

The spectral dimension of simplicial complexes: a renormalization group theory

Ginestra Bianconi^{1,2} and Sergey N Dorogovstev^{3,4}

¹ School of Mathematical Sciences, Queen Mary University of London, London, United Kingdom

² Alan Turing Institute, The British Library, London, United Kingdom

³ Departamento de Física da Universidade de Aveiro & I3N, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal

⁴ A.F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia
E-mail: g.bianconi@qmul.ac.uk

Received 28 October 2019

Accepted for publication 17 November 2019

Published 24 January 2020



Online at stacks.iop.org/JSTAT/2020/014005
<https://doi.org/10.1088/1742-5468/ab5d0e>

Abstract. Simplicial complexes are increasingly used to study complex system structures and dynamics including diffusion, synchronization and epidemic spreading. The spectral dimension of the graph Laplacian is known to determine the diffusion properties at long time scales. Using the renormalization group here we calculate the spectral dimension of the graph Laplacian of two classes of non-amenable d dimensional simplicial complexes: the Apollonian networks and the pseudo-fractal networks. We analyse the scaling of the spectral dimension with the topological dimension d for $d \rightarrow \infty$ and we point out that randomness such as the one present in Network Geometry with Flavor can diminish the value of the spectral dimension of these structures.

Keywords: diffusion, network dynamics, random graphs, networks, renormalisation group

Contents

1. Introduction	2
2. Simplicial complexes under study	4
2.1. Simplicial complexes	4
2.2. Apollonian network	5
2.3. Pseudo-fractal network of any dimension	5
2.4. Network Geometry with Flavor (NGF)	6
3. Laplacian spectrum and the Gaussian model	7
3.1. Spectral dimension.....	7
3.2. Gaussian model.....	8
4. Renormalization group approach	9
5. Apollonian network	11
5.1. General RG equations	11
5.2. $d = 2$ Farey graph.....	13
5.3. $d = 3$ Apollonian graph.....	15
5.4. $d > 3$ dimensional apollonian graph.....	16
5.5. Comparison to NGF with $s = -1$	20
6. Pseudo-fractal networks	20
6.1. RG equations.....	20
6.2. Case $d = 2$	22
6.3. Case $d = 3$	22
6.4. Case $d > 3$	25
6.5. Comparison to NGF with $s = 0, 1$	27
7. Conclusions	29
References	29

1. Introduction

Simplicial complexes [1–4] are generalized network structures that are ideal to investigate network geometry and topology. They are formed by simplices like nodes, links, triangles, tetrahedra etc that describe the higher order interactions between the elements of a complex system. The underlying decomposition of simplicial complexes in their geometric building blocks (the simplices) allows to answer novel questions in network topology and geometry. Network geometry and topology are emergent topics in statistical mechanics and applied mathematics that explore the properties of complex interacting systems using geometrical concepts and methods tailored to

the discrete setting. Novel equilibrium and non-equilibrium modelling frameworks for simplicial complexes have been proposed recently in the literature [5–12], and some classic example of deterministic networks such as Apollonian networks [13, 14] and pseudo-fractal networks [15] can be reinterpreted as skeleton of simplicial complexes. Additionally we note that simplicial complexes have been already extensively used in the quantum gravity literature to describe quantum space-time. They constitute, for instance, the underlying structures of causal dynamical triangulations and of tensor networks [16, 17]. From the applied point of view tools of network topology and geometry span brain research [2, 18], financial networks [19], social science [12] and condensed matter [20].

Network geometry and topology have been recently shown to have a very significant effect on dynamics including synchronization dynamics [21–23], epidemic spreading [24–26] and percolation [27–29]. The spectral properties of network geometries constitute a direct link to the diffusion processes defined on the same structures. The spectral properties of networks have been extensively studied in the literature [30–35], however the study of the spectral properties of discrete network geometries is only at its infancy. Here we focus on the spectral dimension [36–41] of the network geometry, which is known to determine the return distribution of a random walk and define universality classes of the Gaussian model. Recently the spectral dimension has been shown to be key to characterize the stability of the synchronized phase in the Kuramoto model defined on simplicial complexes [21, 22]. The spectral dimension [36] is a concept that extends the notion of dimension for a lattice and in fact it is equal to the lattice dimension for Euclidean lattices. The spectral dimension d_S is however distinct from the Hausdorff dimension d_H for a general network [42, 43]. This concept has been introduced to study the diffusion on fractal structures [36] and it has been then applied to a variety of contexts including the characterization of the stability of the 3D folding of proteins [41]. Interestingly, the notion of spectral dimension is used widely in quantum gravity to compare different models of quantum space-time in search for their universal properties [44–46].

In this work we investigate the spectral properties of the non-amenable skeleton of d -dimensional simplicial complexes generated by deterministic and random models: the Apollonian [13], the pseudo-fractal [15] simplicial complexes and the Network Geometry with Flavor (NGF) [6–8]. The Apollonian simplicial complexes and the NGF with flavor $s = -1$ are hyperbolic manifolds, while the other studied simplicial complexes have a non-amenable hierarchical structure. These discrete network structures are ideal to perform real-space renormalization group calculations revealing the critical properties of percolation [27–29, 48–52] the Ising model [47] and Gaussian model [37, 38, 40]. Here we use the renormalization group technique proposed in [37, 38] to predict the spectral dimension of simplicial complexes of different topological dimension d for the Apollonian and the pseudo-fractal network. Moreover we will compare numerically the spectral properties of these deterministic network models with the spectral properties of the simplicial complexes generated by the model NGF [7] which include some relevant randomness. Our results reveal that the spectral dimension of the deterministic networks can be higher than the topological dimension. Specifically we see that planar Apollonian networks (in $d = 3$) have a spectral dimension $d_S = 3.73\dots$. Additionally we found that the spectral dimension d_S grows asymptotically for large

d as $d_S \simeq 2d \ln d$ for both the Apollonian and the pseudo-fractal network. Finally we show numerically that topological randomness can diminish significantly the spectral dimension of the networks.

The paper is organized as follows: in section 2 we define simplicial complexes and introduce the simplicial complex models investigated in this work. In section 3 we define the spectral dimension and the relation between the spectral dimension and the Gaussian model. In section 4 we introduce the real space renormalization group approach used in this work. In section 5 we derive the RG equations for the Gaussian model on the Apollonian network, we theoretically predict the spectral dimension of Apollonian networks of any dimension $d \geq 2$ and we compare the spectral properties of Apollonian networks with the spectral properties of NGFs with flavor $s = -1$. In section 6 we use the RG approach to predict the spectral dimension of pseudo-fractal networks of any topological dimension $d \geq 2$ and we compare the results with numerical result of both pseudo-fractal networks and NGFs with flavor $s = 0$ and $s = 1$. Finally in section 7 we provide the conclusions.

2. Simplicial complexes under study

2.1. Simplicial complexes

A simplicial complex of N nodes can be used to describe complex interacting systems including higher order interactions. A simplicial complex is formed by simplices glued along their faces. A δ -dimensional simplex is a set of $\delta + 1$ nodes characterizing a single many body interaction. A 0-simplex is a node, a 1-simplex is a link, a 2-simplex is a triangle, and so on. For instance a 2-simplex in a collaboration network can indicate that three authors have co-authored a paper, or a 3-simplex in a face-to-face interaction indicates a group of four people in a conversation. A δ' -dimensional *face* α' of a δ -simplex α , is a simplex formed by a subset of $\delta' + 1$ nodes of α , i.e. $\alpha' \subset \alpha$.

A d -dimensional *simplicial complex* \mathcal{K} is formed by a set of simplices of dimensions $0 \leq \delta \leq d$ (including at least a d -dimensional simplex) that satisfy the two conditions:

- (a) if a simplex α belongs to the simplicial complex, i.e. $\alpha \in \mathcal{K}$ then also all its faces $\alpha' \subset \alpha$ belong to the simplicial complex, i.e. $\alpha' \in \mathcal{K}$;
- (b) if two simplices α and α' belong to the simplicial complex, i.e. $\alpha \in \mathcal{K}$ and $\alpha' \in \mathcal{K}$, then either the two simplices do not intersect $\alpha \cap \alpha' = \emptyset$ or their intersection is a face of the simplicial complex, i.e. $\alpha \cap \alpha' \in \mathcal{K}$.

A *pure* d -dimensional simplicial complex is only formed by d -dimensional simplices and their faces.

The *1-skeleton* of a simplicial complex is the network formed exclusively by the nodes and the links or the simplicial complex.

Here we will focus exclusively on the skeleton of pure d -dimensional simplicial complexes. The simplicial complexes that we will consider are the Apollonian, the

pseudo-fractal simplicial complexes and the NGF. In the following paragraphs we will introduce each one of these models.

2.2. Apollonian network

A d -dimensional Apollonian network [13, 14] (with $d \geq 2$) is the skeleton of a simplicial complex that is generated iteratively by starting from a single d -simplex at iteration $n = 0$ and at each iterations $n > 0$ adding a d -simplex to every $(d - 1)$ -dimensional face introduced at the previous generation. Therefore in these Apollonian networks, at generation $n > 0$ there are N_n nodes, and \mathcal{N}_n nodes added at iteration n with

$$\begin{aligned} N_n &= (d + 1) \frac{d^n + d - 2}{d - 1}, \\ \mathcal{N}_n &= (d + 1) d^{n-1}. \end{aligned} \quad (1)$$

The Apollonian networks are small-world, i.e. their Hausdorff dimension is infinity,

$$d_H = \infty, \quad (2)$$

therefore at each iteration their diameter grows logarithmically with the total number of nodes of the network. Moreover, the Apollonian networks of dimension d are manifolds that define discrete hyperbolic lattices.

Let us add here a pair of additional combinatorial properties of Apollonian networks that will be useful in the future. At each iteration n we call *links of type ℓ* the links added at generation $m = n - \ell$. At generation n , the number of d -simplices of generation n attached to links of type ℓ is given by

$$w_\ell = (d - 1)(d - 2)^{\ell-1}. \quad (3)$$

The number of d -dimensional simplices of generation n incident to nodes added at generation $m = n - \ell$ is given by

$$v_\ell = d(d - 1)^{\ell-1} \quad (4)$$

for $\ell > 0$. Moreover it can be easily shown that the number of links of generation n incident to nodes added at generation $m = n - \ell$ is given by v_ℓ for $\ell > 0$ and $v_0 = d$ for $\ell = 0$.

