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Dual conformable derivative: Variational approach and nonlinear equations

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Abstract – In this letter, we present a variational approach with a recently proposed form of local deformed derivative, the dual conformable derivative, which leads us to obtain a class of nonlinear equations. The ansatz on the solutions can be mathematically inferred by this dual conformable derivative eigenvalue equation and the q-exponential family functions appear naturally. Also, a clear and natural justification for the appearance of the $q \to 2-q$-symmetry is given. To show the potential of our variational approach with this type of dual deformed derivative, we obtain the porous medium equation and present insight for the solution in terms of the q-Gaussian. Also, a new dual conformable wave equation, which is nonlinear, is proposed and a solution is built up in terms of q-plane waves. A dual conformable harmonic oscillator equation is also obtained and promptly solved by the natural ansatz. Aspects of the nonlinear Schroedinger equation are also contemplated and one shows that it can be obtained without the need of an additional Φ-field, from a simple Lagrangian density. The solution to the nonlinear Schroedinger is also expressed in terms of the q-exponential family of functions.

Introduction. – In recent papers [1,2], some of the authors have presented an extension of the standard variational calculus to include the presence of deformed derivatives, both in the Lagrangian of systems of particles and in the Lagrangian density of field-theoretic models. Our first and main focus is on a variational approach with a recently proposed form of local deformed derivative, the dual conformable derivative (DCD) [3]. Following, we show that a certain class of nonlinear equations can be obtained from this variational approach. One also shows that the ansatz on the solutions can be mathematically inferred by an eigenvalue equation along with the appearance of a natural justification for of the $q \to 2-q$-symmetry, present in some versions of the nonlinear Fokker-Plank equation (NFPE) in the context of the nonaddictive statistical mechanics by Tsallis [4]. Considering the DCD, we claim that the justification is based on an eigenvalue equation for this deformed operator. In the light of the DCD approach, we make clear why some nonlinear equations have a common type of solutions [5].

Treading this variational context, as one of the possible applications, we shall obtain the nonlinear parabolic equations for slow diffusion, known as porous media equation (PME), also known as nonlinear heat equation. It describes various diffusion processes, *e.g.*, the flow of a gas through a porous medium. A linear form of PME is the Fourier heat equation. The porous medium equation, also called by some authors NFPE [6], describes the diffusion of the molecules of a gas and fluid particles through porous media [7]. Despite the simplicity of PME, it is important to better understand this equation, because it

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is well known that most of the equations modeling physical phenomena without excessive simplification become nonlinear [8]. Also, in this research we discuss some aspects involved in the solutions to such an equation. In a similar way to what was done in ref. [2], we set out mainly to present developments that go beyond the issue of Lagrangian mechanics in the classical sense. Also, as in ref. [2], we claim that the approach with deformed derivatives, here with DCD, is a simple and efficient option to obtain the equations describing the dynamics for a broad variety of linear and nonlinear systems, particularly complex systems.

As far as detailed solutions to PME and other nonlinear equations are concerned, our focus here is not to solve completely or to obtain rigorous solutions to these equations, but to present the variational approach with DCD embedded into a Lagrangian or Lagrangian density to obtain a class of nonlinear equations and to understand the origin of the ansatz, making available a tool to search for the solutions to a certain class of nonlinear equations. Therefore, the approach itself is of sufficient fundamental or general mathematical interest, especially because it leads to nonlinear equations and indicates possible solutions.

To reinforce and emphasize the potentiality of our approach, some other problems are tackled here. A dual conformable wave equation is proposed and its solution is presented in terms of q-plane waves. A nonlinear harmonic oscillator is promptly obtained and studied. Its solution is presented through a natural ansatz. Finally, with a simple Lagrangian density —it is important to note that here, without the necessity of an additional Φ-field— one is able to obtain a nonlinear Schroedinger equation (NLSE) for a free particle of mass m, as proposed in ref. [5]. The well-known solution to this equation is also formed with the q-exponential family of functions.

