))I - L(0)$. They are holomorphic functions on Γ outside the points above $\lambda = 0$. After normalizing the first component, all other components $c_i(0, P)$ become meromorphic functions on Γ outside the points P_j located above $\lambda = 0$. Their poles are zeros on Γ of the first minor of the matrix $3(z^2 - \wp(\lambda))I - L(0)$, i.e. they are given by common solutions of equation (27) and the equation $\det\Big(3(z^2 - \wp(\lambda))\delta_{ij} - L_{ij}(0)\Big) = 0$, $i, j = 2, \ldots, N$. The location of these poles depends on the initial data.

On all sheets except the lower one the leading term of the matrix $\tilde{L}$ as $\lambda \to 0$ is proportional to $E - I$. Finding explicitly eigenvectors of the matrix $E - I$, one can see that in a neighborhood of the 'points at infinity' P_j ($j = 2, \ldots, 2N$) the functions $c_i(0,P)$ have the form

$$c_i(0, P) = \Big(c_i^{0(j)} + O(\lambda)\Big)\mathrm{e}^{-\zeta(\lambda)(x_i(0) - x_1(0))}, \quad 2 \leqslant i \leqslant N, \quad j = 2, \ldots 2N-1, \tag{30}$$

where $\sum_{i=2}^{N} c_i^{0(j)} = -1$ and

$$c_i(0,P) = \Big(1 + O(\lambda)\Big)\mathrm{e}^{-\zeta(\lambda)(x_i(0)-x_1(0))}, \quad 2 \leqslant i \leqslant N, \quad j = 2N \tag{31}$$

(on the upper sheet). On the lower sheet, the leading term of the matrix $\tilde{L}$ as $\lambda \to 0$ is $O(1)$. Expanding the matrix $\tilde{L}$ in powers of λ, we have

$$\Lambda I - \tilde{L} = 6f_1'(0)E + \dot{X} - 6D - 6Q + O(\lambda),$$

where Q is the matrix with matrix elements $Q_{ij} = (1-\delta_{ij})\wp(x_i - x_j)$. Let $c_i^{0(1)}$ be the eigenvector of the matrix in the right hand side (taken at $t=0$) with zero eigenvalue normalized by the condition $c_1^{0(1)} = 1$, then in a neighborhood of the point P_1 we can write

$$c_i(0,P) = \Big(c_i^{0(1)} + O(\lambda)\Big)\mathrm{e}^{-\zeta(\lambda)(x_i(0)-x_1(0))}, \quad 2 \leqslant i \leqslant N, \quad j = 1. \tag{32}$$

The fundamental matrix $\mathcal{S}(t)$ of solutions to the equation $\partial_t \mathcal{S} = M\mathcal{S}$, $\mathcal{S}(0) = I$, is a regular function of z, λ for $\lambda \neq 0$. Using equation (21) (which plays the role of the Lax equation for our system), we can write

$$\Big(\dot{L} + [L,M] + 12D'(L - \Lambda I)\Big)\mathbf{c}(t) = 0,$$

where $\Lambda = 3(z^2 - \wp(\lambda))$. Substituting $\mathbf{c}(t) = \mathcal{S}(t)\mathbf{c}(0)$ and $M = \dot{\mathcal{S}}\mathcal{S}^{-1}$, we can rewrite this equation as

$$\Big[\partial_t\Big(\mathcal{S}^{-1}(L-\Lambda I)\mathcal{S}\Big) + 12\mathcal{S}^{-1}D'(L-\Lambda I)\mathcal{S}\Big]\mathbf{c}(0) = 0$$

or, equivalently, in the form of the differential equation

$$\partial_t \mathbf{b}(t) = W(t)\mathbf{b}(t), \qquad W(t) = 12\mathcal{S}^{-1}D'\mathcal{S},$$

for the vector $\mathbf{b}(t) = \mathcal{S}^{-1}(L - \Lambda I)\mathbf{c}(t)$ with the initial condition $\mathbf{b}(0) = 0$. This differential equation with zero initial condition has the unique solution $\mathbf{b}(t) = 0$ for all $t > 0$. Therefore, since the matrix $\mathcal{S}$ is non-degenerate, it then follows that $\mathbf{c}(t) = \mathcal{S}(t)\mathbf{c}(0)$ is the common solution of the equations $\dot{\mathbf{c}} = M\mathbf{c}$ and $L\mathbf{c} = \Lambda\mathbf{c}$. Thus the vector $\mathbf{c}(t,P)$ has the same t-independent poles as the vector $\mathbf{c}(0,P)$.