2.3. Pseudo-fractal network of any dimension

The pseudo-fractal network [15] is the skeleton of a simplicial complex constructed iteratively starting at iteration $n = 0$ from a single d -simplex (here and in the following we take $d \geq 2$) At each time $n > 0$ we attach a d -simplex to every $(d - 1)$ -dimensional face introduced a time $n \geq 0$. At iteration $n > 0$ the number of nodes N_n and the number of links $L_n^{(1)}$ added at iteration n is

$$\begin{aligned} N_n &= d + \frac{1}{d} [(d + 1)^n - 1], \\ \mathcal{N}_n &= (d + 1)^n. \end{aligned} \quad (5)$$

The pseudo-fractal networks are small-world, i.e. their Hausdorff dimension is infinity,

$$d_H = \infty. \tag{6}$$

As for the Apollonian network, also for the pseudo-fractal networks, at each iteration n we call *link of type ℓ* the links added at generation $m = n - \ell$. We observe that at generation n the number of d -simplices of generation n attached to links of type ℓ is given by

$$\hat{w}_\ell = (d - 1) \sum_{\ell'=0}^{\ell} (d - 2)^{\ell'-1}. \tag{7}$$

It is also possible to show with straightforward combinatorial arguments that the number of d -dimensional simplices added to nodes of generation $m = n - \ell$ is

$$\hat{v}_\ell = d \sum_{\ell'=0}^{\ell} (d - 1)^{\ell'-1} \tag{8}$$

for $\ell > 0$. Finally the number of links of iteration n added to nodes of generation $m = n - \ell$ is given by \hat{v}_ℓ for $\ell > 0$ and $\hat{v}_0 = d$ for $\ell = 0$.

2.4. Network Geometry with Flavor (NGF)

NGF [6, 7] generates growing d -dimensional simplicial complexes as Apollonian and pseudo-fractal simplicial complexes. However the dynamics of the NGF is not deterministic but random.

We assign to every $(d - 1)$ -dimensional face of a simplex an *incidence number* \hat{n}_α equal to the number of d -dimensional simplices incident to it minus one. Therefore we note that the incidence number can change with time.

The evolution of the NGF is dictated by a parameter called *flavor* $s \in \{-1, 0, 1\}$. The algorithm that determines the NGF evolution assumes that at time $t = 1$ the simplicial complex is formed by a single d -simplex. At each time $t > 1$ a $(d - 1)$ -face α is chosen with probability

$$\Pi_{d,d-1}(\alpha) = \frac{(1 + sn_\alpha)}{Z^{[s]}}, \tag{9}$$

where $Z^{[s]}$ is called the *partition function* of the NGF and is given by

$$Z^{[s]}(t) = \sum_{\alpha' \in S_{d,d-1}} (1 + sn_{\alpha'}). \tag{10}$$

We note here that the Hausdorff dimension of NGFs defined above is always

$$d_H = \infty, \tag{11}$$

as these networks are small-world for any value of the flavor $s \in \{-1, 0, 1\}$. In the case $s = -1$ the NGF evolves as a subgraph of the Apollonian network connected to the initial d -dimensional simplex. In this case we obtain a random Apollonian network [14]. Therefore it is interesting to compare the spectral properties of the NGF with $s = -1$ to the spectral properties of the Apollonian network. The NGF with flavor $s = -1$

describe emergent hyperbolic geometries [6]. In fact they are hyperbolic networks emerging from a fully stochastic dynamics that makes no reference to their underlying geometry. Indeed NGF with flavor $s = -1$ form a subset of the Apollonian networks of the same dimension d . In the case $s = 0$ and $s = 1$ every $(d - 1)$ -dimensional face can be incident to an arbitrary number of d -dimensional simplices. Therefore it is interesting to compare the spectral properties of the NGF with $s = 0$ and $s = 1$ to the spectral properties of the pseudo-fractal network. Note that the NGF [7] was originally defined with an additional dependence to another parameter called *inverse temperature* β . Here we focus only on the case $\beta = 0$, therefore we do not need to introduce this additional parameter in this work.

3. Laplacian spectrum and the Gaussian model

3.1. Spectral dimension

For a network it is possible to defined both a normalized and a un-normalized Laplacian. The un-normalized Laplacian $\hat{\mathbf{L}}$ has elements

$$L_{rq} = k_r \delta_{r,q} - a_{rq}, \tag{12}$$

where a_{rq} is the generic element of their adjacency matrix and k_r is the degree of node $r \in \{1, 2, \dots, N\}$. The normalized Laplacian \mathbf{L} of a network has instead elements

$$L_{rq} = \delta_{r,q} - \frac{a_{rq}}{\sqrt{k_r k_q}}. \tag{13}$$

Their spectrum is in general distinct for non-regular networks having nodes of different degree. However as we will observe later, their spectral dimension is the same in the large network limit.

Here we start from the normalized Laplacian and we predict the spectral dimension analytically. This analytical calculation will be done using the renormalization group which acts on a more general class of graphs in which the links can be weighted, so in general we are interested to study the fixed-point properties of spectrum of weighted normalized Laplacian matrices \mathbf{L} of elements

$$L_{rs} = \delta_{r,q} - \frac{w_{rq}}{\sqrt{s_r s_q}}, \tag{14}$$

where w_{rq} indicates the weight of link (r, q) and s_r indicates the strength of node r , i.e. $s_r = \sum_{q=1}^N w_{rq}$.

The *spectral dimension* d_S determines the scaling of the density of eigenvalues $\rho(\mu)$ of the normalized Laplacian of networks with distinct geometrical properties. In particular, in presence of the spectral dimension for $\mu \ll 1$ we observe the asymptotic behavior

$$\rho(\mu) \cong C \mu^{d_S/2-1}, \tag{15}$$

where C is independent of μ .

In d -dimensional Euclidean lattices the spectral dimension coincides with the Hausdorff dimension $d_S = d_H = d$. More generally, it can be shown that d_S is related to the Hausdorff dimension d_H of the network by the inequalities [42, 43]

$$d_H \geq d_S \geq 2 \frac{d_H}{d_H + 1}. \tag{16}$$

It follows that for small-world networks, having infinite Hausdorff dimension $d_H = \infty$, it is only possible to have finite spectral dimension greater or equal than two, i.e.

$$d_S \geq 2. \tag{17}$$

Additionally we mention here that in presence of a finite spectral dimension, the cumulative distribution $\rho_c(\mu)$ evaluating the density of eigenvalues $\mu' \leq \mu$ follows the scaling

$$\rho_c(\mu) \cong \tilde{C} \mu^{d_S/2}, \tag{18}$$

for $\mu \ll 1$. This relation it is useful to evaluate the spectral dimension numerically, as we will do in order to compare our analytical results with numerical results.

3.2. Gaussian model

In order to predict the spectral dimension of a network it is useful to consider [37] the corresponding Gaussian model whose partition function reads

$$Z(\mu) = \int \mathcal{D}\psi \exp \left[i\mu \sum_r \psi_r^2 - i \sum_{rq} L_{rq} \psi_r \psi_q \right] = \frac{(i\pi)^{N_n/2}}{\sqrt{\prod_r (\mu - \mu_r)}}, \tag{19}$$

where μ_r are the eigenvalue of the normalized Laplacian matrix \mathbf{L} and

$$\mathcal{D}\psi = \prod_{r=1}^{N_n} \left(\frac{d\psi_r}{\sqrt{2\pi}} \right). \tag{20}$$

By changing variables and putting $\phi = \psi/\sqrt{s_r}$ the partition function can be rewritten as

$$Z(\mu) = \prod_r \sqrt{s_r} \int \mathcal{D}\phi \exp \left[\sum_{(r,q) \in E} -i(1 - \mu)w_{rq}(\phi_r^2 + \phi_q^2) - iw_{rq}\phi_r\phi_q \right], \tag{21}$$

where E indicates the set of links of the network. The spectral density $\rho(\mu)$ of the normalized Laplacian matrix can be found using the relation

$$\rho(\mu) = -\frac{2}{\pi} \text{Im} \frac{\partial f}{\partial \mu}, \tag{22}$$

where f is the free-energy density defined as

$$f = - \lim_{n \rightarrow \infty} \frac{1}{N_n} \ln Z(\mu). \tag{23}$$

In fact we can use equation (19) to show that

$$f = - \lim_{n \rightarrow \infty} \frac{1}{N_n} \sum_{r=1}^{N_n} \left[\frac{1}{2} \ln(\mu - \mu_r) + \frac{1}{2} \ln(i\pi) \right]. \quad (24)$$

Using equation (23) we get

$$\rho(\mu) = -\frac{2}{\pi} \text{Im} \frac{\partial f}{\partial \mu} = \frac{1}{\pi} \lim_{n \rightarrow \infty} \frac{1}{N_n} \text{Im} \sum_{r=1}^{N_n} \frac{1}{\mu - \mu_r} = \lim_{n \rightarrow \infty} \frac{1}{N_n} \sum_{r=1}^{N_n} \delta(\mu - \mu_r). \quad (25)$$

4. Renormalization group approach

Under the renormalization flow, p and μ are renormalized. A closer look to the problem reveals that the parameters p and μ are renormalized differently for links of different type ℓ . Therefore we parametrize the partition function $Z_n(\boldsymbol{\omega})$ describing the partition function of the Gaussian model over a network grown up to iteration n with the parameters $\boldsymbol{\omega} = (\{\mu_\ell\}, \{p_\ell\})$, i.e.

$$Z_n(\boldsymbol{\omega}) = \int \mathcal{D}\phi \prod_{\ell=1}^n \prod_{(r,q) \in E_n^{(\ell)}} z_n^{(\ell)}(\phi_r, \phi_q), \quad (26)$$

where

$$z_n^{(\ell)}(\phi_r, \phi_q) = \exp[-i(1 - \mu_\ell) p_\ell(\phi_r^2 + \phi_q^2) + 2ip_\ell \phi_r \phi_q], \quad (27)$$

and where $E_n^{(\ell)}$ indicates the set of links of type ℓ in a network evolved up to iteration n . The Gibbs measure of this Gaussian model reads

$$P_n(\{\phi\}) = \frac{1}{Z(\boldsymbol{\omega})} \prod_{\ell=1}^n \prod_{(r,q) \in E_n^{(\ell)}} z_n^{(\ell)}(\phi_r, \phi_q) = \frac{1}{Z(\boldsymbol{\omega})} e^{-iH(\{\phi\})}, \quad (28)$$

where the *Hamiltonian* $H(\{\phi\})$ is given by

$$H(\{\phi\}) = \sum_{\ell=1}^n \sum_{(r,q) \in E_n^{(\ell)}} [-(1 - \mu_\ell) p_\ell(\phi_r^2 + \phi_q^2) + 2p_\ell \phi_r \phi_q]. \quad (29)$$

Let us indicate with \mathcal{N}_n the nodes added at iteration n . We consider the following real space renormalization group procedure to calculate the free energy of the Gaussian model. We start with initial conditions $\mu_\ell = \mu$ and $p_\ell = 1$ for all values of $\ell > 0$. At each RG iteration, we integrate over the Gaussian variables ϕ_r associated to nodes $r \in \mathcal{N}_n$ and we rescale the remaining Gaussian variables in order to obtain the renormalized Gibbs measure $P(\{\phi'\})$ of the same type as equation (28) but with rescaled parameters $(\{\mu'_\ell\}, \{p'_\ell\})$, i.e.