Again, as in refs. [1,2,9–11], one should consider that justification for the use of deformed derivatives finds its physical basis on the mapping into the fractal continuum [9,12–14]. That is, one considers a mapping from a fractal coarse-grained (fractal porous) space, which is essentially discontinuous in the embedding Euclidean space, to a continuous one [1,2]. A mapping into a continuous fractal space naturally yields the need for modifications in the derivatives and, in connection with the metric, the modifications of the derivatives lead to a change in the algebra involved, which, in turn, may lead to a generalized statistical mechanics with some suitable definitions of entropy [10]. This is a different paradigm as compared to the one in the generalized statistical mechanics' literature, because the deformed derivative is reached from mapping. Therefore, the concept of entropy seems to be adjoint.

Another explanation for deformed derivatives can be expressed in terms of a canonical transformation from one Euclidean space to a deformed space [15]. A number of discussions on the physical interpretation of deformed derivatives in terms of the Gateaux extended derivative may be found in ref. [16].

Our article is outlined as follows. The next section, addresses mathematical aspects and the eigenvalue equation. In the third section, we focus on the variational approach with dual conformable derivatives. In the fourth section, one applies our approach to obtain and solve the porous medium equation, a nonlinear one-dimensional wave equation, a nonlinear harmonic oscillator and a nonlinear Schroedinger equation. Finally, in the last section, we cast our general conclusions and possible paths for further investigations.

Mathematical aspects and the eigenvalue equation. – The DCD was introduced in ref. [3] and the basic eigenvalue equation can be written as

$$[F(x)]^{\alpha-1}\frac{\mathrm{d}F(x)}{\mathrm{d}x} = F(x). \tag{1}$$

Solving the equation, along with the condition $F(0) = 1$, leads to the following important function:

$$F(x) = [1 + (\alpha - 1)x]^{1/(\alpha-1)}. \tag{2}$$

This function is nothing but the reparameterized q-exponential, ubiquitous in one version of generalized statistical mechanics [17]. This can be clearly seen by redefining the relevant parameter α in term of the entropic parameter q, as $\alpha = 2 - q$. With this reparametrization, the solution to eq. (1) becomes

$$F(x) = [1 + (1 - q)x]^{1/(1-q)} = e_q(x), \tag{3}$$

that is exactly the q-exponential $e_q(x)$ [18].

Some mathematical properties. Now, let us present below some important and useful relations, that will be helpful in the forthcoming calculations. These are:

1) First derivative of a q-exponential

$$\frac{\mathrm{d}e_q(\lambda x)}{\mathrm{d}x} = \lambda[e_q(\lambda x)]^q. \tag{4}$$

2) Eigenvalue equation

$$\tilde{D}_x^{\alpha}[e_{2-\alpha}(\lambda x)] = \lambda e_{2-\alpha}(\lambda x), \tag{5}$$

with $\tilde{D}_x^{\alpha}[e_{2-\alpha}(\lambda x)] = [e_{2-\alpha}(\lambda x)]^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}x}[e_{2-\alpha}(\lambda x)]$.

3) Chain rule

$$\begin{aligned}\tilde{D}_x^{\alpha}[u(\lambda(x))] &= u^{\alpha-1}\frac{\mathrm{d}u(\lambda(x))}{\mathrm{d}x}\\ &= u^{\alpha-1}\frac{\mathrm{d}u}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}x} = \frac{\mathrm{d}\lambda}{\mathrm{d}x}(\tilde{D}_x^{\alpha}u).\end{aligned} \tag{6}$$

Equation (5) renders evident the appearance of symmetry $q \to 2 - q$ (or $\alpha \to 2 - \alpha$) in the context of generalized nonaddictive statistical mechanics. It is now clear that this relationship involves the eigenvalue equation and the DCD. Some aspects of this symmetry were also studied in refs. [19–21]. But here, a clear connection appears. The chain rule appears in ref. [3].

DCD variational approach. – Consider now the Lagrangian density $\mathcal{L}(x,t,\phi,\tilde{D}_t^\beta\phi,\tilde{D}_x^\alpha\phi)$, and the action functional

$$J[\phi]=\int_{t_1}^{t_2}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}x\mathcal{L}(x,t,\phi,\tilde{D}_t^\beta\phi,\tilde{D}_x^\alpha\phi). \tag{7}$$

Here, $\phi=\phi(x,t)$ can be the representation of some generic field $\phi(x,t)$.

