

GUP corrections to the Dirac oscillator in the external magnetic field

VISHAKHA TYAGI^{1(a)}, SUMIT KUMAR RAI^{2(b)} and BHABANI PRASAD MANDAL^{3(c)}

¹ Department of Physics, Banasthali Vidyapeeth - Rajasthan, 304022, India

² Sardar Vallabh Bhai Patel College (Veer Kunwar Singh University) - Bhabua, 821101, India

³ Department of Physics, Banaras Hindu University - Varanasi, 221005, India

received 29 September 2019; accepted in final form 17 November 2019

published online 24 January 2020

PACS 03.65.Pm – Relativistic wave equations

PACS 03.65.Ge – Solutions of wave equations: bound states

PACS 04.60.Bc – Phenomenology of quantum gravity

Abstract – We have studied the (2+1)-dimensional Dirac oscillator (DO) in an external magnetic field in the framework of the generalized uncertainty principle (GUP). We have calculated the perturbative corrections for the first few energy levels. We show that the infinite degeneracy of lowest Landau level is partially lifted due to GUP correction and obtain a critical value of the magnetic field for which there is no GUP correction and the DO stops oscillating.

Copyright © EPLA, 2020

Introduction. – According to the Heisenberg Uncertainty Principle (HUP) the position and momentum cannot be measured simultaneously and arbitrarily accurate. In HUP, it is assumed that the position continuously varies from $-\infty$ to $+\infty$. However, with the emergence of various theories on quantum gravity and black hole physics, it has been predicted the existence of a certain minimum measurable length called the Planck length ($l_{pl} \approx 10^{-35}$ m) [1–7]. The restriction $\Delta x \geq l_{pl}$ leads to the modification in HUP, known as the generalized uncertainty principle (GUP) [4,8–12]. The modified commutation relations for the GUP, which are consistent with string theories, black hole physics, doubly special relativity (DSR), etc., theories, have been constructed in various ways [6,11,13–16]. The simplest and commonly used commutation relation in GUP is

$$[x_i, p_j] = i\hbar \left(\delta_{ij} - a \left(p\delta_{ij} + \frac{p_i p_j}{p} \right) + a^2 (p^2 \delta_{ij} + 3p_i p_j) \dots \right), \quad (1)$$

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad (2)$$

where $a = \frac{a_0}{M_{pl}c} = \frac{a_0 l_{pl}}{\hbar}$. M_{pl} is the Planck mass and $M_{pl}c^2 \approx 10^{19}$ GeV is the Planck energy. The order of

a_0 is assumed to be unity. The generalized commutation relation in eq. (1) can be expressed equivalently as [7]

$$[x_i, p_j] = i\hbar(\delta_{ij} + a\delta_{ij}p^2 + 2ap_i p_j), \quad (3)$$

where $p^2 = \sum_{j=1}^3 p_j p_j$. The position and momentum operators in GUP have been further expressed as follows:

$$x_i = x_{0i}, \quad p_i = p_{0i}(1 - ap_0 + 2a^2 p_0^2), \quad (4)$$

where p_{0i} is the momentum operator at low energies. x_{0i} and p_{0i} satisfy the usual canonical commutation relations $[x_{0i}, p_{0i}] = i\hbar\delta_{ij}$. The position operator x_i and momentum operator p_i in GUP are known as high energy, position and momentum operators, respectively. a -dependent terms in eq. (3) become crucial only at momentum (energy) scale comparable to Planck energy scales.