$$P_{n-1}(\{\phi'\}) = \int \mathcal{D}\phi^{(n)} P_n(\{\phi\}) \Big|_{\phi'=\mathbf{F}(\{\phi\})}, \quad (30)$$

where

$$\mathcal{D}\phi^{(n)} = \prod_{r \in \mathcal{N}_n} \left(\frac{d\phi_r}{\sqrt{2\pi}} \right). \quad (31)$$

The rescaling of the field is done in such a way that for a d -dimensional deterministic simplicial complex, $p_1 = 1$ at each iteration of the RG flow. Then at each step of the RG transformation we have

$$H(\{\phi\}) \rightarrow H'(\{\phi'\}), \quad (32)$$

where

$$H'(\{\phi'\}) = \sum_{\ell=1}^{n-1} \sum_{(r,q) \in E_{n-1}^{(\ell)}} \{ -(1 - \mu'_\ell) p'_\ell [(\phi'_r)^2 + (\phi'_q)^2] + 2p'_\ell \phi'_r \phi'_q \}. \quad (33)$$

In this way we define a group transformation acting on the model parameters $\omega = (\{\mu_\ell\}, \{p_\ell\})$ so that

$$\omega' = R\omega. \quad (34)$$

Under this renormalization flow, the partition function transforms in the following way:

$$Z_n(\omega) = e^{-N_n g(\omega)} Z_{n-1}(\omega'). \quad (35)$$

Using equations (1) and (5) for the number of nodes N_n at iteration n respectively for the Apollonian and the pseudo-fractal network, the free energy density

$$f = - \lim_{n \rightarrow \infty} \frac{1}{N_n} \ln Z_n(\omega) \quad (36)$$

can be written as

$$f \simeq \sum_{\tau=0}^{\infty} \frac{g(R^{(\tau)}\omega)}{d^\tau} \quad (37)$$

for the Apollonian network and as

$$f \simeq \sum_{\tau=0}^{\infty} \frac{g(R^{(\tau)}\omega)}{(d+1)^\tau} \quad (38)$$

for the pseudo-fractal network.

Interestingly, we anticipate here that the RG flow will be determined by a fixed point having $\mu^* = 0$. This result reveals that indeed the spectral dimension here calculated for normalized Laplacian is universal, i.e. in the large network limit, the spectral dimension of the normalized Laplacian is the same as the spectral dimension of the un-normalized Laplacian as already observed in [39].

5. Apollonian network

5.1. General RG equations

The renormalization group equations for the Apollonian networks of arbitrary topological dimension d can be obtained using the general renormalization group approach described in the previous paragraph. Therefore at each renormalization group step we need first to integrate over the fields ϕ_r with $r \in \mathcal{N}_n$ and subsequently perform a rescaling of the fields to guarantee that $p_1 = 1$ at the next iteration. Since any node $r \in \mathcal{N}_n$ added at generation n is only connected to nodes at the previous generations the integration over the corresponding field ϕ_r can be done independently for any node $r \in \mathcal{N}_n$.

The integration over a single Gaussian variable ϕ_r with $r \in \mathcal{N}_n$ can be easily done and is given by

$$I = \int d\phi_r \prod_{q=1}^d z^{(1)}(\phi_q, \phi_r) = (-\pi i/d)^{1/2} G(\mu_1)^{-1/2} \exp \left[-i(1 - \mu_1) \sum_{q=1}^d \phi_q^2 \right] \times \exp \left[i \frac{\left(\sum_{q=1}^d \phi_q \right)^2}{d(1 - \mu_1)} \right], \quad (39)$$

where

$$G(\mu) = 1 - \mu. \quad (40)$$

Of course, at each step of the RG procedure we will need to integrate over each node $r \in \mathcal{N}_n$. Each of these integrations will contribute to the Hamiltonian $H'(\{\phi'\})$ by a term

$$\left[\left(2 \frac{1}{d(1 - \mu_1)} \right) \phi_q \phi_{q'} \right] \quad (41)$$

for any link (q, q') incident to the to the d -simplex added at iteration n and including node r . Since in the Apollonian network, there are w_ℓ simplices of iteration n incident to a link added at iteration $m = n - \ell$, the overall contribution to the link is

$$\left[\left(2 \frac{1}{d(1 - \mu_1)} \right) w_\ell \phi_q \phi_{q'} \right]. \quad (42)$$

If we focus on the overall contribution to the Hamiltonian proportional to the product of the two field ϕ_q and $\phi_{q'}$ before rescaling we get

$$\left\{ 2 \left[p_{\ell+1} + \left(\frac{1}{d(1 - \mu_1)} \right) w_\ell \right] \phi_q \phi_{q'} \right\}. \quad (43)$$

After rescaling of the fields $\phi_q \rightarrow \phi'_q$ and defining the Gibbs measure over the renormalized network formed by $n - 1$ generation, i.e. putting $\ell \rightarrow \ell - 1$ we should have

$$\left\{ 2 \left[p_{\ell+1} + \left(\frac{1}{d(1 - \mu_1)} \right) w_\ell \right] \phi_q \phi_{q'} \right\} = \left\{ 2 p'_\ell \phi'_q \phi'_{q'} \right\}. \quad (44)$$

Therefore in order to ensure that the value of the parameter p_1 remains fixed at $p'_1 = p_1 = 1$ after each RG iteration we need to rescale the fields by considering the rescaled variables

$$\phi' = \phi \left[p_2 + \frac{d-1}{d(1-\mu_1)} \right]^{1/2}, \tag{45}$$

where we have used $w_1 = (d-1)$. Finally by using equation (3) for w_ℓ , the RG equation for p'_ℓ reads

$$p'_\ell = \left[p_{\ell+1} + \frac{(d-1)(d-2)^{\ell-1}}{d(1-\mu_1)} \right] \left[p_2 + \frac{d-1}{d(1-\mu_1)} \right]^{-1}. \tag{46}$$

Now we will proceed similarly to find the RG equations for μ'_ℓ . Each integration over the Gaussian variable ϕ_r of a node $r \in \mathcal{N}_n$ contributes a factor

$$\left[\left(-(1-\mu_1) + \frac{1}{d(1-\mu_1)} \right) \phi_q^2 \right] \tag{47}$$

to the Hamiltonian for any node q belonging to the d -simplex added at iteration $m = n - \ell$ and including node r . Since there are v_ℓ simplices of iteration n incident to a node added at iteration $m = n - \ell$ the integration over the Gaussian variable at iteration n provides, for each node q , a contribution to the Hamiltonian given by

$$\left[\left(-(1-\mu_1) + \frac{1}{d(1-\mu_1)} \right) v_\ell \phi_q^2 \right]. \tag{48}$$

By identifying the overall term of the Hamiltonian that is proportional to ϕ_q^2 before and after the rescaling of the fields we obtain the equation

$$\begin{aligned} & \left\{ - \sum_{\ell'=1}^{\ell} (1-\mu_{\ell'+1}) p_{\ell'+1} v_{\ell-\ell'} + \left(-(1-\mu_1) + \frac{1}{d(1-\mu_1)} \right) v_\ell (\phi'_q)^2 \right\} \\ & = \left\{ - \sum_{\ell'=1}^{\ell} (1-\mu'_{\ell'}) p'_{\ell'} v_{\ell-\ell'} (\phi'_q)^2 \right\}. \end{aligned} \tag{49}$$

We now make a useful combinatorial observation and we note that the coefficient v_ℓ can be written as

$$v_\ell = \sum_{\ell'=1}^{\ell} v_{\ell-\ell'} c_{\ell'} \tag{50}$$

where c_ℓ is given by

$$c_\ell = (d-2)^{\ell-1}. \tag{51}$$

In fact, by substituting the explicit expression for v_ℓ equation (50) follows directly from the expression

$$d(d-1)^r = \sum_{k=0}^{r-1} d(d-1)^{r-1-k} (d-2)^k + d(d-2)^r. \tag{52}$$

Using equation (50) and the expression for the rescaled field, equation (45), in equation (49) we get the RG equation for μ_ℓ given by

$$(1 - \mu'_\ell) p'_\ell = \left((1 - \mu_1)(d - 2)^{\ell-1} + (1 - \mu_{\ell+1}) p_{\ell+1} - \frac{(d - 2)^{\ell-1}}{d(1 - \mu_1)} \right) \times \left[p_2 + \frac{d - 1}{d(1 - \mu_1)} \right]^{-1}. \tag{53}$$

In summary, in this paragraph we have derived the RG equation for any d -dimensional Apollonian networks which we rewrite here for completeness,

$$(1 - \mu'_\ell) p'_\ell = \left[(1 - \mu_1)(d - 2)^{\ell-1} + (1 - \mu_{\ell+1}) p_{\ell+1} - \frac{(d - 2)^{\ell-1}}{d(1 - \mu_1)} \right] \times \left[p_2 + \frac{d - 1}{d(1 - \mu_1)} \right]^{-1},$$

$$p'_\ell = \left[p_{\ell+1} + \frac{(d - 1)(d - 2)^{\ell-1}}{d(1 - \mu_1)} \right] \left[p_2 + \frac{d - 1}{d(1 - \mu_1)} \right]^{-1}. \tag{54}$$

Under the renormalization group the partition function follows equation (35) with

$$g(\omega) = \frac{N_n}{2N_n} \ln G(\mu_1) + \frac{N_{n-1}}{2N_n} \ln \left[p_2 + \frac{1}{d(1 - \mu_1)} \right] + c, \tag{55}$$

where c is a constant. The first term comes directly from each integration over the variables ϕ_r with $r \in \mathcal{N}_n$ given by equation (39) and the second term comes from the rescaling of the fields. In the following paragraphs we will first study the RG flows in the cases $d = 2$ and $d = 3$ and afterwards we investigate the RG flow in any arbitrary dimension d .

5.2. $d = 2$ Farey graph

For $d = 2$ the Apollonian network reduces to a Farey graphs and the RG equations (54) simplify greatly. In fact we have

$$\begin{aligned} \mu_\ell &= \mu_2, \\ p_\ell &= p \end{aligned} \tag{56}$$

for all $\ell \geq 2$. The renormalization group transformations read then

$$(1 - \mu'_1) = \left((1 - \mu_1) + (1 - \mu_2) p - \frac{1}{2(1 - \mu_1)} \right) \left[p + \frac{1}{2(1 - \mu_1)} \right]^{-1},$$

$$\mu'_2 = \mu_2,$$

$$p' = p \left[p + \frac{1}{2(1 - \mu_1)} \right]^{-1}. \tag{57}$$

Under the renormalization group the partition function follows equation (35) with $g(\omega)$ given by equation (55).