We shall find the condition for $J[\phi]$ to present a local minimum. To do so, we consider the new fractional function depending on the parameter ε.

Consider the variable $\phi=\phi(x,t)$: $\phi=\phi(x,t)=\bar{\phi}(x,t)+\varepsilon\eta(x,t)$; $\bar{\phi}(x,t)$ is the objective function, and $\eta(x,t_1)=\eta(x,t_2)=0$, ε is a parameter.

We are going to apply the DCD to $\phi(x)$ as $\tilde{D_x^\alpha}\phi(x)\equiv\phi^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}x}\phi(x)$:

$$\tilde{D}_x^\alpha\phi(x)=\phi^{\alpha-1}\frac{\partial\bar{\phi}(x,t)}{\partial x}+\varepsilon\phi^{\alpha-1}\frac{\partial\eta(x,t)}{\partial x}. \tag{8}$$

Analogously for t:

$$\tilde{D}_t^\beta\phi(x,t)=\phi^{\beta-1}\frac{\partial\bar{\phi}(x,t)}{\partial t}+\varepsilon\phi^{\beta-1}\frac{\partial\eta(x,t)}{\partial t}. \tag{9}$$

Now, the usual δ-variational processes related to the ε parameter and the use of the chain rule, leads to

$$\delta_\varepsilon\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi+\frac{\partial\mathcal{L}}{\partial(\tilde{D}_t^\beta\phi)}\delta_\varepsilon(\tilde{D}_t^\beta\phi)+\frac{\partial\mathcal{L}}{\partial(\tilde{D}_x^\alpha\phi)}\delta_\varepsilon(\tilde{D}_x^\alpha\phi). \tag{10}$$

The variation δ_ε can be calculated, remembering that $\phi(x,t)=\bar{\phi}(x,t)+\varepsilon\eta(x,t)$:

$$\begin{aligned}\delta_\varepsilon(\tilde{D}_x^\alpha\phi) &= \delta_\varepsilon\left(\phi^{\alpha-1}\frac{\partial\bar{\phi}(x,t)}{\partial x}+\varepsilon\phi^{\alpha-1}\frac{\partial\eta(x,t)}{\partial x}\right)=\\ &(\alpha-1)\phi^{\alpha-2}\eta(x)\frac{\partial\bar{\phi}(x,t)}{\partial x}+\phi^{\alpha-1}\frac{\partial\eta(x,t)}{\partial x}\\ &+\varepsilon(\alpha-1)\phi^{\alpha-2}\eta(x,t)\frac{\partial\eta(x,t)}{\partial x},\end{aligned} \tag{11}$$

or, by a simple derivative algebra, one can write that

$$\begin{aligned}\delta_\varepsilon(\tilde{D}_x^\alpha\phi) &= \phi^{\alpha-1}\frac{\partial\eta(x,t)}{\partial x}\\ &+(\alpha-1)\phi^{\alpha-2}\eta(x,t)\left[\frac{\partial\bar{\phi}(x,t)}{\partial x}+\frac{\partial\eta(x,t)}{\partial x}\right]=\\ &\phi^{\alpha-1}\frac{\partial\eta(x,t)}{\partial x}+(\alpha-1)\phi^{\alpha-2}\eta(x,t)\frac{\partial\phi(x,t)}{\partial x}=\\ &\phi^{\alpha-1}\frac{\partial\eta(x,t)}{\partial x}+\eta(x,t)\frac{\partial\phi^{\alpha-1}}{\partial x}=\\ &\frac{\partial(\eta(x,t)\phi^{\alpha-1})}{\partial x}.\end{aligned} \tag{12}$$

Analogous expression may be obtained for $\delta_\varepsilon(\tilde{D}_t^\beta\phi)$. The result is given by

$$\delta_\varepsilon(\tilde{D}_t^\beta\phi)=\frac{\partial(\eta(x,t)\phi^{\beta-1})}{\partial t}. \tag{13}$$

The result in eq. (10) is

$$\begin{aligned}\delta_\varepsilon\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\eta(x,t)+\frac{\partial\mathcal{L}}{\partial(\tilde{D}_t^\beta\phi)}\left[\frac{\partial}{\partial t}(\eta(x,t)\phi^{\beta-1})\right]\\ &+\frac{\partial\mathcal{L}}{\partial(\tilde{D}_x^\alpha\phi)}\left[\frac{\partial}{\partial x}(\eta(x,t)\phi^{\alpha-1})\right].\end{aligned} \tag{14}$$

Consider now the variational principle, that is, $\delta_\varepsilon J=0$.