GUP and its consequences have been extensively studied over the last few years [3,14,17–42]. Klein-Gordon [30] and Dirac equations [22,31,36–38] have been modified under GUP corrections. The modified Dirac equation has found its application in graphene [43]. GUP has found wide applications in various non-relativistic quantum systems, such as a particle in a box, simple harmonic oscillator, Landau levels [7,32,39,44], coherent and squeezed states [26–29], PT symmetric non-Hermitian systems [14,23–25], Schwinger's model of angular momentum [45,46], etc. GUP has led to the discretization of time which resembles a crystal lattice in time known as

^(a)E-mail: tyagi.vishakha17@gmail.com

^(b)E-mail: sumitssc@gmail.com

^(c)E-mails: bhabani@bhu.ac.in; bhabani.mandal@gmail.com

time crystals [47]. In the recent years, quantum gravity effects have been studied in the field of astrophysics and cosmology, such as black holes and their thermodynamics [48–54], mass-radius relation of white dwarfs [55], gravitational field [56,57], deformation of Wheeler-DeWitt equation [58,59], de Sitter universe [60,61], non-singular and cyclic universe [62], etc. Its effects have also been studied in the gravitational wave event [63]. The consequences of GUP have been extended to field theories as well [64–68], such as non-local field theories [64], no cloning theorem [65] and Lifshitz field theories [66], supersymmetric field theories [67] and topological defects in deformed gauge theory [68]. The path integral approach to quantization has also been studied [69]. The supersymmetry breaking has been found as a new source of the GUP and the relation between GUP and Lee-Wick field theories has been established [70].

Ito *et al.* [71] studied the Dirac equation by adding a linear harmonic potential $-imc\omega\beta\alpha\cdot\mathbf{r}$ and later it was found that in the non-relativistic limit, it reduces to a harmonic oscillator with a strong spin orbit coupling and so referred to as Dirac oscillator by Moshinsky and Szczepaniak [72]. The Dirac Hamiltonian for a free particle with a linear harmonic potential term is written as

$$H = c\alpha \cdot (\mathbf{p} - im\omega\beta\mathbf{r}) + \beta mc^2. \quad (5)$$

α and β are the usual Dirac matrices. m, c and ω are the rest mass of the particle, speed of light and oscillator frequency, respectively. The Dirac oscillator has wide applications in various branches of physics [73–90]. In the recent years, the Dirac oscillator has been studied in the framework of GUP correction. It has been shown that the Dirac oscillator in a magnetic field with GUP corrections in ordinary quantum mechanics has a single left-right chiral quantum phase transition. The existence of GUP correction modifies the degeneracy of states and some of the states do not exist in the ordinary quantum mechanics limit [22,91].

In this letter, we solve the $(2+1)$ -dimensional Dirac equation in the presence of an external magnetic field with GUP correction using the perturbation theory and we investigate the behaviour of the system at critically high magnetic field. For that purpose, we first calculate the modified Hamiltonian for DO in the framework of GUP up to first-order corrections. The additional momentum-dependent term arises due to GUP correction, then it deals with perturbation. Degeneracy of Landau levels increases due to GUP corrections. We obtain a critical value of magnetic field for which there are no GUP corrections and the dynamics of DO stops.

In an earlier work [92], we showed that the $(2+1)$ -dimensional Dirac oscillator in an external magnetic field can be mapped onto the Dirac oscillator without magnetic field but with the reduced angular frequency given by the

following equation [92,93]:

$$H = c\alpha \cdot (\mathbf{p} - im\tilde{\omega}\beta\mathbf{r}) + \beta mc^2, \quad \text{where} \quad \tilde{\omega} = \omega - \frac{\omega_c}{2}, \quad (6)$$

where the angular frequency $\tilde{\omega}$ is reduced by half the value of cyclotron frequency $\omega_c = \frac{|e|B}{mc}$. The momentum \mathbf{p} in the above equation is the high energy momenta. The magnetic field B is taken along the z -direction for convenience. \mathbf{A} is the vector potential chosen in the symmetric gauge as $\mathbf{A} = (-\frac{B}{2}y, \frac{B}{2}x, 0)$ and e is the charge of the Dirac oscillator. The Dirac Hamiltonian in eq. (6) can be expressed in the matrix form as