By putting in the zero order approximation $\mu_2 = \mu'_2 = 0$ the renormalization group equations (57) have three fixed points:

$$\begin{aligned} (\mu^*, p^*) &= (0, 0), \\ (\mu^*, p^*) &= (0, 1/2), \\ (\mu^*, p^*) &= (3/2, 0). \end{aligned} \tag{58}$$

Since we are interested in the RG flow starting from an initial condition $\mu \ll 1$ we focus on the fixed points with $\mu^* = 0$. The fixed point $(\mu, p) = (0, 0)$ is unstable as it has two eigenvalues given by $\hat{\lambda}' = 4$ and $\hat{\lambda} = 2$. The fixed point $(\mu, p) = (0, 1/2)$ is associated to the eigenvalues $\lambda_1 = \lambda = 2$ and $\lambda_2 = 1/2$. If we have initial condition $\mu_1 = \mu_2 = \mu$ with $0 < \mu \ll 1$ and $p = 1$ the renormalization flow will first move toward the fixed point $(\mu, p) = (0, 1/2)$ and then will move away from it along its repulsive direction. Close to the $(\mu^*, p^*) = (0, 1/2)$ fixed point, putting $\mu_2 = \mu \ll 1$, the linearized RG equations read

$$\begin{pmatrix} \mu'_1 \\ p' - 1/2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1/4 & 1/2 \end{pmatrix} \begin{pmatrix} \mu_1 \\ p - 1/2 \end{pmatrix} + \frac{\mu}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{59}$$

At iteration τ of the RG flow we have

$$\mu^{(\tau)} = (\lambda^\tau - 1) \frac{\mu}{2}, \tag{60}$$

$$p^{(\tau)} = \frac{1}{2} - \frac{1}{4} \sum_{\tau'=1}^{\tau} 2^{-(\tau-\tau')} \mu^{(\tau')} + 2^{-(\tau+1)}. \tag{61}$$

Therefore for τ large the RG flow runs away from the RG fixed point $(0, 1/2)$ and we can approximate

$$\mu^{(\tau)} \simeq \lambda^\tau \frac{\mu}{2}, \tag{62}$$

$$p^{(\tau)} \simeq \frac{1}{2} - \frac{1}{6} \lambda^\tau \mu. \tag{63}$$

The RG flow is shown in figure 1 where we have set initially $\mu_1 = \mu_2 = \mu = 10^{-4}$. Using equation (36) free-energy density can be therefore written as

$$\begin{aligned} f &= \sum_{\tau=0}^{\infty} \frac{g(R^{(\tau)} \omega)}{d^\tau} \\ &\simeq \sum_{\tau=0}^{\infty} \frac{1}{d^\tau} \left\{ \frac{(d-1)}{2d} \ln(1 - \mu_1^{(\tau)}) + \frac{1}{2d} \ln \left[p^{(\tau)} + \frac{1}{d(1 - \mu_1^{(\tau)})} \right] \right\}. \end{aligned} \tag{64}$$

Therefore we have the spectral density $\rho(\mu)$ given by

$$\begin{aligned} \rho(\mu) &\simeq \frac{2}{\pi} \text{Im} \sum_{\tau=0}^{\infty} \frac{1}{d^\tau} \frac{\partial g(\mu_1^\tau, 1)}{\partial \mu} \\ &\simeq \frac{2}{\pi} \text{Im} \sum_{\tau=0}^{\infty} \frac{\lambda^\tau}{d^\tau} \left\{ \frac{(d-1)}{2d} \frac{1}{1 - \mu_1^{(\tau)}} + \frac{1}{2d} \frac{1}{p^{(\tau)} + 1/ [d(1 - \mu_1^{(\tau)})]} \left(-\frac{1}{3} + \frac{1}{d(1 - \mu_1^{(\tau)})^2} \right) \right\}. \end{aligned}$$

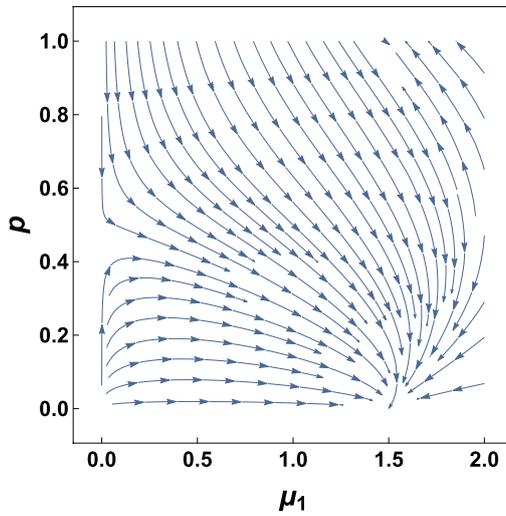


Figure 1. Stream plot of the RG flow for the $d = 2$ Apollonian network given by equations (57) with $\mu_2 = 10^{-4}$.

By approximating the sum over τ with an integral and changing the variable of integration to $z = \lambda^\tau$ upon using the theorem of residues to solve the integral, we can derive the asymptotic scaling of the density of eigenvalues $\rho(\mu)$. This asymptotic scaling for $\mu \ll 1$ is given by

$$\rho(\mu) \simeq C\mu^{d_S/2-1}, \tag{65}$$

where the spectral dimension d_S is given by

$$d_S = 2 \frac{\ln d}{\ln \lambda} = 2. \tag{66}$$

5.3. $d = 3$ Apollonian graph

For $d = 3$ the RG equations (54) simplify significantly. In fact we have

$$\mu_\ell = \mu_1 \tag{67}$$

for all $\ell \geq 1$ and

$$p_\ell = p \tag{68}$$

for all $\ell \geq 2$. The RG equations differ from the ones derived in the case $d = 2$, and they read

$$(1 - \mu'_1) = \left((1 - \mu_1) + (1 - \mu_2)p - \frac{1}{d(1 - \mu_1)} \right) \left[p + \frac{d-1}{2(1 - \mu_1)} \right]^{-1}, \tag{69}$$

$$p' = 1.$$

Under the renormalization group the partition function follows equation (35) with $g(\omega)$ given by equation (55). The renormalization group equations (69) give $p = 1$ and reduce to a single non trivial RG equation for $\hat{\mu} = \mu_1$,

$$(1 - \hat{\mu}') = \left((1 - \hat{\mu}) + (1 - \hat{\mu}) - \frac{1}{d(1 - \hat{\mu})} \right) \left[1 + \frac{d - 1}{d(1 - \hat{\mu})} \right]^{-1}, \quad (70)$$

which has two fixed points:

$$\begin{aligned} \mu^* &= 0 \\ \mu^* &= 4/3. \end{aligned} \quad (71)$$

For $\mu \ll 1$ the relevant fixed point is $\mu^* = 0$ which has a non-trivial associated eigenvalue given by $\lambda = 9/5$. Therefore under the RG flow we have that at iteration τ of the RG flow

$$(\mu_1^{(\tau)}, p^{(\tau)}) = (\lambda^\tau \mu, 1) \quad (72)$$

with μ indicating the initial condition $\mu = \mu_1^{(1)}$. Using equation (36) free-energy density can be therefore written as

$$\begin{aligned} f &= \sum_{\tau=0}^{\infty} \frac{g(R^{(\tau)}\omega)}{d^\tau} \\ &\simeq \sum_{\tau=0}^{\infty} \frac{1}{d^\tau} \left\{ \frac{(d-1)}{2d} \ln(1 - \mu_1^{(\tau)}) + \frac{1}{2d} \ln \left[1 + \frac{d-1}{d(1 - \mu_1^{(\tau)})} \right] \right\}. \end{aligned} \quad (73)$$

Therefore we have the spectral density $\rho(\mu)$ given by

$$\begin{aligned} \rho(\mu) &\simeq \frac{2}{\pi} \text{Im} \sum_{\tau=0}^{\infty} \frac{1}{d^\tau} \frac{\partial g(\mu_1^\tau, 1)}{\partial \mu} \\ &\simeq \frac{2}{\pi} \text{Im} \sum_{\tau=0}^{\infty} \frac{\lambda^\tau}{d^\tau} \left\{ \frac{(d-1)}{2d} \frac{1}{1 - \mu_1^{(\tau)}} + \frac{1}{2d} \frac{1}{\left[d(1 - \mu_1^{(\tau)}) + d - 1 \right]} \frac{d-1}{(1 - \mu_1^{(\tau)})} \right\}. \end{aligned}$$

By proceeding similarly to the case $d = 2$ and approximating the sum over τ with an integral, we obtain the asymptotics

$$\rho(\mu) \cong C \mu^{d_s/2-1} \quad (74)$$

valid for $\mu \ll 1$ with the spectral dimension d_s given by

$$d_s = 2 \frac{\ln d}{\ln \lambda} = 2 \frac{\ln d}{\ln 9/5} = 3.73813\dots \quad (75)$$

Interestingly, Apollonian networks in $d = 3$ are planar. As we will see when comparing the spectral dimension of Apollonian network with the spectral dimension of NGFs, the randomness introduced by the NGF constructions always lower the spectral dimension of the network.

5.4. $d > 3$ dimensional apollonian graph

Let us now determine the RG flow in the general case of a d -dimensional Apollonian network. By putting

$$x_\ell = (1 - \mu_\ell) p_\ell, \tag{76}$$

the RG equations (54) relating the parameters $(\{x_\ell^{(\tau)}\}, \{p_\ell^{(\tau)}\})$ at iteration τ of the RG flow with the parameters $(\{x_\ell^{(\tau+1)}\}, \{p_\ell^{(\tau+1)}\})$ at iteration $\tau + 1$ of the RG flow read

$$\begin{aligned} x_\ell^{(\tau+1)} &= \left[x_\ell^{(\tau)} + \left(x_1^{(\tau)} - \frac{1}{dx_1^{(\tau)}} \right) (d-2)^{\ell-1} \right] \left[p_2^{(\tau)} + \frac{d-1}{dx_1^{(\tau)}} \right]^{-1}, \\ p_\ell^{(\tau+1)} &= \left[p_{\ell+1}^{(\tau)} + \frac{(d-1)(d-2)^{\ell-1}}{dx_1^{(\tau)}} \right] \left[p_2^{(\tau)} + \frac{d-1}{dx_1^{(\tau)}} \right]^{-1}. \end{aligned} \tag{77}$$

In order to solve these equations we use the auxiliary variable $y_1^{(\tau)}$ defined as

$$y_1^{(\tau+1)} = p_2^{(\tau+1)} + \frac{d-1}{dx_1^{(\tau+1)}}. \tag{78}$$

The explicit solution of the RG equations (77) equations read

$$\begin{aligned} p_2^{(\tau+1)} &= \prod_{m=1}^{\tau} \frac{1}{y_1^{(m)}} + \frac{d-1}{d} \sum_{m=1}^{\tau} \frac{(d-2)^{\tau-m+1}}{x^{(m)}} \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}}, \\ y_1^{(\tau+1)} &= p_2^{(\tau+1)} + \frac{(d-1)}{dx_1^{(\tau+1)}} \\ &= \prod_{m=1}^{\tau} \frac{1}{y_1^{(m)}} + \frac{d-1}{d} \sum_{m=1}^{\tau} \frac{(d-2)^{\tau-m+1}}{x^{(m)}} \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}} + \frac{(d-1)}{dx_1^{(\tau+1)}}, \\ x_1^{(\tau+1)} &= x_1^{(1)} \prod_{m=1}^{\tau} \frac{1}{y_1^{(m)}} + \sum_{m=1}^{\tau} \left(x_1^{(m)} - \frac{1}{dx_1^{(m)}} \right) (d-2)^{\tau-m} \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}}. \end{aligned}$$

