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PreHamiltonian and Hamiltonian operators for differential-difference equations

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Abstract

In this paper we are developing a theory of rational (pseudo) difference Hamiltonian operators, focusing in particular on its algebraic aspects. We show that a pseudo-difference Hamiltonian operator can be represented as a ratio AB^{-1} of two difference operators with coefficients from a difference field F, where A is preHamiltonian. A difference operator A is called preHamiltonian if its image is a Lie subalgebra with respect to the Lie bracket of evolutionary vector fields on F. The definition of a rational Hamiltonian operator can be reformulated in terms of its factors which simplifies the theory and makes it useful for applications. In particular we show that for a given rational Hamiltonian operator H in order to find a second Hamiltonian operator K compatible with H one only needs to find a preHamiltonian pair A and B such that $K = AB^{-1}H$ is skew-symmetric. We apply our theory to study multi-Hamiltonian structures of Narita–Itoh–Bogayavlensky and Adler–Postnikov equations.

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1. Introduction

Poisson brackets play a fundamental role in the study of Hamiltonian systems of ordinary differential equations (ODE). For an ODE defined by a Poisson structure and a Hamiltonian function, its first integrals (conserved quantities) form a subalgebra of functions commuting with respect to the Poisson bracket. When this subalgebra is large enough, the ODE is Liouville integrable and it can be integrated in quadratures [1]. As for their infinite dimensional analogues, such as systems of partial differential equations or of differential-difference equations, the notion of Poisson bracket is equivalently given in terms of the so-called Hamiltonian operators. The prototypical example is the Korteweg–de-Vries (KdV) equation

$$\partial_t u = 6uu_x + u_{xxx},$$

where the dependent variable u is a function of two variables: time t and space variable x. It is Hamiltonian with respect to the operator $H_1 = \partial_x$ and the local functional

$$f = \int \left(u^3 - \frac{u_x^2}{2} \right),$$

that is, it can be written as

$$\partial_t u = H_1 \delta_u f. \tag{1}$$

Here $\delta_u f$ is the variational derivative. The Hamiltonian operator H_1 defines a Poisson bracket $\{F, G\}_1 = \int \delta_u F\, H_1 \delta_u G$ (see [2, 3]). Zakharov and Faddeev have shown that the KdV equation can be viewed as a completely integrable Hamiltonian system in the sense of Liouville for Hamiltonian ODEs [2].

The concept of compatible pair of Poisson brackets (Hamiltonian structure) was introduced by Magri [4]. Equations which admit two compatible Hamiltonian structures are called bi-Hamiltonian. For bi-Hamiltonian systems one can construct infinite hierarchies of symmetries and conservation laws [5, 6].

For example, the KdV equation is a bi-Hamiltonian system. Its the second Hamiltonian structure is of the form

$$H_2 = 4u\partial_x + 2u_x + \partial_x^3.$$

Starting from the translation symmetry $u_{t_1} = u_x$, one can generate the infinite hierarchy of symmetries

$$u_{t_{2n+1}} = K_n(u, u_x, \ldots, \partial_x^{2n+1} u) = R^n u_x, \quad n = 0, 1, 2, \ldots$$

of the KdV equation applying a Nijenhuis recursion operator $R = H_2 H_1^{-1}$ which is a rational (pseudo) differential operator. It can be shown that all $K_n \in \operatorname{Im} \partial_x$, thus the action of R on K_n can be defined in the ring of polynomials of $u, u_x, \ldots$ and all K_n are differential polynomials.

A natural extension from the continuous to the discrete setting is to study differential-difference systems. The foundations of calculus for difference operators have been developed by Kupershmidt [7]. In all known examples of scalar differential-difference bi-Hamiltonian equations, except the Volterra chain, one or both Hamiltonian operators are rational. This justifies the necessity to develop a rigorous theory of rational Hamiltonian and recursion operators. We have made the first step in this direction in our paper [8] where we extended the results obtained in the differential setting [9] to the difference case. In [8], we have introduced the concept of preHamiltonian pairs of difference operators and demonstrated their connections with Nijenhuis operators. In particular, we have shown that rational recursion operators generating the symmetries of an integrable differential-difference equation must be factorisable as a ratio of two compatible preHamiltonian difference operators.

In this paper we develop the theory of rational difference Hamiltonian operators, and study their interrelations with preHamiltonian operators. In particular we show that in the minimal decomposition of a pseudo-difference Hamiltonian operator $H = AB^{-1}$ as a ratio of two difference operators A and B the operator A is preHamiltonian while B satisfies certain operator identity. Together with the skew-symmetry of H these properties of difference operators A, B can be taken as a definition of a rational Hamiltonian operator. It enables us to develop a theory of rational difference Hamiltonian operators and Hamiltonian pairs. We will illustrate our results using the Adler–Postnikov integrable differential-difference equation, whose Hamiltonian structure was not known previously [10].

Let us consider the well-known modified Volterra chain [11, 12]

$$u_t = u^2(u_1 - u_{-1}), \tag{2}$$

where u is a function of a lattice variable $n \in \mathbb{Z}$ and continuous time variable t. Here we use the standard notations

$$u_t = \partial_t(u), \quad u_j = \mathcal{S}^j u(n,t) = u(n+j,t)$$

and $\mathcal{S}$ is the shift operator. The right hand side of the equation (2) belongs to the difference field $\mathrm{F} = \mathbb{C}(..., u_{-1}, u, u_1, ...)$ of rational functions in the generators u_i, $i \in \mathbb{Z}$.

Equation (2) possesses a rational recursion operator

$$R = u^2\mathcal{S} + 2uu_1 + u^2\mathcal{S}^{-1} + 2u^2(u_1 - u_{-1})(\mathcal{S} - 1)^{-1}\frac{1}{u}$$

where $(\mathcal{S} - 1)^{-1}$ is standing for the inverse of $\mathcal{S} - 1$. This recursion operator is only defined on the space $u\mathrm{Im}\,(\mathcal{S} - 1)$. The operator R can be written as a ratio of two difference operators

$$R = AB^{-1}, \text{ where } A = u^2(\mathcal{S} - \mathcal{S}^{-1})u(\mathcal{S} + 1) \text{ and } B = u(\mathcal{S} - 1). \tag{3}$$

The pair of difference operator A and B *generates* the hierarchy of symmetries of the modified Volterra chain. We have shown in [8] that the difference operators A and B must then form a preHamiltonian pair, that is, any linear combination $C = A + \lambda B$, $\lambda \in \mathbb{C}$ satisfies

$$[\mathrm{Im}\ C, \mathrm{Im}\ C] \subseteq \mathrm{Im}\ C,$$

where the Lie bracket on F is given by $[a, b] = X_a(b) - X_b(a)$ for $a, b \in \mathrm{F}$ and $X_a = \sum_{n\in\mathbb{Z}} \mathcal{S}^n(a)\frac{\partial}{\partial u_n}$ is the evolutionary derivation with characteristic function a of the difference field F.

The recursion operator R can also be presented as $R = H_2H_1^{-1}$ with

$$H_1 = u(\mathcal{S} - 1)(\mathcal{S} + 1)^{-1}u \quad \text{and} \quad H_2 = u^2(\mathcal{S} - \mathcal{S}^{-1})u^2. \tag{4}$$

The operators H_1 and H_2 are Hamiltonian operators. The difference operator H_2 induces a Poisson bracket $\{f, g\}_2 = \int \delta_u(f)H_2\delta_u(g)$ on the space of functionals $f, g \in \mathrm{F}' = \mathrm{F}/(\mathcal{S} - 1)\mathrm{F}$, where δ_u denotes the variational derivative with respect to the dependent variable u

$$\delta_u(a) = \sum_{n\in\mathbb{Z}} \mathcal{S}^{-n}\frac{\partial a}{\partial u_n}, \qquad a \in \mathrm{F}.$$

The Hamiltonian operator H_1 is rational, it can be represented as $H_1 = \hat{A}\hat{B}^{-1}$, where $\hat{A} = u(\mathcal{S} - 1)$ is preHamiltonian and $\hat{B} = \frac{1}{u}(\mathcal{S} + 1)$. It induces a Poisson bracket on a smaller space $F'_{\hat{B}} = \{f \in \mathrm{F}' | \delta_u f \in Im\hat{B}\}$. The modified Volterra chain (2) is a bi-Hamiltonian system for the pair of compatible Hamiltonian operators H_1, H_2

$$u_t = H_1\, \delta_u(uu_1) = H_2\, \delta_u \ln u.$$

It follows from theorem 4 in section 3.2 that the sequence $R^n H_1$, $n \in \mathbb{Z}$, form a family of compatible rational Hamiltonian operators for equation (2).

The Modified Volterra equation is also known as discrete modified KdV. Under the Miura transformation $w = uu_1$, it can be transformed into the Volterra lattice

$$w_t = w(w_1 - w_{-1}). \tag{5}$$

The integrability of the Volterra system was discovered by Manakov [13] and independently by Kac and van Moerbeke [14]. Manakov called it the Langmuir chain since the system (5) appeared in the description of the fine structure of the spectra of Langmuir oscillations in a plasma by Zakharov *et al* [15]. One soliton solution of equation (5) was also found in [15]. The Volterra lattice can be viewed as a discretisation of the KdV equation and as a special case of the generalised Lotka–Volterra model for predator-prey interactions. The Volterra system (5) is exceptional, it is the only known scalar bi-Hamiltonian equation which possesses two difference Hamiltonian operators (see examples 1 and 2 in section 3.1 of this paper). All other known bi-Hamiltonian scalar integrable differential-difference evolutionary equations have at least one rational operator in a Hamiltonian pair.

The arrangement of this paper is as follows: in section 2 we introduce notations and recall some algebraic properties of the noncommutative ring of difference operators and the skew field of rational (pseudo) difference operators, i.e. operators of the form AB^{-1}, where A and B are difference operators (a detail description of the properties with proofs the reader can find in our recent paper [8]). We then define the Fréchet derivative of rational operators, introduce the notion of bi-difference operators and prove two technical lemmas which are used in the other sections of the paper.

The main results are presented in section 3 where we explore the interrelations between preHamiltonian and Hamiltonian operators. In section 3.1. we show that a Hamiltonian operator is a skew-symmetric preHamiltonian operator with simple conditions on its coefficients:

A difference operator $H = \sum_{i=1}^{N} h^{(i)} S^i - S^{-i} h^{(i)}$ is Hamiltonian $\Longleftrightarrow$ H is preHamiltonian and $h^{(i)} = h^{(i)}(u, ..., u_i)$ for all $i = 1, ..., N$.

In section 3.2 we discuss the extension of the definition of Hamiltonian difference operators to the case of rational (pseudo difference) operators. We re-cast the definition of rational Hamiltonian operator $H = AB^{-1}$ in terms of its factors (definition 8). It enables us to replace a tedious verification of the Jacobi identity for pseudo-difference operators by a relatively simple and algorithmic confirmation of two identities for the factors which are difference operators. In particular, we need to check that the first factor A is preHamiltonian. We demonstrate that preHamiltonian pairs provide us with a method to find compatible Hamiltonian rational operators to a given (rational) Hamiltonian operator. We have shown the following

Let H be a rational Hamiltonian operator and A, B be a preHamiltonian pair of difference operators. Then the rational operator $K = AB^{-1}H$ is Hamiltonian if and only if it is skew-symmetric. Moreover, it is compatible with H.

In section 4, we apply our theory to a new integrable equation derived by Adler and Postnikov [10]:

$$u_t = u^2(u_2 u_1 - u_{-1} u_{-2}) - u(u_1 - u_{-1}). \tag{6}$$

We show that equation (6) is a Hamiltonian system

$$u_t = H\delta_u \ln u$$

with the rational Hamiltonian operator

$$H = u^2 u_1 u_2^2 \mathcal{S}^2 - \mathcal{S}^{-2} u^2 u_1 u_2^2 + \mathcal{S}^{-1} u u_1 (u + u_1) - u u_1 (u + u_1)\mathcal{S}$$
$$+ u(1 - \mathcal{S}^{-1})(1 - u u_1)(\mathcal{S}u - u\mathcal{S}^{-1})^{-1}(1 - u u_1)(\mathcal{S} - 1)u\,.$$

In [8] we have found a rational Nijenhuis recursion operator R for equation (6). We show that the sequence $R^n H, n \in \mathbb{Z}$ forms a family of compatible rational Hamiltonian operators for (6).