Integrating by parts the second and the third terms in eq. (14) and using the usual transversality condition for an extreme value, one obtains the deformed Euler-Lagrange (EL) equation as

$$\left(\frac{\partial\mathcal{L}}{\partial\phi}-\phi^{\beta-1}\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\tilde{D}_t^\beta\phi}\right)-\phi^{\alpha-1}\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\tilde{D}_x^\alpha\phi}\right)\right)=0, \tag{15}$$

For purposes that shall become clear when we will present our applications, one can also consider the presence of complementary field $\phi^*(x,t)$, independent of $\phi(x,t)$. In this case, the Lagrangian reads as $\mathcal{L}(x,t,\phi,\phi^*,\tilde{D}_t^\beta\phi,\tilde{D}_x^\alpha\phi,\tilde{D}_t^\beta\phi^*,\tilde{D}_x^\alpha\phi^*)$ and the action will become

$$\begin{aligned}J[\phi] &= \int_{t_1}^{t_2}\mathrm{d}t\\ &\times\left\{\int_{-\infty}^{\infty}\mathrm{d}x\mathcal{L}(x,t,\phi,\phi^*,\tilde{D}_t^\beta\phi,\tilde{D}_x^\alpha\phi,\tilde{D}_t^\beta\phi^*,\tilde{D}_x^\alpha\phi^*)\right\}.\end{aligned} \tag{16}$$

Following the analogous variational processes, two EL equations can be obtained, one for each independent field, ϕ and ϕ^* that are analogous to eq. (15).

Note the presence of similar operators in ref. [22].

The approach above can also be followed for the Lagrangian L, instead of the Lagrangian density $\mathcal{L}$. The EL resultant is similar, by replacing $\mathcal{L}$ by L.

Applications. –

General dual conformable porous medium equation. Consider now the Lagrangian density

$$\begin{aligned}&\mathcal{L}(x,t,\phi,\phi^*,\tilde{D_t^m}\phi,\tilde{D_x^m}\phi,\tilde{D_t^m}\phi^*,\tilde{D_x^m}\phi^*)=\\ &\frac{\phi}{2}\tilde{D_t^m}(\phi^*)-\frac{\phi^*}{2}\tilde{D_t^m}(\phi)+m(\tilde{D_x^m}\phi)(\tilde{D_x^m}\phi^*).\end{aligned} \tag{17}$$

The EL equation for ϕ is what we can call dual conformable porous media equation:

$$\tilde{D_t^m}\phi-m\phi^{m-1}\frac{\partial(\tilde{D_x^m}\phi)}{\partial x}=0. \tag{18}$$

Expanding the DCD, we can write the EL equation as

$$\phi^{m-1}\frac{\partial\phi}{\partial t}-m\phi^{m-1}\frac{\partial\left(\phi^{m-1}\frac{\partial\phi}{\partial x}\right)}{\partial x}=0. \tag{19}$$

Observe that one could define a sequential dual derivative as, for example, $\tilde{D_x^m}(\tilde{D_x^m}\phi)=m\phi^{m-1}\frac{\partial(\phi^{m-1}\frac{\partial\phi}{\partial x})}{\partial x}$. However, to avoid confusions, we will not follow this path.