This explicit solution reveals that $p_2^{(\tau+1)}, y_1^{(\tau+1)}$ and $x_1^{(\tau+1)}$ depend on the dimension d and on all the values that the parameters $p_2^{(\tau')}, y_1^{(\tau')}$ and $x_1^{(\tau')}$ take for $\tau' \leq \tau$. However if we introduce a set of additional auxiliary variables called $A^{(\tau)}, B^{(\tau)}$ and $C^{(\tau)}$ we can express the variables $p_2^{(\tau+1)}, y_1^{(\tau+1)}$ and $x_1^{(\tau+1)}$ only as functions of the value of the additional auxiliary variable at time τ . In fact we have

$$\begin{aligned} y_1^{(\tau+1)} &= A^{(\tau)} + B^{(\tau)}, \\ x_1^{(\tau+1)} &= (1 - \mu)A^{(\tau)} + C^{(\tau)}, \end{aligned} \tag{79}$$

where $A^{(\tau)}, B^{(\tau)}$ and $C^{(\tau)}$ are defined as

$$\begin{aligned} A^{(\tau)} &= \prod_{m=1}^{\tau} \frac{1}{y_1^{(m)}}, \\ B^{(\tau)} &= \frac{d-1}{d} \sum_{m=1}^{\tau} \frac{(d-2)^{\tau-m+1}}{x^{(m)}} \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}} + \frac{(d-1)}{dx_1^{(\tau+1)}}, \\ C^{(\tau)} &= \sum_{m=1}^{\tau} \left(x_1^{(m)} - \frac{1}{dx_1^{(m)}} \right) (d-2)^{\tau-m} \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}}. \end{aligned} \tag{80}$$

The RG equations can be then solved by writing the recursive equations for $A^{(\tau)}$, $B^{(\tau)}$ and $C^{(\tau)}$ that read

$$\begin{aligned} A^{(\tau+1)} &= \frac{1}{y_1^{(\tau+1)}} A^{(\tau)}, \\ B^{(\tau+1)} &= \frac{d-2}{y_1^{(\tau+1)}} B^{(\tau)} + \frac{d-1}{d} \frac{1}{x_1^{(\tau+2)}}, \\ C^{(\tau+1)} &= \frac{(d-2)}{y_1^{(\tau+1)}} C^{(\tau)} + \frac{1}{y_1^{(\tau+1)}} \left(x_1^{(\tau+1)} - \frac{1}{dx_1^{(\tau+1)}} \right). \end{aligned} \tag{81}$$

By using equation (79) this set of equations can be written as a closed set of equations for $A^{(\tau)}$, $B^{(\tau)}$ and $C^{(\tau)}$ only, i.e.

$$\begin{aligned} A^{(\tau+1)} &= \frac{1}{A^{(\tau)} + B^{(\tau)}} A^{(\tau)}, \\ B^{(\tau+1)} &= \frac{d-2}{A^{(\tau)} + B^{(\tau)}} B^{(\tau)} + \frac{d-1}{d} (A^{(\tau)} + B^{(\tau)}) \\ &\quad \times \left[2(1-\mu)A^{(\tau)} + (d-1)C^{(\tau)} - \frac{1}{d} \frac{1}{(1-\mu)A^{(\tau)} + C^{(\tau)}} \right]^{-1}, \\ C^{(\tau+1)} &= \frac{1}{A^{(\tau)} + B^{(\tau)}} \left((d-1)C^{(\tau)} + (1-\mu)A^{(\tau)} - \frac{1}{d[(1-\mu)A^{(\tau)} + C^{(\tau)}]} \right) \end{aligned}$$

with initial conditions $A^{(0)} = 1, B^{(0)} = (d-1)/(d[1-\mu]), C^{(0)} = 0$.

The relevant fixed point of these RG equations for $\mu = 0$ is

$$\begin{aligned} A^* &= 0, \\ B^* &= \frac{d^2 - d - 1}{d}, \\ C^* &= 1. \end{aligned} \tag{82}$$

If we consider the Jacobian matrix of the RG transformation we get the three eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$ given by

$$\begin{aligned} \lambda_1 &= \frac{d^2}{d^2 - d - 1}, \\ \lambda_2 &= \frac{d}{d^2 - d - 1}, \\ \lambda_3 &= 0 \end{aligned} \tag{83}$$

with $\lambda_1 > 1$ and $\lambda_2 < 1$. The right eigenvectors corresponding to these eigenvalues are

$$\begin{aligned} \mathbf{u}_1 &= \frac{1}{c_1} (d+1, -d, d^2 - d + 1), \\ \mathbf{u}_2 &= (1, 0, 0), \\ \mathbf{u}_3 &= \frac{1}{c_3} (d^2 - d - 1, d, d - 1), \end{aligned} \tag{84}$$

where c_1 and c_3 are normalization constants. The left eigenvectors corresponding to these eigenvalues are

$$\begin{aligned} \mathbf{v}_1 &= \frac{1}{d_1} (0, d+1, -d), \\ \mathbf{v}_2 &= \frac{1}{d_2} (-1, d-2, 1), \\ \mathbf{v}_3 &= \frac{1}{d_3} (0, d^2-d+1, d), \end{aligned} \tag{85}$$

where d_1, d_2, d_3 are normalization constants. We call $\mathbf{X}^{(\tau)}$ the column vector

$$\mathbf{X}^{(\tau)} = (A^{(\tau)}, B^{(\tau)}, C^{(\tau)}). \tag{86}$$

Then near the fixed point \mathbf{X}^* given by

$$\mathbf{X}^* = (A^*, B^*, C^*), \tag{87}$$

we have

$$\mathbf{X}^{(\tau)} = \mathbf{X}^* + \sum_{m=1}^3 \lambda_m^\tau \mathbf{v}_m \langle \mathbf{u}_m, \mathbf{X}^{(0)} - \mathbf{X}^* \rangle. \tag{88}$$

To the leading term, we have

$$\mathbf{X}^{(\tau)} = \mathbf{X}^* + \lambda_1^\tau \mathbf{v}_1 \langle \mathbf{u}_1, \mathbf{X}^{(0)} - \mathbf{X}^* \rangle, \tag{89}$$

where the scalar product

$$\langle \mathbf{u}_1, \mathbf{X}^{(0)} - \mathbf{X}^* \rangle \propto (d-1) \frac{\mu}{(1-\mu)}. \tag{90}$$

Under the renormalization group transformations the partition function follows equation (35) with $g(\boldsymbol{\omega})$ given by equation (55). We calculate the leading terms of the density of eigenvalues $\rho(\mu)$ following similar steps used in the case $d=2$ and $d=3$, and find

$$\rho(\mu) \simeq C \mu^{d_s/2-1} \tag{91}$$

for $\mu \ll 1$ with the spectral dimension d_s given by

$$d_s = 2 \frac{\ln d}{\ln \lambda} = -2 \frac{\ln d}{\ln[1 - 1/d - 1/d^2]}. \tag{92}$$

For large d , the spectral dimension d_s scales as

$$d_s \simeq 2 \ln(d) \left[d - \frac{3}{2} + O(1/d) \right]. \tag{93}$$

Therefore the spectral dimension grows with the topological dimension faster than linearly.

5.5. Comparison to NGF with $s = -1$

In this paragraph we conduct the numerical check of our analytical predictions for the spectral dimension of Apollonian networks and we compare the spectral dimension of the Apollonian network with the numerically observed spectral dimension in NGF with flavor $s = -1$ which are also called random Apollonian networks. In figure 2 we compare the spectrum of the Apollonian network of dimension $d = 2$, $d = 3$ and $d = 4$ to the analytical predictions for the spectral dimension finding very good agreement. Additionally we compare the spectrum of the Apollonian network of dimension $d = 2$, $d = 3$ and $d = 4$ to the spectrum of NGFs with flavor $s = -1$ and dimensions $d = 2, 3$ and $d = 4$. From our numerical study of the spectrum of the Apollonian networks and the NGF with $s = -1$ we draw a series of observations.

- Apollonian networks are highly symmetrical structures which implies that the spectrum has many degeneracies that complies with a finite spectral dimension. The randomness present in NGFs reduces the relevance of these degeneracies of eigenvalues as the symmetries are not exact any more for these random structures.
- Any randomness in the generation of the manifolds present in the NGFs can significantly change the spectral dimension. In particular it seems always to lower the spectral dimension. This is in agreement with the intuition that a random walker moving on a hyperbolic manifold like the Apollonian network will ‘experience’ a higher spectral dimension than a random walker moving on a subgraph of this manifold, such as the NGFs $s = -1$.

It would be interesting to have also a RG approach to predict the spectral dimension of NGF. However the extension of the RG approach to random and disordered systems is challenging and it has been so far addressed in the literature only in rare cases (see for instance [53]).

6. Pseudo-fractal networks

6.1. RG equations

In a pseudo-fractal network evolved until generation n , we indicate as type ℓ all links added at iteration $m = n - \ell$. At each iteration n we add a new d -dimensional simplex to every $(d - 1)$ -dimensional face present at the previous iteration, i.e. we attach a new d -dimensional simplex to every face added at any iteration $m < n$. Therefore the RG equation can be written directly starting from the equations valid for the Apollonian graphs by taking into account that each link of type ℓ receives the sum of the contributions coming from the integration of the Gaussian variables associated to nodes added at the last generation. This considerations lead immediately to the RG equations