2. Difference and rational difference operators

In this section, we briefly recall some notations and statements that were introduced and discussed in detail in section 2 of our paper [8]. In the end of this section, we prove two lemmas on (bi)difference operators, which we are going to use in the next section. Although in this paper we consider only the scalar case, most of our results can be generalised to the case of rational matrix operators.

Let k be a zero characteristic base field, such as $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$. We define the polynomial ring

$$\mathrm{K} = \mathrm{k}[\ldots, u_{-1}, u_0, u_1, \ldots]$$

in the infinite set of variables $\{u\} = \{u_k;\ k \in \mathbb{Z}\}$ and the corresponding field of fractions

$$\mathrm{F} = \mathrm{k}(\ldots, u_{-1}, u_0, u_1, \ldots).$$

Note that every element of K and F depends on a finite number of variables only.

There is a natural automorphism $\mathcal{S}$ of the field F, which we call the shift operator, defined as

$$\mathcal{S} : a(u_k, \ldots, u_r) \mapsto a(u_{k+1}, \ldots, u_{r+1}), \quad \mathcal{S} : \alpha \mapsto \alpha, \qquad a(u_k, \ldots, u_r) \in \mathrm{F}, \ \ \alpha \in \mathrm{k}.$$

We often use the shorthand notation $a_i = \mathcal{S}^i(a) = a(u_{k+i}, \ldots, u_{r+i}), i \in \mathbb{Z}$, and omit the index zero in a_0 or u_0 when there is no ambiguity. The field F equipped with the automorphism $\mathcal{S}$ is a difference field and the base field k is its subfield of constants.

The partial derivatives $\frac{\partial}{\partial u_i},\ i \in \mathbb{Z}$ are commuting derivations of F satisfying the conditions

$$\mathcal{S}\frac{\partial}{\partial u_i} = \frac{\partial}{\partial u_{i+1}}\mathcal{S}. \tag{7}$$

A derivation of F is said to be evolutionary if it commutes with the shift operator $\mathcal{S}$. Such a derivation is completely determined by one element of $a \in \mathrm{F}$ and is of the form

$$X_a = \sum_{i\in\mathbb{Z}} \mathcal{S}^i(a)\frac{\partial}{\partial u_i}, \qquad a \in \mathrm{F}. \tag{8}$$

The element a is called the characteristic of the evolutionary derivation X_a. For $b \in \mathrm{F}$ element $X_a(b) \in \mathrm{F}$ can also be represented in the form

$$X_a(b) = b_*[a],$$

where $b_*[a]$ is the Fréchet derivative of $b = b(u_p, \ldots, u_q)$ in the direction a, which is defined as

$$b_*[a] := \frac{\mathrm{d}}{\mathrm{d}\epsilon} b(u_p + \epsilon a_p, \ldots, u_q + \epsilon a_q)|_{\epsilon=0} = \sum_{i=p}^{q} \frac{\partial b}{\partial u_i} a_i.$$

The Fréchet derivative of $b = b(u_p, \ldots, u_q)$ is a difference operator represented by a finite sum

$$b_* = \sum_{i=p}^{q} \frac{\partial b}{\partial u_i} \mathcal{S}^i. \tag{9}$$

Evolutionary derivations form a Lie k-subalgebra $\mathcal{A}$ in the the Lie algebra of derivations of the field F. Indeed,

$$\alpha X_a + \beta X_b = X_{\alpha a+\beta b}, \qquad \alpha, \beta \in \mathrm{k},$$
$$[X_a, X_b] = X_{[a,b]},$$

where $[a, b] \in \mathrm{F}$ denotes the Lie bracket

$$[a, b] = X_a(b) - X_b(a) = b_*[a] - a_*[b]. \tag{10}$$

The bracket (10) is k-bilinear, antisymmetric and satisfies the Jacobi identity. Thus F, equipped with the bracket (10), has a structure of a Lie algebra over k.

Definition 1. A difference operator B of order $\operatorname{ord} B := (M, N)$ is a finite sum of the form

$$B = b^{(N)}\mathcal{S}^N + b^{(N-1)}\mathcal{S}^{N-1} + \cdots + b^{(M)}\mathcal{S}^M, \qquad b^{(k)} \in \mathrm{F}, \;\; M \leqslant N, \;\; N, M \in \mathbb{Z}, \tag{11}$$

where $b^{(N)}$ and $b^{(M)}$ are non-zero. The term $b^{(N)}\mathcal{S}^N$ is called the leading monomial of B. The total order of B is defined as $\mathrm{Ord}B = N - M$. The total order of the zero operator is defined as $\mathrm{Ord}\, 0 := \{\infty\}$.

The Fréchet derivative (9) is an example of a difference operator of order (p, q) and total order $\mathrm{Ord}\, b_* = q - p$. For an element $a \in \mathrm{F}$ the order and total order are defined as $\operatorname{ord} a_*$ and $\mathrm{Ord}\, a_*$ respectively.

Difference operators form a unital ring $\mathcal{R} = \mathrm{F}[\mathcal{S}, \mathcal{S}^{-1}]$ of Laurent polynomials in $\mathcal{S}$ with coefficients in F, equipped with the usual addition and multiplication defined by

$$a\mathcal{S}^n \cdot b\mathcal{S}^m = a\mathcal{S}^n(b)\mathcal{S}^{n+m}, \quad a, b \in \mathrm{F}, \quad n, m \in \mathbb{Z}. \tag{12}$$

This multiplication is associative, but non-commutative. It follows from (12) that zero total order operators are invertible.

The ring $\mathcal{R}$ is a right and left Euclidean domain and it satisfies the right (left) Ore property, that is, for any $A, B \in \mathcal{R}$ their exist A_1, B_1, not both equal to zero, such that $AB_1 = BA_1$, (resp. $B_1A = A_1B$). In other words, the right (left) ideal $A\mathcal{R} \cap B\mathcal{R}$ (resp. $\mathcal{R}A \cap \mathcal{R}B$) is nontrivial. Its generator M has total order $\mathrm{Ord}A + \mathrm{Ord}B - \mathrm{Ord}D$, where D is the greatest left (resp. right) common divisor of A and B. The domain $\mathcal{R}$ can be naturally embedded in the skew field of rational pseudo-difference operators.

Definition 2. Rational pseudo-difference operators are elements of

$$\mathcal{Q} = \{AB^{-1} \,|\, A, B \in \mathcal{R}, \; B \neq 0\}.$$

We shall call them rational operators for simplicity.

Any rational operator $L = AB^{-1}$ can also be written in the form $L = \hat{B}^{-1}\hat{A}$, $\hat{A}, \hat{B} \in \mathcal{R}$ and $\hat{B} \neq 0$. Thus any statement for the representation $L = AB^{-1}$ can be easily reformulated to the representation $L = \hat{B}^{-1}\hat{A}$. In particular, we have shown in [8] that rational operators $\mathcal{Q}$ form a skew field with respect to usual addition and multiplication. The decomposition $L = AB^{-1}$, $A, B \in \mathcal{R}$ of an element $L \in \mathcal{Q}$ is unique if we require that B has a minimal possible total order with the leading monomial being 1.

The definition of the total order for difference operators (definition 1) can be extended to rational operators:

$$\mathrm{Ord}\,(AB^{-1}) := \mathrm{Ord}\,A - \mathrm{Ord}\,B, \quad A, B \in \mathcal{R}.$$

Definition 3. A formal adjoint operator $A^\dagger$ for any $A \in \mathcal{Q}$ can be defined recursively by the rules: $a^\dagger = a$ for any $a \in \mathrm{F}$, $\mathcal{S}^\dagger = \mathcal{S}^{-1}$, $(A+B)^\dagger = A^\dagger + B^\dagger$ and $(A \cdot B)^\dagger = B^\dagger \cdot A^\dagger$ for any $A, B \in \mathcal{Q}$. In particular $(A^{-1})^\dagger = (A^\dagger)^{-1}$ and $(a\mathcal{S}^n)^\dagger = \mathcal{S}^{-n}a = a_{-n}\mathcal{S}^{-n}$.

A rational operator $K \in \mathcal{Q}$ is called skew-symmetric if $K^\dagger = -K$.

While difference operators act naturally on elements of the field F, rational operators cannot be *a priori* applied to elements of F. Similarly to the theory of rational differential operators [16] for $L = AB^{-1} \in \mathcal{Q}$ and $a, b \in \mathrm{F}$ we define the *correspondence* $a = Lb$ when there exists $c \in \mathrm{F}$ such that $a = Ac$ and $b = Bc$.

Finally we define the Fréchet derivative of difference operators and rational difference operators.

Definition 4. The Fréchet derivative of a difference operator B (11) in the direction of $a \in \mathrm{F}$ is defined as

$$B_*[a] = b_*^{(N)}[a]\mathcal{S}^N + b_*^{(N-1)}[a]\mathcal{S}^{N-1} + \cdots + b_*^{(M)}[a]\mathcal{S}^M. \tag{13}$$

Here we can also view B_* as a bi-difference operator in the sense that, for a given $a \in \mathrm{F}$, both $B_*[\bullet](a)$ and $B_*[a]$ are in $\mathcal{R}$, i.e. difference operators on F. For convenience, we introduce the notation D_B as the following bi-difference operator:

$$(D_B)_a(b) = B_*[b](a) \quad \text{for all} \quad a, b \in \mathrm{F}. \tag{14}$$

This definition can be naturally extended to rational operators:

$$(AB^{-1})_*[c] = A_*[c]B^{-1} - AB^{-1}B_*[c]B^{-1}.$$

We complete this section by proving two lemmas on (bi)difference operators, which we are going to use in section 3. For a bi-difference operator M and an element $a \in \mathrm{F}$ we denote the difference operator $M(a, \bullet)$ by M_a.

Lemma 1. *Let C and D be two difference operators and P, Q be two bi-difference operators on* F *such that $CP_a = Q_aD$ for all $a \in \mathrm{F}$. Then there exists a bi-difference operator M such that $P_a = M_aD$ for all $a \in \mathrm{F}$.*

Proof. There exist two bi-difference operators M and R such that

$$P_a = M_aD + R_a \quad \text{and} \quad \mathrm{Ord}\,R_a < \mathrm{Ord}\,D \quad \text{for all } a \in \mathrm{F}.$$

Indeed, if we write $P_a = \sum_{n=L_P}^{U_P} P^{(n)}(a)\mathcal{S}^n$, $P^{(n)} \in \mathcal{R}$ and $D = \sum_{k=L_D}^{U_D} d^{(k)}\mathcal{S}^k$, $d^{(k)} \in \mathrm{F}$ and if $U_P - L_P > \mathrm{Ord}D = U_D - L_D$, then one can define $\tilde{P}_a = P_a - P^{(U_P)}(a)\mathcal{S}^{U_P - U_D}(d^{(U_D)})^{-1}D$.