$$H = \begin{pmatrix} mc^2 & 2cp_z + im\tilde{\omega}c\bar{z} \\ 2cp_{\bar{z}} + im\tilde{\omega}cz & -mc^2 \end{pmatrix}, \quad (7)$$

where the momenta p_z and $p_{\bar{z}}$ are defined as follows:

$$p_z = -i\hbar \frac{d}{dz} = \frac{1}{2}(p_x - ip_y), \quad (8)$$

$$p_{\bar{z}} = -i\hbar \frac{d}{d\bar{z}} = \frac{1}{2}(p_x + ip_y), \quad (9)$$

with $[z, p_z] = i\hbar = [\bar{z}, p_{\bar{z}}]$, $[z, p_{\bar{z}}] = 0 = [\bar{z}, p_z]$ and hence $p^2 = 4p_z p_{\bar{z}}$. The GUP modified Dirac oscillator in eq. (6) up to the first order in a can be expressed as [94]

$$\begin{aligned} H\psi &= (c\alpha \cdot (\mathbf{p} - im\tilde{\omega}\beta\mathbf{r}) + \beta mc^2)\psi, \\ &= [c(\alpha \cdot \mathbf{p}_0 - a(\alpha \cdot \mathbf{p}_0)(\alpha \cdot \mathbf{p}_0) \\ &\quad - \alpha \cdot (im\tilde{\omega}\beta\mathbf{r})) + \beta mc^2]\psi, \\ &= E\psi. \end{aligned} \quad (10)$$

We drop the subscript from the position and momentum operator and henceforth all the variables are low-energy variables which satisfy usual commutation relations.

The usual Hamiltonian in eq. (6) with the GUP correction can be expressed as the following Hamiltonian:

$$\begin{aligned} H &= \begin{pmatrix} mc^2 - acp^2 & 2cp_z + im\tilde{\omega}c\bar{z} \\ 2cp_{\bar{z}} + im\tilde{\omega}cz & -mc^2 - acp^2 \end{pmatrix} \\ &= \begin{pmatrix} mc^2 & 2cp_z + im\tilde{\omega}c\bar{z} \\ 2cp_{\bar{z}} + im\tilde{\omega}cz & -mc^2 \end{pmatrix} - a \begin{pmatrix} cp^2 & 0 \\ 0 & cp^2 \end{pmatrix}, \end{aligned} \quad (11)$$

where the two component $|\psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}$. The above Hamiltonian can be expressed as

$$H = H_0 + H', \quad (12)$$

where H_0 is the usual Hamiltonian. The Dirac equation for H_0 has been solved by writing $|\psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}$ and

$$H' = - \begin{pmatrix} acp^2 & 0 \\ 0 & acp^2 \end{pmatrix} \quad (13)$$

is dealt with perturbative techniques. The normalized negative and positive energy states for H_0 are given by [92]

$$|\psi_n^\pm\rangle = c_n^\pm \left| n; \frac{1}{2} \right\rangle + d_n^\pm \left| n-1; \frac{-1}{2} \right\rangle; \quad (14)$$

$$\begin{pmatrix} \langle \psi_{200} | H' | \psi_{200} \rangle & \langle \psi_{200} | H' | \psi_{210} \rangle & \langle \psi_{200} | H' | \psi_{211} \rangle & \langle \psi_{200} | H' | \psi_{21-1} \rangle \\ \langle \psi_{210} | H' | \psi_{200} \rangle & \langle \psi_{210} | H' | \psi_{210} \rangle & \langle \psi_{210} | H' | \psi_{211} \rangle & \langle \psi_{210} | H' | \psi_{21-1} \rangle \\ \langle \psi_{211} | H' | \psi_{200} \rangle & \langle \psi_{211} | H' | \psi_{210} \rangle & \langle \psi_{211} | H' | \psi_{211} \rangle & \langle \psi_{211} | H' | \psi_{21-1} \rangle \\ \langle \psi_{21-1} | H' | \psi_{200} \rangle & \langle \psi_{21-1} | H' | \psi_{210} \rangle & \langle \psi_{21-1} | H' | \psi_{211} \rangle & \langle \psi_{21-1} | H' | \psi_{21-1} \rangle \end{pmatrix} \\ = -\frac{acm\tilde{\omega}\hbar}{2} \begin{pmatrix} 11 & -5 & -5 & -5 \\ -5 & 11 & -5 & -5 \\ -5 & -5 & 13 & -5 \\ -5 & -5 & -5 & 9 \end{pmatrix} \quad (23)$$