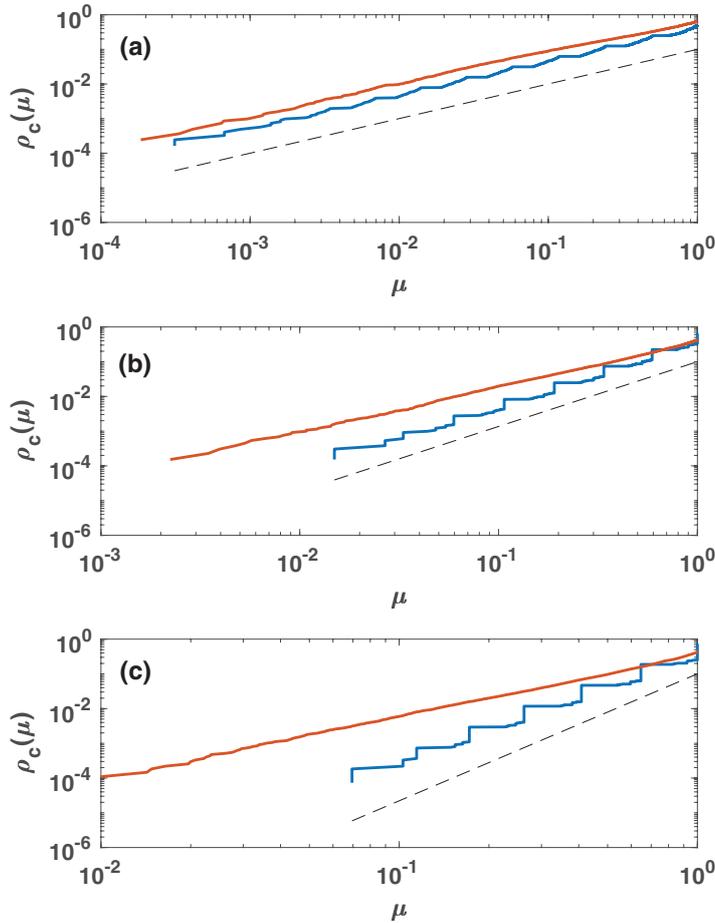


Figure 2. Cumulative distribution of the eigenvalues $\rho_c(\mu)$ for the Apollonian network in $d = 2$ (panel a), $d = 3$ (panel b) and $d = 4$ (panel c) (shown in blue) are compared with the theoretical predictions of the spectral dimension (black dashed line) and with the cumulative distribution of the eigenvalues for the NGF with flavor $s = -1$ (red lines).

$$\begin{aligned}
 x'_\ell &= \left[x_{\ell+1} + \left(x_1 - \frac{1}{dx_1} \right) \sum_{\ell'=0}^{\ell-1} (d-2)^{\ell'} \right] \left[p_2 + \frac{d-1}{dx_1} \right]^{-1}, \\
 p'_\ell &= \left[p_{\ell+1} + \frac{(d-1) \sum_{\ell'=0}^{\ell-1} (d-2)^{\ell'}}{dx_1} \right] \left[p_2 + \frac{d-1}{dx_1} \right]^{-1},
 \end{aligned}
 \tag{94}$$

where x_ℓ are defined in equation (76), $x_\ell = (1 - \mu_\ell) p_\ell$. The free energy is given by equation (38) where by using a procedure similar to the one used to derive the corresponding expression for the Apollonian network we easily find

$$g(\omega) = \frac{\mathcal{N}_n}{2N_n} \ln G(\mu_1) + \frac{N_{n-1}}{2N_n} \ln \left[p_2 + \frac{1}{d(1 - \mu_1)} \right] + c,
 \tag{95}$$

where c is a constant. In the following paragraphs we will study the RG flow and predict the spectral dimension for pseudo-fractal networks for dimension $d = 2, d = 3$ and $d > 3$.

6.2. Case $d = 2$

For $d = 2$ the RG equations (94) for the pseudo-fractal network simplify significantly. We have

$$\mu_\ell = \mu_1 \tag{96}$$

for $\ell \geq 1$ and

$$p_\ell = p \tag{97}$$

for all $\ell \geq 2$. In particular, the RG equations reduce to the same equations found for the Apollonian network in dimension $d = 3$ but with the difference in the value of d . The resulting equations are

$$(1 - \mu'_1) = \left((1 - \mu_1) + (1 - \mu_2)p - \frac{1}{d(1 - \mu_1)} \right) \left[p + \frac{d - 1}{2(1 - \mu_1)} \right]^{-1}, \tag{98}$$

$$p' = 1.$$

The fixed point is $\mu_1^* = 0$ and $p^* = 1$ with eigenvalue $\lambda = 2$. A straightforward calculation of the leading term of the density of eigenvalues for $\mu \ll 1$ leads to the spectral dimension

$$d_s = 2 \frac{\ln(d + 1)}{\ln \lambda} = 2 \frac{\ln 3}{\ln 2} = 3.16993\dots \tag{99}$$

We note here that the pseudo-fractal networks of dimension $d = 2$ are planar. However here they are found to have a smaller spectral dimension of the planar Apollonian networks in dimension $d = 3$. One might naively think that the spectral dimension of the $d = 3$ Apollonian network is the largest spectral dimension of planar networks. However a slight modification of the $d = 2$ pseudo-fractal network in which at each iteration every link is attached to k new triangles gives a spectral dimension that diverges for $k \rightarrow \infty$. So planar networks can have unbounded spectral dimension.

6.3. Case $d = 3$

In the case $d = 3$ the RG equations (94) relating the parameter values at iteration $\tau + 1$ with the parameter values at iteration τ are given by

$$x_\ell^{(\tau+1)} = \left[x_{\ell+1}^{(\tau)} + \left(x_1^{(\tau)} - \frac{1}{dx_1^{(\tau)}} \right) \ell \right] \left[p_2^{(\tau)} + \frac{d - 1}{dx_1^{(\tau)}} \right]^{-1},$$

$$p_\ell^{(\tau+1)} = \left[p_{\ell+1}^{(\tau)} + \frac{(d - 1)\ell}{dx_1^{(\tau)}} \right] \left[p_2^{(\tau)} + \frac{d - 1}{dx_1^{(\tau)}} \right]^{-1}, \tag{100}$$

where x_ℓ is defined in equation (76). In fact these equations can be directly derived from equations (94) by performing the sum over ℓ' in the case $d = 3$. By defining the auxiliary variable

$$y_1^{(\tau+1)} = p_2^{(\tau+1)} + \frac{d-1}{d} \frac{1}{x_1^{(\tau+1)}}, \tag{101}$$

we can express the explicit solution of the RG equations (100) as

$$\begin{aligned} x_1^{(\tau+1)} &= (1-\mu) \prod_{m=1}^{\tau} \frac{1}{y_1^{(m)}} + \sum_{m=1}^{\tau} \left(x_1^{(m)} - \frac{1}{dx_1^{(m)}} \right) (\tau+1-m) \prod_{m'=m}^{\tau} \frac{1}{y^{(m')}}, \\ p_2^{(\tau+1)} &= \prod_{m=1}^{\tau} \frac{1}{y_1^{(m)}} + \frac{d-1}{d} \sum_{m=1}^{\tau} \frac{1}{x_1^{(m)}} (\tau+2-m) \prod_{m'=m}^{\tau} \frac{1}{y^{(m')}}. \end{aligned} \tag{102}$$

We now write this expression in terms of the auxiliary variables $A^{(\tau)}, B^{(\tau)}, C^{(\tau)}$ as

$$\begin{aligned} y_1^{(\tau+1)} &= A^{(\tau)} + B^{(\tau)} + \frac{d-1}{d} \frac{1}{x_1^{(\tau+1)}}, \\ x_1^{(\tau+1)} &= (1-\mu)A^{(\tau)} + C^{(\tau)}, \end{aligned} \tag{103}$$

where we have put

$$\begin{aligned} A^{(\tau)} &= \prod_{m=1}^{\tau} \frac{1}{y_1^{(m)}}, \\ B^{(\tau)} &= \frac{d-1}{d} \sum_{m=1}^{\tau} \frac{1}{x_1^{(m)}} (\tau+2-m) \prod_{m'=m}^{\tau} \frac{1}{y^{(m')}}, \\ C^{(\tau)} &= \sum_{m=1}^{\tau} \left(x_1^{(m)} - \frac{1}{dx_1^{(m)}} \right) (\tau+1-m) \prod_{m'=m}^{\tau} \frac{1}{y^{(m')}}. \end{aligned} \tag{104}$$

In this case it is impossible to write a recursive equation for $A^{(\tau+1)}, B^{(\tau+1)}, C^{(\tau+1)}$ depending only on the variables $A^{(\tau)}, B^{(\tau)}, C^{(\tau)}$. It is possible however to circumvent this difficulty by defining a further pair of auxiliary variables $D^{(\tau)}$ and $E^{(\tau)}$ given by

$$\begin{aligned} D^{(\tau)} &= + \sum_{m=1}^{\tau} \left(x_1^{(m)} - \frac{1}{dx_1^{(m)}} \right) \prod_{m'=m}^{\tau} \frac{1}{y^{(m')}}, \\ E^{(\tau)} &= \frac{d-1}{d} \sum_{m=1}^{\tau} \frac{1}{x_1^{(m)}} \prod_{m'=m}^{\tau} \frac{1}{y^{(m')}}. \end{aligned} \tag{105}$$

In this way we can study the RG flow by studying the behavior of the following set of recursive equations close to their relevant fixed point. These equations read

$$\begin{aligned}
 x_1^{(\tau+1)} &= (1 - \mu)A^{(\tau)} + C^{(\tau)}, \\
 y_1^{(\tau+1)} &= A^{(\tau)} + B^{(\tau)} + \frac{d-1}{d} \frac{1}{[(1 - \mu)A^{(\tau)} + C^{(\tau)}]}, \\
 A^{(\tau+1)} &= \frac{1}{y_1^{(\tau+1)}} A^{(\tau)}, \\
 B^{(\tau+1)} &= \frac{1}{y_1^{(\tau+1)}} \left[B^{(\tau)} + E^{(\tau)} + 2 \frac{d-1}{d} \frac{1}{x_1^{(\tau+1)}} \right], \\
 C^{(\tau+1)} &= \frac{1}{y_1^{(\tau+1)}} \left[C^{(\tau)} + D^{(\tau)} + \left(x_1^{(\tau+1)} - \frac{1}{dx_1^{(\tau+1)}} \right) \right], \\
 D^{(\tau+1)} &= \frac{1}{y_1^{(\tau+1)}} \left[D^{(\tau)} + \left(x_1^{(\tau+1)} - \frac{1}{dx_1^{(\tau+1)}} \right) \right], \\
 E^{(\tau+1)} &= \frac{1}{y_1^{(\tau+1)}} \left[E^{(\tau)} + \frac{d-1}{d} \frac{1}{x_1^{(\tau+1)}} \right],
 \end{aligned} \tag{106}$$

with initial conditions $A^{(1)}, B^{(1)}, C^{(1)}, D^{(1)}, E^{(1)}$ which can be found by inserting in equations (104) and (105) $x_1^{(1)} = 1 - \mu$ and $y_1^{(1)} = [1 + \frac{d+1}{d(1-\mu)}]$. The relevant fixed point of these equations is

$$\begin{aligned}
 A^* &= 0, \\
 B^* &= \frac{1}{6} (1 + 3 + \sqrt{28}), \\
 C^* &= 1, \\
 D^* &= 1/2 \left(-1 + \frac{1}{3} + \frac{\sqrt{28}}{3} \right), \\
 E^* &= 1/2 \left(-1 + \frac{1}{3} + \frac{\sqrt{28}}{3} \right).
 \end{aligned} \tag{107}$$