$\mathrm{Ord}\tilde{P}_a < U_P - L_P$ for all $a \in \mathrm{F}$ and we conclude by induction on $U_P - L_P$. We know that $CP_a = Q_a D$, that is $CR_a = N_a D$, where $N_a = Q_a - CM_a$ for all $a \in \mathrm{F}$. Let us assume that $R_a \neq 0$. There exist difference operators $R^{(j)}$ and $N^{(i)}$ such that for all $a \in \mathrm{F}$,

$$R_a = \sum_{j=l}^{k} a_j R^{(j)}, \qquad N_a = \sum_{i=m}^{n} a_i N^{(i)}. \tag{15}$$

In particular $\mathrm{Ord}R^{(j)} < \mathrm{Ord}D$ for all $j = l, ..., k$. If fS^r is the leading term of C we must have

$$N^{(n)} D = fS^r R^{(k)}, \tag{16}$$

which implies that $\mathrm{Ord}R^{(k)} \geqslant \mathrm{Ord}D$ contradicting to $\mathrm{Ord}\,R_a < \mathrm{Ord}\,D$. □

Lemma 2. *Let C, D and E be non-zero difference operators such that $C + \lambda D$ divides E on the right for all $\lambda \in \mathrm{k}$. Then there exists $a \in \mathrm{F}$ and a difference operator X such that $XC = aXD$ and $E = XD$.*

Proof. We first prove the statement in the case where $\mathrm{Ord}\,C = \mathrm{Ord}\,D = 0$. Since C and D are invertible difference operators, we can assume that $C = 1$ and $D = bS^n$. We want to show that if a difference operator E is divisible on the right by $1 + \lambda b\mathcal{S}^n$ for all $\lambda \in \mathrm{k}$, then $n = 0$. Assume that $n \neq 0$ and define the difference operator M_λ for $\lambda \in \mathrm{k}$ uniquely by

$$E = M_\lambda(1 + \lambda b\mathcal{S}^n). \tag{17}$$

It is clear since $n \neq 0$ that the coefficients of M_λ are elements of $\mathrm{F}[\lambda, \lambda^{-1}]$. In other words, M_λ is an element of $\mathcal{R}[\lambda, \lambda^{-1}]$. We get a contradiction looking at (17) in $\mathcal{R}[\lambda, \lambda^{-1}]$ since we assumed $E \neq 0$. Hence $n = 0$.

We now prove the lemma in the general case by induction on $\mathrm{Ord}\,E$. If $\mathrm{Ord}\,E = 0$ then $\mathrm{Ord}\,(C + \lambda D) = 0$ for all $\lambda \in \mathrm{k}$, which implies that $\mathrm{Ord}\,C = \mathrm{Ord}\,D = 0$, which we have treated already. Assume then that $\mathrm{Ord}\,E > 0$ and that $C + \lambda D$ divides E on the right for all $\lambda \in \mathrm{k}$. Let $MC = ND$ be the left least common multiple (llcm) of C and D. Both C and D divide E on the right, hence so does their llcm. Therefore there exists a difference operator G such that $E = GMC = GND$. As earlier we define for all $\lambda \in \mathrm{k}$ the operator M_λ by

$$E = M_\lambda(C + \lambda D). \tag{18}$$

Substituting $E = GMC$ in (18) and using the definition of the llcm there exist $P_\lambda \in \mathcal{R}$ for all $\lambda \in \mathrm{k}$ such that

$$GM - M_\lambda = P_\lambda M; \quad \lambda M_\lambda = P_\lambda N. \tag{19}$$

Similarly there exist $Q_\lambda \in \mathcal{R}$ for all $\lambda \in \mathrm{k}$ such that

$$GN - \lambda M_\lambda = Q_\lambda N; \quad M_\lambda = Q_\lambda M. \tag{20}$$

From (19) and (20) we can see that for all $\lambda \in \mathrm{k}$, $G = P_\lambda + Q_\lambda$ and

$$GN = Q_\lambda(N + \lambda M); \quad \lambda GM = P_\lambda(N + \lambda M). \tag{21}$$

If $\mathrm{Ord}\,C = \mathrm{Ord}\,D = 0$ we have nothing to prove. Otherwise, without loss of generality, we can assume that $\mathrm{Ord}\,D > 0$. Hence $\mathrm{Ord}\,GN < \mathrm{Ord}\,E$. We see from (21) that $N + \lambda M$ divides

GN on the right for all $\lambda \in \mathrm{k}$. By the induction hypothesis, one can find a difference operator Y and an element $a \in \mathrm{F}$ such that $YN = aYM$ and $GN = YM$. Let $X = YM$. We have $XC = YM$ $C = YND = aYMD = aXD$, which concludes the proof. □

3. PreHamiltonian and Hamiltonian operators

In this section, we start by recalling the definitions of preHamiltonian and Hamiltonian difference operators and pairs. Then we extend the definition to the class of rational (pseudo-difference) Hamiltonian operators and study their relations with preHamiltonian difference operators. In particular, we prove that given a (rational) Hamiltonian operator H, to find a Hamiltonian operator compatible to H is the same as to find a preHamiltonian pair A and B such that the operator $AB^{-1}H$ is skew-symmetric.

3.1. Definitions and interrelations with examples

Definition 5. A difference operator A is called preHamiltonian if $\operatorname{Im} A$ is a Lie subalgebra, i.e.

$$[\operatorname{Im} A,\ \operatorname{Im} A] \subseteq \operatorname{Im} A. \tag{22}$$

By direct computation, it is easy to show that (22) is equivalent to the existence of a two-form ω_A on F, such that (see [17])

$$A_*[Aa](b) - A_*[Ab](a) = A(\omega_A(a,b)) \qquad \text{for all} \qquad a, b \in \mathrm{F}, \tag{23}$$

where A_* denotes the Fréchet derivative of the operator A. More precisely, ω_A is a bi-difference operator, i.e. a finite sum of the form

$$\omega_A(a,b) = \sum \omega_A^{(i,j)} \mathcal{S}^i(a)\mathcal{S}^j(b), \qquad \omega_A^{(i,j)} \in \mathrm{F}.$$

Using the notation introduced in (14), the identity (23) is equivalent to

$$A_*[Aa] - (D_A)_a A = A\omega_A(a, \bullet) \qquad \text{for all} \qquad a \in \mathrm{F}. \tag{24}$$

The preHamiltonian operator A defines a Lie algebra on $\mathrm{F}/\ker A$ with the Lie bracket

$$A([a,b]_A) = [Aa, Ab].$$

The bracket $[a, b]_A$ is anti-symmetric, k-linear and satisfies the Jacobi identity. The latter follows from the fact that $A(\mathrm{F})$ is a Lie subalgebra with respect to the standard Lie bracket (10).

We can construct higher order preHamiltonian operators from known ones using the following two lemmas. The first one appeared in [17] in the context of scalar preHamiltonian differential operators of arbitrary order.

Lemma 3. *Assume that A is a preHamiltonian difference operator. For any difference operator C, the operator AC is preHamiltonian if and only if*

$$\omega_A(Ca, Cb) + C_*[ACa](b) - C_*[ACb](a) \in \operatorname{Im} C$$

for all $a, b \in \mathrm{F}$.

Proof. On F it should exist a bi-linear from ω_{AC} such that

$$AC\left(\omega_{AC}(a,b)\right) = (AC)_*[ACa](b) - (AC)_*[ACb](a) = A\left(\omega_A(Ca,Cb) + C_*[ACa](b) - C_*[ACb](a)\right)$$

for all $a, b \in \mathrm{F}$. □

Remark 1. If A is a preHamiltonian operator with associated form ω_A and Q is an invertible difference operator then $B = AQ$ is also preHamiltonian. The proof of lemma 3 provides us with an explicit expression for ω_B

$$\omega_B(a,b) = Q^{-1}(\omega_A(Qa,Qb) + Q_*[Ba](b) - Q_*[Bb](a)).$$

Lemma 4. *If A and B are preHamiltonian difference operators, then their right least common multiple is also preHamiltonian.*

Proof. Let $M = AD = BC$ be the right least common multiple (rlcm) of A and B. Then $[\mathrm{Im}\, M,\ \mathrm{Im}\, M] = [\mathrm{Im}\, AD,\ \mathrm{Im}\, AD] \subseteq [\mathrm{Im}\, A,\ \mathrm{Im}\, A] \subseteq \mathrm{Im}\, A$ since A is a preHamiltonian operator. Similarly, $[\mathrm{Im}\, M,\ \mathrm{Im}\, M] \subseteq \mathrm{Im}\, B$. Moreover we have $\mathrm{Im}\, M = \mathrm{Im}\, A \cap \mathrm{Im}\, B$ (lemma 10 in [8]). Therefore, $[\mathrm{Im}\, M,\ \mathrm{Im}\, M] \subseteq \mathrm{Im}\, M$. □

Similarly to Hamiltonian operators, in general, a linear combination of two preHamiltonian operators is no longer preHamiltonian. This naturally leads to the following definition:

Definition 6. We say that two difference operators A and B form a preHamiltonian pair if $A + \lambda B$ is preHamiltonian for all constant $\lambda \in \mathrm{k}$.

A preHamiltonian pair A and B implies the existence of two-forms ω_A, ω_B and $\omega_{A+\lambda B} = \omega_A + \lambda\omega_B$. They satisfy

$$A_*[Ba](b) + B_*[Aa](b) - A_*[Bb](a) - B_*[Ab](a) = A\omega_B(a,b) + B\omega_A(a,b) \quad \text{for all} \quad a, b \in \mathrm{F}. \tag{25}$$

Using the notation introduced by (14), equation (25) is equivalent to

$$A_*[Ba] + B_*[Aa] - (D_A)_a B - (D_B)_a A = A\omega_B(a,\bullet) + B\omega_A(a,\bullet) \quad \text{for all} \quad a \in \mathrm{F}. \tag{26}$$

Proposition 1. *Let A and B be a preHamiltonian pair. If there exists an operator C such that AC and BC are both preHamiltonian, then they again form a preHamiltonian pair.*

Proof. Let ω_A and ω_B be the 2-form associated to preHamiltonian operators A and B, that is,

$$A_*[Aa] = (D_A)_a A + A\omega_A(a,\bullet), \qquad B_*[Ba] \quad = (D_B)_a B + B\omega_B(a,\bullet)$$

for all $a \in \mathrm{F}$. The forms ω_A and ω_B satisfy (25) since A and B form a preHamiltonian pair. According to lemma 3, we know that there exist two bi-difference operators M and N such that for all $a, b \in \mathrm{F}$

$$\omega_A(Ca,Cb) + C_*[ACa](b) - C_*[ACb](a) = CM(a,b);$$
$$\omega_B(Ca,Cb) + C_*[BCa](b) - C_*[BCb](a) = CN(a,b).$$

Substituting them into (25) for Ca and Cb, we get

$$(AC)_*[BCa](b) + (BC)_*[ACa](b) - (AC)_*[BCb](a) - (BC)_*[ACb](a) = ACN(a,b) + BCM(a,b),$$

which implies that AC and BC for a preHamiltonian pair. □

Before we move on to justify the terminology *preHamiltonian*, we first recall the definition of a Hamiltonian difference operator.

For any element $a \in \mathrm{F}$, we define an equivalent class (or a functional) $\int a$ by saying that two elements $a, b \in \mathrm{F}$ are equivalent if $a - b \in \mathrm{Im}(\mathcal{S} - 1)$. The space of functionals is denoted by F'. For any functional $\int f \in \mathrm{F}'$ (simply written $f \in \mathrm{F}'$ without confusion), we define its difference variational derivative (Euler operator) denoted by $\delta_u f \in \mathrm{F}$ (here we identify the dual space with itself) as

$$\delta_u f = \sum_{i \in \mathbb{Z}} \mathcal{S}^{-i} \frac{\partial f}{\partial u_i} = \frac{\partial}{\partial u} \left(\sum_{i \in \mathbb{Z}} \mathcal{S}^{-i} f \right).$$

Definition 7. A difference operator H is Hamiltonian if the bracket

$$\{f, g\}_H := \int \delta_u f \cdot H(\delta_u g) \tag{27}$$

defines a Lie bracket on F'.

As in the differential case [18] this definition can be re-cast purely in terms of operators acting on the difference field F and avoiding computations on the quotient space F' of functionals.

Theorem 1. *A difference operator H is Hamiltonian if and only if H is skew-symmetric and*

$$H_*[Ha] - (D_H)_a H = H (D_H)_a{}^{\dagger} \text{ for all } a \in \mathrm{F}, \tag{28}$$

where $(D_H)_a{}^{\dagger}$ is the adjoint operator of $(D_H)_a$ defined in (14).

