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Long hitting times for expanding systems

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Abstract

We prove a new result in the area of hitting time statistics. Currently, there is a lot of papers showing that the first entry times into cylinders or balls are often faster than the Birkhoff's Ergodic theorem would suggest. We provide an opposite counterpart to these results by proving that the hitting times into shrinking balls are also often much larger than these theorems would suggest, by showing that for many expanding dynamical systems

$$\limsup_{r\to 0} \tau_{B(y,r)}(x)\mu(B(y,r)) = +\infty,$$

for an appropriately large, at least of full measure, set of points y and x.

We first do this for all transitive open distance expanding maps and Gibbs/equilibrium states of Hölder continuous potentials; in particular for all irreducible subshifts of finite type with a finite alphabet. Then we prove such result for all finitely irreducible subshifts of finite type with a countable alphabet and Gibbs/equilibrium states for Hölder continuous summable potentials. Next, we show that the *limsup* result holds for all graph directed Markov systems (far going natural generalizations of iterated function systems) and projections of aforementioned Gibbs states on their limit sets. By utilizing the first return map techniques, we then prove the *limsup* result for all tame topological Collect–Eckmann multimodal maps of an interval, all tame topological Collect–Eckmann rational functions of the Riemann sphere, and all dynamically semi-regular transcendental meromorphic functions from $\mathbb{C}$ to $\widehat{\mathbb{C}}$.

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1. Introduction

In the recent years there has been a growing interest in the topic of quantitative recurrence and hitting (also called *entry*) times. There is currently a number of ways in which we may estimate the *speed* of entry/return times into various sets. The historically first such result is the well-known Kac's lemma. However, the in-depth study of the field seems to have started with the paper by Boshernitzan [3]. It asserts that if (X, d) is a separable metric space and (T, μ) is any transformation preserving a Borel, probability measure, then

$$\liminf_{n\to\infty} n^{1/\beta} d(T^n(x), x) < +\infty, \quad \mu - \text{a.e.} \tag{1.1}$$

if $\mathrm{H}_\beta(X)$, the β-dimensional Hausdorff measure of X, is σ-finite.

Another way of quantifying the recurrence (resp. hitting) speed utilizes the notion of lower and upper recurrence (resp. hitting) rates. The hitting rates of a point y into neighbourhoods of x are defined as follows

$$\underline{E}(x, y) := \liminf_{r\to 0} \frac{\log \tau_{B(x,r)}(y)}{-\log(r)} \quad \text{and} \quad \overline{E}(x, y) := \liminf_{r\to 0} \frac{\log \tau_{B(x,r)}(y)}{-\log(r)},$$

where

$$\tau_U(x) := \inf\{k \geqslant 1 : T^k(x) \in U\},$$

is the first entry time of point x into U. The recurrence rates are defined on the diagonal, i.e $\underline{R}(x) = \underline{E}(x, x)$ and $\overline{R}(x) = \overline{E}(x, x)$.

For many systems exhibiting some kind of hyperbolic behaviour we have

$$\underline{R}(x) = \underline{d}_\mu(x) := \liminf_{r\to 0} \frac{\log \mu(B(x, r))}{\log(r)} \quad \text{and} \quad \overline{R}(x) = \overline{d}_\mu(x) := \limsup_{r\to 0} \frac{\log \mu(B(x, r))}{\log(r)},$$

for μ-a.e. x. The respective quantities $\underline{d}_\mu(x)$ and $\overline{d}_\mu(x)$ are commonly called the lower and upper pointwise dimension of the invariant measure μ at the point x, for more details see [1].

A similar result holds for the entry times:

$$\underline{E}(x, y) = \underline{d}_\mu(x) \quad \text{and} \quad \overline{E}(x, y) = \overline{d}_\mu(x),$$

for μ-a.e. x and y. See theorem 4 from [8] for precise assumptions.

It is worth mentioning that the two notions are usually related (at least for ergodic systems). For some general relations between the hitting and entry times, see [11]

Calculating the recurrence/hitting time rates is thus related to calculating the Hausdorff (or packing) dimension of the ambient space. A more subtle task, which is a reminiscent of finding the value of the (appropriate dimensional) Hausdorff measure of the space, is to study the expression

$$R_r(x) := \tau_{B(x,r)}(x) \cdot \mu(B(x, r))$$

for the return time, and

$$E_r(x) := \tau_{B(y,r)}(x) \cdot \mu(B(y,r))$$

for the entry time.

If we take a sequence $(r_n)_{n=1}^{\infty}$ of radii, typically converging to 0, then $\left(R_{r_n}\right)_{n=1}^{\infty}$ may be viewed as a sequence of random variables on the probability space (X, μ). The entry times E_{r_n}, $n \geqslant 1$, may be also treated as real-valued random variables whose distribution is determined by conditional measures $\mu_{B(y,r_n)}$ on $B(y,r_n)$.

Returning to the beginning: it is easy to see that the aforementioned Boshernitzan's result gives the following implication:

$$\mu(B(x,r)) \leqslant Cr^{\beta} \quad \mu - \text{a.e.} \quad \Longrightarrow \quad \liminf_{r \to 0} R_r(x) < +\infty \quad \mu - \text{a.e.}.$$

There are several results improving or widening this implication. See for example [13] for results on generic maps on a manifold, or [2], where the the limit in (1.1) is proved to be always finite for some $\beta \in (0, +\infty)$. Also, the first author of the current paper proved both a strengthening of the recurrence result (1.1) and an analogous result for the hitting times in [20].

Another way of looking at the hitting times statistics is to consider the *shrinking target* setup. Indeed, given $(r_n)_{n=1}^{\infty}$, decreasing sequence of positive real numbers, define

$$E\left(x, (r_n)_{n=1}^{\infty}\right) := \left\{ y \in X : d(T^n x, y) < r_n \quad \text{for infinitely many } n \right\}.$$

The most natural question is then about the value of $\mu\left(E\left(x, (r_n)_{n=1}^{\infty}\right)\right)$. It is fairly obvious that if the sequence r_n converges to zero sufficiently fast, then usually $\mu\left(E\left(x, (r_n)_{n=1}^{\infty}\right)\right) = 0$. Then the next natural question is about the Hausdorff dimension of this set. For example, the authors in [22] consider the sets $E\left(x, (r_n)_{n=1}^{\infty}\right)$ for some fairly general maps of an interval and provide a closed formula for their Hausdorff dimension depending on the sequence $(r_n)_{n=1}^{\infty}$.

It should be emphasized at this point that most of the results cited above pertain to the lower limit (1.1), i.e. show that the recurrence is (may be) significantly faster than the one suggested by the average. Boshernitzan's formulation of the rate of recurrence is however unsuitable for the question of how slow the recurrence can be since replacing the lower limit by the upper limit in (1.1) trivially gives $+\infty$ for all non-fixed points x of the map T. On the other hand, studying the upper limits of $R_r(x)$ and $E_r(x)$ as $r \to 0$ makes perfect sense and constitutes a very natural problem.

There is also a pure probabilistic counterpart to such questions. For example, taking a sequence of independent coin flips, one may ask about the longest time it will take for n consecutive *heads* to appear. Asking such questions started in the 1970s by, amongst others, Erdös in [6]. This is still a popular topic in probability theory.

In the current paper we identify a large class of naturally defined dynamical systems for which

$$\limsup_{r \to 0} E_r(x) = +\infty, \tag{1.2}$$

for almost every x and y. Since for these systems the lower limit is equal to 0, the deviation from the expected value is as large as possible in both directions.

There are only a few results in the literature pertaining to long hitting times. As mentioned above, the recurrence rates (defined above) are often equal to the pointwise dimension, giving a *logarythmic* upper and lower bound on the recurrence rate. Notably *et al* [7] constructed a very special system on the three dimensional torus for which both the *liminf* and the *limsup* are infinite μ-a.e. In fact, the correct scaling in their example is not the inverse of the measure. However, as it is shown in their paper, such phenomena may appear only for slowly mixing systems. Similar results have been proved for systems with a random component, see [8].

Our paper is organized as follows. Dealing all the time with limsup, in the next section, section 2, we prove our main theorem, i.e. that (1.2) holds, in the case of all open transitive distance expanding maps and Gibbs/equilibrium states of all Hölder continuous potentials; in particular for all irreducible subshifts of finite type with a finite alphabet. But we go beyond this case. In section 3 we show this result also holds for symbolic systems. In section 4 we prove (1.2) for all graph directed Markov systems; in particular for all finitely irreducible subshifts of finite type with a countable alphabet. Then, in section 5, we show that the upper limit of hitting times does not change when passing to the first return map. This leads, see section 6, to a multitude of examples such as tame topological Collect–Eckmann multimodal maps of an interval, tame topological Collect–Eckmann rational functions of the Riemann sphere $\widehat{\mathbb{C}}$, and dynamically semi-regular transcendental meromorphic functions from $\mathbb{C}$ to $\widehat{\mathbb{C}}$.

2. Distance expanding maps

Let (X, ρ) be a compact metric space and let $T\colon X \to X$ be an open topologically transitive Lipschitz continuous distance expanding map in the sense of [25]. Recall that being distance expanding means that there exist $\delta > 0$ and $\lambda > 1$ such that

$$\rho(T(x), T(y)) \geqslant \lambda\rho(x, y)$$

for all points x and y in X such that $\rho(x, y) \leqslant \delta$. Taking $\delta > 0$ sufficiently small we will also have that for every integer $n \geqslant 0$ and every $x \in X$, there exists a unique continuous map

$$T_x^{-n}\colon B(T^n(x), 4\delta) \to X$$

such that

$$T^n \circ T_x^{-n} = \mathrm{id}_{B(T^n(x),4\delta)}$$

and

$$T_x^{-n}(T^n(x)) = x.$$

In addition, by taking $\delta > 0$ small enough, we will have that

$$\rho\big(T_x^{-n}(z), T_x^{-n}(w)\big) \leqslant \lambda^{-n}\rho(z, w) \leqslant 8\delta\lambda^{-n} \tag{2.1}$$

for all $z, w \in B(T^n(x), 4\delta)$. The map T_x^{-n} will be referred in the sequel as the unique continuous inverse branch of T^n defined on $B(T^n(x), 4\delta)$ and sending $T^n(x)$ to x.