Close to this fixed point, the RG equations (106) have the relevant eigenvalue

$$\lambda = 1.68471. \tag{108}$$

All other eigenvalues are real non-negative and smaller than one. Therefore they are negligible. By performing the study of the density of eigenvalues $\rho(\mu)$ for $\mu \ll 1$ we can derive the value of the spectral dimension d_s given by

$$d_s = 2 \frac{\ln(d+1)}{\ln \lambda} = 5.31562\dots \tag{109}$$

6.4. Case $d > 3$

For the pseudo-fractal network of dimension $d > 3$, the RG equations (94) relating the parameter values at the iteration $\tau + 1$ of the RG flow to the parameter values at iteration τ , are given by

$$\begin{aligned} x_\ell^{(\tau+1)} &= \left[x_{\ell+1}^{(\tau)} + \left(x_1^{(\tau)} - \frac{1}{dx_1^{(\tau)}} \right) \frac{[(d-2)^\ell - 1]}{d-3} \right] \left[p_2^{(\tau)} + \frac{d-1}{dx_1^{(\tau)}} \right]^{-1}, \\ p_\ell^{(\tau+1)} &= \left[p_{\ell+1}^{(\tau)} + \frac{(d-1)[(d-2)^\ell - 1]}{dx_1^{(\tau)}(d-3)} \right] \left[p_2^{(\tau)} + \frac{d-1}{dx_1^{(\tau)}} \right]^{-1}, \end{aligned} \tag{110}$$

where x_ℓ is defined in equation (76). In order to solve these equations we put

$$y_1^{(\tau+1)} = p_2^{(\tau+1)} + \frac{d-1}{dx_1^{(\tau+1)}}. \tag{111}$$

The solution of the equations (110) reads

$$\begin{aligned} p_2^{(\tau+1)} &= \prod_{m=1}^{\tau} \frac{1}{y_1^{(m)}} + \frac{(d-1)}{d(d-3)} \sum_{m=1}^{\tau} \frac{[(d-2)^{\tau-m+2} - 1]}{x^{(m)}} \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}}, \\ y_1^{(\tau+1)} &= p_2^{(\tau+1)} + \frac{(d-1)}{dx_1^{(\tau+1)}} \\ &= \prod_{m=1}^{\tau} \frac{1}{y_1^{(m)}} + \frac{(d-1)}{d(d-3)} \sum_{m=1}^{\tau} \frac{[(d-2)^{\tau-m+2} - 1]}{x^{(m)}} \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}} + \frac{(d-1)}{dx_1^{(\tau+1)}}, \\ x_1^{(\tau+1)} &= x_1^{(1)} \prod_{m=1}^{\tau} \frac{1}{y_1^{(m)}} + \frac{1}{d-3} \sum_{m=1}^{\tau} \left(x_1^{(m)} - \frac{1}{dx_1^{(m)}} \right) [(d-2)^{\tau+1-m} - 1] \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}}. \end{aligned}$$

Let us now put

$$\begin{aligned} y_1^{(\tau+1)} &= A^{(\tau)} + B^{(\tau)} - D^{(\tau)} + \frac{d-1}{dx_1^{(\tau+1)}}, \\ x_1^{(\tau+1)} &= (1-\mu)A^{(\tau)} + C^{(\tau)} - E^{(\tau)} \end{aligned}$$

with $A^{(\tau)}, B^{(\tau)}, C^{(\tau)}, D^{(\tau)}, E^{(\tau)}$ given by

$$\begin{aligned} A^{(\tau)} &= \prod_{m=1}^{\tau} \frac{1}{y_1^{(m)}}, \\ B^{(\tau)} &= \frac{(d-1)}{d(d-3)} \sum_{m=1}^{\tau} \frac{(d-2)^{\tau-m+2}}{x^{(m)}} \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}}, \\ C^{(\tau)} &= \frac{1}{d-3} \sum_{m=1}^{\tau} \left(x_1^{(m)} - \frac{1}{dx_1^{(m)}} \right) (d-2)^{\tau+1-m} \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}}, \\ D^{(\tau)} &= \frac{(d-1)}{d(d-3)} \sum_{m=1}^{\tau} \frac{1}{x^{(m)}} \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}}, \\ E^{(\tau)} &= \frac{1}{d-3} \sum_{m=1}^{\tau} \left(x_1^{(m)} - \frac{1}{dx_1^{(m)}} \right) \prod_{m'=m}^{\tau} \frac{1}{y_1^{(m')}}. \end{aligned} \tag{112}$$

The RG flow can be cast in a set of recursive equations for $A^{(\tau)}, B^{(\tau)}, C^{(\tau)}, D^{(\tau)}, E^{(\tau)}$. In fact we have

$$\begin{aligned}
 y_1^{(\tau+1)} &= A^{(\tau)} + B^{(\tau)} - D^{(\tau)} + \frac{d-1}{d[(1-\mu)A^{(\tau)} + C^{(\tau)} - E^{(\tau)}]}, \\
 x_1^{(\tau+1)} &= (1-\mu)A^{(\tau)} + C^{(\tau)} - E^{(\tau)}, \\
 A^{(\tau+1)} &= \frac{1}{y_1^{(\tau+1)}}A^{(\tau)}, \\
 B^{(\tau+1)} &= \frac{d-2}{y_1^{(\tau+1)}}B^{(\tau)} + \frac{(d-1)(d-2)^2}{d(d-3)}\frac{1}{y_1^{(\tau+1)}x_1^{(\tau+1)}}, \\
 C^{(\tau+1)} &= \frac{(d-2)}{y_1^{(\tau+1)}}C^{(\tau)} + \frac{(d-2)}{(d-3)}\frac{1}{y_1^{(\tau+1)}}\left(x_1^{(\tau+1)} - \frac{1}{dx_1^{(\tau+1)}}\right), \\
 D^{(\tau+1)} &= \frac{1}{y_1^{(\tau+1)}}D^{(\tau)} + \frac{(d-1)}{d(d-3)}\frac{1}{y_1^{(\tau+1)}x_1^{(\tau+1)}}, \\
 E^{(\tau+1)} &= \frac{1}{y_1^{(\tau+1)}}E^{(\tau)} + \frac{1}{(d-3)}\frac{1}{y_1^{(\tau+1)}}\left(x_1^{(\tau+1)} - \frac{1}{dx_1^{(\tau+1)}}\right)
 \end{aligned} \tag{113}$$

with initial conditions $A^{(1)}, B^{(1)}, C^{(1)}, D^{(1)}, E^{(1)}$ which can be found by inserting $x_1^{(1)} = 1 - \mu$ and $y_1^{(1)} = [1 + \frac{d+1}{d(1-\mu)}]$ in equation (112). The RG flow resulting from these equations can be studied numerically for any finite dimension d . In particular we can find the relevant fixed point and the maximum eigenvalue λ of the Jacobian of the RG equation at the relevant fixed point. Finally by studying the density of eigenvalues $\rho(\mu)$ for $\mu \ll 1$ we can derive the spectral dimension

$$d_s = 2\frac{\ln(d+1)}{\ln \lambda}. \tag{114}$$

The spectral dimension d_s of d -dimensional pseudo-fractal networks with $2 \leq d \leq 20$ is reported in table 1 together with the spectral dimension of the Apollonian networks of the same topological dimension d . Finally we observe that in the large d limit the RG equations (110) for the pseudo-fractal network have the same leading term as the RG equations (77) for the Apollonian networks. Therefore for $d \gg 1$, the largest eigenvalue λ close to the non-trivial fixed point will have the same leading behavior. Indeed we can check numerically (see figure 3) that the largest eigenvalue λ satisfy

$$\lambda = \frac{d^2}{d^2 - d - 1} + \mathcal{O}(d^{-2}), \tag{115}$$

where $d^2/(d^2 - d - 1)$ is the leading eigenvalue of the RG flow for the Apollonian network. It follows that for the pseudo-fractal network the leading term for the spectral dimension is

$$d_s \simeq 2d \ln(d+1). \tag{116}$$

This is confirmed by numerical results shown in figure 4.

Table 1. Numerical values for the spectral dimension d_s of the d -dimensional Apollonian network and of the d -dimensional pseudo-fractal network.

d	d_s Apollonian network	d_s pseudo-fractal network
2	2	3.16993
3	3.73813	5.31562
4	7.39962	8.3761
5	11.729	12.7543
6	16.5732	17.8447
7	21.8337	23.421
8	27.4423	29.3665
9	33.3496	35.610
10	39.5179	42.104
11	45.9179	48.8146
12	52.5262	55.7169
13	59.3233	62.7913
14	66.2933	70.0226
15	73.4226	77.3979
16	80.6995	84.9067
17	88.1141	92.5401
18	95.6574	100.29
19	103.322	108.150
20	111.100	116.114

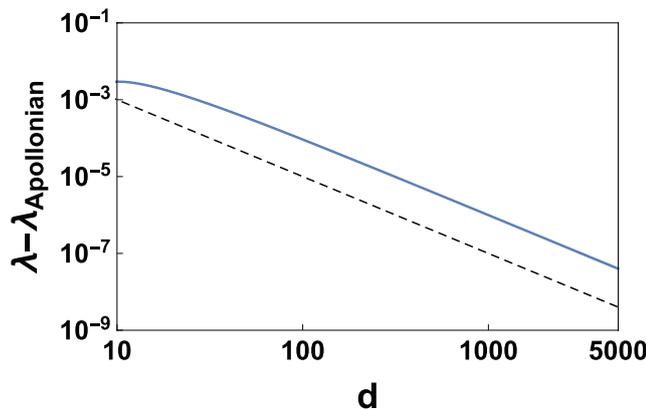


Figure 3. The difference between the leading eigenvalue λ of the RG flow of the pseudo-fractal network and the leading eigenvalue $\lambda_{\text{Apollonian}} = d^2 / (d^2 - d - 1)$ is plotted versus the topological dimension d (solid line). The dashed line with slope -2 is a guide to the eye.

6.5. Comparison to NGF with $s = 0, 1$

In this paragraph we compare the results obtained numerically for the pseudo-fractal networks with the theoretical predictions for $d = 2, d = 3$ and $d = 4$. The numerical calculation of the cumulative distribution $\rho_c(\mu)$ clearly show that the pseudo-fractal networks of topological dimension $d = 2, 3$ and $d = 4$ display the predicted spectral dimension (see figure 5). Moreover, in figure 5 we also compare the results for the cumulative distribution of the eigenvalues $\rho_c(\mu)$ of the pseudo-fractal network to the one obtained for NGFs with flavor $s = 0$ and $s = 1$. We found that the randomness present in the NGF significantly lowers the spectral dimension of the NGF. At the same time

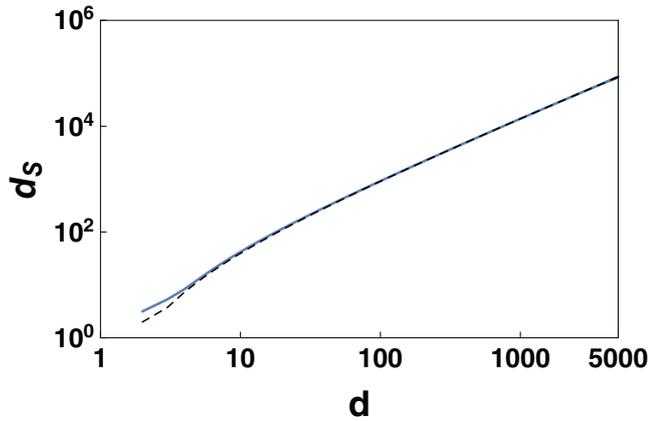


Figure 4. The spectral dimension d_s of the d -dimensional pseudo-fractal network calculated numerically from the RG equations is plotted versus d (solid line) and compared with the spectral dimension of the d -dimensional Apollonian networks (dashed line).