Proof. We first prove the following: if $a \in \mathrm{F}$ is such that $\int a \cdot \delta_u f = 0$ for all $f \in \mathrm{F}'$, then $a = 0$. Since $(\mathcal{S} - 1)\mathrm{F} \subset \ker \delta_u$, we have $\delta_u(a \cdot \delta_u f) = 0$ for all $f \in \mathrm{F}'$. In particular we can consider $f = u u_k$ for $k \in \mathbb{Z}$. Let (m, p) be the order of a. We have for all $k \geqslant 0$,

$$a_k + a_{-k} + \sum_{n=m}^{p} (u_{k-n} + u_{-k-n}) S^{-n}\left(\frac{\partial a}{\partial u_n}\right) = 0. \tag{29}$$

For a given n and for k large enough in (29), after applying $\frac{\partial}{\partial u_{n-k}}$ we get

$$S^k\left(\frac{\partial a}{\partial u_{-n}}\right) = S^{-n}\left(\frac{\partial a}{\partial u_n}\right). \tag{30}$$

Since (30) holds for all k large enough, we deduce that $\frac{\partial a}{\partial u_n} = 0$. Hence $a = 0$.

The anti-symmetry of (27) is equivalent to the skew-symmetry of the operator H. Indeed, (27) is anti-symmetric if and only if

$$\int \delta_u f \cdot (H + H^{\dagger})(\delta_u g) = 0 \text{ for all } f, g \in \mathrm{F}'. \tag{31}$$

From what we just proved, this is equivalent to say that $(H + H^{\dagger})(\delta_u f) = 0$ for all $f \in \mathrm{F}'$, hence that $H + H^{\dagger} = 0$ since nonzero difference operators have finite dimensional kernel over the constants.

Finally we look at the Jacobi identity. For this, we take $a=\delta_u f, b=\delta_u g$ and $c=\delta_u h$, where $f,g,h\in \mathrm{F}'$. Note that

$$\begin{aligned}\{f,\{g,h\}_H\}_H &= \textstyle\int a\cdot H(\delta_u(b\cdot Hc)) = -\int Ha\cdot\delta_u(b\cdot Hc) = \int Ha\cdot\delta_u(c\cdot Hb)\\ &= \textstyle\int (c\cdot Hb)_*[Ha] = \int c\cdot (Hb)_*[Ha] + \int Hb\cdot c_*[Ha].\end{aligned} \tag{32}$$

Similarly, we have

$$\{g,\{h,f\}_H\}_H = -\{g,\{f,h\}_H\}_H = -\textstyle\int c\cdot (Ha)_*[Hb] - \int Ha\cdot c_*[Hb]. \tag{33}$$

As for the third term, we simply write

$$\{h,\{f,g\}_H\}_H = \textstyle\int c\cdot H(\delta_u(a\cdot Hb)). \tag{34}$$

Since $c_* = c_*^\dagger$, this leads to

$$\begin{aligned}&\{f,\{g,h\}_H\}_H + \{g,\{h,f\}_H\}_H + \{h,\{f,g\}_H\}_H\\ &= \textstyle\int c\cdot\big((Hb)_*[Ha] - (Ha)_*[Hb] + H(a_*^\dagger(Hb) + (Hb)_*^\dagger(a))\big) = 0,\end{aligned}$$

which itself is equivalent to

$$[Hb,Ha] = (Ha)_*[Hb] - (Hb)_*[Ha] = H(a_*^\dagger(Hb) + (Hb)_*^\dagger(a)). \tag{35}$$

Using the notation introduced in (14), we have $(Hb)_* = (D_H)_b + Hb_*$ for all $b\in\mathrm{F}$, which leads to $(Hb)_*^\dagger = (D_H)_b^\dagger + b_*^\dagger H^\dagger = (D_H)_b^\dagger - b_*^\dagger H$. Since a_* and b_* are self-adjoint, we can write

$$a_*^\dagger(Hb) + (Hb)_*^\dagger(a) = a_*(Hb) - b_*(Ha) + (D_H)_b^\dagger(a).$$

Moreover, $(Ha)_*[Hb] = H_*[Hb](a) + H(a_*[Hb]) = (D_H)_a Hb + H(a_*[Hb])$. Therefore from (35) we deduce that (28) holds on $\delta_u\mathrm{F}\times\delta_u\mathrm{F}$. We proved that equation (35) holds for any $(a,b)\in\mathrm{F}\times\mathrm{F}$ since it is enough to check that it holds for any $(a,b)\in V\times V$, where V is a subspace of F infinite dimensional over the constants, and $V=\delta_u\mathrm{F}$ provides us with such a subspace. □

This theorem immediately implies that a Hamiltonian operator H is preHamiltonian with

$$\omega_H(a,b) = (D_H)_a^\dagger(b). \tag{36}$$

Note that the skew-symmetry of operator H is a necessary condition since ω_H is a two-form. This can be used as a criteria to determine whether an operator is Hamiltonian. Using (36), we have the following result for scalar difference operators:

Theorem 2. *A skew-symmetric operator* $H=\sum_{i=1}^k \left(h^{(i)}\mathcal{S}^i - \mathcal{S}^{-i}h^{(i)}\right)$ *of total order* $2k$ $(k>0)$ *is Hamiltonian if and only if it is preHamiltonian and its coefficients* $h^{(i)}$ *only depend on* $u,\ldots,u_i$ *for all* $i=1,\ldots,k$.

Proof. First we assume that H is a Hamiltonian operator, and show that its coefficients $h^{(i)}$ only depend on $u,\ldots,u_i$. It follows that H satisfies (28), that is,

$$H_*[Ha] = (D_H)_a H + H(D_H)_a{}^\dagger \text{ for all } a\in\mathrm{F}. \tag{37}$$

This identity is an equality between bi-difference operators, that is between summands of the form $ba_n\mathcal{S}^m$ for $b \in \mathrm{F}$ and $n, m \in \mathbb{Z}$. The left hand side of (37) is a difference operator in $\mathcal{S}$ of order $(-k, k)$, or in other words a sum of terms of the form $ba_n\mathcal{S}^m$ with $|m| \leqslant k$. Hence so must be the right hand side of (37) (RHS). We can rewrite the RHS as

$$\sum_{i=1}^{k} (a_i - a_{-i}\mathcal{S}^{-i})h^{(i)}{}_*H - \sum_{i=1}^{k} H{h^{(i)}}_*^\dagger(a\mathcal{S}^i - a_i). \tag{38}$$

In the second term of (38), it is clear that every summand $ba_n\mathcal{S}^m$ is such that $|m - n| \leqslant k$. Combining this remark with the fact that as a difference operator in $\mathcal{S}$ (38) has order $(-k, k)$, we deduce that any subterm $ba_n\mathcal{S}^m$ appearing in the first term of (38) must be such that $|m| \leqslant k$ or $|m - n| \leqslant k$. Therefore, given i such that $1 \leqslant i \leqslant k$, as a difference operator in $\mathcal{S}$, $a_{-i}\mathcal{S}^{-i}h^{(i)}{}_*H$ cannot involve powers of $\mathcal{S}$ below S^{-i-k}. This implies that $h^{(i)}{}_*$ does not depend on negative powers of $\mathcal{S}$ (recall that H has order $(-k, k)$). Similarly, the operator $a_ih^{(i)}{}_*H$ cannot involve powers of $\mathcal{S}$ strictly bigger than $\mathcal{S}^{k+i}$, which implies that $h^{(i)}{}_*$ can only depend on $1, ..., \mathcal{S}^i$.

Conversely, we need to show that a skew-symmetric preHamiltonian operator H such that all its coefficients $h^{(i)}$ depend only on u,...,u_i is Hamiltonian. For any $a \in \mathrm{F}$, we write

$$P_a = H_*[Ha] - (D_H)_aH - H{(D_H)_a}^\dagger.$$

We want to prove that P_a is identically zero. Under the assumption, we have that P_a is skew-symmetric and its total order is at most $4k$. We also know since H is preHamiltonian that H divides P_a on the left for all $a \in \mathrm{F}$. Of course H must also divide P_a on the right since P_a and H are both skew-symmetric. Therefore by lemma 1 there exists Q bi-difference operator such that $P_a = HQ_aH$ for all $a \in \mathrm{F}$. Moreover, Q_a is skew-symmetric, hence its total order is at least 2 if it is non-zero. Therefore $Q = 0$. □

A recent classification of low order scalar Hamiltonian operators in the framework of multiplicative Poisson λ-brackets [19] is consistent with this theorem.

Example 1. Consider the well-known Hamiltonian operator $H = u(\mathcal{S} - \mathcal{S}^{-1})u = uu_1\mathcal{S} - \mathcal{S}^{-1}uu_1$ of the Volterra equation $u_t = u(u_1 - u_{-1})$. Obviously, H is skew-symmetric and its coefficient $h^{(1)} = uu_1$ only depends on u,u_1. To conclude that it is indeed Hamiltonian using theorem 2, one needs to check that H is preHamiltonian. Indeed, for all $a, b \in \mathrm{F}$:

$$H_*[Ha](b) - H_*[Hb](a) = H\left(\frac{1}{u}(bH(a) - aH(b))\right). \tag{39}$$

Example 2. We can do the same for the second Hamiltonian operator of the Volterra equation

$$\begin{aligned} K &= u(\mathcal{S}u\mathcal{S} + u\mathcal{S} + \mathcal{S}u - u\mathcal{S}^{-1} - \mathcal{S}^{-1}u - \mathcal{S}^{-1}u\mathcal{S}^{-1})u \\ &= uu_1u_2\mathcal{S}^2 + (u^2u_1 + uu_1^2)\mathcal{S} - \mathcal{S}^{-1}(u^2u_1 + uu_1^2) - \mathcal{S}^{-2}uu_1u_2. \end{aligned}$$

Note that it is skew-symmetric and its coefficients $h^{(1)} = u^2u_1 + uu_1^2$ depending on u,u_1 and $h^{(2)} = uu_1u_2$ depending on u,u_1 and u_2. To check that K is preHamiltonian, we denote $A = K\frac{1}{u}$ and it follows from

$$A_*[Aa](b) - A_*[Ab](a) = A(u(a_1b_{-1} + a_1b + ab_{-1} - a_{-1}b - ab_1 - a_{-1}b_1)) \quad \text{for all} \quad a, b \in \mathrm{F}.$$

In the same manner, we can use (36) to determine a Hamiltonian pair. The operators H and K form a Hamiltonian pair if and only if

$$\omega_H(a,b) = (D_H)_a^\dagger(b), \omega_K(a,b) = (D_H)_a^\dagger(b) \text{ and } \omega_{H+\lambda K}(a,b) = (D_{H+\lambda K})_a^\dagger(b) \text{ for all } a,b \in \mathrm{F}.$$

Moreover, we are able to prove the statement on the relation between perHamiltonian and Hamiltonian pairs.

Theorem 3. *Let A and B be a preHamiltonian pair. Assume that there exists a difference operator C such that AC is skew-symmetric and BC is Hamiltonian. Then AC is also Hamiltonian and forms a Hamiltonian pair with BC.*

In the next section we shall give a more general result in theorem 4 and the proof of the above theorem will be a simple corollary. A special case of theorem 3 is when the operator $C = 1$, which leads to the following result.

Corollary 1. *Let A and B be a preHamiltonian pair such that A is skew-symmetric and B is Hamiltonian. Then A is also Hamiltonian and forms a Hamiltonian pair with B.*

Example 3. Consider the Volterra chain $u_t = u(u_1 - u_{-1})$. It possesses a recursion operator

$$R = AB^{-1}, \text{ where } A = u(\mathcal{S}+1)(u\mathcal{S} - \mathcal{S}^{-1}u), B = u(\mathcal{S}-1),$$

and A, B form a preHamiltonian pair. Take $C = (1 + \mathcal{S}^{-1})u$. In example 1, we verified that BC is Hamiltonian. Notice that AC is skew-symmetric. Using the above theorem, we obtain that it is a Hamiltonian operator and forms a Hamiltonian pair with BC.

