

Generalized Heisenberg Algebras: periodicity and finite representation

Abdessamad Belfakir¹  and Yassine Hassouni

Equipe des Sciences de la Matière et du Rayonnement(ESMAR), Faculty of sciences, Mohammed V University. Av. Ibn Battouta, B.P. 1014, Agdal, Rabat, Morocco

E-mail: abdobelfakir01@gmail.com and yassine.hassouni@gmail.com

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Abstract

We consider the Generalized Heisenberg Algebra (GHA) and the deformed GHA with finite representations. We provide the necessary restriction on the characteristic function of the periodic GHA and that of the periodic deformed GHA and study particular examples. Then, we give a deformation of GHA raising operator of the Morse system and conclude that this system can be described by a periodic deformed GHA as it can be deduced from a nilpotent GHA.

Keywords: Generalized Heisenberg Algebra (GHA), deformed GHA, Morse potential, finite representation

1. Introduction

In the last decade, the Generalized Heisenberg Algebra (GHA) [1, 2] has been introduced and applied to the solution of many quantum physical systems such as a free particle in a square well potential [3], Pöschl–Teller potential [4] and Morse oscillator [5]. Given a physical system, the GHA depends on an analytical function of its Hamiltonian H , $f(H)$ called the characteristic function of the algebra and on the ladder operators A and A^\dagger where $A = (A^\dagger)^\dagger$. The function f lies between two successive energy levels E_n and E_{n+1} as $E_{n+1} = f(E_n)$. In other words, any physical system having $E_{n+1} = f(E_n)$ can be described by GHA [3]. It has been shown that by choosing the appropriate characteristic functions, the GHA contains the standard harmonic oscillator algebra [3, 6] and the q -deformed harmonic oscillator algebra [7, 8], as particular cases.

The GHA plays also a crucial role in the construction of coherent states for several systems in an easier way. These states have been introduced as the eigenvectors of the GHA lowering operator A and it has been shown that they satisfy the set of conditions required to obtain Klauder's coherent states [4, 9–14].

Recently, the deformed Generalized Heisenberg Algebra was introduced [15] in order to find new solvable models from old ones and it was linked with quons and the \mathcal{D} -pseudo-bosons [15, 16].

In the other hand, the Morse potential [17] is a well known solvable model and different realizations of its ladder operators have been provided [18–23]. The GHA describing the Morse oscillator and the representation of the corresponding GHA generators have also been provided [12]. This work aims to show a deformed GHA from which the Morse potential can be deduced and give the restriction on the characteristic function of periodic GHA and that of deformed GHA. The representation of GHA in these cases is finite.

This paper is organized as follows. In section 2, we briefly recall the GHA and the deformed GHA. In section 3 we present the condition on the characteristic function describing the periodic GHA and give some examples. Then, in section 4, we provide the periodic deformed GHA characteristic function and give the periodic deformed GHA describing the Morse potential. Finally, section 5 contains our conclusions.

2. An introduction to GHA and deformed GHA

Let $A = (A^\dagger)^\dagger$ and $J = J^\dagger$ be two operators, the Generalized Heisenberg algebra (GHA) is defined by the following commutation relations

$$JA^\dagger = A^\dagger f(J), \quad (1)$$

$$AJ = f(J)A, \quad (2)$$

and

¹ Author to whom any correspondence should be addressed.

$$[A, A^\dagger] = f(J) - J, \quad (3)$$

where $[A, A^\dagger] = AA^\dagger - A^\dagger A$ and $f(x)$ is an increasing function of a variable x , called the characteristic function of GHA. In most of cases, the hermitian operator J is the dimensionless Hamiltonian of the system under consideration [1–3].

Given an eigenvector $|n\rangle$ of the generator J , the irreducible representation of GHA is given as follows [1–3]

$$J|n\rangle = E_n|n\rangle, \quad (4)$$

$$A^\dagger|n\rangle = \sqrt{E_{n+1} - E_0}|n+1\rangle, \quad (5)$$

$$A|n\rangle = \sqrt{E_n - E_0}|n-1\rangle, \quad (6)$$

where E_{n+1} and E_n are two successive eigenvalues of J such that $E_{n+1} = f(E_n)$ and E_0 is the eigenvalue corresponding to the vacuum state $|0\rangle$ ($A|0\rangle = 0$). It follows that $E_n = f^n(E_0)$ which means that E_n is the n th iterate of E_0 under f . The Casimir operator of GHA is defined as

$$C = A^\dagger A - J. \quad (7)$$

For the particular linear function $f(J) = qJ + 1$ where q is a parameter, the GHA defined in equations (1)–(3) reduces to

$$[J, A^\dagger]_q = A^\dagger, \quad (8)$$

$$[J, A]_{q^{-1}} = -\frac{1}{q}A, \quad (9)$$

and

$$[A, A^\dagger] = (q - 1)J + 1, \quad (10)$$

where $[a, b]_q = ab - qba$ is the q -deformed commutation of two operators a and b . The relations given in equations (8)–(10) contain the q -deformed harmonic oscillator algebra [2]. In the limit $q \rightarrow 1$, the relations defined in equations (8)–(10) recover the standard harmonic oscillator algebra. Thus, the GHA contains the harmonic oscillator algebra and the q -deformed harmonic oscillator algebra by considering $f(J) = J + 1$ and $f(J) = qJ + 1$, respectively [2, 7].