where the coefficients $c_n^\pm = \pm \sqrt{\frac{E_n^\pm \pm mc^2}{2E_n^\pm}}$ and $d_n^\pm = \sqrt{\frac{E_n^\pm \mp mc^2}{2E_n^\pm}}$. In eq. (14), the notation adopted is $|n, \frac{1}{2}m_s\rangle \approx \psi_n(z, \bar{z})\Phi_{m_s}$. $\psi_n(z, \bar{z})$ is the spatial part of the wave function, whereas Φ_{m_s} is the spin part of the wave function. n is the eigenvalue of the number operator $a^\dagger a$ and $m_s = \pm 1$ are the eigenvalues of the spin operator σ_z . When the creation and annihilation operators are defined as [92]

$$a = \frac{1}{\sqrt{m\tilde{\omega}\hbar}}p_{\bar{z}} - \frac{i}{2}\sqrt{\frac{m\tilde{\omega}}{\hbar}}z, \quad (15)$$

$$a^\dagger = \frac{1}{\sqrt{m\tilde{\omega}\hbar}}p_z - \frac{i}{2}\sqrt{\frac{m\tilde{\omega}}{\hbar}}\bar{z},$$

the relativistic Landau levels due to the unperturbed Hamiltonian H_0 are given by [92]

$$E_n^\pm = \pm mc^2 \sqrt{1 + \frac{4\hbar\tilde{\omega}}{mc^2}n}, \quad n = 0, 1, 2, 3, \dots \quad (16)$$

For $n = 0$, the state is $|\psi_0^\pm\rangle = c_0^\pm|0, \frac{1}{2}\rangle$. The GUP corrected part H' can be dealt with perturbative techniques as it is proportional to a which is extremely small. Using eq. (8) and eq. (15), p^2 can be expressed as

$$4p_z p_{\bar{z}} = 2m\tilde{\omega}\hbar \left[a^\dagger a + a a^\dagger - \frac{m\tilde{\omega}}{2\hbar} z \bar{z} + \frac{\hat{L}_z}{\hbar} \right]. \quad (17)$$

The first-order correction to the n -th state is given by

$$\begin{aligned} \langle \psi_n | 4p_z p_{\bar{z}} | \psi_n \rangle &= 2m\tilde{\omega}\hbar \langle \psi_n | a^\dagger a + a a^\dagger | \psi_n \rangle \\ &- (m\tilde{\omega})^2 \langle \psi_n | z \bar{z} | \psi_n \rangle + 2m\tilde{\omega} \langle \psi_n | \hat{L}_z | \psi_n \rangle. \end{aligned} \quad (18)$$

It is difficult to obtain a general expression for the correction in the n -th level due to degeneracy of the levels $n \geq 2$.

The ground-state energy correction arising out of the perturbative Hamiltonian H' for $n = 0$ in eq. (18) can be obtained as

$$\begin{aligned} \langle \psi_0 | acp^2 | \psi_0 \rangle &= ac[2m\tilde{\omega}\hbar \langle \psi_0 | a^\dagger a + a a^\dagger | \psi_0 \rangle \\ &- (m\tilde{\omega})^2 \langle \psi_0 | z \bar{z} | \psi_0 \rangle]. \end{aligned} \quad (19)$$