3.2. Generalisation to rational difference operators

In examples 1–3 we illustrated our theory using the Hamiltonian structure of the Volterra hierarchy. Actually, the Volterra equation is the only example known to us of a scalar nonlinear difference equation possessing a compatible pair of difference Hamiltonian operators[4]. For all other integrable differential-difference equations known to us at least one Hamiltonian is a rational (pseudo-difference) operator. In this section we give all required definitions, develop the theory of rational Hamiltonian operators and study their relations with pairs of preHamiltonian difference operators.

Let H be a skew-symmetric operator with a decomposition $H = AB^{-1}$. It is defined on the following subspace of F' denoted by F'_B, that is,

$$\mathrm{F}'_B = \{f \in \mathrm{F}' | \delta_u f \in \operatorname{Im} B\}. \tag{40}$$

Note that if a difference operator C divides B on the left, then $\mathrm{F}'_B \subseteq \mathrm{F}'_C$ since $\operatorname{Im} B \subseteq \operatorname{Im} C$.

Example 4. The domain of the rational operator H_1 with the decomposition $u(\mathcal{S}-1)(\frac{1}{u}(\mathcal{S}+1))^{-1}$ introduced for the modified Volterra chain (4) is

$$\{f \in \mathrm{F}' | \sum_n \frac{\partial f_n}{\partial u} \in \frac{1}{u}(\mathcal{S}+1)\mathrm{F}\}.$$

[4] In the recent preprint [19] the authors classified difference Hamiltonian operators of total order less or equal to 10. However, it turned out that for the Hamiltonian pairs appeared in this classification, the hierarchies obtained through the Lenard scheme techniques were all equivalent to the Volterra chain [20].

It follows from $H^\dagger = -H$ that

$$B^\dagger A = -A^\dagger B. \tag{41}$$

The pair A, B naturally defines an anti-symmetric bracket $\{\bullet,\bullet\}_{A,B} : \mathrm{F}'_B \times \mathrm{F}'_B \mapsto \mathrm{F}'_B$. For $f, g \in \mathrm{F}'_B$ there exist $a, b \in \mathrm{F}$ such that $Ba = \delta_u f$ and $Bb = \delta_u g$. Then the bracket $\{f,g\}_{A,B}$ can be defined as follows (see (27))

$$\{f,g\}_{A,B} = \textstyle\int Ba \cdot Ab. \tag{42}$$

It is independent on the choice of a and b. Indeed,

$$\textstyle\int \delta_u f \cdot Ab = \int Ba \cdot Ab = \int a \cdot B^\dagger Ab = -\int a \cdot A^\dagger Bb = -\int Aa \cdot \delta_u g,$$

since $A^\dagger B$ is skew-symmetric (41). This also implies that the bracket $\{\bullet,\bullet\}_{A,B}$ itself is anti-symmetric:

$$\{f,g\}_{A,B} = \textstyle\int \delta_u f \cdot Ab = -\int Aa \cdot \delta_u g = -\{g,f\}_{A,B}.$$

Proposition 2. *Let A and B be two difference operators such that their ratio AB^{-1} is skew-symmetric and such that the bracket $\{\bullet,\bullet\}_{A,B}$ is a Lie bracket on F'_B. Assume that the form $\int(r \cdot \delta_u f)$, where $r \in \mathrm{F}$, $f \in \mathrm{F}'_B$ is non-degenerate. Then the operator A is preHamiltonian satisfying*

$$A(\omega_A(a,\bullet)) = A_*[Aa] - (D_A)_a A, \qquad \forall a \in \mathrm{F}, \tag{43}$$

and the operator B satisfies

$$B_*[Aa] - (D_B)_a A + (D_B)_a^\dagger A + (D_A)_a^\dagger B = B(\omega_A(a,\bullet)), \qquad \forall a \in \mathrm{F}. \tag{44}$$

Proof. We know that $\{\bullet,\bullet\}_{A,B}$ is a Lie bracket on F'_B, which implies that $\{f,g\}_{A,B} \in \mathrm{F}'_B$ for all $f, g \in \mathrm{F}'_B$ and thus $\delta_u\{f,g\}_{A,B} \in B(\mathrm{F})$. Let W be the k-linear space $W = \{a \in \mathrm{F} \,|\, (Ba)_* = (Ba)_*^\dagger\}$, or in other words for any element $a \in W$ there exists $f \in \mathrm{F}'_B$ such that $Ba = \delta_u f$. The space W is infinite dimensional over k since the form $\int(r \cdot \delta_u f)$ is non-degenerate. For all $a, b \in W$ we have

$$\begin{aligned}
\delta_u(Ba \cdot Ab) &= (Ba)_*^\dagger(Ab) + (Ab)_*^\dagger(Ba) = (Ba)_*[Ab] + (Ab)_*^\dagger(Ba)\\
&= B_*[Ab](a) + Ba_*[Ab] + b_*^\dagger A^\dagger B(a) + (D_A)_c^\dagger(Ba)\\
&= B_*[Ab](a) + Ba_*[Ab] - b_*^\dagger B^\dagger A(a) + (D_A)_b^\dagger(Ba)\\
&= B_*[Ab](a) + Ba_*[Ab] - (Bb)_*^\dagger(Aa) + (D_B)_b^\dagger(Aa) + (D_A)_b^\dagger(Ba)\\
&= B_*[Ab](a) + Ba_*[Ab] - (Bb)_*(Aa) + (D_B)_b^\dagger(Aa) + (D_A)_b^\dagger(Ba)\\
&= \left(B_*[Ab] - (D_B)_b A + (D_B)_b^\dagger A + (D_A)_b^\dagger B\right)(a) + B\left(a_*[Ab] - b_*[Aa]\right).
\end{aligned}$$

This implies the existence of a form ω such that for all $a \in \mathrm{F}$,

$$B_*[Aa] - (D_B)_a A + (D_B)_a^\dagger A + (D_A)_a^\dagger B = B(\omega(a,\bullet)).$$

Indeed, if M is a bi-difference operator such that $M(a,b) \in \operatorname{Im} B$ for all $a, b \in V$, where V is a subspace of F infinite-dimensional over k, then there exists a bi-difference operator N such that $M(a,b) = B(N(a,b))$ for all $a, b \in \mathrm{F}$. In terms of ω we have for all $a, b \in W$

$$B(\omega(a,b)+b_*[Aa]-a_*[Ab])=\delta_u(Bb\cdot Aa). \tag{45}$$

Let $f,g,h\in \mathrm{F}'_B$ be such that $\delta_u f=Ba$, $\delta_u g=Bb$, and $\delta_u h=Bc$ for some $a,b,c\in W$. The first term in the Jacobi identity is

$$\{f,\{g,h\}_{A,B}\}_{A,B}=-\textstyle\int B(a)\cdot A(\omega(b,c)+c_*[Ab]-b_*[Ab]).$$

The second term is:

$$\begin{aligned}\{g,\{h,f\}_{A,B}\}_{A,B}&=\textstyle\int Ab\cdot\delta_u(Bc\cdot Aa)=-\int Ab\cdot\delta_u(Ba\cdot Ac)\\&=-\textstyle\int Ab\cdot(Ba)_*^\dagger(Ac)-\int Ab\cdot(Ac)_*^\dagger(Ba)=-\int Ab\cdot(Ba)_*[Ac]-\int Ba\cdot(Ac)_*[Ab]\end{aligned}$$

and similarly, the third term is

$$\{h,\{f,g\}_{A,B}\}_{A,B}=\textstyle\int Ac\cdot\delta_u(Ba\cdot Ab)=\int Ac\cdot(Ba)_*[Ab]+\int Ba\cdot(Ab)_*[Ac].$$

Hence we get

$$\textstyle\int Ba\cdot(A(\omega(b,c)+c_*[Ab]-b_*[Ac])+(Ab)_*[Ac]-(Ac)_*[Ab])=0. \tag{46}$$

Therefore

$$A\,(\omega(a,b))=A_*[Aa](b)-A_*[Ab](a) \tag{47}$$

for all $a,b\in W$. Since W is infinite-dimensional over k, (47) holds for all $a,b\in\mathrm{F}$, which is to say that A is preHamiltonian. □

The converse statement is also true:

Proposition 3. *Let A and B be two difference operators such that their ratio $H=AB^{-1}$ is skew-symmetric. Assume that the operator A is preHamiltonian , i.e. there exists a 2-form ω_A such that (43) holds and that the operator B satisfy the equation (44). Then the bracket $\{\bullet,\bullet\}_{A,B}$ is a Lie bracket on F'_B.*

Proof. The bracket $\{\bullet,\bullet\}_{A,B}$ is well-defined on F'_B. Indeed, for all $a,b\in W$ equation (45) holds. Moreover, since A is preHamiltonian, equation (46) is satisfied for all $a,b,c\in W$. Therefore the bracket $\{\bullet,\bullet\}_{A,B}$ satisfies the Jacobi identity. □

Proposition 3 can be seen as an analogue of proposition 7.8 in [21] in the case of rational differential Hamiltonian operators, which has been proven by methods of Poisson vertex algebras. Note that in the proof of proposition 3 we do not make any assumptions on the dimension of the space F'_B. In particular we do not require the form $\int(r\cdot\delta_u f)$ to be non-degenerate. Although the properties of the Poisson bracket, such as anti-symmetry and Jacobi identity have to be verified only on the elements of F'_B, the operator identities obtained are satisfied on all elements of F. This reflects the *Substitution Principle* (see [22], exercise 5.42). In general it is very difficult to characterise the space F'_B, the substitution principle enables us to check the identities over the difference field F. Having it in mind and as well the propositions 2 and 3 we can give a new and easily verifiable definition of a rational Hamiltonian operator.

Definition 8. Let H be a skew-symmetric rational operator. We say that H is Hamiltonian if there exists a decomposition $H=AB^{-1}$ such that the operator A is preHamiltonian, i.e. if there is a 2-form ω_A such that for all $a \in \mathrm{F}$

$$A(\omega_A(a, \bullet)) = A_*[Aa] - (D_A)_a A \tag{48}$$

and if the operators A and B satisfy

$$B_*[Aa] - (D_B)_a A + (D_B)_a^\dagger A + (D_A)_a^\dagger B = B(\omega_A(a, \bullet)) \text{ for all } a \in \mathrm{F}. \tag{49}$$

Remark 2. Note that if a decomposition $H = AB^{-1}$ satisfies equations (48) and (49), then so does a minimal decomposition of $H = A_0 {B_0}^{-1}$. Indeed if a pair of difference operator A, B such that A is prehamiltonian and equation (49) is satisfied has a common right factor, i.e. $A = A_0 C$ and $B = B_0 C$, then A_0 is preHamiltonian and the pair A_0, B_0 satisfies (49) as well.

Remark 3. Taking $B = 1$ in definition 8 of rational Hamiltonian operators, one recovers the definition 7 of Hamiltonian difference operator. In the sequel, we will say Hamiltonian operator to refer to a (*a priori* rational) operator in $\mathcal{Q}$ satisfying definition 8.

Definition 8 can also be viewed as direct generalisation of theorem 1 as explained in the following statement.