For a particular choice of nonzero field, $\phi^{m-1} \neq 0$ (Note that the equation becomes degenerate at $\phi = 0$, resulting to the phenomenon of finite speed of propagation.), the equation can be simplified to

$$\frac{\partial \phi}{\partial t} - m \frac{\partial \left(\phi^{m-1} \frac{\partial \phi}{\partial x} \right)}{\partial x} = 0. \tag{20}$$

Equation (20) can be simplified to the well-known porous medium equation:

$$\frac{\partial \phi}{\partial t} - \frac{\partial^2 \phi^m}{\partial x^2} = 0. \tag{21}$$

For the complementary field ϕ^*, the equation reads as below

$$\frac{\partial \phi^*}{\partial t} + \frac{\partial^2 \phi^{*m}}{\partial x^2} = 0. \tag{22}$$

The plus sign in the second term of the left-hand side may be interpreted as related to the external part of system. Consider that a possible explanation for the complementary field ϕ^* can be done as a field related to the border line of the system. Since we are here dealing with open systems, there is a field flow from the system to the border line. That flux, in turn, is going towards the exterior region. Probably, there are other possible interpretations of the auxiliary field, but the result given here obtained is undeniable.

Natural ansatz. Now, it seems natural to propose an ansatz for the solution of porous media eq. (21), but with eq. (18) being the starting point. Since eq. (18) contains dual deformed derivatives in time and space coordinates, a natural ansatz can be

$$\phi(x,t) = \Gamma(t) e_{q=2-m}(\lambda(x), \beta(t)), \tag{23}$$

where we have defined a generalized two-variable q-exponential as

$$e_q(\lambda(x), \beta(t)) \equiv [1 + (1-q)\lambda(x)\beta(t)]^{1/(1-q)} \tag{24}$$

and $\lambda(x), \beta(t), \Gamma(t)$ will be determined further.

The reasoning here is to consider the meaning of the eigenvalue equation (5), including two variables, x, t, in the new eigenfunction $e_q(\lambda(x), \beta(t))$. But there is a close resemblance with the ansatz for solutions of nonlinear Fokker-Planck equation, in the context of anomalous diffusion [4,23].

Using the eigenvalue equation, property 2), and the chain rule for the derivative, property 3), one can readily obtain that

$$\tilde{D_t^m}[e_{q=2-m}(\lambda(x), \beta(t))] = \frac{\mathrm{d}\beta(t)}{\mathrm{d}t} \lambda(x) e_{q=2-m}(\lambda(x), \beta(t)), \tag{25}$$

and

$$\tilde{D_x^m}[e_{q=2-m}(\lambda(x), \beta(t))] = \frac{\mathrm{d}\lambda(x)}{\mathrm{d}x} \beta(t) e_{q=2-m}(\lambda(x), \beta(t)). \tag{26}$$

But, we have to remember that eq. (18) is a nonlinear equation and the ansatz given by eq. (23) contains a pre-factor $\Gamma(t)$ that has to be considered. In this way, considering the natural anzats and the properties in the second section, we can write

$$\begin{aligned} &\tilde{D_t^m}\{\Gamma(t)[e_{q=2-m}(\lambda(x), \beta(t))]\} = \\ &[\Gamma(t)]^{m-1} \left\{ \frac{\mathrm{d}\Gamma(t)}{\mathrm{d}t} [e_{q=2-m}(\lambda(x), \beta(t))]^m \right. \\ &\left. + \Gamma(t) \frac{\mathrm{d}\beta(t)}{\mathrm{d}t} \lambda(x) e_{q=2-m}(\lambda(x), \beta(t)) \right\}. \end{aligned} \tag{27}$$

Analogously for x, $\tilde{D_x^m}\phi(x,t)$:

$$\begin{aligned} &\tilde{D_x^m}\{\Gamma(t)[e_{q=2-m}(\lambda(x), \beta(t))]\} = \\ &\Gamma(t) \frac{\mathrm{d}\lambda(x)}{\mathrm{d}x} \beta(t) e_{q=2-m}(\lambda(x), \beta(t)). \end{aligned} \tag{28}$$

In the sequence

$$\begin{aligned} &\phi^{m-1} \frac{\partial}{\partial x} \{\tilde{D_x^m}\{\{\Gamma(t)[e_{q=2-m}(\lambda(x), \beta(t))]\}\} = \\ &\Gamma(t)]^{m-1} \beta(t) \frac{\mathrm{d}^2\lambda(x)}{\mathrm{d}x^2} [e_{q=2-m}(\lambda(x), \beta(t))]^m \\ &+ [\Gamma(t)]^{2m-2} \beta^2(t) \left(\frac{\mathrm{d}\lambda(x)}{\mathrm{d}x} \right)^2 e_{q=2-m}(\lambda(x), \beta(t)). \end{aligned} \tag{29}$$