For the ground state $l = 0$, hence there is no contribution from the last term. The space part of the ground-state

wave function $\psi_0 = C_0 \exp^{-\frac{m\tilde{\omega}}{2\hbar} z \bar{z}}$ is used to calculate the middle term in eq. (18). The correction to the ground state is then given by

$$E'_0 = -acm\tilde{\omega}\hbar. \quad (20)$$

Using eqs. (14) and (18), one can obtain the energy correction for the first excited state as

$$\langle \psi_1 | 4p_z p_{\bar{z}} | \psi_1 \rangle = -\frac{5m\tilde{\omega}\hbar}{2} - 2m\tilde{\omega} \langle \psi_1 | \hat{L}_z | \psi_1 \rangle, \quad (21)$$

$$E'_1 = -\frac{5acm\tilde{\omega}\hbar}{2}. \quad (22)$$

The second excited state is a 4-fold degenerate, therefore we need to use the degenerate perturbation theory to calculate the correction for these states. We write the matrix as

see eq. (23) above

The perturbative energy corrections are obtained by diagonalizing the above matrix. The energy eigenvalues obtained are

$$\begin{aligned} E_2'^{(1)} &= -8.7308acm\tilde{\omega}\hbar; & E_2'^{(2)} &= -8acm\tilde{\omega}\hbar, \\ E_2'^{(3)} &= -7.3192acm\tilde{\omega}\hbar; & E_2'^{(4)} &= 2.05acm\tilde{\omega}\hbar. \end{aligned} \quad (24)$$

The corresponding eigenvectors are as follows:

$$\begin{pmatrix} |\psi_2\rangle_1 \\ |\psi_2\rangle_2 \\ |\psi_2\rangle_3 \\ |\psi_2\rangle_4 \end{pmatrix} = \begin{pmatrix} 2.36839 & 2.36839 & -6.42909 & 1 \\ -1 & 1 & 0 & 0 \\ -0.468884 & -0.468884 & -0.189917 & 1 \\ 0.900498 & 0.900498 & 0.819005 & 1 \end{pmatrix} \times \begin{pmatrix} |\psi_{200}\rangle \\ |\psi_{210}\rangle \\ |\psi_{211}\rangle \\ |\psi_{21-1}\rangle \end{pmatrix}. \quad (25)$$

In conclusion, we have studied the consequences of the generalized uncertainty principle in the modified $(2+1)$ -dimensional Dirac oscillator in the presence of a magnetic field. The perturbative corrections up to the first few energy levels have been explicitly calculated. The degeneracy of Landau levels are partially lifted due to GUP corrections. An interesting situation occurs for a critical value of the magnetic field, i.e., $B = \frac{2\omega mc}{|e|}$, when $\tilde{\omega} = 0$.

The Dirac oscillator stops oscillating even with the GUP corrections.

Recently the discreteness of space is shown to arise in the Schwarzschild metric due to the GUP correction but still we do not have it in the most generic curved space time [95,96]. The Dirac oscillator in the context of the gravity rainbow has been studied where the modified energy levels have been obtained [97]. It would be very interesting to extend our approach to the most generic curved space time and to the gravity rainbow for future studies.

* * *

BPM acknowledges the support from MATRIX project (Grant No. MTR/2018/000611), SERB, DST Govt. of India.