Proposition 4. *Let H be a skew-symmetric rational operator with minimal decomposition $H = AB^{-1}$. If H satisfies (28) for all a in the images of operator B, then there is a 2-form ω_A satisfying (48) and (49) for all $a \in \mathrm{F}$.*

Proof. For $H = AB^{-1}$, we have $H_* = A_* B^{-1} - AB^{-1} B_* B^{-1}$. Taking $a = Bb, b \in \mathrm{F}$, we get

$$(D_H)_a = (D_A)_b - AB^{-1}(D_B)_b.$$

Thus identity (28) leads to

$$A_*[Ab] - AB^{-1}B_*[Ab] - (D_A)_b A + AB^{-1}(D_B)_b A = AB^{-1}\left((D_A)_b{}^\dagger B + (D_B)_b{}^\dagger A\right),$$

where we used H being skew-symmetric, that is,

$$A_*[Ab] - (D_A)_b A = AB^{-1}\left(B_*[Ab] - (D_B)_b A + (D_A)_b{}^\dagger B + (D_B)_b{}^\dagger A\right). \tag{50}$$

Let $CA = DB$ be the left least common multiple of the pair A and B. It is also the right least common multiple of the pair C and D since AB^{-1} is minimal. It follows from (50) that

$$C\left(A_*[Ab] - (D_A)_b A\right) = D\left(B_*[Ab] - (D_B)_b A + (D_A)_b{}^\dagger B + (D_B)_b{}^\dagger A\right).$$

Therefore there exists a two-form denoted by ω_A satisfying (48) and (49). □

Example 5. We check that the operator H_1 defined by (4) is indeed Hamiltonian. Note that

$$H_1 = AB^{-1},\ A = u(\mathcal{S} - 1),\ B = \frac{1}{u}(\mathcal{S} + 1).$$

It is obviously skew-symmetric. For any $a, b \in \mathrm{F}$ we have $A_*[Aa](b) = u(a_1 - a)(b_1 - b)$.

Hence A is preHamiltonian with $\omega_A = 0$. We have $(D_A)_a = a_1 - a$ and $(D_B)_a = -\frac{1}{u^2}(a_1 + a)$. Thus we have

$$B_*[Aa] = -\frac{1}{u}(a_1 - a)(S+1), (D_B)_a A = \frac{1}{u}(a_1 + a)(S-1),$$
$$(D_B)_a^\dagger A = \frac{1}{u}(a_1 + a)(S-1), (D_A)_a^\dagger B = \frac{1}{u}(a_1 - a)(S+1).$$

Therefore, (49) is satisfied and H_1 is a Hamiltonian operator.

We now investigate how preHamiltonian pairs relate to Hamiltonian pairs.

Proposition 5. *Let A and B be compatible preHamiltonian operators. Assume that there exists a difference operator C such that BC^{-1} is skew-symmetric, B and C satisfy (49) and AC^{-1} is skew-symmetric. Then the operators A and C satisfy (49). In particular, the rational operator AC^{-1} is Hamiltonian.*

Proof. Since the difference operators A and B form a preHamiltonian pair, for all $a \in \mathrm{F}$ we have

$$A_*[Aa] - (D_A)_a A = AM_a; \tag{51}$$

$$B_*[Ba] - (D_B)_a B = BN_a; \tag{52}$$

$$A_*[Ba] + B_*[Aa] - (D_B)_a A - (D_A)_a B = AN_a + BM_a, \tag{53}$$

where $M_a = \omega_A(a, \bullet)$ and $N_a = \omega_B(a, \bullet)$. From the assumption, we know that

$$C_*[Ba] + (D_B)_a^\dagger C + (D_C)_a^\dagger B - (D_C)_a B = CN_a. \tag{54}$$

We need to prove that the operators A and C satisfy (49), that is, for all $a \in \mathrm{F}$,

$$C_*[Aa] + (D_A)_a^\dagger C + (D_C)_a^\dagger A - (D_C)_a A = CM_a. \tag{55}$$

Let Σ be the difference of the LHS with the RHS of (55). We are going to show that both $A^\dagger\Sigma$ and $B^\dagger\Sigma$ are skew-symmetric. We know that the rational operators AC^{-1} and BC^{-1} are skew-symmetric, that is, $A^\dagger C$ and $B^\dagger C$ are skew-symmetric. We first prove that $A^\dagger\Sigma$ is skew-symmetric. In the following we use the notation $\equiv$ to say that two operators are equal up to adding an skew-symmetric operator. We have

$$\begin{aligned}
A^\dagger\Sigma &= A^\dagger C_*[Aa] + A^\dagger(D_A)_a^\dagger C + A^\dagger(D_C)_a^\dagger A - A^\dagger(D_C)_a A - A^\dagger CM_a \\
&\equiv A^\dagger C_*[Aa] + A^\dagger(D_A)_a^\dagger C - A^\dagger CM_a \\
&\equiv -A_*^\dagger[Aa]C + A^\dagger(D_A)_a^\dagger C + C^\dagger AM_a \\
&\equiv -M_a^\dagger A^\dagger C + C^\dagger AM_a \equiv 0
\end{aligned}$$

since $A^\dagger C$ is a skew-symmetric operator and A is a preHamiltonian operator. We now check that $B^\dagger\Sigma$ is also skew-symmetric:

$$\begin{aligned}
B^\dagger\Sigma &= B^\dagger C_*[Aa] + B^\dagger(D_A)_a^\dagger C + B^\dagger(D_C)_a^\dagger A - B^\dagger(D_C)_a A - B^\dagger C M_a \\
&\equiv -B_*^\dagger[Aa]C + B^\dagger(D_A)_a^\dagger C + B^\dagger(D_C)_a^\dagger A - B^\dagger(D_C)_a A - B^\dagger C M_a \\
&\equiv A_*[Ba]^\dagger C - A^\dagger(D_B)_a^\dagger C - N_a^\dagger A^\dagger C - M_a^\dagger B^\dagger C \\
&\quad + B^\dagger(D_C)_a^\dagger A - B^\dagger(D_C)_a A + C^\dagger B M_a \\
&\equiv -A^\dagger C_*[Ba] - A^\dagger(D_B)_a^\dagger C + N_a^\dagger C^\dagger A + B^\dagger(D_C)_a^\dagger A - B^\dagger(D_C)_a A \\
&\equiv -A^\dagger C_*[Ba] - A^\dagger(D_B)_a^\dagger C + C_*[Ba]^\dagger A + C^\dagger(D_B)_a A \\
&\equiv 0.
\end{aligned}$$

We used relations (52)–(54) and the fact that $A^\dagger C$ and $B^\dagger C$ are skew-symmetric operators. By now, we have proved that

$$A^\dagger\Sigma = -\Sigma^\dagger A \quad \text{and} \quad B^\dagger\Sigma = -\Sigma^\dagger B. \tag{56}$$

This leads to that for all $\lambda \in \mathrm{k}$ we have $(A+\lambda B)^\dagger\Sigma = -(\Sigma)^\dagger(A+\lambda B)$. By lemma 1 it implies that $(A+\lambda B)$ divides Σ on the right for all $\lambda \in \mathrm{k}$. If $\Sigma \neq 0$, it follows from lemma 2 that there exists $b \in \mathrm{F}$ and $X \in \mathcal{R}$ such that $XB = bXA$. Hence we have $H = AC^{-1}$ and $X^{-1}bXH = BC^{-1}$ are both skew-symmetric, that is $b\tilde{H} = \tilde{H}b$ and $\tilde{H} = XHX^\dagger$. This can only be the case if $b \in \mathrm{k}$ is a constant. But in this case we have nothing to prove. Thus $\Sigma = 0$ implying that AC^{-1} is a Hamiltonian operator by definition 8. □

The above proposition shows that for a preHamiltonian pair A and B, if there is a difference operator C such that the ratio with one of them is a Hamiltonian operator, so is the ratio with another one if it is skew-symmetric. We will give much stronger result in the following theorem:

Theorem 4. *Let A and B be compatible preHamiltonian operators and H be a rational Hamiltonian operator. Then, provided that $K = AB^{-1}H$ is skew-symmetric, it is Hamiltonian and compatible with H.*

Proof. Let CD^{-1} be a minimal decomposition of H. We start by writing $B^{-1}C$ as a right fraction using the Ore condition $BG = CP$. We only need to check that AG and BG are compatible preHamiltonian operators and that the pair CP and DP satisfies (49). Since $K = (AG)(DP)^{-1}$, we will then be able to conclude using proposition 5.

We are going to prove that AG is preHamiltonian by making use of lemma 4: if two difference operators are preHamiltonians, then their rlcm is preHamiltonian as well. The key is to write AG as the rlcm of two preHamiltonian operators. *A priori* B and C do not need to be left coprime. Let us write $B = E\tilde{B}$ and $C = E\tilde{C}$, where $\tilde{B}$ and $\tilde{C}$ are left coprime. Since K and CD^{-1} are skew-symmetric, we have $P^\dagger D^\dagger AG = -G^\dagger A^\dagger DP$ and $C^\dagger D = -D^\dagger C$. Therefore $D^\dagger AG = \tilde{C}^\dagger X$ and $A^\dagger DP = -\tilde{B}^\dagger X$ for some difference operator X since we have $G^\dagger B^\dagger = P^\dagger C^\dagger$. C and D are right coprime, hence *a fortiori* $\tilde{C}$ and D are right coprime. It follows that $D^\dagger$ and $\tilde{C}^\dagger$ are left coprime. Therefore there exist two difference operators Y and Z with $\operatorname{Ord} Y = \operatorname{Ord} \tilde{C}$ and $\operatorname{Ord} Z = \operatorname{Ord} D$ such that $D^\dagger Y = \tilde{C}^\dagger Z$ is the rlcm of $D^\dagger$ and $\tilde{C}^\dagger$. From $C^\dagger D = -D^\dagger C$ we see that Y divides C on the left hence that it is preHamiltonian (any left factor of a preHamiltonian operator is preHamiltonian). AG is the rlcm of Y and A. Indeed, from $D^\dagger AG = \tilde{C}^\dagger X$ we see that Y divides AG on the left. Moreover $\operatorname{Ord} Y = \operatorname{Ord} \tilde{C} = \operatorname{Ord} G$ by definition of G, Y and $\tilde{C}$. AG is the rlcm of two preHamiltonian operators, hence it is preHamiltonian.

The exact same argument to get AG being preHamiltonian can be applied to $K + \lambda CD^{-1}$ for any $\lambda \in \mathrm{k}$. It amounts to replace AG by $AG + \lambda BG$. Therefore, we have proved that the two difference operators AG and BG form a compatible pair of preHamiltonian operators. Let us call N the bi-difference operator associated to $BG = CP$ (that is to say $\omega_{BG}(a, \bullet) = N_a$ for all $a \in \mathrm{F}$).

Next we want to check that operators CP and DP satisfies (49). We already know that $CP = BG$ is preHamiltonian, with bi-difference operator N. Hence, we need to verify that for all $a \in \mathrm{F}$

$$(DP)_*[CPa] + (D_{CP})_a^\dagger DP + (D_{DP})_a^\dagger CP - (D_{DP})_a CP = DPN_a, \tag{57}$$

which follows from

$$C_{Pa}P = PN_a + (D_P)_a CP - P_*[CPa], \tag{58}$$

where C_a is the bi-difference operator associated to the preHamiltonian C (i.e. $C_a = \omega_C(a, \bullet)$ for all $a \in \mathrm{F}$). Indeed (recall that $BG = CP$), we have

$$\begin{aligned} CPN_a &= (CP)_*[CPa] - (D_{CP})_a CP \\ &= C_*(CPa)P - (D_C)_{Ca}CP + CP_*(CPa) - C(D_P)_a CP \\ &= C(C_{Pa}P + P_*(CPa) - (D_P)_a CP) \end{aligned}$$

and we can simplify on the left by C since $\mathcal{R}$ is a domain. One deduces (57) from (58) multiplying on the left (58) by D and using the fact that the operators C and D satisfy (49).

By proposition 5, we obtain that AG and DP satisfy (49). In other words, $K = (AG)(DP)^{-1}$ is a Hamiltonian operator under the assumption that it is skew-symmetric. The same proof holds when replacing A by $A + \lambda B$ for $\lambda \in \mathrm{k}$ and thus H and K are compatible. $\square$

This result is very strong. Theorem 3 corresponds to the special case when the Hamiltonian operator $H = BC$.