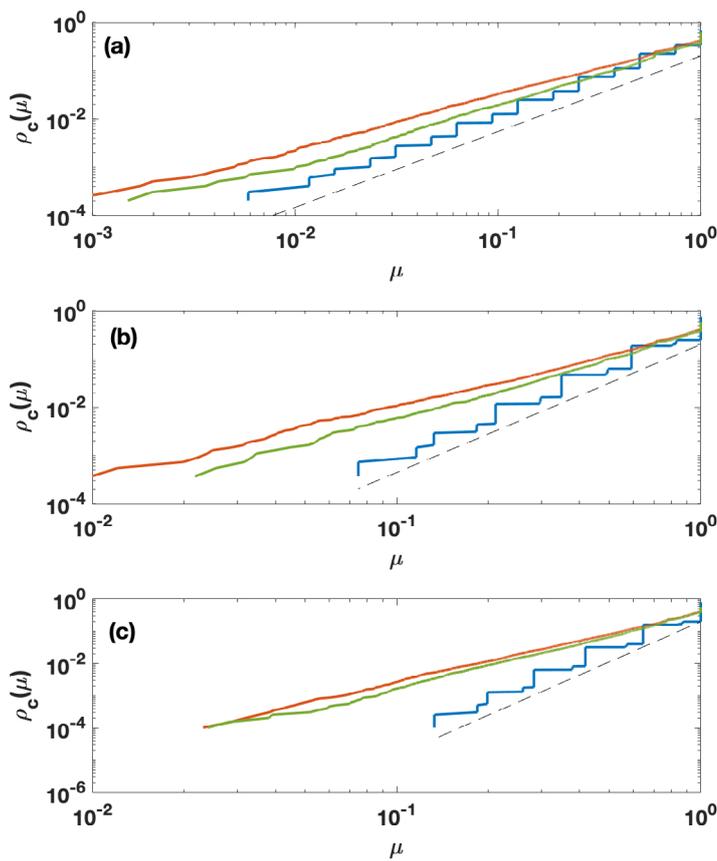


Figure 5. Cumulative distribution of the eigenvalues $\rho_c(\mu)$ for the pseudo-fractal network in (a) $d = 2$, (b) $d = 3$ and (c) $d = 4$ (shown in blue) are compared with the theoretical predictions of the spectral dimension (black dashed line) and with the cumulative distribution of the eigenvalues for the NGF with flavor $s = 0$ (red lines) and $s = 1$ (green lines).

we observe that the degeneracies of eigenvalues present in the pseudo-fractal network are less pronounced for the NGFs.

7. Conclusions

In this work we have studied the spectral dimension of the skeleton of simplicial complexes with distinct geometrical properties: the Apollonian networks, the pseudo-fractal networks and the Network Geometry with Flavor (NGF). The Apollonian networks are non-amenable hyperbolic manifolds, the pseudo-fractal networks are non-amenable branching networks, and the NGF are non-amenable network structures that can be either be hyperbolic manifolds (for $s = -1$) or branching (for $s = 0, s = 1$) simplicial complexes. However while Apollonian networks and pseudo-fractal networks are deterministic the NGF describes a stochastic model. We used the RG approach to predict the spectral dimension of Apollonian and pseudo-fractal networks. We have obtained the functional dependence of the spectral dimension d_S on the topological dimension d of their underlying simplicial complex structure and have found that for large value of d , the spectral dimension d_S scales like $d_S \simeq 2d \ln d$ for both the Apollonian and the pseudo-fractal networks. We have shown that the spectral dimension of the planar Apollonian network of dimension $d = 3$ is $d_S = 2 \ln 3 / \ln(9/5)$. Finally we have studied the effect of randomness on the spectral properties of the networks by comparing the spectrum of the Apollonian network to the spectrum of the NGF with $s = -1$ and the spectrum of the pseudo-fractal network to the spectrum of the NGF with $s = 0$ and $s = 1$. We have found numerically the intuitively reasonable result that randomness can only reduce the spectral dimension of the underlying lattice where some nodes and links are removed. We hope that this work will stimulate further interest in the relations between network geometry and spectral dimension.

Finally we observe that this work can be extended in different directions. On one side, an extension of the RG technique to address the spectral dimension of random topologies might be challenging but would be very much welcome, as real networks are typically driven by a stochastic evolution. On the other side it would be very interesting to investigate further the spectrum of highly symmetrical network structures like the one considered in this paper and predict the degeneracies of eigenvalues.

References

- [1] Bianconi G 2015 *Europhys. Lett.* **111** 56001
- [2] Giusti C, Ghrist R and Bassett D S 2016 *J. Comput. Neurosci.* **41** 1
- [3] Salnikov V, Cassese D and Lambiotte R 2018 *Eur. J. Phys.* **14** 014001
- [4] Kahle M 2014 *AMS Contemp. Math.* **620** 201
- [5] Wu Z, Menichetti G, Rahmede C and Bianconi G 2014 *Sci. Rep.* **5** 10073
- [6] Bianconi G and Rahmede C 2017 *Sci. Rep.* **7** 41974
- [7] Bianconi G and Rahmede C 2016 *Phys. Rev. E* **93** 032315
- [8] Mulder D and Bianconi G 2018 *J. Stat. Phys.* **73** 783
- [9] Courtney O T and Bianconi G 2016 *Phys. Rev. E* **93** 062311
- [10] da Silva D C, Bianconi G, da Costa R A, Dorogovtsev S N and Mendes J F F 2018 *Phys. Rev. E* **97** 032316
- [11] Dorogovtsev S N, Mendes J F F and Samukhin A N 2001 *Phys. Rev. E* **63** 062101

- [12] Petri G and Barrat A 2018 *Phys. Rev. Lett.* **121** 228301
- [13] Andrade J S Jr, Herrmann H J, Andrade R F S and Da Silva L R 2005 *Phys. Rev. Lett.* **94** 018702
- [14] Zhang Z, Rong L and Comellas F 2006 *Physica A* **364** 610
- [15] Dorogovtsev S N, Goltsev A V and Mendes J F F 2002 *Phys. Rev. E* **65** 066122
- [16] Ambjørn J, Jurkiewicz J and Loll R 2005 *Phys. Rev. D* **72** 064014
- [17] Bonzom V, Gurau R, Riello A and Rivasseau V 2011 *Nucl. Phys. B* **853** 174
- [18] Petri P *et al* 2014 *J. R. Soc. Interface* **11** 20140873
- [19] Tumminello M, Aste T, Di Matteo T and Mantegna R N 2005 *Proc. Natl Acad. Sci.* **102** 10421
- [20] Šuvakov M, Andjelković M and Tadić B 2018 *Sci. Rep.* **8** 1987
- [21] Millán A P A P, Torres J J and Bianconi G 2018 *Sci. Rep.* **8** 9910
- [22] Millán A P, Torres J J and Bianconi G 2019 *Phys. Rev. E* **99** 022307
- [23] Skardal P S and Arenas A 2019 *Phys. Rev. Lett.* **122** 248301
- [24] Iacopini I, Petri G, Barrat A and Latora V 2019 *Nat. Commun.* **10** 2485
- [25] Jhun B, Jo M and Kahng B 2019 *J. Stat. Mech.* 123207
- [26] Matamalas J T, Gómez S and Arenas A 2019 (arXiv:1910.03069)
- [27] Kryven I, Ziff R M and Bianconi G 2019 *Phys. Rev. E* **100** 022306
- [28] Bianconi G and Ziff R M 2018 *Phys. Rev. E* **98** 052308
- [29] Bianconi G, Kryven I and Ziff R M 2019 *Phys. Rev. E* **100** 062311
- [30] Dorogovtsev S N, Goltsev A V, Mendes J F F and Samukhin A N 2003 *Phys. Rev. E* **68** 046109
- [31] Samukhin A N, Dorogovtsev S N and Mendes J F F 2008 *Phys. Rev. E* **77** 036115
- [32] Dorogovtsev S N and Mendes J F F 2003 *Evolution of Networks: from Biological Nets to the Internet and WWW* (Oxford: Oxford University Press)
- [33] Aygün E and Erzan A 2011 *J. Phys.: Conf. Ser.* **319** 012007
- [34] Bradde S, Caccioli F, Dall'Asta L and Bianconi G 2010 *Phys. Rev. Lett.* **104** 218701
- [35] Flanagan R, Lacasa L and Nicosia V 2019 (arXiv:1903.00754)
- [36] Rammal R and Toulouse G 1983 *J. Phys. Lett.* **44** 1
- [37] Hwang S, Yun C-K, Lee D-S, Kahng B and Kim D 2010 *Phys. Rev. E* **82** 056110
- [38] Kim D 1984 *J. Korean Phys. Soc.* **17** 3
- [39] Burioni R and Cassi D 1996 *Phys. Rev. Lett.* **76** 1091
- [40] Burioni R, Cassi D and Vezzani A 1999 *Phys. Rev. E* **60** 1500
- [41] Burioni R, Cassi D, Cecconi F and Vulpiani A 2004 *Proteins* **55** 529
- [42] Jonsson T and Wheeler J F 1998 *Nucl. Phys. B* **515** 549
- [43] Durhuus B, Jonsson T and Wheeler J F 2007 *J. Stat. Phys.* **128** 1237
- [44] Ambjørn J, Jurkiewicz J and Loll R 2005 *Phys. Rev. Lett.* **95** 171301
- [45] Benedetti D 2009 *Phys. Rev. Lett.* **102** 111303
- [46] Benedetti D and Henson J 2009 *Phys. Rev. D* **80** 124036
- [47] Boettcher S and Brunson C T 2011 *Front. Physiol.* **2** 102
- [48] Boettcher S, Singh V and Ziff R M 2012 *Nat. Commun.* **3** 787
- [49] Boettcher S, Cook J L and Ziff R M 2009 *Phys. Rev. E* **80** 041115
- [50] Auto D M, Moreira A A, Herrmann H J and Andrade J S Jr 2008 *Phys. Rev. E* **78** 066112
- [51] Rozenfeld H D, Havlin S and Ben-Avraham D 2007 *New J. Phys.* **9** 175
- [52] Rozenfeld H D and Ben-Avraham D 2007 *Phys. Rev. E* **75** 061102
- [53] Samukhin A N, Prigodin V N and Jastrabik L 1997 *Phys. Rev. Lett.* **78** 326