In order to find $c_i(t,P)$ near the pre-images of the point $\lambda = 0$ it is convenient to pass to the gauge equivalent pair $\tilde{L}$, $\tilde{M}$, where

$$\tilde{L} = G^{-1}LG, \quad \tilde{M} = -G^{-1}\partial_t G + G^{-1}MG$$

with the same diagonal matrix G as before. Let $\tilde{\mathbf{c}} = G^{-1}\mathbf{c}$ be the gauge-transformed vector $\mathbf{c} = (c_1, \ldots, c_N)^T$, then our linear system is

$$\tilde{L}\tilde{\mathbf{c}} = 3(z^2 - \wp(\lambda))\tilde{\mathbf{c}}, \qquad \partial_t\tilde{\mathbf{c}} = \tilde{M}\tilde{\mathbf{c}}.$$

By a straightforward calculation one can check that the following relation holds:

$$\tilde{M} = -\lambda^{-1}\tilde{L} + (3z\lambda^{-2} - 4\lambda^{-3})I + 6(z - \lambda^{-1})(Q - D) + O(1). \tag{33}$$

(It should be taken into account that z is of order $O(\lambda^{-1})$, see (29), so the terms proportional to z have to be kept in the expansion.) Applying the both sides to an eigenvector $\tilde{\mathbf{c}}$ of $\tilde{L}$ with the eigenvalue $3(z^2 - \wp(\lambda)) = 3(z^2 - \lambda^{-2}) + O(\lambda^2)$, we get

$$\partial_t\tilde{\mathbf{c}} = -z^3\tilde{\mathbf{c}} + (z - \lambda^{-1})^3\tilde{\mathbf{c}} + 6(z - \lambda^{-1})(Q - D)\tilde{\mathbf{c}} + O(1). \tag{34}$$

Therefore, since $z = \lambda^{-1} + O(1)$ on all sheets except the upper one, we have

$$\partial_t \tilde{\mathbf{c}}^{(j)} = -(z^3 + O(1))\tilde{\mathbf{c}}^{(j)}, \quad j = 1, \ldots, 2N-1, \tag{35}$$

so

$$\tilde{\mathbf{c}}^{(j)}(t, P) = (\mathbf{c}^{0(j)} + O(\lambda))\mathrm{e}^{-z^3 t}, \quad j = 1, \ldots, 2N-1.$$

In order to find the time dependence of the vector $\tilde{\mathbf{c}}^{(2N)}$ on the upper sheet, we note that the corresponding eigenvector of the matrix $\tilde{L}$ is proportional to the vector $\mathbf{e} = (1, 1, \ldots, 1)^T$ with an addition of terms of order $O(1)$ and also note that $(Q - D)\mathbf{e} = 0$. Therefore, since $z = -(2N-1)\lambda^{-1} + f_{2N}$ on the upper sheet, we have

$$\partial_t \tilde{\mathbf{c}}^{(2N)} = \Big(-z^3 + k^3(\lambda) + O(1)\Big)\tilde{\mathbf{c}}^{(2N)}, \tag{36}$$

where

$$k(\lambda) = -2N\lambda^{-1} + f_{2N},$$

so

$$\tilde{\mathbf{c}}^{(2N)}(t, P) = (\mathbf{e} + O(\lambda))\mathrm{e}^{(-z^3 + k^3(\lambda))t}.$$

Coming back to the vector $\mathbf{c}(t, P)$, we obtain after normalization

$$c_i^{(j)}(t, P) = c_{ij}(\lambda)\mathrm{e}^{-\zeta(\lambda)(x_i(t) - x_1(0)) + \nu_j(\lambda)t}, \tag{37}$$

where $\nu_j = -z^3$ for $j = 1, \ldots, 2N-1$, $\nu_{2N} = -z^3 + k^3(\lambda)$ and $c_{ij}(\lambda)$ are regular functions in a neighborhood of $\lambda = 0$. Their values at $\lambda = 0$ are

$$c_{1j}(0) = 1, \quad j = 1, \ldots, 2N, \qquad c_{ij}(0) = c_i^{0(j)}, \quad i \geqslant 2,\ j \neq 2N, \qquad c_{i\,2N}(0) = 1, \tag{38}$$

with $\sum_{i=2}^{N} c_i^{0(j)} = -1$ for $j = 2, \ldots, 2N-1$.