Substituting into eq. (18), considering a nondegenerate solution, that is,

$$\phi = \Gamma(t)[e_{q=2-m}(\lambda(x), \beta(t))] \neq 0, \tag{30}$$

and, simplifying, it gives

$$\begin{aligned} &\frac{\mathrm{d}\Gamma(t)}{\mathrm{d}t} e_{q=2-m}(\lambda(x), \beta(t)) \\ &+ \Gamma(t) \frac{\mathrm{d}\beta(t)}{\mathrm{d}t} \lambda(x) [e_{q=2-m}(\lambda(x), \beta(t))]^{2-m} \\ &- m\beta(t) \frac{\mathrm{d}^2\lambda(x)}{\mathrm{d}x^2} e_{q=2-m}(\lambda(x), \beta(t)) \\ &- m\beta^2(t) \left(\frac{\mathrm{d}\lambda(x)}{\mathrm{d}x} \right)^2 [\Gamma(t)]^{m-1} [e_{q=2-m}(\lambda(x), \beta(t))]^{2-m} = 0. \end{aligned} \tag{31}$$

Two equations can be obtained from the equations above, by taking the corresponding powers of $[e_{q=2-m}(\lambda(x), \beta(t))]^{2-m}$ and $e_{q=2-m}(\lambda(x), \beta(t))$, equal to zero, in a similar way to that of ref. [24]. The result is given by

$$\begin{cases} \dfrac{\mathrm{d}\Gamma(t)}{\mathrm{d}t} - m[\Gamma(t)]^{m-1} \beta(t) \dfrac{\mathrm{d}^2\lambda(x)}{\mathrm{d}x^2} = 0, \\ \Gamma(t) \dfrac{\mathrm{d}\beta(t)}{\mathrm{d}t} \lambda(x) - m[\Gamma(t)]^{m-1} \beta^2(t) \left(\dfrac{\mathrm{d}\lambda(x)}{\mathrm{d}x} \right)^2 = 0. \end{cases} \tag{32}$$

From the first equation, one can write

$$m[\Gamma(t)]^{m-1} = \frac{\frac{\mathrm{d}\Gamma(t)}{\mathrm{d}t}}{\beta(t) \frac{\mathrm{d}^2\lambda(x)}{\mathrm{d}x^2}}, \tag{33}$$

which, by substituting into the second equation of the system and simplifying, yields

$$\Gamma(t)\frac{\mathrm{d}\beta(t)}{\mathrm{d}t}\lambda(x)-\frac{\mathrm{d}\Gamma(t)}{\mathrm{d}t}\beta(t)\frac{(\frac{\mathrm{d}\lambda(x)}{\mathrm{d}x})^2}{\frac{\mathrm{d}^2\lambda(x)}{\mathrm{d}x^2}}=0. \tag{34}$$

Now, let us suppose simple possibilities for the function $\lambda(x)$.

For $\lambda(x)$ considered as a constant or a simple function $\lambda(x)=x$, an indeterminate result is obtained.

Considering the spatial invariance $x\to -x$, a simple choice can be $\lambda(x)=x^2$. With this choice, the obtained equation is

$$\frac{\Gamma(t)}{\beta(t)}=2\frac{\frac{\mathrm{d}\Gamma(t)}{\mathrm{d}t}}{\frac{\mathrm{d}\beta(t)}{\mathrm{d}t}}. \tag{35}$$

Integration of last equation gives

$$\frac{\Gamma^2(t)}{\beta(t)}=\frac{\Gamma^2(t_0)}{\beta(t_0)}=k, \tag{36}$$

where k is a constant. This equation is connected with the normalization [24,25].

In so doing, one can write a simple relation between $\Gamma(t)$ and $\beta(t)$ as $\Gamma(t)=\sqrt{k\beta(t)}$.