Let $\mathcal{R}$ be a Markov partition for T (see section 4.5 from [25] for definitions and the proof of existence of partitions with arbitrarily small diameters) with

$$\mathrm{diam}(\mathcal{R}) < \delta. \tag{2.2}$$

For every integer $n \geqslant 1$ let

$$\mathcal{R}^n := \mathcal{R} \vee T^{-1}(\mathcal{R}) \vee \ldots \vee T^{-(n-1)}(\mathcal{R}).$$

The elements of the cover $\mathcal{R}^n$ will be called in the sequel the cells of order n generated by the partition $\mathcal{R}$. Also, any union U of elements of $\mathcal{R}^n$ which cannot be represented as a union of elements of $\mathcal{R}^{n-1}$ will be referred to as a set of order n generated by the partition $\mathcal{R}$. We will then write that

$$n = \mathrm{ord}(U).$$

If we do not want/need to specify the order of U, we will just say that the set U is generated by the Markov partition $\mathcal{R}$. Because of (2.1) and (2.2), we have that

$$\mathrm{diam}(\mathcal{R}^n) < \delta\lambda^{-(n-1)}. \tag{2.3}$$

It is known (see [25]) that for every Hölder continuous function $f\colon X \to \mathbb{R}$, following tradition, called a potential in the sequel, there exists a unique equilibrium measure (state) μ_f on X. Being an equilibrium state means that

$$\mathrm{h}_{\mu_f}(T) + \int_X f\,\mathrm{d}\mu_f = \mathrm{P}(f) := \sup\left\{\mathrm{h}_\mu(T) + \int_X f\,\mathrm{d}\mu\right\},$$

where the supremum is taken over all Borel probability T-invariant (ergodic) measures μ on X and $\mathrm{h}_\mu(T)$ is the Kolmogorov–Sinai metric entropy of T with respect to μ. The measure μ_f is also called, for reasons explained e.g. in [25], a Gibbs state for T and f. We should also note that the quantity $\mathrm{P}(f)$, called the topological pressure of f with respect to T, has a purely topological characterization with no measures involved. It is also known from [25] that there exist two constants $\alpha > 0$ and $C \geqslant 1$, depending on T and f such that

$$\mu_f(B(z,r)) \leqslant Cr^\alpha \tag{2.4}$$

for all $z \in X$ and all radii $r > 0$. We now shall prove the following technical but very useful result.

Lemma 1. *Assume that X is a compact subset of some (finitely dimensional) Euclidean space and that $T\colon X \to X$ is an open topologically transitive distance expanding map. Assume also that μ is a Gibbs/equilibrium state for a Hölder continuous potential. If y is an arbitrary point of X, then there exists a Lebesgue measurable set $\Delta \subset (0,1)$ with the following properties*

(a) $\displaystyle\lim_{\Delta \ni r \to 0} \frac{Leb(\Delta \cap (0,r))}{r} = 1.$

(b) For every $r \in \Delta$ there exists a set R_r, generated by the Markov partition $\mathcal{R}$, satisfying

- *(b1)* $B(y,r) \subset R_r$,
- *(b2)* $\displaystyle\frac{\mu(R_r)}{\mu(B(y,r))} \leqslant 2,$
- *(b3)* $\displaystyle\lim_{r\to 0} \mathrm{ord}(R_r)\mu(B(y,r)) = 0.$

Proof. Because of lemma 3.6 in [21], applied with κ_y identically equal to 2, there exists a Lebesgue measurable set $\Delta \subset (0,1)$ such that item (a) above holds and

$$\frac{\mu(B(y,r+r^2))}{\mu(B(y,r))} \leqslant 2 \tag{2.5}$$

for all $r \in \Delta$. For every $r \in \Delta$ let $n(r) \geqslant 1$ be the least integer such that

$$\delta\lambda^{-(n(r)-1)} \leqslant r^2. \tag{2.6}$$

Let

$$R_r := \bigcup \left\{R \in \mathcal{R}^{n(r)} : R \cap B(y,r) \neq \emptyset\right\}.$$

Then item (b1) holds trivially and also R_r is a set generated by the Markov partition $\mathcal{R}$ with

$$\operatorname{ord}(R_r) \leqslant n(r).$$

Invoking (2.6) and (2.3), we see that

$$R_r \subset B(y, r + r^2).$$

Because of this and (2.5) we have (b2). It follows from the definition of $n(r)$ that

$$\delta\lambda^{-(n(r)-2)} > r^2.$$

Taking logarithms, yields

$$n(r) < \frac{\log(\delta\lambda^2) - 2\log r}{\log\lambda}.$$

Therefore, using also (2.4), we get that

$$\begin{aligned} 0 \leqslant \limsup_{r\to 0} \operatorname{ord}(R_r)\mu(B(y,r)) &\leqslant \limsup_{r\to 0} n(r)\mu(B(y,r)) \\ &\leqslant \frac{C}{\log\lambda} \limsup_{r\to 0} \Big(r^\alpha \big(\log(\delta\lambda^2) - 2\log r \big) \Big) = 0. \end{aligned}$$

This means that (b3) holds and the proof is complete. □

The main result of this section, and one of the main theorems of this paper, is the following.

Theorem 2. *Assume that (X, ρ) is a compact metric space and $T\colon X \to X$ is an open topologically transitive distance expanding map. If μ is a Gibbs/equilibrium state for a Hölder continuous potential, then for every $y \in X$ we have that*

$$\limsup_{r\to 0} \tau_{B(y,r)}(x) \cdot \mu(B(y,r)) = +\infty, \quad \text{for } \mu - \text{a.e. } x \in X. \tag{2.7}$$

Proof. Fix a point $y \in X$. To ease notation, we will denote the ball $B(y, r)$ by B_r and the complement of any set Z in X by Z^c.

Fix $M > 0$ and a radius $r > 0$ belonging to Δ, the set produced in lemma 1. Define

$$A_r := \left\{ x \in X : \tau_{B_r}(x) > \frac{M}{\mu(B_r)} \right\}. \tag{2.8}$$

In order to prove our result it suffices to show that for every $M > 0$ sufficiently large and for any $\delta > 0$ we may define a decreasing sequence r_n, $n = 1, 2, \ldots, \Omega$ (Ω to be chosen later) such that

$$\mu\left(\bigcap_{n=0}^{\Omega} A^c_{r_n} \right) \leqslant \delta. \tag{2.9}$$

The definition of A_r coupled with the results on the set R_r, namely (b1) and (b2) from lemma 1 leads to

$$\bigcap_{n=0}^{\Omega} A_{r_n}^c = \left\{x \in X : \forall_{0 \leqslant i \leqslant \Omega}\ \tau_{B_{r_i}}(x) \leqslant \frac{M}{\mu(B_{r_i})}\right\} \subset \left\{x \in X : \forall_{0 \leqslant i \leqslant \Omega}\ \tau_{R_{r_i}}(x) \leqslant \frac{M}{\mu(R_{r_i})} \frac{\mu(R_{r_i})}{\mu(B_{r_i})}\right\}$$
$$\subset \left\{x \in X : \forall_{0 \leqslant i \leqslant \Omega}\ \tau_{R_{r_i}}(x) \leqslant \frac{2M}{\mu(R_{r_i})}\right\}, \tag{2.10}$$

where the last inequality holds assuming that $M > 0$ is sufficiently large so that then $r_1 > 0$ is sufficiently small. Let us introduce the following notation. Given an integer $k \geqslant 1$ let

$$\begin{aligned} a_r^{(k)} &:= \mu(\{x \in X : \exists_{1 \leqslant l \leqslant k}\ T^l x \in B_r\}) = \mu(\{x \in X : \tau_{B_r}(x) \leqslant k\}), \\ q_r^{(k)} &:= \mu(\{x \in X : \exists_{1 \leqslant l \leqslant k}\ T^l x \in R_r\}) = \mu(\{x \in X : \tau_{R_r}(x) \leqslant k\}). \end{aligned} \tag{2.11}$$

Trivially, $a_r^{(k)} \leqslant q_r^{(k)}$; and from the inclusions above

$$\mu(A_r^c) = a_r^{([\frac{M}{\mu(B_r)}])} \leqslant q_r^{([\frac{2M}{\mu(R_r)}])}.$$

Since the measure μ is T-invariant we get the estimate

$$q_r^{(k)} \leqslant k\mu(R_r). \tag{2.12}$$

Additionally, also because the measure μ is T-invariant, we get for all integers $0 \leqslant k_1 \leqslant k_2$ that

$$\mu(\{x \in X : \exists_{k_1 \leqslant l \leqslant k_2}\ T^l x \in R_r\}) = \mu(\{x \in X : \exists_{0 \leqslant l \leqslant k_2 - k_1}\ T^l x \in R_r\}). \tag{2.13}$$

Now, we need a well-known upper bound of the measure of the intersection of two Markov sets. Indeed, there exists a constant $C \in [1, +\infty)$ such that if U and V are arbitrary sets generated be the Markov partition $\mathcal{R}$, then for every integer $k \geqslant \mathrm{ord}(U)$ we have that

$$\mu(U \cap T^{-k}(V)) \leqslant C\mu(U)\mu(V). \tag{2.14}$$

Put

$$o(r) := \mathrm{ord}(R_r).$$

Then

$$\mathrm{ord}\left(\bigcap_{i=0}^{k-o(r)} T^{-i}(R_r^c)\right) \leqslant k$$

and we have

$$\mu\left(T^{-(k+1)}(R_r) \cap \bigcap_{i=0}^{k-o(r)} T^{-i}(R_r^c)\right) \leqslant C\mu\left(T^{-(k+1)}(R_r)\right) \cdot \mu\left(\bigcap_{i=0}^{k-o(r)} T^{-i}(R_r^c)\right)$$
$$= C\mu\left(R_r\right) \cdot \mu\left(\bigcap_{i=0}^{k-o(r)} T^{-i}(R_r^c)\right), \tag{2.15}$$

where the inequality comes from (2.14) and the equality from T being measure-preserving.

Using the estimate above we may find a satisfactory estimate on $q_r^{(k)}$. Observe that

$$\begin{aligned} q_r^{(k+1)} &= q_r^{(k)} + \mu(x \in X : \tau_{R_r}(x) = k+1) \\ &= q_r^{(k)} + \mu(x \in X : T^{k+1}(x) \in R_r \wedge \forall_{i \leqslant k} T^i(x) \notin R_r) \\ &\leqslant q_r^{(k)} + \mu(x \in X : T^{k+1}(x) \in R_r \wedge \forall_{i \leqslant k-o(r)} T^i(x) \notin R_r) \\ &= q_r^{(k)} + \mu\left(T^{-(k+1)}(R_r) \cap \bigcap_{i=0}^{k-o(r)} T^{-i}(R_r^c)\right) \\ &\leqslant q_r^{(k)} + C\mu(R_r) \cdot \mu\left(\bigcap_{i=0}^{k-o(r)} T^{-i}(R_r^c)\right) = q_r^{(k)} + C\mu(R_r)(1 - q_r^{(k-o(r))}). \end{aligned} \tag{2.16}$$

The first line trivially yields an estimate $q_r^{(k+1)} \leqslant q_r^{(k)} + \mu(R_r)$ and using this $o(r)$ times, yields

$$q_r^{(k)} \leqslant q_r^{(k-o(r))} + o(r)\mu(R_r). \tag{2.17}$$

Observe that by (2.12) this inequality also holds if $k \leqslant o(r)$. We put this back into (2.16), arriving at

$$q_r^{(k+1)} \leqslant q_r^{(k)} + C\mu(R_r)(1 - q_r^{(k)} + o(r)\mu(R_r)) = q_r^{(k)}(1 - C\mu(R_r)) + C\mu(R_r) + Co(r)\mu(R_r)^2. \tag{2.18}$$

We apply this inductively k times, and by observing that $q_r^{(1)} = \mu(R_r)$ we arrive at

$$\begin{aligned} q_r^{(k)} &\leqslant (1 - C\mu(R_r))^{k-1} q_r^{(1)} + C\mu(R_r)\big(1 + o(r)\mu(R_r)\big) \sum_{i=0}^{k-2} (1 - C\mu(R_r))^i \\ &= (1 - C\mu(R_r))^{k-1}\mu(R_r) + C\mu(R_r)(1 + o(r)\mu(R_r)) \frac{1 - (1 - C\mu(R_r))^{k-1}}{C\mu(R_r)}. \end{aligned}$$