REFERENCES

- [1] GROSS D. J. and MENDE P. F., *Phys. Lett. B*, **197** (1987) 129.
- [2] GROSS D. J. and MENDE P. F., *Nucl. Phys. B*, **303** (1988) 407.
- [3] TAWFIK A. and DIAB A., *Int. J. Mod. Phys. D*, **23** (2014) 1430025.
- [4] AMATI D., CIAFALONI M. and VENEZIANO G., *Phys. Lett. B*, **216** (1989) 41.
- [5] KONISHI K., PAFFUTI G. and PROVERO P., *Phys. Lett. B*, **234** (1990) 276.
- [6] ALI A. F., DAS S. and VAGENAS E. C., *Phys. Lett. B*, **678** (2009) 497.
- [7] ALI A. F., DAS S. and VAGENAS E. C., *Phys. Rev. D*, **84** (2011) 044013.
- [8] MAGGIORE M., *Phys. Lett. B*, **304** (1993) 65.
- [9] GARAY L. J., *Int. J. Mod. Phys. A*, **10** (1995) 145.
- [10] CAMELLA G. A., *Int. J. Mod. Phys. D*, **11** (2002) 1643.
- [11] SCARDIGLI F., *Phys. Lett. B*, **452** (1999) 39.
- [12] MAGGIORE M., *Phys. Lett. B*, **304** (1993) 65.
- [13] KEMPF A., MANGANO G. and MANN R. B., *Phys. Rev. D*, **52** (1995) 1108.
- [14] BHAT A., DEY S., FAIZAL M., HOU C. and ZHAO Q., *Phys. Lett. B*, **770** (2017) 325.
- [15] HINRICHSSEN H. and KEMPF A., *J. Math. Phys.*, **37** (1996) 2121.
- [16] MUREIKA J. R., *Phys. Lett. B*, **789** (2019) 88.
- [17] TAWFIK A. N. and DIAB A. M., *Rep. Prog. Phys.*, **78** (2015) 126001.
- [18] SCARDIGLI F., LAMBIASE G. and VAGENAS E., *Phys. Lett. B*, **767** (2017) 242.
- [19] MAJUMDER B., *Phys. Lett. B*, **699** (2011) 315.
- [20] MAZIASHVILI M., *Phys. Lett. B*, **635** (2006) 232.
- [21] FAIZAL M., ALI A. F. and NASSAR A., *Phys. Lett. B*, **765** (2017) 238.
- [22] MENCULINI L., PANELLA O. and ROY P., *Phys. Rev. D*, **87** (2013) 065017.
- [23] PEDRAM P., *Phys. Lett. B*, **718** (2012) 638.
- [24] FAIZAL M. and MANDAL B. P., *Gravit. Cosmol.*, **21** (2015) 270.
- [25] BAGCHI B. and FRING A., *Phys. Lett. A*, **373** (2009) 4307.
- [26] GHOSH S. and ROY P., *Phys. Lett. B*, **711** (2012) 423.
- [27] DEY S. and HUSSIN V., *Phys. Rev. D*, **91** (2015) 124017.
- [28] DEY S., FRING A., GOUBA L. and CASTRO P. G., *Phys. Rev. D*, **87** (2013) 084033.
- [29] DEY S. and FRING A., *Phys. Rev. D*, **86** (2012) 064038.
- [30] CHEN D., WU H., YANG H. and YANG S., *Int. J. Mod. Phys. A*, **29** (2014) 1430054.
- [31] NOZARI K. and KARAMI M., *Mod. Phys. Lett. A*, **20** (2005) 3095.