After investigating the analytic properties of the vector $\mathbf{c}(t, P)$ let us turn to the function ψ:

$$\psi(x, t, P) = \sum_{i=1}^{N} c_i(t, P)\Phi(x - x_i, \lambda)\mathrm{e}^{zx + z^3 t}.$$

The function $\Phi(x - x_i, \lambda)$ has essential singularities at all points P_j located above $\lambda = 0$. It follows from (37) that in the function ψ these essential singularities cancel on all sheets except the upper one, where $\psi \propto \mathrm{e}^{k(\lambda)x + k^3(\lambda)t}\mathrm{e}^{\zeta(\lambda)x_1(0)}$. From (38) it follows that ψ has simple poles at the points P_1, P_{2N} (the two fixed points of the involution ι) and no poles at the points P_j for $j = 2, \ldots 2N-1$. The residue at the pole at P_1 is constant as a function of x, t. This is in agreement with the fact that the differential operators B_3, B_5 (3) have no free terms, and so the result of their action to a constant vanishes.

The function ψ also has other poles in the finite part of the curve Γ, which do not depend on x, t. Presumably, their number is $2N-2$ but the argument which allows one to count the number of poles of the ψ-function in the KP case (see [7]) does not work for BKP.

6. Conclusion

In this paper we have derived equations of motion for poles of double-periodic (elliptic) solutions to the BKP equation (equations (9)). In contrast to the equations of motion for poles of elliptic solutions to the KP equation, where interaction between 'particles' (poles) is pairwise, in the BKP case there are both two-body and three-body interactions. To the best of our knowledge, many-body integrable systems with three-body interaction were never mentioned in the literature (see, however, [25, 26], where some three-body integrable systems with three-body interaction were discussed). Instead of the Lax representation, the equations of motion admit a kind of the Manakov's triple representation.

There are some problems which require further investigation. First, the Hamiltonian structure of equations (9) is not known. Besides, since the Lax representation is missing, integrability of equations (9) is not clear. Nevertheless, we believe that the system is integrable since the equation of the spectral curve depending on the spectral parameter provides a large supply of independent conserved quantities. Three of them are known explicitly for any N. Another problem is to complete the proof that the ψ-function (11) is the Baker–Akiezer function on the spectral curve. To do that, one should invent a way to count the number of poles of the ψ-function in the finite part of the spectral curve.

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Appendix

A.1. Proof of equation (18)

Here we prove the main identity (18). Using the explicit form of the matrices L, M (15) and (16), we write

$$
\begin{aligned}
\dot{L} + [L, M] \quad = \quad & 36z^2\Big([A, B] + [A, D]\Big) \\
& -6z\Big(\dot{A} - [\dot{X}, B]\Big) \\
& +36z\Big([A, C] - [A, D'] + 2[B, D]\Big) \\
& -6\Big(\dot{B} - [\dot{X}, C]\Big) - \ddot{X} + 6\dot{D} \\
& +36\Big([B, C] - [B, D'] + [C, D]\Big).
\end{aligned}
$$

First of all we notice that $\dot{A}_{ik} = (\dot{x}_i - \dot{x}_k)\Phi'(x_i - x_k)$, $\dot{B}_{ik} = (\dot{x}_i - \dot{x}_k)\Phi''(x_i - x_k)$, and, therefore, we have $\dot{A} = [\dot{X}, B]$, $\dot{B} = [\dot{X}, C]$. To transform the commutators $[A, B] + [A, D]$, we use the identity

$$
\Phi(x)\Phi'(y) - \Phi(y)\Phi'(x) = \Phi(x + y)(\wp(x) - \wp(y)) \tag{A.1}
$$

which, in turn, follows from the easily proved identity

$$\Phi(x,\lambda)\Phi(y,\lambda) = \Phi(x+y,\lambda)\Big(\zeta(x)+\zeta(y)-\zeta(x+y+\lambda)+\zeta(\lambda)\Big). \tag{A.2}$$