The deformed GHA introduced in [15] is defined through the set of the following commutation rules

$$hb = bf(h), \quad (11)$$

$$ah = f(h)a, \quad (12)$$

and

$$[a, b] = f(h) - h, \quad (13)$$

where a , b and h are three operators defined on a dense domain \mathcal{D} of a given Hilbert space \mathcal{H} and f is some increasing function (we adopt here the same notations as in [15]). We note that if $b = a^\dagger$ and $h = h^\dagger$ the deformed GHA given in equations (11)–(13) becomes the GHA defined in equations (1)–(3). These hermitian relations do not necessarily hold in equations (11)–(13).

Let ϕ_0 and ψ_0 be the eigenvectors of h and h^\dagger , respectively, corresponding to the same eigenvalue ε_0 , $h\phi_0 = \varepsilon_0\phi_0$ and $h^\dagger\psi_0 = \varepsilon_0\psi_0$. The eigenvalues and the eigenvectors of h and h^\dagger are

$$h\phi_n = \varepsilon_n\phi_n, \quad h^\dagger\psi_n = \varepsilon_n\psi_n, \quad (14)$$

where

$$\phi_n = \frac{1}{\sqrt{(\varepsilon_n - \varepsilon_0)!}}(b)^n\phi_0, \quad \psi_n = \frac{1}{\sqrt{(\varepsilon_n - \varepsilon_0)!}}(a^\dagger)^n\psi_0, \quad (15)$$

and $\varepsilon_{n+1} = f(\varepsilon_n)$. The eigenvectors ϕ_n and ψ_n satisfy the following relation

$$\langle\psi_n|\phi_m\rangle = \delta_{n,m}\langle\psi_n|\phi_n\rangle. \quad (16)$$

The action of a , b on a given eigenvector ϕ_n of h (with $n \geq 0$) is

$$a\phi_n = \sqrt{\varepsilon_n - \varepsilon_0}\phi_{n-1}, \quad b\phi_n = \sqrt{\varepsilon_{n+1} - \varepsilon_0}\phi_{n+1}, \quad (17)$$

and the action of a^\dagger , b^\dagger on an eigenvector ψ_n (with $n \geq 0$) of h^\dagger is

$$a^\dagger\psi_n = \sqrt{\varepsilon_{n+1} - \varepsilon_0}\psi_{n+1}, \quad b^\dagger\psi_n = \sqrt{\varepsilon_n - \varepsilon_0}\psi_{n-1}. \quad (18)$$

Let $\mathcal{F}_\phi = \{\phi_n, n \geq 0\}$ and $\mathcal{F}_\psi = \{\psi_n, n \geq 0\}$. Then, b and a^\dagger are creation operators for the vectors in \mathcal{F}_ϕ and \mathcal{F}_ψ , respectively [15].

From equations (17) and (18), it is easily seen that

$$h\phi_n = (ba + \varepsilon_0)\phi_n, \quad h^\dagger\psi_n = (a^\dagger b^\dagger + \varepsilon_0)\psi_n. \quad (19)$$

An easy method to deforming the GHA generators was given in [15], showing how one can deduce other solvable models and deduce the eigenvalues and the eigenfunctions of new non-hermitian Hamiltonians.

3. Periodic GHA

In this section, we provide the added restriction on the characteristic function f defined in equations (1)–(3) for which the cyclic condition $A^n = \underbrace{AA \cdots A}_{n \text{ times}} = A$ holds (with $n \geq 0$).

If $A^n = A$ we have from equation (2)

$$A^n J = f(J)A^n, \quad (20)$$

and

$$J(A^\dagger)^n = (A^\dagger)^n f(J). \quad (21)$$

If we apply equation (20) on a given eigenvector of J , $|m\rangle$ (where $m \geq 0$) we find that

$$\begin{aligned} & \sqrt{(E_{m-n+1} - E_0)(E_{m-n+2} - E_0) \cdots (E_m - E_0)} E_m |m - n\rangle \\ &= E_{m-n+1} \sqrt{(E_{m-n+1} - E_0)(E_{m-n+2} - E_0) \cdots (E_m - E_0)} |m - n\rangle, \end{aligned} \quad (22)$$

it follows then that for $n \geq 0$, $E_m = E_{m-n+1}$. This implies that $E_{m+n} = E_{m+1}$ which implies that $f^n(E_m) = f(E_m)$ (with $(n, m) \geq 0$). Thus, the restriction on the characteristic function for which $A^n = A$ holds is simply

$$f^n(J) = f(J). \quad (23)$$

Now, Let us consider particular functions satisfying equation (23) and give the corresponding representation of GHA generators.