Now, a simple choice for $\beta(t)$ leads to a vanishing function $\Gamma(t)$, for $t\to\infty$. So, one can choose $\beta(t)=C(t)^{-2\alpha}$, where α, C are constants. Then, the solution to the dual deformed porous media equation (18), that is the nondegenerate solution of the porous medium equations (20), can be written as the well-known Barenblatt–Pattle's-like solution [7] and is also referred to as a q-Gaussian-like solution. The q-Gaussian distribution generalizes the standard Gaussian [5],

$$\phi(x,t)=\sqrt{k}\frac{1}{t^{\alpha}}\left[1-C_0(q-1)\frac{x^2}{(t)^{2\alpha}}\right]^{\frac{1}{1-q}} \qquad (q=2-m). \tag{37}$$

Dual conformable one-dimensional wave equation: q-plane wave solution. Consider a string which has constant tension and density ρ. In equilibrium, it is stretched along the x-axis, and we consider small displacements away from this equilibrium position, $\eta(x,t)$. The Lagrangian density may be written as

$$\mathcal{L}(x,t,\eta,\tilde{D_t^m}\eta,\tilde{D_x^m}\eta)=\frac{\rho}{2}(\tilde{D_t^m}(\eta))^2-\frac{\kappa}{2}(\tilde{D_x^m}\eta)^2. \tag{38}$$

Here κ is the Young'modulus.

The resulting EL is the new dual conformable wave equation (DCWE), that is a nonlinear equation:

$$\eta^{m-1}\frac{\partial}{\partial t}(\tilde{D_t^m}\eta)-c^2\eta^{m-1}\frac{\partial}{\partial x}(\tilde{D_x^m}\eta)=0, \tag{39}$$

where $c=\sqrt{\frac{\kappa}{\rho}}$.

Following the same steps as in the previous section, one can show that the ansatz $\eta(x,t)=N.e_q[i(kx-\omega t)]$ is a solution for this type of nonlinear equation, with $c=\sqrt{\frac{\kappa}{\rho}}=\frac{\omega}{k}$; N is some normalizing factor. The ansatz proposed is also a solution to the simple and standard one-dimensional linear wave equation [5]. Here, our nonlinear wave equation is different from that given in ref. [26].

Dual conformable harmonic oscillator. Consider now a Lagrangian similar to one-dimensional harmonic oscillator, but here with the DCD substituting the first-order derivative in space x as

$$L=\frac{1}{2}m(\tilde{D_t^{\alpha}}x)^2-\frac{1}{2}kx^2. \tag{40}$$

Again, following the approach, one can show that the natural ansatz $x(t)=A.e_q[i(\omega t+\varphi)]$ is a solution of this oscillator, with $\omega=\sqrt{A^{1-\alpha}k/m}$. A very interesting result is that the angular frequency, ω, now depends on the amplitude A. Note that for $\alpha\to 1\Rightarrow q=2-\alpha\to 1$, what implies that e_q becomes the usual exponential and the angular frequency becomes independent of the amplitude, equal to $\omega=\sqrt{k/m}$.

Nonlinear Schroedinger equation. A nonlinear Schroedinger equation (NLSE) for a free particle of mass m was proposed in ref. [5] and it can in fact be expressed in terms of DCD. Here we will show that one can obtain this equation with our variational approach, without the necessity of an additional field Φ, with a simple Lagrangian density.

Let us now consider the Lagrangian density,

$$\begin{aligned}\mathcal{L}=&\frac{i\hbar}{2}\Psi(\vec{x},t)^*D_t^{\alpha}\Psi(\vec{x},t)-\frac{i\hbar}{2}\Psi(\vec{x},t)D_t^{\alpha}\Psi(\vec{x},t)^*\\&-\frac{\hbar^2}{2m}\tilde{\nabla}^{\alpha}\Psi(\vec{x},t)^*\cdot\tilde{\nabla}^{\alpha}\Psi(\vec{x},t)\end{aligned} \tag{41}$$

where we have defined the spatial DCD gradient as

$$\tilde{\nabla}^{\alpha}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]=\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{\alpha-1}\nabla\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]. \tag{42}$$

The EL results in

$$i\hbar\tilde{D_t^{\alpha}}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]=-\frac{\hbar^2}{2m}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{\alpha-1}\nabla\left\{\tilde{\nabla}^{\alpha}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]\right\}, \tag{43}$$

or making the operators explicit,

$$\begin{aligned}&i\hbar\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{\alpha-1}\frac{\partial}{\partial t}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]=\\&-\frac{\hbar^2}{2m}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{\alpha-1}\nabla\left\{\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{\alpha-1}\nabla\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]\right\}.\end{aligned} \tag{44}$$