Let us return to the task of estimating $\mu(A_r^c)$. Applying the estimate on $q_r^{(k)}$ and lemma 1, making also a trivial simplification, and using common estimates on e^x, we get for all $r \in \Delta$ small enough, that

$$\begin{aligned} q_r^{([\frac{2M}{\mu(R_r)}])} &\leqslant (1 - C\mu(R_r))^{\frac{2M}{\mu(R_r)} - 2}\mu(R_r) + (1 + o(r)\mu(R_r))(1 - (1 - C\mu(R_r))^{\frac{2M}{\mu(R_r)} - 2}) \\ &\leqslant e^{-2MC}(1 - C\mu(R_r))^{-2}\mu(R_r) + (1 + o(r)\mu(R_r))\left(1 - e^{-3MC}(1 - C\mu(R_r))^{-2}\right) \\ &\leqslant \Gamma(M) := 1 - e^{-4MC} < 1. \end{aligned}$$

Observe that this estimate also gives

$$k \leqslant \left[\frac{2M}{\mu(R_r)}\right] \Longrightarrow q_r^k \leqslant \Gamma(M). \tag{2.19}$$

We will additionally use a stronger mixing result, namely theorem 5.4.10. from [25], which applied to our situation means that

$$\mu\left(T^{-n}(A) \cap B\right) \leqslant (1 + D\gamma^{n-k})\mu(A)\mu(B), \tag{2.20}$$

where $D > 0$ and $\gamma < 1$ are some constants, A is an arbitrary measurable set and $B \in \mathcal{F}_0^k$. Find $s \in \mathbb{N}$ such that

$$\Gamma(M) \cdot (1 + D\gamma^s) < 1. \tag{2.21}$$

Denote $W := 1 + D\gamma^s$. We are finally ready to show that the required measure of the intersections is small, i.e. to prove (2.9).

For brevity, denote

$$R_i := R_{r_i}, \quad k_i := \frac{2M}{\mu(R_{r_i})}, \quad \text{and} \quad \tau_i := \tau_{R_{r_i}}(x).$$

Because of lemma 1 there exists a decreasing sequence $(r_i)_{i=1}^{\infty}$ of positive radii, all belonging to Δ, such that

$$\Omega \text{ may be taken so big that} \qquad (W\Gamma)^{\Omega+1} \leqslant \frac{\delta}{2}, \tag{2.22}$$

$$r_i \text{ decrease so fast that} \qquad k_{i+1} \geqslant 2(s + k_i), \quad \text{and} \tag{2.23}$$

$$\mu(R_{i+1}) \leqslant \frac{\delta}{2\Omega} \frac{1}{k_i + s}. \tag{2.24}$$

Using (2.10) first, and then dividing the set into 2^{Ω} subsets depending on the behaviour of τ_i gives the following.

$$\begin{aligned}
\mu\Big(\bigcap_{n=0}^{\Omega} A_{r_n}^c\Big) &\leqslant \mu\left(\{\forall_{0 \leqslant i \leqslant \Omega}\ \tau_i \leqslant k_i\}\right) \\
&= \mu\left(\left\{\tau_0 \leqslant k_0 \wedge \forall_{1 \leqslant i \leqslant \Omega}\ \left(\tau_i \leqslant k_{i-1} + s \vee \exists_{k_{i-1}+s<u\leqslant k_i} T^u x \in R_i\right)\right\}\right) \\
&\leqslant \mu\left(\left\{\tau_0 \leqslant k_0 \wedge \forall_{1 \leqslant i \leqslant \Omega} \exists_{k_{i-1}+s<u\leqslant k_i} T^u x \in R_i\right\}\right) + \mu\left(\exists_{1 \leqslant i \leqslant \Omega}\left\{\tau_i < k_{i-1} + s\right\}\right) \\
&\leqslant \mu\left(\left\{\tau_0 \leqslant k_0 \wedge \forall_{1 \leqslant i \leqslant \Omega} \exists_{k_{i-1}+s<u\leqslant k_i} T^u x \in R_i\right\}\right) + \Omega \max_{1 \leqslant i \leqslant \Omega} \mu\left(\{\tau_i < k_{i-1} + s\}\right).
\end{aligned} \tag{2.25}$$

A series of estimates follows below. For the second summand, use the easy estimate on the entry time (2.12) and then apply (2.24). For the first summand, apply the estimate on the intersection (2.20) Ω times, use the definitions of $\Gamma(M)$ and W, then use the estimate (2.19) on $q_r^{(k)}$ along with (2.13), and finally invoke (2.22). Using symbols:

$$\begin{aligned}
&\mu\Big(\bigcap_{n=0}^{\Omega} A_{r_n}^c\Big) \\
&\leqslant \Omega \max_{1 \leqslant i \leqslant \Omega} (k_{i-1} + s)\mu(R_i) + (1 + D\gamma^s)\mu\left(\{\tau_0 \leqslant k_0\}\right) \mu\left(\left\{\forall_{1 \leqslant i \leqslant \Omega} \exists_{k_{i-1}+s<u\leqslant k_i} T^u x \in R_i\right\}\right) \\
&\leqslant \frac{\delta}{2} + W\Gamma(M)\mu\left(\left\{\forall_{1 \leqslant i \leqslant \Omega} \exists_{k_{i-1}+s<u\leqslant k_i} T^u x \in R_i\right\}\right) \\
&\leqslant \frac{\delta}{2} + W^{\Omega+1}\Gamma(M) \prod_{i=1}^{\Omega} \mu\left(\left\{\exists_{k_{i-1}+s<u\leqslant k_i} T^u x \in R_i\right\}\right) \leqslant \frac{\delta}{2} + W^{\Omega+1}\Gamma(M) \prod_{i=1}^{\Omega} \mu\left(\{\tau_i \leqslant k_i\}\right) \\
&\leqslant \frac{\delta}{2} + W^{\Omega+1}\Gamma(M)^{\Omega+1} \leqslant \delta.
\end{aligned}$$

This ends the proof. □

Remark 3. What we have actually proved is that if $T: X \to X$ is an open topologically transitive distance expanding map of a compact metric space (X, ρ) and μ is a Borel probability T-invariant measure on X such that (2.20) holds and (2.4) holds for some point $y \in X$, then (2.7) holds for μ-a.e. $x \in X$ with that point y. We also note that compactness of the metric space X was not essential for the proof.

3. Thermodynamic formalism of subshifts of finite type with countable alphabet; preliminaries

In this section we introduce the basic symbolic setting in which we will be working in the sequel. We will describe some fundamental thermodynamic concepts, ideas and results, particularly those used in later sections for applications.

Let $\mathbb{N} = \{1, 2, \ldots\}$ and let E be a countable set, either finite or infinite, called in the sequel an alphabet. Let

$$\sigma: E^{\mathbb{N}} \to E^{\mathbb{N}}$$

be the shift map. It is given by the formula

$$\sigma\big((\omega_n)_{n=1}^{\infty}\big) = \big((\omega_{n+1})_{n=1}^{\infty}\big).$$

We also put

$$E^* = \bigcup_{n=0}^{\infty} E^n,$$

to be the set of finite strings. For every $\omega \in E^*$, we denote by $|\omega|$ the unique integer $n \geqslant 0$ such that $\omega \in E^n$. We call $|\omega|$ the length of ω. We make the convention that $E^0 = \{\emptyset\}$. If $\omega \in E^{\mathbb{N}}$ and $n \geqslant 1$, we put

$$\omega|_n = \omega_1 \ldots \omega_n \in E^n.$$

If $\tau \in E^*$ and $\omega \in E^* \cup E^{\mathbb{N}}$, we define the concatenation of τ and ω by:

$$\tau\omega := \begin{cases} \tau_1 \ldots \tau_{|\tau|}\omega_1\omega_2 \ldots \omega_{|\omega|} & \text{if } \omega \in E^*, \\ \tau_, \ldots \tau_{|\tau|}\omega_1\omega_2 \ldots & \text{if } \omega \in E^{\mathbb{N}}. \end{cases}$$

Given $\omega, \tau \in E^{\mathbb{N}}$, we define $\omega \wedge \tau \in E^{\mathbb{N}} \cup E^*$ to be the longest initial block common to both ω and τ. For each $\alpha > 0$, we define a metric d_α on $E^{\mathbb{N}}$ by setting

$$d_\alpha(\omega, \tau) = \mathrm{e}^{-\alpha|\omega\wedge\tau|}. \tag{3.1}$$

All these metrics induce the same topology, known to be the product (Tichonov) topology. A real or complex valued function defined on a subset of $E^{\mathbb{N}}$ is Hölder with respect to one of these metrics if and only if it is Hölder with respect to all of them, although, of course, the Hölder exponent depends on the metric. If no metric is specifically mentioned, we take it to be d_1.

Now consider an arbitrary matrix $A: E \times E \to \{0, 1\}$. Such a matrix will be called the incidence matrix in the sequel. Set

$$E_A^{\infty} := \{\omega \in E^{\mathbb{N}} : \ A_{\omega_i\omega_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\}.$$

Elements of E_A^∞ are called *A-admissible*. We also set

$$E_A^n := \{\omega \in E^{\mathbb{N}} :\ A_{\omega_i\omega_{i+1}} = 1 \text{ for all } 1 \leqslant i \leqslant n-1\},\ n \in \mathbb{N},$$

and

$$E_A^* := \bigcup_{n=0}^{\infty} E_A^n.$$

The elements of these sets are also called *A-admissible*. For every $\omega \in E_A^*$, we put

$$[\omega] := \{\tau \in E_A^\infty :\ \tau_{||\omega|} = \omega\}.$$

The set $[\omega]$ is called the cylinder generated by the word ω. The collection of all such cylinders forms a base for the product topology relative to E_A^∞. The following fact is obvious.

Proposition 4. *The set E_A^∞ is a closed subset of $E^{\mathbb{N}}$, invariant under the shift map $\sigma\colon E^{\mathbb{N}} \to E^{\mathbb{N}}$, the latter meaning that*

$$\sigma(E_A^\infty) \subset E_A^\infty.$$

The matrix A is said to be *finitely irreducible* if there exists a finite set $\Lambda \subset E_A^*$ such that for all $i, j \in E$ there exists $\omega \in \Lambda$ for which $i\omega j \in E_A^*$. If all elements of some such Λ are of the same length, then A is called finitely primitive (or aperiodic).

The topological pressure of a continuous function $f : E_A^\infty \to \mathbb{R}$ with respect to the shift map $\sigma : E_A^\infty \to E_A^\infty$ is defined to be

$$\mathrm{P}(f) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in E_A^n} \exp\left(\sup_{\tau\in[\omega]} \sum_{j=0}^{n-1} f(\sigma^j(\tau))\right). \tag{3.2}$$

The existence of this limit, following from the observation that the 'log' above forms a subadditive sequence, was established in [16], comp. [17]. Following the common usage we abbreviate

$$S_n f := \sum_{j=0}^{n-1} f \circ \sigma^j$$

and call $S_n f(\tau)$ the nth Birkhoff's sum of f evaluated at a word $\tau \in E_A^\infty$.