- [32] DAS S. and VAGENAS E. C., *Can. J. Phys.*, **87** (2009) 233.
- [33] BOSSO P., DAS S. and MANN R. B., *Phys. Rev. D*, **96** (2017) 066008.
- [34] BOSSO P. and DAS S., *Ann. Phys.*, **383** (2017) 416.
- [35] DAS S., ROBBINS M. P. G. and WALTON M. A., *Can. J. Phys.*, **94** (2016) 139.
- [36] OAKES T. L. A., FRANSISCO R. O., FABRIS J. C. and NOGUEIRA J. A., *Eur. Phys. J. C*, **73** (2013) 2495.
- [37] PEDRAM P., AMIRFAKHRIAN M. and SHABABI H., *Int. J. Mod. Phys. D*, **24** (2015) 1550016.
- [38] FAIZAL M. and KRUGLOV S. I., *Int. J. Mod. Phys. D*, **25** (2016) 1650013.
- [39] MASOOD S., FAIZAL M., ZAZ Z., ALI A. F., RAZA J. and SHAH M. B., *Phys. Lett. B*, **763** (2016) 218.
- [40] KHODADI M., NOZARI K., DEY S., BHAT A. and FAIZAL M., *Sci. Rep.*, **8** (2018) 1659.
- [41] BRAIDOTTI M. C., CONTI C., FAIZAL M., DEY S., ALASFAR L. and ALASALEH S., *EPL*, **124** (2018) 44001.
- [42] FAIZAL M., *Int. J. Geom. Methods Mod. Phys.*, **12** (2015) 1550022.
- [43] IORIO A., PAIS P., ELMASHAD I. A., ALI A. F., FAIZAL M. and ABOU-SALEM L. I., *Int. J. Mod. Phys. D*, **27** (2018) 1850080.
- [44] CASTRO L. B. and OBISPO A. E., *J. Phys. A*, **50** (2017) 285202.
- [45] SAKURAI J. J., *Modern Quantum Mechanics*, Vol. 1 (Pearson) 1994, p. 229.
- [46] VERMA H., MITRA T. and MANDAL B. P., *EPL*, **123** (2018) 30009.
- [47] FAIZAL M., KHALIL M. M. and DAS S., *Eur. Phys. J. C*, **76** (2016) 30.
- [48] GECIM G. and SUCU Y., *Phys. Lett. B*, **773** (2017) 391.
- [49] ONG Y. C., *Phys. Lett. B*, **785** (2018) 217.
- [50] KHODADI M., NOZARI K. and A. HAJIZADEH, *Phys. Lett. B*, **770** (2017) 556.
- [51] CHUNG W. S. and HASSANABADI H., *Phys. Lett. B*, **793** (2019) 451.
- [52] FAIZAL M. and KHALIL M. M., *Int. J. Mod. Phys. A*, **30** (2015) 1550144.
- [53] GANGOPADHYAY S., DUTTA A. and FAIZAL M., *EPL*, **112** (2015) 20006.
- [54] GECIM G. and SUCU Y., *Gen. Relativ. Gravit.*, **50** (2018) 152.
- [55] MATHEW A. and NANDY M. K., *Ann. Phys.*, **393** (2018) 184.
- [56] DEB S., DAS S. and VAGENAS E. C., *Phys. Lett. B*, **755** (2016) 17.
- [57] ZHAO Q., FAIZAL M. and ZAZ Z., *Phys. Lett. B*, **770** (2017) 564.
- [58] FAIZAL M., *Int. J. Mod. Phys. A*, **29** (2014) 1450106.