With the help of (A.1) we get for $i \neq k$

$$\begin{aligned}&\Big([A,B]+[A,D]\Big)_{ik}\\ &= \sum_{j\neq i,k}\Phi(x_i-x_j)\Phi'(x_j-x_k) - \sum_{j\neq i,k}\Phi'(x_i-x_j)\Phi(x_j-x_k)\\ &+\Phi(x_i-x_k)\Big(\sum_{j\neq k}\wp(x_j-x_k) - \sum_{j\neq i}\wp(x_i-x_j)\Big) = 0,\end{aligned}$$

so $[A,B]+[A,D]$ is a diagonal matrix. To find its matrix elements, we use the limit of (A.1) at $y=-x$:

$$\Phi(x)\Phi'(-x) - \Phi(-x)\Phi'(x) = \wp'(x) \tag{A.3}$$

which leads to

$$\begin{aligned}&\Big([A,B]+[A,D]\Big)_{ii}\\ &= \sum_{j\neq i}\Big(\Phi(x_i-x_j)\Phi'(x_j-x_i) - \Phi'(x_i-x_j)\Phi(x_j-x_i)\Big) = \sum_{j\neq i}\wp'(x_i-x_j) = D'_{ii},\end{aligned}$$

so we finally obtain the matrix identity

$$[A,B]+[A,D] = D'. \tag{A.4}$$

Combining the derivatives of (A.1) w.r.t. x and y, we obtain the identities

$$\Phi(x)\Phi''(y) - \Phi(y)\Phi''(x) = 2\Phi'(x+y)(\wp(x)-\wp(y)) + \Phi(x+y)(\wp'(x)-\wp'(y)), \tag{A.5}$$

$$\Phi'(x)\Phi''(y) - \Phi'(y)\Phi''(x) = \Phi''(x+y)(\wp(x)-\wp(y)) + \Phi'(x+y)(\wp'(x)-\wp'(y)). \tag{A.6}$$

Their limits as $y \to -x$ are

$$\Phi(x)\Phi''(-x) - \Phi(-x)\Phi''(x) = 0, \tag{A.7}$$

$$\Phi'(x)\Phi''(-x) - \Phi'(-x)\Phi''(x) = -\frac{1}{6}\wp'''(x) + 2\alpha_1\wp'(x). \tag{A.8}$$

Using these formulas, it is easy to prove the following matrix identities:

$$[A,C] = 2[D,B] + D'A + AD', \tag{A.9}$$

$$[B,C] = [D,C] + D'B + BD' - \frac{1}{6}D''' + 2\alpha_1 D' \tag{A.10}$$

which are used to transform $\dot{L} + [L,M]$ to the form (18).

A.2. Some useful identities

Apart from already mentioned identities for the Φ-function for the calculations in section 3 we need the following ones:

$$\Phi(x)\Phi(-x) = \wp(\lambda) - \wp(x), \tag{A.11}$$

$$\Phi'(x)\Phi(-x) + \Phi'(-x)\Phi(x) = \wp'(\lambda), \tag{A.12}$$

$$\Phi'(x)\Phi'(-x) = \wp^2(x) + \wp(\lambda)\wp(x) + \wp^2(\lambda) - \frac{1}{4}\, g_2, \tag{A.13}$$

$$\Phi(x)\Phi''(-x) = \wp^2(\lambda) + \wp(\lambda)\wp(x) - 2\wp^2(x), \tag{A.14}$$

$$\Phi'(x)\Phi''(-x) = \Big(\wp'(\lambda) - \wp'(x)\Big)\Big(\wp(x) + \frac{1}{2}\,\wp(\lambda)\Big). \tag{A.15}$$

They eventually follow from the basic identity (A.2). We also need some identities for the Weierstrass functions:

$$2\zeta(\lambda) - \zeta(\lambda + x) - \zeta(\lambda - x) = \frac{\wp'(\lambda)}{\wp(x) - \wp(\lambda)}, \tag{A.16}$$

$$\wp'^2(x) = 4\wp^3(x) - g_2\wp(x) - g_3, \tag{A.17}$$

$$\wp(x+\lambda) - \wp(x-\lambda) = -\frac{\wp'(\lambda)\wp'(x)}{(\wp(x) - \wp(\lambda))^2}, \tag{A.18}$$

$$\wp(x+\lambda) + \wp(x-\lambda) = \frac{1}{2}\,\frac{\wp'^2(x) + \wp'^2(\lambda)}{(\wp(x) - \wp(\lambda))^2} - 2\Big(\wp(x) + \wp(\lambda)\Big), \tag{A.19}$$

$$\begin{aligned} &2\wp(x)\Big(\wp(x-a) + \wp(a) + \wp(x)\Big) - \wp'(x)\Big(\zeta(x-a) + \zeta(a) - \zeta(x)\Big) \\ &\quad = \wp(x)\wp(a) + \wp(x)\wp(x-a) + \wp(a)\wp(x-a) + \tfrac{1}{4}\, g_2. \end{aligned} \tag{A.20}$$

The last identity can be proved by expanding the both sides near the poles at $x = 0$ and $x = a$.

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