A function $f\colon E_A^\infty \longrightarrow \mathbb{R}$ is called (locally) Hölder continuous with an exponent $\alpha > 0$ if

$$V_\alpha(f) := \sup_{n\geqslant 1} \{V_{\alpha,n}(f)\} < +\infty,$$

where

$$V_{\alpha,n}(f) = \sup\{|f(\omega) - f(\tau)|\mathrm{e}^{\alpha(n-1)} : \omega, \tau \in E_A^\infty \text{ and } |\omega \wedge \tau| \geqslant n\}.$$

A function $f\colon E_A^\infty \to \mathbb{R}$ is called summable if

$$\sum_{e\in E} \exp(\sup(f|_{[e]})) < +\infty. \tag{3.3}$$

We note that if f has a Gibbs state, then f is summable. The following theorem has been proved in [16], comp. [17], for the class of acceptable functions defined there. Since Hölder continuous ones are among them, we have the following.

Theorem 5 (Variational principle). *If the incidence matrix $A\colon E\times E\to\{0,1\}$ is finitely irreducible and if $f\colon E_A^\infty\to\mathbb{R}$ is Hölder continuous, then*

$$\mathrm{P}(f)=\sup\left\{\mathrm{h}_\mu(\sigma)+\int f\,\mathrm{d}\mu\right\},$$

where the supremum is taken over all σ-invariant (ergodic) Borel probability measures μ such that $\int f\,\mathrm{d}\mu>-\infty$.

We call a σ-invariant probability measure μ on E_A^∞ an equilibrium state of a Hölder continuous function $f\colon E_A^\infty\to\mathbb{R}$ if $\int -f\,\mathrm{d}\mu<+\infty$ and

$$\mathrm{h}_\mu(\sigma)+\int f\,\mathrm{d}\mu=\mathrm{P}(f). \tag{3.4}$$

It was proved in [16] that if the matrix A is finitely irredicible and $f\colon E_A^\infty\to\mathbb{R}$ is a Hölder continuous summable potential, then there exists a unique equilibrium state of f. We denote it by μ_f. For the reasons explained in [16] and [17] it is also called a Gibbs state for f and, crucially, it satisfies (2.20). We have the following.

Theorem 6. *Let E be a countable set, either finite or infinite, and let $A\colon E\times E\to\{0,1\}$ be a finitely irreducible incidence matrix. If $f\colon E_A^\infty\to\mathbb{R}$ is a Hölder continuous summable potential, then, given any $\alpha>0$, we have for the dynamical system $\left(\sigma\colon E_A^\infty\to E_A^\infty,\mu_f\right)$ that*

$$\limsup_{r\to0}\tau_{B_\alpha(\rho,r)}(\omega)\cdot\mu_f(B_\alpha(\rho,r))=+\infty, \tag{3.5}$$

for every $\rho\in E_A^\infty$ and μ_f-a.e. $\omega\in E_A^\infty$, where $B_\alpha(\gamma,r)$ denotes the ball, with respect to the metric d_α defined in (3.1), centred at $\gamma\in E_A^\infty$ with radius r.

If E is a finite set, then this theorem is an immediate consequence of theorem 2 once one observes that then $\sigma\colon E_A^\infty\to E_A^\infty$ is an open topologically transitive distance expanding map on the compact space E_A^∞ with respect to every metric d_α, $\alpha>0$. In order to see that this theorem holds in its full generality, i.e. for all countable sets E, it suffices to note the following:

(1) Given $\alpha>0$, for every $r\in(0,1)$ there exists an integer $n_r\asymp-\frac{1}{\alpha}\log r\geqslant0$ such that

$$B_\alpha(\tau,r)=\left[\tau|_{n_r}\right]$$

for every $\tau\in E_A^\infty$ and

$$\lim_{r\to0}n_r=+\infty.$$

(2) There exists $\beta>0$ (see [17]) such that

$$\mu_f\left([\tau|_n]\right)\leqslant e^{-\beta n}$$

for every $\tau\in E_A^\infty$ and every integer $n\geqslant0$.

(3) Because of (1) and (2), and remark 3, the proof of theorem 2 goes through in the current setting.

In fact, as an immediate consequence of theorem 6 and items (1) and (2) above, we get the following version of this theorem, independent of any metric d_α:

Theorem 7. *Let E be a countable set, either finite or infinite, and let $A\colon E\times E\to\{0,1\}$ be a finitely irreducible incidence matrix. If $f\colon E_A^\infty\to\mathbb{R}$ is a Hölder continuous summable* *potential, then we have for the dynamical system* $\left(\sigma\colon E_A^\infty \to E_A^\infty, \mu_f\right)$ *that*

$$\limsup_{n\to\infty} \tau_{[\rho|_n]}(\omega)\cdot\mu_f([\rho|_n]) = +\infty, \tag{3.6}$$

for every $\rho \in E_A^\infty$ *and* μ_f*-a.e.* $\omega \in E_A^\infty$.

4. Graph directed Markov systems

Our goal now is to go beyond (uniformly) expanding maps. It will be accomplished in two major steps. First, in the current section, we will extend the previous results to some class of maps of infinite degree. These will be the maps naturally resulting from graph directed Markov systems with a countable infinite alphabet. The second major step, carried on in the next section, will be to employ the techniques of the first return maps. We will first relate, see lemma 15, hitting times for a given system and an induced one. Then, as a consequence, we will be able to prove an analog of theorem 2 for all systems that allow inducings having structure of graph directed Markov systems. Finally, in section 6, we will provide several classes of examples.

We now define a graph directed Markov system (*abbr.* GDMS) relative to a directed multigraph (V, E, i, t) with an incidence matrix A. A *directed multigraph* consists of

- a finite set V of vertices,
- a countable (either finite or infinite) set E of directed edges,
- a map $A\colon E\times E \to \{0, 1\}$ called an *incidence matrix* on (V, E),
- two functions $i, t\colon E \to V$, such that $A_{ab} = 1$ implies $t(b) = \mathrm{i}(a)$.

In addition, we have a collection of non-empty compact metric spaces $\{X_v\}_{v\in V}$ and a number $s \in (0, 1)$, and for every $e \in E$, we have a 1-to-1 contraction $\phi_e\colon X_{t(e)} \to X_{i(e)}$ with Lipschitz constant $\leqslant s$. Then the collection

$$\mathcal{S} := \{\phi_e\colon X_{t(e)} \longrightarrow X_{i(e)}\}_{e\in E}$$

is called a GDMS. We now describe the limit set of the system $\mathcal{S}$. For each $n \geqslant 1$ and $\omega \in E_A^n$, we consider the map coded by ω

$$\phi_\omega := \phi_{\omega_1}\circ\cdots\circ\phi_{\omega_n}\colon X_{t(\omega)} \longrightarrow X_{i(\omega)}.$$

For $\omega \in E_A^\infty$, the sets $\{\phi_{\omega|_n}\left(X_{t(\omega_n)}\right)\}_{n\geqslant 1}$ form a descending sequence of non-empty compact sets and therefore $\bigcap_{n\geqslant 1}\phi_{\omega|_n}\left(X_{t(\omega_n)}\right) \neq \emptyset$. Since for every $n \geqslant 1$,

$$\mathrm{diam}\left(\phi_{\omega|_n}\left(X_{t(\omega_n)}\right)\right) \leqslant s^n \mathrm{diam}\left(X_{t(\omega_n)}\right) \leqslant s^n \max\{\mathrm{diam}(X_v) : v \in V\},$$

we conclude that the intersection

$$\bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}\left(X_{t(\omega_n)}\right)$$

is a singleton and we denote its only element by $\pi(\omega)$. In this way we have defined the map

$$\pi\colon E_A^\infty \longrightarrow X := \bigoplus_{v\in V} X_v$$

from E_A^∞ to $\bigoplus_{v\in V} X_v$, the disjoint union of the compact sets X_v. The set

$$J = J_{\mathcal{S}} = \pi(E_A^\infty)$$

will be called the limit set of the GDMS S.

A GDMS is called an iterated function system (*abbr.* IFS) if V, the set of vertices, is a singleton and the incidence matrix A consists of 1s only, i.e. $A(E\times E) = \{1\}$.

Definition 8. We call the GDMS $\mathcal{S}$ and its incidence matrix A *finitely irreducible* if there exists a finite set $\Lambda \subset E_A^*$ such that for all $a, b \in E$ there exists a word $\omega \in \Lambda$ such that the concatenation $a\omega b$ is in E_A^*.

Given an integer $n \geqslant 1$ and a set $F \subset E_A^n$, we call the set

$$U := \bigcup_{\omega\in F} \phi_\omega\big(X_{t(\omega)}\big)$$

a set of order $\leqslant n$ generated by the GDMS S. If in addition U cannot be represented as a union of sets of the form $\phi_\tau\big(X_{t(\tau)}\big)$, $\tau \in E_A^{n-1}$, then R will be called of order n, and we will write

$$n = \mathrm{ord}(U).$$

Assume now that for some integer $d \geqslant 1$, X_v is a subset of $\mathbb{R}^d$ for every vertex $v \in V$. Assume further that

$$\overline{\mathrm{Int}(X_v)} = X_v$$

and

$$\phi_a\big(\mathrm{Int}\big(X_{t(a)}\big)\big) \cap \phi_b\big(\mathrm{Int}\big(X_{t(b)}\big)\big) = \emptyset \tag{4.1}$$

whenever $a, b \in E$ and $a \neq b$. This assumption is commonly called the open set condition (*abbr.* OSC).

A GDMS $S = \{\phi_e\}_{e\in E}$ is said to satisfy the strong open set condition (*abbr.* SOSC) if it satisfies the OSC and

$$J_S \cap \mathrm{Int}X \neq \emptyset.$$

We want to define an ordinary dynamical system out of the GDMS S. The problem is that the map projection map $\pi\colon E_A^\infty \longrightarrow X$ need not be 1-to-1. In order to remedy this problem (i.e. with non-unique coding), we introduce the set

$$\mathring{J}_{\mathcal{S}} := J_{\mathcal{S}} \setminus \bigcup_{\omega\in E_A^*} \phi_\omega(\partial X_{t(\omega)}).$$

Set $\mathring{E}_A^\infty := \pi_{\mathcal{S}}^{-1}\big(\mathring{J}\big)$ and notice that for every $z \in \mathring{J}_{\mathcal{S}}$ there exists a unique $\omega(z) \in E_A^\infty$ such that $z = \pi(\omega(z))$. Moreover, $\omega(z) \in \mathring{E}_A^\infty$ and we simply denote it by $\pi^{-1}(z)$. Note that

$$\sigma\big(\mathring{E}_A^\infty\big) \subset \mathring{E}_A^\infty$$

and this restricted shift map induces a map

$$T_{\mathcal{S}}\colon \mathring{J}_{\mathcal{S}} \longrightarrow \mathring{J}_{\mathcal{S}}$$

by the formula

$$T_{\mathcal{S}}(z) = \pi \circ \sigma(\pi^{-1}(z)) \in \mathring{J}_{\mathcal{S}},$$

so that the following diagram commutes

$$\begin{array}{ccc} \mathring{E}_A^\infty & \xrightarrow{\sigma} & \mathring{E}_A^\infty \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathring{J}_{\mathcal{S}} & \xrightarrow[T_{\mathcal{S}}]{} & \mathring{J}_{\mathcal{S}} \end{array}$$

and the map $\pi\colon \mathring{E}_A^\infty \longrightarrow \mathring{J}_{\mathcal{S}}$ is a continuous bijection. The proof of the following result can be found in [14].