- [59] GARATTINI R. and FAIZAL M., *Nucl. Phys. B*, **905** (2016) 313.
- [60] MOHAMMADI A., ALI A. F., GOLANBARI T., AGHAMOHAMMADI A., SAAIDI KH. and FAIZAL M., *Ann. Phys.*, **385** (2017) 214.
- [61] ALI A. F., FAIZAL M. and KHALIL M. M., *JCAP*, **09** (2015) 025.
- [62] SALAH M., HAMMAD F., FAIZAL M. and ALI A. F., *JCAP*, **02** (2017) 035.
- [63] FENG ZHONG-WEN, YANG SHU-ZHENG, LI HUI-LING and ZU XIAO-TAO, *Phys. Lett. B*, **768** (2017) 81.
- [64] GIM Y., UM H. and KIM W., *Phys. Lett. B*, **784** (2018) 206.
- [65] VAGENAS E. C., ALI A. F. and ALSHAL H., *Eur. Phys. J. C*, **79** (2019) 276.
- [66] FAIZAL M. and MAJUMDER B., *Ann. Phys.*, **357** (2015) 49.
- [67] ZHAO Q., FAIZAL M., SHAH M. B., BHAT A., GANAI P. A., ZAZ Z., MASOOD S., RAZA J. and IRFAN R. M., *Eur. Phys. J. C*, **77** (2017) 612.
- [68] FAIZAL M. and TSUN T. S., *Nucl. Phys. B*, **924** (2017) 588.
- [69] PRAMANIK S., FAIZAL M., MOUSSA M. and ALI A. F., *Ann. Phys.*, **362** (2015) 24.
- [70] FAIZAL M., *Phys. Lett. B*, **757** (2016) 244.
- [71] ITO D., MORI K. and CARRIERI E., *Nuovo Cimento A*, **51** (1967) 1119.
- [72] MOSHINSKY M. and SZCZEPANIAK A., *J. Phys. A*, **22** (1989) L817.
- [73] LISBOA R., MALHEIRO M., DE CASTRO A. S., ALBRETO P. and FOILHAIS M., *Phys. Rev. C*, **69** (2004) 024319.
- [74] BERMUDEZ A., MARTIN DELGADO M. A. and LUIS A., *Phys. Rev. A*, **77** (2008) 063815.
- [75] SADURNI E., TORRES J. MAURICIO and SELIGMAN T. H., *J. Phys. A*, **43** (2010) 285204.
- [76] FERKOUS N. and BOUNAMES A., *Phys. Lett. A*, **325** (2004) 21.
- [77] JELLAL AHMED, ALHAIDARI ABDULAZIZ D. and BAHLOULI HOCINE, *Phys. Rev. A*, **80** (2009) 012109.
- [78] KULIKOV D. A. and TUTIK R. S., in *Proceedings of International Conference on Mathematical Methods in Electromagnetic Theory, (MMET*06), Kharkiv, 26–28 June* (IEEE) 2006, pp. 554–556 (arXiv:physics/0608277 [physics.atom-ph]).
- [79] VILLALBA VICTOR M., *Phys. Rev. A*, **49** (1994) 586.
- [80] MORENO M. and ZENTELLA A., *J. Phys. A*, **22** (1989) L821.
- [81] VILLALBA V. M. and MAGGIOLO A. RINCON, *Eur. Phys. J. B*, **22** (2001) 31.
- [82] QUESNE C. and TKACHUK V. M., *J. Phys. A*, **38** (2005) 1747.
- [83] MOHADESI M. and MIRZA B., *Commun. Theor. Phys.*, **42** (2004) 664.
- [84] ALHAIDARI A. D., *Found. Phys. Lett.*, **18** (2005) 651.
- [85] BRIHAYE Y. and NININHAZWE A., *Mod. Phys. Lett. A*, **20** (2005) 1875.
- [86] DE LIMA RODRIGUES R., *Phys. Lett. A*, **372** (2008) 2587.
- [87] MARTINEZ-Y-ROMERO R. P., NÚÑEZ-YÉPEZ H. N. and SALAS-BRITO A. L., *Eur. J. Phys.*, **16** (1995) 135.
- [88] ROZMEJ P. and ARVIEU R., *J. Phys. A*, **32** (1999) 5367.
- [89] BERMUDEZ A, MARTIN-DELGADO M. A. and SOLANO E., *Phys. Rev. A*, **76** (2007) 041801(R).
- [90] BENÍTEZ J., MARTINEZ Y ROMERO R. P., NÚÑEZ-YÉPEZ H. N. and SALAS-BRITO A. L., *Phys. Rev. Lett.*, **64** (1990) 1643.
- [91] MENCULINI L., PANELLA O. and ROY P., *Phys. Rev. D*, **91** (2015) 045032.
- [92] MANDAL B. P. and VERMA S., *Phys. Lett. A*, **374** (2010) 1021.
- [93] RAI S. K. and MANDAL B. P., *Phys. Lett. A*, **376** (2012) 2467.
- [94] DAS S., VAGENAS E. C. and ALI A. F., *Phys. Lett. B*, **690** (2010) 407.
- [95] ABUTALEB A. A., *Adv. High Energy Phys.*, **2013** (2013) 124543.
- [96] BARROS C. C. jr., *Eur. Phys. J. C*, **42** (2005) 119.
- [97] BAKKE K. and MOTA H., *Eur. Phys. J. Plus*, **133** (2018) 409.