Example 6. Consider the Narita–Itoh–Bogayavlensky lattice [23–25] of the form

$u_t = u(u_1u_2 - u_{-1}u_{-2})$. It possesses a Nijenhuis recursion operator [26]

$$\begin{aligned} R &= u(\mathcal{S}^2 - 1)^{-1}(\mathcal{S} - \mathcal{S}^{-2})(\mathcal{S}^2 uu_1 - uu_1\mathcal{S}^{-1})(\mathcal{S}uu_1 - uu_1\mathcal{S}^{-1})^{-1}(u_1u_2\mathcal{S}^3 - uu_1)(\mathcal{S} - 1)^{-1}\frac{1}{u} \\ &= AB^{-1}, \qquad B = u(\mathcal{S} - 1)(\mathcal{S}a_1 + b - \mathcal{S}^{-1}a)(\mathcal{S}\Delta_1 + \Delta), \end{aligned}$$

where

$$a = uu_{-1} - u_{-1}u_{-2}; \qquad b = u_1u_{-2} - uu_{-1};$$
$$\Delta = u_{-3}^2u_{-2}u - u_{-3}u_{-1}u^2 + u_{-4}u_{-3}u_{-2}u_{-1} - u_{-4}u_{-3}u_{-1}u + u_{-3}u_{-2}uu_1 - u_{-2}u_{-1}uu_1$$

and a rational Hamiltonian operator

$$H = u\mathcal{S}^{-1}(\mathcal{S}^3 - 1)(\mathcal{S} + 1)^{-1}u,$$

which can be proved as in example 5 following definition 8. Using the procedure described in the proof of proposition 6 in the next section we show that the operator B is preHamiltonian. Since R is Nijenhuis, thus A and B form a preHamiltonian pair [8]. It is easy to verify that RH is skew-symmetric, hence by theorem 4 the rational operator RH is a Hamiltonian operator.

Theorem 5. *Let H and K be two compatible rational Hamiltonian operators. Then there exist two compatible preHamiltonian operators A and B such that $HK^{-1} = AB^{-1}$.*

Proof. Let CD^{-1} (resp. PQ^{-1}) be a minimal presentation of H (resp. K). Let $DM = QN$ be the least right common multiple of D and Q and $\lambda \in \mathrm{k}$. Then $H + \lambda K$ which by hypothesis is Hamiltonian can be rewritten as $(CM + \lambda PN)(DM)^{-1}$. For infinitely many λ, $CM + \lambda PN$ and $DM = QN$ are right coprime. Hence $CM + \lambda PN$ is preHamiltonian for infinitely many constants $\lambda \in \mathrm{k}$. Notice that $HK^{-1} = (CM)(PN)^{-1}$. We conclude the proof letting $A = CM$ and $B = PN$. □

Combining theorem 4 and 5, we are able to prove the following known statement:

Corollary 2. *Let H and K be two rational compatible Hamiltonian operators. Define $L = HK^{-1}$. Then operator $L^n H$ is Hamiltonian for all $n \in \mathbb{Z}$.*

4. An application to Hamiltonian integrable equations

In our recent paper [8] we constructed a recursion operator for the Adler–Postnikov equation [10]

$$u_t = u^2(u_2u_1 - u_{-1}u_{-2}) - u(u_1 - u_{-1}) := c \tag{59}$$

using its (rational) Lax representation. In this section, we show that it is a Hamiltonian system. We start by introducing some relevant basic definitions for differential-difference equations.

There is a bijection between evolutionary equations

$$u_t = a, \qquad a \in \mathrm{F} \tag{60}$$

and evolutionary derivations of F. With equation (60) we associate the vector field X_a.

Definition 9. An evolutionary vector field with characteristic $b \in \mathrm{F}$ is a symmetry of equation (60) if $[b,\ a] = 0$.

The k-linear space of symmetries of an equation forms a Lie algebra. The existence of an infinite dimensional commutative Lie algebra of symmetries is a characteristic property of an integrable equation and it can be taken as a definition of integrability.

Often symmetries of integrable equations can be generated by recursion operators [27]. Roughly speaking, a recursion operator is a linear operator $R : \mathrm{F} \to \mathrm{F}$ mapping a symmetry to a new symmetry. For an evolutionary equation (60), it satisfies the following rational operator equation in $\mathcal{Q}$

$$R_t = R_*[a] = [a_*,\ R]. \tag{61}$$

It was shown in [9] for the differential case and in [8] for the difference case that a necessary condition for a rational operator R to generate an infinite dimensional commutative Lie algebra of symmetries is to admit a decomposition $R = AB^{-1}$ where A and B are compatible preHamiltonian operators. It follows then that R is Nijenhuis, and in particular that R is also a recursion operator for each of the evolutionary equations in the hierarchy $u_t = R^k(a)$, where $k = 0, 1, 2, \ldots$. An alternative method for proving the locality of the hierarchy generated by a Nijenhuis operator is given in [26].

Definition 10. An evolutionary equation (60) is said to be a Hamiltonian equation if there exists a Hamiltonian operator H and a Hamiltonian functional $\int f \in \mathrm{F}'$ such that $u_t = a = H\delta_u \int f$.

This is the same as to say that the evolutionary vector field a is a Hamiltonian vector field and thus the Hamiltonian operator is invariant along it, that is,

$$H_t = H_*[a] = a_* H + H a_*^\dagger, \tag{62}$$

which follows immediately from equation (28) and the fact that for $b \in \mathrm{F}$, $b_* = {b_*}^\dagger$ if and only if b is a variational derivative.

We now recall some relevant results for the equation (59) in [8]. The equation (59) possesses a recursion operator:

$$R = u\left(u(\mathcal{S}^2 - \mathcal{S}^{-1})u + \mathcal{S}^{-1} - 1\right)(\mathcal{S}u - u\mathcal{S}^{-1})^{-1}\left(u(\mathcal{S} - \mathcal{S}^{-2})u(\mathcal{S}^2 + \mathcal{S} + 1) + \mathcal{S}^2 - \mathcal{S}\right)(\mathcal{S}^2 - 1)^{-1}\frac{1}{u}$$
$$+ u(2\mathcal{S}^{-1}u - \mathcal{S}^{-2}u - \mathcal{S}u + u - u\mathcal{S})(\mathcal{S}^2 + \mathcal{S} + 1)(\mathcal{S}^2 - 1)^{-1}\frac{1}{u}. \tag{63}$$

The rational operator R can be factorised as $R = AB^{-1}$, where the operators A and B form a preHamiltonian pair. We have proved the following statement:

> There exists $d^{(n)} \in \mathrm{F}$, $n \geqslant 1$ such that $c^{(n+1)} = B(d^{(n+1)}) = A(d^{(n)}) \in \mathrm{K}$ for all n and $[c^{(n)},\ c^{(m)}] = 0$ for all $n, m \geqslant 1$.

Thus $c^{(n)} = R^{n-1}c,\ \ n \in \mathbb{N}$ is a well defined hierarchy of local symmetries of equation (59).

In what follows, we show that the system (59) is Hamiltonian. Let H be the following skew-symmetric rational operator

$$\begin{aligned} H = {} & u^2u_1u_2^2\mathcal{S}^2 - \mathcal{S}^{-2}u^2u_1u_2^2 + \mathcal{S}^{-1}uu_1(u + u_1) - uu_1(u + u_1)\mathcal{S} \\ & + u(1 - \mathcal{S}^{-1})(1 - uu_1)(\mathcal{S}u - u\mathcal{S}^{-1})^{-1}(1 - uu_1)(\mathcal{S} - 1)u. \end{aligned} \tag{64}$$

Note that the equation (59) can be written in the form $u_t = H\delta_u \ln u$. We are going to prove that H is a Hamiltonian operator.

The operator (64) can be represented in the factorised form $H = CG^{-1}$, where C, G are difference operators. Indeed, it is easy to verify that

$$(1 - uu_1)(\mathcal{S} - 1)uG = (\mathcal{S}u - u\mathcal{S}^{-1})E$$

where

$$G = u_1v_2(u_2v_1 - u_1v_3)\mathcal{S} - (u^2v_2v_{-1} - u_1u_{-1}v_1v) + u_{-1}v_{-1}(u_{-2}v - u_{-1}v_{-2})\mathcal{S}^{-1},$$
$$E = v_2vu(u_{-1}v_1 - uv_{-1}) + v_2vu_1(u_2v_1 - u_1v_3)\mathcal{S}$$

and $v = 1 - u_{-1}u$. Thus $H = CG^{-1}$, where

$$C = (u^2u_1u_2^2\mathcal{S}^2 - \mathcal{S}^{-2}u^2u_1u_2^2 + \mathcal{S}^{-1}uu_1(u + u_1) - uu_1(u + u_1)\mathcal{S})G + u(1 - \mathcal{S}^{-1})(1 - uu_1)E.$$

We have $C = C^{(3)}\mathcal{S}^3 + \cdots + C^{(-3)}\mathcal{S}^{-3}$, where

$$C^{(3)} = u^2u_1u_2^2u_3(1 - u_3u_4)(u_4 - u_3 - u_2u_3u_4 + u_3u_4u_5).$$

Proposition 6. *The operator H given by (64) is a Hamiltonian operator.*

Proof. We prove the statement by a direct computation. First we need to show that C is a preHamiltonian difference operator. Namely, we need to prove the existence of the form $\omega_C(a, b) = \sum_{n>m} \omega_{n,m}(\mathcal{S}^n(a)\mathcal{S}^m(b) - \mathcal{S}^n(b)\mathcal{S}^m(a)),\ \ \omega_{n,m} \in \mathrm{F}$ satisfying the equation

$$C(\omega_C(a,b)) = C_*[C(a)](b) - C_*[C(b)](a), \qquad \forall a,b \in \mathrm{F} \tag{65}$$

and find its entries $\omega_{n,m}$ explicitly. The order of the operator C is $(-3,3)$. The right hand side of (65) is a difference operator of order $(-8,8)$ acting on a (same for b). Thus $\omega(a,b)$ should be a difference operator of order $(-5,5)$ acting on a (same for b). Equation (65) represents the over-determined system of 50 linear difference equations on 20 non-zero entries $\omega_{n,m}$. We order this system of equations according to the lexicographic ordering for products of variables $a_i = \mathcal{S}^i(a), b_j = \mathcal{S}^j(b)$, namely $a_ib_j > a_nb_m$ if $i > n$ or if $i = n$ and $j > m$. In this ordering of the basis the equations on $\omega_{n,m}$ have a triangular form and can be solved consequently. The highest equation, corresponding to a_8b_3, is of the form

$$C^{(3)}\mathcal{S}^3(\omega_{5,0}) = u^2u_1u_2^2u_3^2u_4(-1+u_3u_4)u_5^2u_6u_7^2u_8(-1+u_8u_9)(u_8-u_9+u_7u_8u_9-u_8u_9u_{10}).$$

Thus

$$\omega_{5,0} = \frac{uu_1u_2^2u_3u_4^2u_5(1-u_5u_6)(u_4u_5u_6-u_5u_6u_7+u_5-u_6)}{u_{-1}u_1u-u_1u_2u+u-u_1}.$$

Then we eliminate $\omega_{5,0}$ and its shifts from the system obtained. Consequently we can find all twenty nonzero entries $\omega_{n,m} = -\omega_{m,n}$, where

$$-5 \leqslant m \leqslant 0 \leqslant n \leqslant 5, \qquad n \leqslant n-m \leqslant 5$$

and check the consistency of the system (65). In order to complete the proof we need to verify the identity

$$G_*[Ca] - (D_G)_aC + (D_G)_a^\dagger C + (D_C)_a^\dagger G = G(\omega_C(a,\bullet)) \text{ for all } a \in \mathrm{F},$$

which we have done by a direct substitution. □

Theorem 6. *Let $K = RH$. Then K is skew-symmetric and hence K is a Hamiltonian operator.*