Theorem 9. *If $S = \{\phi_e\}_{e\in E}$ is a GDMS satisfying SOSC and μ is a Borel probability σ-invariant ergodic measure on E_A^∞ with full topological support, then*

$$\mu \circ \pi^{-1}(\mathring{J}_{\mathcal{S}}) = 1 \tag{4.2}$$

and

$$\mu \circ \pi^{-1} \circ T_{\mathcal{S}}^{-1} = \mu \circ \pi^{-1}. \tag{4.3}$$

If $f\colon E_A^\infty \longrightarrow \mathbb{R}$ is a Hölder continuous summable potential, then denote

$$\hat{\mu}_f := \mu_f \circ \pi^{-1}.$$

As an immediate consequence of theorem 9 and bijectivity of the map $\pi\colon \mathring{E}_A^\infty \to \mathring{J}_{\mathcal{S}}$, we get the following.

Corollary 10. *Let $\mathcal{S}$ be a finitely irreducible GDMS satisfying the SOSC. Let $f\colon E_A^\infty \longrightarrow \mathbb{R}$ be a Hölder continuous summable potential. Denote by μ_f its unique σ-invariant Gibbs/equilibrium state. Then*

$$\mu_f\big(\mathring{E}_A^\infty\big) = 1 \ \text{ and } \ \hat{\mu}_f\big(\mathring{J}_{\mathcal{S}}\big) = 1.$$

Moreover, the projection $\pi\colon \mathring{E}_A^\infty \longrightarrow \mathring{J}_{\mathcal{S}}$ establishes a measure-preserving isomorphism between measure-preserving dynamical systems $\big(\sigma\colon \mathring{E}_A^\infty \to \mathring{E}_A^\infty, \mu_f\big)$ and $\big(T_{\mathcal{S}}\colon \mathring{J}_{\mathcal{S}} \to \mathring{J}_{\mathcal{S}}, \hat{\mu}_f\big)$.

The following lemma is entirely analogous to lemma 1. The proof is also analogous. Since however, the proof is short, we provide it here for the sake of completeness and convenience of the reader.

Lemma 11. *Assume that S is GDMS and all spaces X_v, $v \in V$, are compact subsets of some (finitely dimensional) Euclidean space. Assume also that μ is a Borel probability measure on the limit set J_S. If $y \in J_S$ and*

$$\mu(B(y,r)) \leqslant Cr^\alpha \tag{4.4}$$

for some constant $C \geqslant 1$, $\alpha > 0$, and all radii $r > 0$, then there exists a Lebesgue measurable set $\Delta \subset (0,1)$ with the following properties *(a)* $\lim_{\Delta \ni r \to 0} \frac{Leb(\Delta \cap (0,r))}{r} = 1.$

(b) For every $r \in \Delta$ *there exists a set* R_r*, generated by S, satisfying*

(b1) $B(y,r) \subset R_r$,

(b2) $\frac{\mu(R_r)}{\mu(B(y,r))} \leqslant 2$,

(b3) $\lim_{r\to 0} \mathrm{ord}(R_r)\mu(B(y,r)) = 0$.

Proof. Lemma 3.6 from [21], applied with $\kappa_y = 2$, gives a Lebesgue measurable set $\Delta \subset (0,1)$ for which item (a) above holds and

$$\frac{\mu(B(y, r + r^2))}{\mu(B(y,r))} \leqslant 2 \tag{4.5}$$

for all $r \in \Delta$. Let $n(r) \geqslant 1$ be the least integer such that

$$\delta s^{(n(r)-1)} \leqslant r^2. \tag{4.6}$$

Let

$$R_r := \bigcup \left\{ \phi_\omega(X_{t(\omega)}) : \omega \in E_A^{n(r)} \text{ and } B(y,r) \cap \phi_\omega\left(X_{t(\omega)}\right) \neq \emptyset \right\}.$$

Item (b1) holds trivially, and R_r is a set generated by the Markov partition $\mathcal{R}$ with

$$\mathrm{ord}(R_r) \leqslant n(r).$$

Invoking (4.6) and (2.3), we see that $R_r \subset B(y, r + r^2)$. This and (4.5) yields (b2). It follows from the definition of $n(r)$ that

$$\delta s^{(n(r)-2)} > r^2.$$

Taking logarithms gives

$$n(r) < \frac{\log(\delta s^{-2}) - 2\log r}{-\log s}.$$

Combining all that with (4.4), we get that

$$\begin{aligned} 0 \leqslant \limsup_{r\to 0} \mathrm{ord}(R_r)\mu(B(y,r)) &\leqslant \limsup_{r\to 0} n(r)\mu(B(y,r)) \\ &\leqslant \frac{C}{\log(1/s)} \limsup_{r\to 0} \left(r^\alpha \left(\log(\delta s^{-2}) - 2\log r \right) \right) = 0, \end{aligned}$$

thus (b3) holds and the proof is complete. □

Having this lemma, corollary 10, and already knowing that (2.20) holds for all equilibrium states of Hölder continuous summable potentials, the same proof (see remark 3) as the one of theorem 2, gives the following.

Theorem 12. *Let* $\mathcal{S}$ *be a finitely irreducible GDMS satisfying the SOSC. Let* $f: E_A^\infty \longrightarrow \mathbb{R}$ *be a Hölder continuous summable potential. If* $y \in \mathring{J}_{\mathcal{S}}$ *and there exist constants* $\alpha > 0$ *and* $C \geqslant 1$ *such that*

$$\hat{\mu}_f(B(y,r)) \leqslant Cr^{\alpha} \tag{4.7}$$

for all radii $r > 0$, then for the dynamical system $\left(T_{\mathcal{S}} : \mathring{J}_{\mathcal{S}} \to \mathring{J}_{\mathcal{S}}, \hat{\mu}_f\right)$ we have that

$$\limsup_{r\to 0} \tau_{B(y,r)}(x) \cdot \hat{\mu}_f(B(y,r)) = +\infty, \quad \text{for } \hat{\mu}_f - \text{a.e. } x \in \mathring{J}_{\mathcal{S}}. \tag{4.8}$$

In order to meaningfully apply this theorem, we shall prove the following.

Proposition 13. *Let $\mathcal{S}$ be be a finitely irreducible conformal GDMS satisfying the SOSC. If $f : E_A^{\infty} \longrightarrow \mathbb{R}$ is a Hölder continuous summable potential, then there exists Λ, a Borel subset of E_A^{∞} such that $\mu_f(\Lambda) = 1$ and for every $\omega \in \Lambda$ there exist $C \geqslant 1$ and $\alpha > 0$ such that*

$$\mu_f(B(\pi(\omega), r)) \leqslant Cr^{\alpha}$$

for all radii $r \geqslant 0$.

Proof. Since $\mathcal{S}$ satisfies the SOSC and since the measure μ_f is of full topological support, there exists $v \in V$, $\xi \in X_v$, and $R \in (0,1)$ such that

$$B(\xi, 4R) \subset X_v \tag{4.9}$$

and

$$\mu_f\left(\pi^{-1}(B(\xi, R))\right) = \hat{\mu}_f(B(\xi, R)) > 0.$$

Let $\Lambda_1 \subset E_A^{\infty}$ be the set of all points ω such that

$$\sigma^n(\omega) \in \pi^{-1}(B(\xi, R)) \tag{4.10}$$

for infinitely many positive n's. By Birkhoff's Ergodic theorem and ergodicity of measure μ_f, we have that

$$\mu_f(\Lambda_1) = 1.$$

Take any $\omega \in \Lambda_1$. Let $(n_k(\omega))_{k=1}^{\infty}$ be the infinite strictly increasing sequence of positive integers for which (4.10) holds. Abbreviate $n_k := n_k(\omega)$. Then, because of (4.9), we have for every $k \geqslant 1$ that

$$B\big(\pi(\sigma^{n_k}(\omega)), 2R\big) \subset \mathrm{Int}(X_v)$$

and for some fixed $K \geqslant 1$

$$B\Big(\pi(\omega), K^{-1}\Big|\phi'_{\omega|_{n_k}}\big(\pi(\sigma^{n_k}(\omega))\big)\Big|R\Big) \subset \phi_{\omega|_{n_k}} B\big(\pi(\sigma^{n_k}(\omega)), R\big).$$

μ_f is a Gibbs measure, allowing us to write, for some constant $Q \in [0, +\infty)$, that

$$\begin{aligned}\mu_f\Big(B\Big(\pi(\omega), K^{-1}\big|\phi'_{\omega|_{n_k}}\big(\pi(\sigma^{n_k}(\omega))\big)\big|R\Big)\Big) &\leqslant \mu_f\Big(\phi_{\omega|_{n_k}} B\big(\pi(\sigma^{n_k}(\omega)), R\big)\Big) \\ &\leqslant Q \exp\big(S_{n_k} f(\omega) - \mathrm{P}(f) n_k\big).\end{aligned}$$

Now, given $r > 0$, there exist a largest integer $k_r^- = k_r^-(\omega) \geqslant 1$ and a smallest integer $k_r^+ = k_r^+(\omega) \geqslant 1$ such that respectively

$$K^{-1}\left|\phi'_{\omega|_{n_{k_r^+}}}\left(\pi(\sigma^{n_{k_r^+}}(\omega))\right)\right|R \leqslant r \quad \text{and} \quad r \leqslant K^{-1}\left|\phi'_{\omega|_{n_{k_r^-}}}\left(\pi(\sigma^{n_{k_r^-}}(\omega))\right)\right|R. \tag{4.11}$$

Then

$$B(\pi(\omega), r) \subset B\Big(\pi(\omega), K^{-1}\left|\phi'_{\omega|_{n_{k_r^-}}}\left(\pi(\sigma^{n_{k_r^-}}(\omega))\right)\right|R\Big),$$

and, consequently

$$\mu_f(B(\pi(\omega), r)) \leqslant Q\exp\left(S_{n_{k_r^-}}f(\omega) - \mathrm{P}(f)n_{k_r^-}\right).$$

But, by Birkhoff's Ergodic theorem, there exists a measurable set $\Lambda_2 \subset \Lambda_1$ such that $\mu_f(\Lambda_2) = 1$ and

$$\lim_{n\to\infty}\frac{1}{n}S_nf(\omega) = \mu_f(f) := \int f\,\mathrm{d}\mu_f$$

for every $\omega \in \Lambda_2$. Therefore, for every $\omega \in \Lambda_2$ there exists $N_1(\omega) \geqslant 1$ such that if $n \geqslant N_1(\omega)$, then, using the fact that the entropy $\mathrm{h}_{\mu_f}(\sigma)$ is positive, we get that

$$\frac{1}{n}S_nf(\omega) \leqslant \mu_f(f) + \frac{1}{2}\mathrm{h}_{\mu_f}(\sigma).$$

Therefore, for all $\omega \in \Lambda_2$ and all radii $r > 0$ small enough, say $0 < r \leqslant r_\omega$, we get that

$$\mu_f(B(\pi(\omega), r)) \leqslant Q\exp\left(\left(\mu_f(f) + \frac{1}{2}\mathrm{h}_{\mu_f}(\sigma) - \mathrm{P}(f)\right)n_{k_r^-}\right) = Q\exp\left(-\frac{1}{2}\mathrm{h}_{\mu_f}(\sigma)n_{k_r^-}\right), \tag{4.12}$$

where writing the equality sign we used the fact that μ_f an (in fact unique) equilibrium state, i.e.