For nontrivial solutions, that is, for $\frac{\Psi(\vec{x},t)}{\Psi_0}\neq 0$, one can also rewrite this equations as

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]=-\frac{1}{2-q}\frac{\hbar^2}{2m}\nabla^2\left[\frac{\Psi(\vec{x},t)}{\Psi_0}\right]^{2-q}, \tag{45}$$

where we have divided by $\frac{\Psi(\vec{x},t)}{\Psi_0}$ and renamed $\alpha = 2 - q$. The equation above is exactly the same NLSE presented in refs. [5,22,27].

In this form, we render evident and natural a certain definition of energy $\hat{E}$ and momentum $\hat{p}$ operators applied to a function $[\frac{\Psi(\vec{x},t)}{\Psi_0}]$, which is taken in an *ad hoc* way in ref. [5]. Here, based on our proposed Lagrangian density, we can write naturally it in terms of DCD as

$$\begin{cases} \hat{E}\left[\dfrac{\Psi(\vec{x},t)}{\Psi_0}\right] \equiv i\hbar \left[\dfrac{\Psi(\vec{x},t)}{\Psi_0}\right]^{\alpha-1} \frac{\partial}{\partial t}\left[\dfrac{\Psi(\vec{x},t)}{\Psi_0}\right], \\ \hat{p}\left[\dfrac{\Psi(\vec{x},t)}{\Psi_0}\right] \equiv -i\hbar \left[\dfrac{\Psi(\vec{x},t)}{\Psi_0}\right]^{\alpha-1} \nabla\left[\dfrac{\Psi(\vec{x},t)}{\Psi_0}\right]. \end{cases} \tag{46}$$

The q-exponential-like solution (a q-plane wave) is also admissible [5]. Other details can be found in ref. [28]. We emphasize that, with a q-exponential-like ansatz, also using the form of equation with DCD, eq. (43), and the mathematical properties for the DCD previously presented, the solution of eq. (45) can be more easily and naturally found. Note that in ref. [22] the Lagrangian density is a complicated one and contains an additional field.

Conclusions and outlook for further investigations. – In this work, we present some important and innovative results.

A variational approach with DCD is presented, the relevance and potentiality of this operator for nonlinear problems have been put into evidence here.

We have applied this variational approach to obtain: the porous medium equation, that is a well-known form of nonlinear heat transfer equation, a dual conformable wave equation, a dual conformable harmonic oscillator and the nonlinear Schroedinger equation. All of those are nonlinear equations and have in common the fact that they can be written in terms of DCD and, as a consequence of an eigenvalue equation, they have solutions expressed in terms of q-exponential family of functions.

The search for an ansatz on the solutions now becomes more natural and related to the eigenvalue of DCD and one shows the natural justification for the appearance of the symmetry $q \to 2-q$, which is present in the literature, in the context of nonaddictive statistical mechanics [17]. This justification is also based on an eigenvalue equation for the DCD.

A nonlinear wave equation is obtained from a simple Lagrangian density and the solution in terms of q-plane waves are presented. Following the same path, a nonlinear harmonic oscillator is also approached and an interesting solution was worked out. The solution indicates an angular frequency ω that is dependent on the oscillation's amplitude, A.

Finally, without the necessity of an additional heuristic field and with a simple Lagrangian density, we have built up a nonlinear Schroedinger equation. The equation agrees with the one presented in refs. [5,22,27]. In a nutshell, we have presented the variational approach with DCD embedded into a Lagrangian or Lagrangian density to obtain a class of nonlinear equation and to understand the origin of the ansatz, making available a mathematical tool to search for the solutions to a certain class of nonlinear equations.

For future studies, as glimpse of the task ahead, we can list some possible applications: a new formulation of a nonlinear electrodynamics and field theory, a nonlinear form for the Dirac equation, different from refs. [5,29], nonlinear Fokker-Planck equation, nonlinear models of population growth [30], population biology nonlinear equations [31], soil physics equations [32] and so on.

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