Proof. R is a recursion operator for $u_t = c$, which means that

$$R_*[c] = c_*R - Rc_*. \tag{66}$$

H is a Hamiltonian operator and $u_t = c$ is Hamiltonian for H with density $\ln u$, which means that

$$H_*[c] = c_*H + Hc_*{}^\dagger.$$

It is immediate that

$$K_*[c] = c_*K + Kc_*{}^\dagger.$$

Let $L = K + K^\dagger$. We want to check that $L = 0$. We have

$$L_*[c] = c_*L + Lc_*{}^\dagger.$$

If we consider the degree of u and its shifts, we can write $K = K^{(0)} + K^{(2)} + K^{(4)} + K^{(6)} + K^{(8)}$. Moreover, $K^{(0)} = R^{(-1)}H^{(1)}$ and $K^{(8)} = R^{(3)}H^{(5)}$ are obviously skew-symmetric since $R^{(-1)}$ (resp. $R^{(3)}$) is recursion for $u_t = u(u_1 - u_{-1})$ (resp. $u_t = u^2(u_1u_2 - u_{-1}u_{-2})$) and $u_t = u(u_1 - u_{-1})$ (resp. $u_t = u^2(u_1u_2 - u_{-1}u_{-2})$) is Hamiltonian for $H^{(1)}$ (resp. $H^{(5)}$). Therefore we can write $L = L^{(2)} + L^{(4)} + L^{(6)}$. Let $a = u^2(u_1u_2 - u_{-1}u_{-2})$. Then

$$L^{(6)}{}_*[a] = a_*L^{(6)} + L^{(6)}a_*{}^\dagger.$$

If P is a Laurent series in S^{-1} such that its coefficients are homogeneous of degree n and

$$P_*[a] = a_*P + Pa_*{}^\dagger$$

it is straightforward looking at the leading term of P to see that $n = 3k + 2$ for some integer k. In that case, the order of P is $2k$. Therefore $L^{(6)} = L^{(4)} = 0$ and $L = L^{(2)}$, with L of order 0. But we must also have

$$L_*[b] = b_*L + Lb_*{}^\dagger,$$

where $b = u(u_1 - u_{-1})$. But the only solution to this equation is $L = u(S - S^{-1})u$ which has order 1. Therefore $L = 0$. We know that $R = AB^{-1}$ and A, B form a preHamiltonian pair. It follows from theorem 4 that the operator $K = RH$ is Hamiltonian. □

Theorem 7. *Let $\phi \in \mathrm{k}[X, X^{-1}]$. Then $\phi(R)H$ is a Hamiltonian operator.*

Proof. We know from [8] that R generates the hierarchy of (59). therefore $\phi(R)$ also generates an arbitrary large set of commuting flows. From theorem 5 in [8] we see that a minimal decomposition of $\phi(R)$ must come from a pair of compatible preHamiltonian operators. Moreover, $\phi(R)H$ is skew-symmetric. We conclude with theorem 4. □

Theorem 8. *Every element in the hierarchy $u_{t_n} = c^{(n)}$ is a Hamiltonian system with respect to the Hamiltonian operator H.*

Proof. We proceed by induction on $n \geqslant 1$. We know that the first equation can be written as $u_t = H(1/u)$. Let us assume that the first n equation are Hamiltonian for H. Let $u_t = a$ be the nth equation of the hierarchy and $u_t = b$ be the $(n+1)$th equation. By the induction hypothesis, $a = H(f)$ for some variational derivative f. Moreover, $b = R(f)$. Since $RH = HR^\dagger$ and the total order of a reduced denominator for RH is 6, b is in the image of a reduced numerator for H. As was noticed at the end of section 4, since b is in the image of a reduced numerator of H, it is equivalent to say that $u_t = b$ is Hamiltonian for H and to say that

$$H_*[b] = b_*H + Hb_*{}^\dagger. \tag{67}$$

Since R^{n-1} is recursion for $u_t = b$, the previous equation is equivalent to

$$(R^{n-1}H)_*[b] = b_*R^{n-1}H + R^{n-1}Hb_*{}^\dagger. \tag{68}$$

But this holds thanks to the same principle: $R^{n-1}H$ is a Hamiltonian rational operator and $u_t = g = R^{n-1}c$ is Hamiltonian for $R^{n-1}H$. □

Remark 4. For instance, the second equation of the hierarchy is

$$u_{t_2} = RH(\delta_u \log u) = H\delta_u(uu_1u_2 - u). \tag{69}$$

Moreover, it is easy to check that

$$\delta_u(uu_1u_2 - u) = u_1u_2 + u_{-1}u_1 + u_{-1}u_{-2} - 1 = R^\dagger(\delta_u \log u).$$

5. Concluding remarks

In this paper, we have developed the theory of Poisson brackets, Hamiltonian rational operators and difference preHamiltonian operators associated with a difference field $(\mathrm{k}, \{u\}, \mathcal{S})$, where k is a zero characteristic base field of constants, $\{u\} = \{\dots u_{-1}, u = u_0, u_1, \dots\}$ is a sequence of a single 'dependent' variable and $\mathcal{S}$ is the shift automorphism of infinite order such that $\mathcal{S} : u_k \mapsto u_{k+1}$. This formalism is suitable for the description of scalar Hamiltonian dynamical systems.

It can be extended to the case of several dependent variables, i.e. the case when $\mathbf{u} = (u^1, \dots, u^N)$ is a vector. Some of the definitions concerning the algebra of difference and rational operators in the vector case were presented in [8]. The majority of the definitions and results of the current paper can be extended to the vector case. This includes the crucial propositions 2 and 3, theorems 4 and 5 as well as definitions 1–8. Rational matrix difference operators consist of ratios AB^{-1}, where B is a regular matrix difference operator, that is not a zero-divisor. The fact that the ring of matrix difference operators is not a domain leads to technical difficulties. In particular, a generalisation of theorem 2 to the matrix case is not straightforward.

Not all integrable systems of differential-difference equations are bi-Hamiltonian. Some systems do possess an infinite hierarchy of commuting symmetries generated by a recursion operator which is a ratio of compatible preHamiltonian operators, but cannot be cast in a Hamiltonian form for any Hamiltonian operator. For example, let us consider the equation

$$u_{t_1} = u(u_1 - u) := f^{(1)}. \tag{70}$$

It can be linearised to $v_{t_1} = v_1$ using the substitution $u = v_1/v := \phi$, from which we find its hierarchy of commuting symmetries, corresponding to $v_{t_n} = v_n,\ n \in \mathbb{Z}$:

$$u_{t_0} = 0, \qquad u_{t_n} = (u_n - u)\prod_{k=0}^{n-1} u_k := f^{(n)}, \quad u_{t_{-n}} = \frac{u_{-n} - u}{\prod_{k=1}^{n} u_{-k}} := f^{(-n)}, \qquad n \geqslant 1.$$

The recursion operator for the linearised equation is $\mathcal{S}$. Thus the recursion operator for (70) is

$$R = \phi_*\mathcal{S}\phi_*^{-1} = u(\mathcal{S} - 1)u(\mathcal{S} - 1)^{-1}\mathcal{S}\frac{1}{u}, \quad \phi_* = \frac{1}{v}\mathcal{S} - \frac{v_1}{v^2} = u(\mathcal{S} - 1)\frac{1}{v}.$$

It generates the hierarchy of symmetries of the system (70) as follows:

$$R^{-1}(f^{(1)}) = R(f^{(-1)}) = 0, \quad f^{(n+1)} = R^n(f^{(1)}), \quad f^{(-n-1)} = R^{-n}(f^{(-1)}), \qquad n > 0.$$

A minimal decomposition for the recursion operator is given by $R = AB^{-1}$ with

$$A = u(\mathcal{S} - 1)u\mathcal{S}, \qquad B = u(\mathcal{S} - 1)$$

and difference operators A, B form a preHamiltonian pair. Indeed, the operator B is the same as in example 3 and it is preHamiltonian with the form $\omega_B = 0$. The difference operator $A = BQ$, where $Q = u\mathcal{S}$ is invertible operator. Thus A is also a preHamiltonian operator with the form $\omega_A(a, b) = u(a_1 b - b_1 a)$ (see remark 1). It is easy to check that A and B are compatible.

However, the system (70) cannot be cast into a Hamiltonian form for any Hamiltonian rational operator H. Indeed, equation (62)

$$H_t = f_*^{(1)} H + H f_*^{(1)\dagger}$$

has no solutions for $H \in \mathcal{Q}$, since the order of $f_*^{(1)}$ is $(0, 1)$ while the order of $f_*^{(1)\dagger}$ is $(-1, 0)$.

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References

[1] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (New York: Springer)
[2] Zaharov V E and Faddeev L D 1971 The Korteweg–de Vries equation is a fully integrable Hamiltonian system *Funct. Anal. Appl.* **5** 280–7
[3] Gardner C S 1971 Korteweg–de Vries equation and generalizations. IV. The Korteweg–de Vries equation as a Hamiltonian system *J. Math. Phys.* **12** 1548–51
[4] Magri F 1978 A simple model of integrable Hamiltonian equation *J. Math. Phys.* **19** 1156–62
[5] Gel'fand I M and Dorfman I Ya 1979 Hamiltonian operators and algebraic structures related to them *Funct. Anal. Appl.* **13** 248–62
[6] Fuchssteiner B and Fokas A S 1981 Symplectic structures, their Bäcklund transformations and hereditary symmetries *Physica* D **4** 47–66
[7] Kupershmidt B A 1985 *Discrete Lax Equations and Differential-Difference Calculus* (*Astérisque* vol 123) (Paris: Société Mathématique de France)
[8] Carpentier S, Mikhailov A V and Wang J P 2019 Rational recursion operators for integrable differential-difference equations *Commun. Math. Phys.* **370** 807–51
[9] Carpentier S 2017 A sufficient condition for a rational differential operator to generate an integrable system *Japan. J. Math.* **12** 33–89
[10] Adler V E and Postnikov V V 2011 Differential difference equations associated with the fractional Lax operators *J. Phys. A: Math. Theor.* **44** 415203
[11] Hirota R 1973 Exact *n*-soliton solution of a nonlinear lumped self-dual network equations *J. Phys. Soc. Japan* **35** 289–94
[12] Yamilov R I 2006 Symmetries as integrability criteria for differential difference equations *J. Phys. A: Math. Gen.* **39** R541–623
[13] Manakov S V 1974 Complete integrability and stochastization of discrete dynamical systems *Zh. Eksp. Teor. Fiz.* **67** 543–55
[14] Kac M and van Moerbeke P 1975 On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices *Adv. Math.* **16** 160–9
[15] Zakharov V E, Musher S L and Rubenchik A M 1974 Nonlinear stage of parametric wave excitation in a plasma *JETP Lett.* **19** 151–2

[16] Carpentier S, De Sole A and Kac V G 2015 Singular degree of a rational matrix pseudodifferential operator *Int. Math. Res. Not.* **2015** 5162–95

[17] Sanders J A and Wang J P 2002 On a family of operators and their Lie algebras *J. Lie Theory* **12** 503–14

[18] Dorfman I 1993 *Dirac Structures and Integrability of Nonlinear Evolution Equations* (New York: Wiley)

[19] De Sole A, Kac V G, Valeri D and Wakimoto M 2018 Poisson λ-brackets for differential-difference equations *Int. Math. Res. Not.* (https://doi.org/10.1093/imrn/rny242)

[20] Casati M and Wang J P 2019 A Darboux–Getzler theorem for scalar difference Hamiltonian operators *Commun. Math. Phys.* 1–3

[21] Kac V G and De Sole A 2013 Non-local Poisson structures and applications to the theory of integrable systems *Japan. J. Math.* **8** 233–347

[22] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (*Graduate Texts in Mathematics* vol 107) 2nd edn (New York: Springer)

[23] Narita K 1982 Soliton solution to extended Volterra equation *J. Math. Soc. Japan* **51** 1682–5

[24] Itoh Y 1987 Integrals of a Lotka–Volterra system of odd number of variables *Prog. Theor. Phys.* **78** 507–10

[25] Bogoyavlensky O I 1988 Integrable discretizations of the KdV equation *Phys. Lett.* A **134** 34–8

[26] Wang J P 2009 Lenard scheme for two-dimensional periodic Volterra chain *J. Math. Phys.* **50** 023506

[27] Olver P J 1977 Evolution equations possessing infinitely many symmetries *J. Math. Phys.* **18** 1212–5