$$\mathrm{P}(f) = \mathrm{h}_{\mu_f}(\sigma) + \mu_f(f).$$

On the other hand, it follows from Birkhoff's Ergodic theorem that there exists a measurable set $\Lambda_3 \subset \Lambda_2$ such that $\mu_f(\Lambda_3) = 1$ and

$$-\lim_{n\to\infty}\frac{1}{n}\log\left|\phi'_{\tau|_n}\left(\pi(\sigma^n(\tau))\right)\right| = \chi_{\mu_f} := -\int_{E_A^\infty}\log\left|\phi'_{\rho_1}\left(\pi(\sigma(\rho))\right)\right|\mathrm{d}\mu_f(\rho) > 0$$

for all $\tau \in \Lambda_3$. Hence, if $\tau \in \Lambda_3$, then for every integer $n \geqslant 0$ large enough,

$$\left|\phi'_{\tau|_n}\left(\pi(\sigma^n(\tau))\right)\right| \geqslant \exp\Big(-\frac{1}{2}\chi_{\mu_f}n\Big).$$

In conjunction with (4.11), this implies that

$$\exp\Big(-\frac{1}{2}\chi_{\mu_f}n_{k_r^+}\Big) \leqslant KR^{-1}r$$

for all $r > 0$ small enough. Combining this with (4.12), we obtain:

$$\begin{aligned}
\mu_f(B(\pi(\omega),r)) &\leqslant Q\exp\left(-\frac{1}{2}\chi_{\mu_f}n_{k_r^+}\frac{\mathrm{h}_{\mu_f}(\sigma)}{\chi_{\mu_f}}\frac{n_{k_r^-}}{n_{k_r^+}}\right) = Q\left(\exp\left(-\frac{1}{2}\chi_{\mu_f}n_{k_r^+}\right)\right)^{\frac{\mathrm{h}_{\mu_f}(\sigma)}{\chi_{\mu_f}}\frac{n_{k_r^-}}{n_{k_r^+}}} \\
&\leqslant Q(KR^{-1})^{\frac{\mathrm{h}_{\mu_f}(\sigma)}{\chi_{\mu_f}}\frac{n_{k_r^-}}{n_{k_r^+}}} r^{\frac{\mathrm{h}_{\mu_f}(\sigma)}{\chi_{\mu_f}}\frac{n_{k_r^-}}{n_{k_r^+}}} \\
&\leqslant Q(KR^{-1})^{\frac{\mathrm{h}_{\mu_f}(\sigma)}{\chi_{\mu_f}}} r^{\frac{\mathrm{h}_{\mu_f}(\sigma)}{\chi_{\mu_f}}\frac{n_{k_r^-}}{n_{k_r^+}}},
\end{aligned} \tag{4.13}$$

where we wrote the last inequality since $KR^{-1} \geqslant 1$ and $n_{k_r^-}/n_{k_r^+} \leqslant 1$. Since $k_r^+ - k_r^- \leqslant 1$ and since, by Birkhoff's Ergodic theorem, there exists a measurable set $\Lambda_4 \subset \Lambda_3$ such that $\mu_f(\Lambda_4) = 1$ and

$$\lim_{j\to\infty}\frac{n_{j+1}(\tau)}{n_j(\tau)} = 1$$

for all $\tau \in \Lambda_4$, we conclude that if $\omega \in \Lambda_4$, then

$$\frac{n_{k_r^+(\omega)}(\omega)}{n_{k_r^-(\omega)}(\omega)} \leqslant 2$$

for all $r > 0$ small enough. Inserting this into (4.13), we get for all $r > 0$ small enough that

$$\mu_f(B(\pi(\omega),r)) \leqslant Cr^{\alpha},$$

where $C = Q(KR^{-1})^{\frac{\mathrm{h}_{\mu_f}(\sigma)}{\chi_{\mu_f}}}$ and $\alpha = \frac{1}{2}\frac{\mathrm{h}_{\mu_f}(\sigma)}{\chi_{\mu_f}}$. The proof of proposition 13 is thus complete. □

As an immediate consequence of this proposition and theorem 12, we get the following.

Theorem 14. *Let $\mathcal{S}$ be a finitely irreducible conformal GDMS which satisfies the SOSC. If $f : E_A^\infty \longrightarrow \mathbb{R}$ is a Hölder continuous summable potential, then for the dynamical system $(T_{\mathcal{S}} : \mathring{J}_{\mathcal{S}} \longrightarrow \mathring{J}_{\mathcal{S}}, \hat{\mu}_f)$ we have that*

$$\limsup_{r\to 0}\tau_{B(y,r)}(x)\cdot\hat{\mu}_f(B(y,r)) = +\infty, \tag{4.14}$$

for $\hat{\mu}_f$-a.e. $y \in \mathring{J}_{\mathcal{S}}$ and $\hat{\mu}_f$-a.e. $x \in \mathring{J}_{\mathcal{S}}$.

5. First return map techniques

The following result states that the upper limit, the subject of this paper, does not change when we go to an induced system. As we have already announced in the previous section, this is our crucial step to apply the results of the previous section concerning graph directed Markov systems to many other classes of non-uniformly expanding systems.

Lemma 15. *Consider an ergodic, metric, measure preserving, dynamical system (X, T, μ, d); in particular μ is a Borel probability measure on X. Let $\widehat{X} \subset X$ be a measurable subset of X with positive measure μ. Define (in the standard way) the first return map (induced system) $(\widehat{X}, \widehat{T}, \widehat{\mu}, d)$. Assume that $y \in \mathrm{Int}\widehat{X}$. Then the upper limits of entry times, i.e. (2.7), for the original and the induced system coincide. Likewise for lower limits. In other words,*

$$\limsup_{r\to 0} \tau_{B(y,r)}(x)\cdot\mu(B(y,r)) = \limsup_{r\to 0} \widehat{\tau}_{B(y,r)}(x)\cdot\widehat{\mu}(B(y,r)) \tag{5.1}$$

and

$$\liminf_{r\to 0} \tau_{B(y,r)}(x)\cdot\mu(B(y,r)) = \liminf_{r\to 0} \widehat{\tau}_{B(y,r)}(x)\cdot\widehat{\mu}(B(y,r)) \tag{5.2}$$

for $\widehat{\mu}$*-a.e.* $x\in\widehat{X}$*, where* $\widehat{\tau}_U$ *is the first entry time into a set* $U\subset\widehat{X}$ *for the induced system* $(\widehat{X},\widehat{T})$.

Proof. Having taken $x\in\widehat{X}$, we start by defining the sequence of subsequent *closest* approaches to y. Inductively:

$$n_1 := 1,$$
$$n_{k+1} := \min\{n : d(T^n x, y) < d(T^{n_k}x, y)\}.$$

Also, denote $r_k := d(T^{n_k}x, y)$. Using this notation we have that

$$\tau_{B(y,r)}(x) = n_k \quad \text{if} \quad r_k < r \leqslant r_{k-1}.$$

For brevity, denote the first return map into $\widehat{X}$ by $t(x) := \tau_{\widehat{X}}(x)$, and define:

$$A_l(x) := \sum_{i=0}^{l-1} t(\widehat{T}^i(x)).$$

By the definition of the induced map we have

$$\widehat{T}^l(x) = T^{A_l(x)}(x), \tag{5.3}$$

and therefore, by ergodicity of μ, along with Birkhoff's Ergodic theorem and Kac's lemma, we get that

$$\lim_{l\to\infty}\frac{1}{l}A_l(x) = \int_{\widehat{X}} t\,\mathrm{d}\widehat{\mu} = \frac{1}{\mu(\widehat{X})} \tag{5.4}$$

for μ-a.e. $x\in X$, say $x\in Z\subset X$, where Z is measurable and $\mu(Z)=1$. Since $y\in\mathrm{Int}\widehat{X}$, there exists $r_y>0$ so small that $B(y,r_y)\subset\widehat{X}$. Consequently, for all $r\in(0,r_y]$, the *closest* approaches to y using the map $\widehat{T}$ are the same points as for the map T and only the numbers of the iterates may, and usually do, differ. This observation together with (5.3) gives

$$\widehat{\tau}_{B(y,r)}(x) = \tau_{B(y,r)}(x)\frac{l}{A_l(x)}$$

for all $r>0$ small enough. Because of (5.4), for every $\varepsilon>0$ and every $x\in Z$, there exists $r_y(x,\varepsilon)\in(0,r_y]$ such that

$$\widehat{\tau}_{B(y,r)}(x)\left(\frac{1}{\mu(\widehat{X})}-\varepsilon\right) \leqslant \tau_{B(y,r)}(x) \leqslant \widehat{\tau}_{B(y,r)}(x)\left(\frac{1}{\mu(\widehat{X})}+\varepsilon\right)$$

for all $r\in(0,r_y(x,\varepsilon)]$. Multiplying both sides of this inequality by $\mu(B(y,r))$ and recalling that $\widehat{\mu}(B(y,r)) = \mu(B(y,r))/\mu(\widehat{X})$, gives the following

$$\begin{aligned}
&\widehat{\tau}_{B(y,r)}(x)\widehat{\mu}(B(y,r))\left(1-\varepsilon\mu(\widehat{X})\right)=\widehat{\tau}_{B(y,r)}(x)\left(\frac{1}{\mu(\widehat{X})}-\varepsilon\right)\widehat{\mu}(B(y,r))\mu(\widehat{X})\\
&\leqslant \tau_{B(y,r)}(x)\mu(B(y,r))\\
&\leqslant \widehat{\tau}_{B(y,r)}(x)\left(\frac{1}{\mu(\widehat{X})}+\varepsilon\right)\widehat{\mu}(B(y,r))\mu(\widehat{X})=\widehat{\tau}_{B(y,r)}(x)\widehat{\mu}(B(y,r))\left(1+\varepsilon\mu(\widehat{X})\right).
\end{aligned}$$

Thus, letting $\varepsilon \to 0$ finishes the proof. □

As an immediate consequence of this lemma, we get the following.

Corollary 16. *Consider an ergodic, metric, measure preserving, dynamical system (X, T, μ, d); in particular μ is a Borel probability measure on X. Assume that μ is not supported on any periodic orbit of T; equivalently, μ is atomless. Let $\widehat{X} \subset X$ be a measurable subset of X with positive measure μ. Assume that $y \in \mathrm{Int}\widehat{X}$. Then*

$$\limsup_{r\to 0}\tau_{B(y,r)}(x)\cdot\mu(B(y,r))=\limsup_{r\to 0}\widehat{\tau}_{B(y,r)}(x)\cdot\widehat{\mu}(B(y,r))=\limsup_{r\to 0}\widehat{\tau}_{B(y,r)}(\widehat{T}(x))\cdot\widehat{\mu}(B(y,r)) \tag{5.5}$$

and

$$\liminf_{r\to 0}\tau_{B(y,r)}(x)\cdot\mu(B(y,r))=\liminf_{r\to 0}\widehat{\tau}_{B(y,r)}(x)\cdot\widehat{\mu}(B(y,r))=\liminf_{r\to 0}\widehat{\tau}_{B(y,r)}(\widehat{T}(x))\cdot\widehat{\mu}(B(y,r)) \tag{5.6}$$

for μ-a.e. $x \in X$, where $\widehat{T}(x)$, the first entry time to $\widehat{X}$, defined as $T^n(x)$, where $n \geqslant 1$ is a minimal integer such that $T^n(x) \in \widehat{X}$, is well defined for a μ-a.e. $x \in X$ since the measure μ is ergodic and $\mu(\widehat{X}) > 0$.

In particular, if

$$\limsup_{r\to 0}\widehat{\tau}_{B(y,r)}(x)\cdot\widehat{\mu}(B(y,r))=+\infty$$

for $\widehat{\mu}$-a.e. $x \in \widehat{X}$, then

$$\limsup_{r\to 0}\tau_{B(y,r)}(x)\cdot\mu(B(y,r))=+\infty$$

for μ-a.e. $x \in X$.

We now shall take fruits of this corollary and the previous section. Let (X, d) be a metric space and let $T\colon X \to X$ be a continuous map. Fix $z \in X$. Assume that z has a compact neighborhood, call it $W(z)$. We say that the map T is of local IFS type at z if there exists a closed set $\Gamma_z \subset W(z)$ such that

(1) $z \in \mathrm{Int}(\Gamma_z)$.

(2) For every integer $n \geqslant 1$ there exists α_n, a countable partition of $\tau_{\Gamma_z}^{-1}(n)$ ($\tau_{\Gamma_z}(x) := \min\{n \in \mathbb{N} \cup \{+\infty\} : T^n(x) \in \Gamma_z\}$ is now the first return time of $x \in \Gamma_z$ to Γ_z under T), such that

 (a) For each $A \in \alpha_n$ the map $T^n|_A\colon A \to \Gamma_z$ is a homeomorphism,

(b) There exists $s \in (0, 1)$ such that for every integer $n \geqslant 1$ and every $A \in \alpha_n$, the map $\phi_A := \left(T^n|_A\right)^{-1} : \Gamma_z \to A$ is a contraction and its Lipschitz constant does not exceed s.

Obviously the maps

$$\mathcal{S} := \left\{\phi_A : n \in \mathbb{N},\, A \in \tau_{\Gamma_z}^{-1}(n)\right\}$$

form an IFS. Therefore, as an immediate consequence of theorem 12 and corollary 16, we get the following.

Theorem 17. *Let (X, d) be a metric space and let $T\colon X \to X$ be a continuous map. Fix $z \in X$. Assume that the map T is of local IFS type at z. Assume that μ is a Borel probability T-invariant measure on X such that*

(1) $\mu\left(\Gamma_z\right) > 0$ and
(2) $\mu_{\Gamma_z} = \hat{\mu}_f$ for some Hölder continuous summable potential f defined on the symbol space $E^{\mathbb{N}}$ generated by the IFS $\mathcal{S}$.
(3) There exist constants $\alpha > 0$ and $C \geqslant 1$ such that

$$\mu(B(z, r)) \leqslant Cr^{\alpha} \tag{5.7}$$

for all radii $r > 0$.

Then for the dynamical system $\left(T, \mu\right)$ we have that

$$\limsup_{r \to 0} \tau_{B(z,r)}(x) \cdot \mu(B(z, r)) = +\infty, \quad \text{for } \hat{\mu}_f - \text{a.e. } x \in \mathring{J}_{\mathcal{S}}. \tag{5.8}$$

6. Examples

In this section we describe several classes of examples that fulfil the hypotheses of theorem 17. Given a measurable dynamical system $T\colon X \to X$ preserving a probability measure μ, a measurable set $F \subset X$ with $\mu(F) > 0$ and a function $g\colon X \to \mathbb{R}$, we define the function

$$g_F\colon X \to \mathbb{R}$$

by the formula

$$g_F(x) := \sum_{k=0}^{\tau_F(x)-1} g \circ T^k(x). \tag{6.1}$$

Example A. The first class of examples is formed by distance expanding maps T considered in section 2 along with invariant measures being equilibrium/Gibbs states of Hölder continuous potentials ψ. Indeed, fixing an element R of a Markov partition and taking as the function f of theorem 17 the function $\left(\psi - \mathrm{P}(\psi)_R\right)$ given by (6.1), because of formula (2.4) the hypotheses of theorem 17 are satisfied for all points z, and this theorem applies to give us the desired limsup result. So, we have rediscovered theorem 2 from a more general, but simultaneously, a more complex perspective.

Example B. We now shall describe a large class of dynamical systems being multimodal smooth maps of an interval for which theorem 17 applies.

We start with the definition of the class of dynamical systems and potentials we consider.

Definition 18. Let $I = [0, 1]$ be the closed interval. Let $T\colon I \to I$ be a C^3 differentiable map with the following properties:

(a) T has only finitely many maximal closed intervals of monotonicity; or equivalently $\mathrm{Crit}(T) = \{x \in I\colon\ T'(x) = 0\}$, the set of all critical points of T, is finite.
(b) The dynamical system $T\colon I \to I$ is topologically exact, meaning that for every non-empty subset U of I there exists an integer $n \geqslant 0$ such that $T^n(U) = I$.
(c) All critical points are non-flat.
(d) T is a *topological Collet–Eckmann map*, meaning that

$$\inf\left\{ \left(|(T^n)'(x)|\right)^{1/n} : T^n(x) = x \text{for } n \geqslant 1\right\} > 1$$

where the infimum is taken over all integers $n \geqslant 1$ and all fixed points of T^n.

We then call T a topologically exact topological Collet–Eckmann map (*abbr.* teTCE). If (c) and (d) are relaxed and only (a) and (b) are assumed then T is called a topologically exact multimodal map.

We next recall the following definition.

Definition 19 An interval $V \subset I$ is called a *nice set* for a multimodal map $T\colon I \to I$ if

$$\mathrm{int}(V) \cap \bigcup_{n=0}^{\infty} T^n(\partial V) = \emptyset.$$

The proof of the following theorem is both standard and straightforward; it can be found in many sources, see e.g. [23].

Theorem 20. *If $T\colon I \to I$ is topologically exact multimodal map then for every point $\xi \in (0, 1)$ and every $R > 0$ there exists a nice set $V \subset I$ such that $\xi \in V \subset B(\xi, R)$.*

We set

$$\mathrm{PC}(T) := \bigcup_{n=1}^{\infty} T^n(\mathrm{Crit}(T))$$

and call this the *postcritical* set of T. We say that the map $T\colon I \to I$ is *tame* if

$$\overline{\mathrm{PC}(T)} \neq I.$$

Given a set $F \subset I$ and an integer $n \geqslant 0$, we denote by $\mathcal{C}_F(n)$ the collection of all connected components of $T^{-n}(F)$. From their definitions, nice sets enjoy the following property.

Theorem 21. *If V is a nice set for a multimodal map $T\colon I \to I$, then for every integer $n \geqslant 0$ and every $U \in \mathcal{C}_V(n)$ either $U \cap V = \emptyset$* or *$U \subset V$.*

From now on throughout this section we assume that $T\colon I \to I$ is a tame teTCE map. Fix a point $\xi \in I \backslash \overline{\mathrm{PC}(T)}$. By virtue of theorem 20 there is a nice set V such that

$$\xi \in V \text{ and } 2V \cap \overline{\mathrm{PC}(T)} = \emptyset.$$

The nice set V canonically gives rise to a countable alphabet conformal IFS in the sense considered in the previous sections of the present paper. Namely, put

$$\mathcal{C}_V^* := \bigcup_{n=1}^{\infty} \mathcal{C}_V(n).$$

For every $U \in \mathcal{C}_V^*$ let $\tau_V(U) \geqslant 1$ the unique integer $n \geqslant 1$ such that $U \in \mathcal{C}_V(n)$. Put further

$$\phi_U := f_U^{-\tau_V(U)} \colon V \to U$$

and keep in mind that

$$\phi_U(V) = U.$$

Denote by E_V the subset of all elements U of $\mathcal{C}_V^*$ such that

(a) $\phi_U(V) \subset V$,
(b) $f^k(U) \cap V = \emptyset$ for all $k = 1, 2, \ldots, \tau_V(U) - 1$.

Theorem 22. *If V is a nice set for a tame teTCE map $T \colon I \to I$ with $\overline{V} \cap \overline{\mathrm{PC}(T)} = \emptyset$, then the collection*

$$\mathcal{S}_V := \{\phi_U \colon V \to V\}$$

of all such inverse branches obviously forms a conformal IFS satisfying the strong open set condition. In particular, the elements of $\mathcal{S}_V$ are formed by all inverse branches of the first return map $f_V \colon V \to V$.

The following theorem collects together some fundamental results of [9, 10, 15] telling us how nice is the dynamical system generated by the map T and potential ψ.

Theorem 23. *If $T \colon I \to I$ is a topologically exact multimodal map and $\psi \colon I \to \mathbb{R}$ is a Hölder continuous potential, then*

(a) there exists a Borel probability eigenmeasure m_ψ for the dual operator $\mathcal{L}_\psi^*$ whose corresponding eigenvalue is equal to $e^{\mathrm{P}(\psi)}$. It then follows that $\mathrm{supp}(m_\psi) = I$.
(b) there exists a unique Borel T-invariant probability measure μ_ψ on I absolutely continuous with respect to m_ψ. Furthermore, μ_ψ is equivalent to m_ψ.

Of our most direct interest is item (b) of this theorem which induces a metric dynamical system $(T \colon I \to I, \mu_\psi)$. Because of theorem 22 above and proposition 5.4.8 in [23], our theorem 17 applies for this dynamical system, and along with proposition 13, it yields the following.

Theorem 24. *If $T \colon I \to I$ is a tame teTCE map and $\psi \colon I \to \mathbb{R}$ is a Hölder continuous potential, then for the dynamical system $\big(T \colon I \to I, \mu_\psi\big)$, we have that*

$$\limsup_{r \to 0} \tau_{B(y,r)}(x) \cdot \mu_\psi(B(y,r)) = +\infty, \tag{6.2}$$

μ_ψ-a.e. $y \in I$ and μ_ψ-a.e. $x \in I$.

Example C. We now shall describe a large class of dynamical systems being rational functions of the Riemann sphere $\widehat{\mathbb{C}}$ for which theorem 17 applies. Let $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree $d \geqslant 2$. Let $J(f)$ denote the Julia set of f. Let $\psi \colon \widehat{\mathbb{C}} \to \mathbb{R}$ be a Hölder continuous potential. We say that $\psi\colon \widehat{\mathbb{C}} \to \mathbb{R}$ has a *pressure gap* if

$$n\mathrm{P}(\psi) - \sup\left(\psi_n\right) > 0 \tag{6.3}$$

for some integer $n \geqslant 1$, where $\mathrm{P}(\psi)$ denotes the ordinary topological pressure of $\psi|_{J(f)}$ and the Birkhoff's nth sum ψ_n is also considered as restricted to $J(f)$.

We would like to mention that (6.3) always holds (with all $n \geqslant 0$ sufficiently large) if the function $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ restricted to its Julia set is expanding, which is then also frequently referred to as hyperbolic.

The probability invariant measure we are interested in comes from the following.

Theorem 25 ([4]). *If $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function of degree $d \geqslant 2$ and if $\psi\colon \widehat{\mathbb{C}} \to \mathbb{R}$ is a Hölder continuous potential with a pressure gap, then ψ admits a unique equilibrium state μ_ψ, i.e. a unique Borel probability f-invariant measure on $J(f)$ such that*

$$\mathrm{P}(\psi) = \mathrm{h}_{\mu_\psi}(f) + \int_{J(f)} \psi \, \mathrm{d}\mu_\psi.$$

In addition,

(a) The measure μ_ψ is ergodic, in fact K-mixing, and (see [28]) enjoys further finer stochastic properties.
(b) The Jacobian

$$J(f) \ni z \longmapsto \frac{\mathrm{d}\mu_\psi \circ f}{\mathrm{d}\mu_\psi}(z) \in (0, +\infty)$$

is a Hölder continuous function.

Let

$$\mathrm{Crit}(f) := \{c \in \widehat{\mathbb{C}} : f'(c) = 0\}$$

be the set of all critical (branching) points of f. As in the case of interval maps set

$$\mathrm{PC}(f) := \bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}(f))$$

and call it the postcritical set of f.

In [24], as in the previous class of examples, a rational function $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ was called *tame* if

$$J(f) \setminus \overline{\mathrm{PC}(f)} \neq \emptyset.$$

Likewise, following [27], we adopt the same definition for (transcendental) meromorphic functions $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$.

Remark 26. Tameness is a very mild hypothesis and there are many classes of maps for which these hold. These include:

(1) Quadratic maps $z \mapsto z^2 + c$ for which the Julia set is not contained in the real line;
(2) Rational maps for which the restriction to the Julia set is expansive which includes the case of expanding rational functions;

(3) Misiurewicz maps, where the critical point is not recurrent.

The main advantage of dealing with tame functions is that these admit nice sets. Analogously as in the case of interval maps, given a set $F \subset \widehat{\mathbb{C}}$ and $n \geqslant 0$, we denote by $\mathcal{C}_F(n)$ the collection of all connected components of $f^{-n}(F)$. Rivera–Letelier introduced in [26] the concept of nice sets in the realm of the dynamics of rational maps of the Riemann sphere. In [5] N. Dobbs proved their existence for tame meromorphic functions from $\mathbb{C}$ to $\widehat{\mathbb{C}}$. We quote now his theorem.

Theorem 27. *Let $f: \mathbb{C} \to \widehat{\mathbb{C}}$ be a tame meromorphic function. Fix a non-periodic point $z \in J(f) \setminus \overline{\mathrm{PC}(f)}$, $\kappa > 1$, and $K > 1$. Then for all $L > 1$ and for all $r > 0$ sufficiently small there exists an open connected set $V = V(z, r) \subset \mathbb{C} \setminus \overline{\mathrm{PC}(f)}$ such that*

(a) If $U \in \mathcal{C}_V(n)$ and $U \cap V \neq \emptyset$, then $U \subseteq V$.
(b) If $U \in \mathcal{C}_V(n)$ and $U \cap v \neq \emptyset$, then, for all $w, w' \in U$,

$$|(f^n)'(w)| \geqslant L \text{ and } \frac{|(f^n)'(w)|}{|(f^n)'(w')|} \leqslant K.$$

(c) $\overline{B(z, r)} \subset U \subset B(z, \kappa r) \subset \mathbb{C} \setminus \overline{\mathrm{PC}(f)}$.

We now follow the same procedure as in the previous example, see paragraph leading to theorem 22. Define $\mathcal{C}_V^*$, $\tau_V(U)$ and finally ϕ_U. Identically as before we arrive at

Theorem 28. *With hypotheses of theorem 27, the collection*

$$\mathcal{S}_V := \{\phi_U : V \to V\}$$

of all inverse branches forms a conformal IFS. In other words the elements of $\mathcal{S}_V$ are formed by all holomorphic inverse branches of the first return map $f_V : V \to V$. In particular, $\tau_V(U)$ is the first return time of all points in $U = \phi_U(V)$ to V.

As in the case of interval maps, we call a rational function $f: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ topological Collet–Eckmann (*abbr.* TCE) if

$$\inf\left\{ \left(|(f^n)'(x)|\right)^{1/n} : f^n(x) = x \text{ for all } n \geqslant 1\right\} > 1$$

where the infimum is taken over all integers $n \geqslant 1$ and all fixed points of T^n. There are several other useful characterizations of TCE rational functions, most notably the one commonly referred to as the exponential shrinking property, but we do not really need them in this paper. We can now easily prove the following.

Theorem 29. *If $f: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a tame TCE rational function and $\psi: \widehat{\mathbb{C}} \to \mathbb{R}$ is a Hölder continuous potential then, for the dynamical system $\left(f: J(f) \to J(f), \mu_\psi\right)$, we have that*

$$\limsup_{r \to 0} \tau_{B(y,r)}(x) \cdot \mu_\psi(B(y, r)) = +\infty \tag{6.4}$$

for μ_ψ-a.e. $y \in J(f)$ and μ_ψ-a.e. $x \in J(f)$.

Proof. Since f is a TCE rational function and $\psi: \widehat{\mathbb{C}} \to \mathbb{R}$ is a Hölder continuous potential, this potential has a pressure gap because of corollary 1.2 in [12]. So, because of theorems 28 and 25 above, and also because of proposition 5.4.8 in [23], our theorem 17 applies for the dynamical system $\left(f: J(f) \to J(f), \mu_\psi\right)$. Along with proposition 13, this completes the proof. □

Example D. Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function. Let $\mathrm{Sing}(f^{-1})$ be the set of all singular points of f^{-1}, i.e. the set of all points $w \in \widehat{\mathbb{C}}$ such that if W is any open connected neighborhood of w, then there exists a connected component U of $f^{-1}(W)$ such that the map $f\colon U \to W$ is not bijective. Of course, if f is a rational function, then $\mathrm{Sing}(f^{-1}) = f(\mathrm{Crit}(f))$. As in the case of rational functions, we define

$$\mathrm{PS}(f) := \bigcup_{n=0}^{\infty} f^n(\mathrm{Sing}(f^{-1})).$$

The function f is called *topologically hyperbolic* if

$$\mathrm{dist}_{\mathrm{Euclid}}(J_f, \mathrm{PS}(f)) > 0,$$

and it is called *expanding* if there exist $c > 0$ and $\lambda > 1$ such that

$$|(f^n)'(z)| \geqslant c\lambda^n$$

for all integers $n \geqslant 1$ and all points $z \in J_f \setminus f^{-n}(\infty)$. Note that every topologically hyperbolic meromorphic function is tame. A meromorphic function that is both topologically hyperbolic and expanding is called *hyperbolic*. The meromorphic function $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is called dynamically *semi-regular* if it is of finite order, commonly denoted by ρ_f, and satisfies the following rapid growth condition for its derivative.

$$|f'(z)| \geqslant \kappa^{-1}(1+|z|)^{\alpha_1}(1+|f(z)|)^{\alpha_2}, \quad z \in J_f, \tag{6.5}$$

with some constant $\kappa > 0$ and α_1, α_2 such that $\alpha_2 > \max\{-\alpha_1, 0\}$. Set $\alpha := \alpha_1 + \alpha_2$.

Remark 30. A particularly simple example of such maps are entire functions $f_\lambda(z) = \lambda e^z$ where $\lambda \in (0, 1/e)$ since these maps have an attracting periodic point. A good reference is [18].

Let $h\colon J_f \to \mathbb{R}$ be a weakly Hölder continuous function in the sense of [19]. The definition, introduced therein, is somewhat technical and we will not provide it here; the simplest example of a weakly Hölder continuous function is the function identically equal to zero. The corresponding function $\psi_{t,0}$ is by no means trivial. Furthermore, each bounded, uniformly locally Hölder function $h\colon J_f \to \mathbb{R}$ is weakly Hölder. Fix $\tau > \alpha_2$ as required in [19]. For $t \in \mathbb{R}$, let

$$\psi_{t,h} := -t \log |f'|_\tau + h \tag{6.6}$$

where $|f'(z)|_\tau$ is the norm, or—equivalently—the scaling factor, of the derivative of f evaluated at a point $z \in J_f$ with respect to the Riemannian metric

$$|\mathrm{d}\tau(z)| = (1+|z|)^{-\tau}|\mathrm{d}z|.$$

Following [19] functions (potentials) of the form (6.6) are called *loosely tame*. Let $\mathcal{L}_{t,h}\colon C_b(J_f) \to C_b(J_f)$ be the corresponding *Perron–Frobenius operator* given by the formula

$$\mathcal{L}_{t,h}g(z) := \sum_{w \in f^{-1}(z)} g(w)e^{\psi_{t,h}(w)}.$$

It was shown in [19] that, for every $z \in J_f$ and for the function $\mathbb{1}\colon z \mapsto 1$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_{t,h} \mathbb{1}(z)$$

exists and takes on the same common value, which we denote by $\mathrm{P}(t)$ and call *the topological pressure* of the potential ψ_t. The following theorem was proved in [19].

Theorem 31. *If $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is a dynamically semi-regular meromorphic function and $h\colon J_f \to \mathbb{R}$ is a weakly Hölder continuous potential, then for every $t > \rho_f/\alpha$ there exist uniquely determined Borel probability measures $m_{t,h}$ and $\mu_{t,h}$ on J_f with the following properties.*

(a) $\mathcal{L}^*_{t,h} m_{t,h} = m_{t,h}$.

(b) $\mathrm{P}\big(\psi_{t,h}\big) = \sup\left\{\mathrm{h}_\mu(f) + \int \psi_{t,h}\, \mathrm{d}\mu : \mu \circ f^{-1} = \mu \quad \text{and} \quad \int \psi_{t,h}\, \mathrm{d}\mu > -\infty\right\}$.

(c) $\mu_{t,h} \circ f^{-1} = \mu_{t,h}$, $\int \psi_{t,h}\, \mathrm{d}\mu_{t,h} > -\infty$, *and* $\mathrm{h}_{\mu_{t,h}}(f) + \int \psi_{t,h}\, \mathrm{d}\mu_{t,h} = \mathrm{P}\big(\psi_{t,h}\big)$.

(d) The measures $\mu_{t,h}$ and $m_{t,h}$ are equivalent and the Radon–Nikodym derivative $\frac{\mathrm{d}\mu_{t,h}}{dm_{t,h}}$ has a nowhere-vanishing Hölder continuous version which is bounded above.

Theorem 27 of course holds and so do the analogs of theorem 28 and proposition 5.5.7 in [23]. Thus our theorem 17 applies for the dynamical system $\big(f\colon J(f) \to J(f), \mu_{t,h}\big)$, and along with proposition 13, it yields the following.

Theorem 32. *Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a dynamically semi-regular meromorphic function and let $h\colon J_f \to \mathbb{R}$ be a weakly Hölder continuous potential. If $t > \rho_f/\alpha$, then for the dynamical system $\big(f\colon J(f) \to J(f), \mu_{t,h}\big)$, we have that*

$$\limsup_{r\to 0} \tau_{B(y,r)}(x) \cdot \mu_{t,h}(B(y,r)) = +\infty, \tag{6.7}$$

for $\mu_{t,h}$-a.e. $y \in J(f)$ and $\mu_{t,h}$-a.e. $x \in J(f)$.

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