Example 1 Let $f(J) = AA^\dagger = \gamma - J$ and $J = A^\dagger A$, where γ is a complex number. The commutation relations defined in equations (1)–(3) become

$$AJ = (\gamma - J)A, \quad (24)$$

$$JA^\dagger = A^\dagger(\gamma - J), \quad (25)$$

and

$$[A, A^\dagger] = \gamma - 2J. \quad (26)$$

Let us note the eigenvalues of J by λ and the corresponding eigenvectors by $|\lambda\rangle$, i.e.

$$J|\lambda\rangle = \lambda|\lambda\rangle. \quad (27)$$

Since $AA^\dagger \geq 0$ (AA^\dagger is a positive operator), it implies that $\gamma - J \geq 0$. Then, the eigenvalues λ are bounded by γ , $\lambda \leq \gamma$. And since $J = A^\dagger A \geq 0$. Then, $\lambda \geq 0$. Now, let us provide the action of A and A^\dagger on $|\lambda\rangle$. From equation (24) we have $J(A|\lambda\rangle) = A(\gamma - J)|\lambda\rangle = (\gamma - \lambda)(A|\lambda\rangle)$, this implies that

$$A|\lambda\rangle = c(\lambda)|\gamma - \lambda\rangle, \quad (28)$$

where $c(\lambda)$ is a complex number. Similarly, from equation (25) we find that

$$A^\dagger|\lambda\rangle = d(\lambda)|\gamma - \lambda\rangle, \quad (29)$$

where $d(\lambda)$ is a complex number. From equations (28)–(29), it seems that both A and A^\dagger have the same affect on $|\lambda\rangle$.

Let us consider that the eigenvectors of J are normalized, $\langle\lambda|\lambda\rangle = 1$. From equation (28) we have $\langle\lambda|A^\dagger A|\lambda\rangle = |c(\lambda)|^2$. Remembering that $J = A^\dagger A$. Then,

$$\langle\lambda|A^\dagger A|\lambda\rangle = \langle\lambda|J|\lambda\rangle = |c(\lambda)|^2 = \lambda. \quad (30)$$

Then, $c(\lambda)$ can be written as $c(\lambda) = \sqrt{\lambda}e^{i\theta_\lambda}$, where $\theta_\lambda \in \mathbb{R}$. Thus, from equation (28) it follows that

$$A|\lambda\rangle = \sqrt{\lambda}e^{i\theta_\lambda}|\gamma - \lambda\rangle, \quad (31)$$

On the other hand, we have $AA^\dagger = \gamma - J$. Since we have $\langle\gamma - \lambda|\gamma - \lambda\rangle = 1$, it follows from equation (29) that

$$\langle\lambda|AA^\dagger|\lambda\rangle = \langle\lambda|(\gamma - J)|\lambda\rangle = |d(\lambda)|^2 = \gamma - \lambda, \quad (32)$$

which implies that $d(\lambda) = \sqrt{\gamma - \lambda}e^{i\nu_\lambda}$ with $\nu_\lambda \in \mathbb{R}$. Consequently, equation (29) becomes

$$A^\dagger|\lambda\rangle = \sqrt{\gamma - \lambda}e^{i\nu_\lambda}|\gamma - \lambda\rangle. \quad (33)$$

From equations (31) and (33) we have

$$J|\lambda\rangle = A^\dagger A|\lambda\rangle = \lambda e^{i(\theta_\lambda + \nu_{\gamma-\lambda})}|\lambda\rangle, \quad (34)$$

then $\theta_\lambda + \nu_{\gamma-\lambda} = 0$. The equations (31) and (33) together with the relation $J|\lambda\rangle = \lambda|\lambda\rangle$ define the representation of GHA for $f(J) = AA^\dagger = \gamma - J$, $J = A^\dagger A$ and $A^n = A$. Now, let $n = 3$, the constraint $A^3 = A$ implies that $A^2 = \mathbb{I}_2$ where \mathbb{I}_2 denotes the identity operator acting on the space \mathbb{C}^2 . For a given eigenvector $|\lambda\rangle$, we have $A^2|\lambda\rangle = \sqrt{\lambda(\gamma - \lambda)}e^{i(\theta_\lambda + \theta_{\gamma-\lambda})}|\lambda\rangle$. The constraint $A^2 = \mathbb{I}_2$ implies that

$$\begin{cases} \lambda(\gamma - \lambda) = 1, \\ \theta_\lambda + \theta_{\gamma-\lambda} = 0. \end{cases} \quad (35)$$

Consequently, we have two eigenvalues $\lambda_1 = \frac{\gamma + \sqrt{\gamma^2 - 4}}{2}$ and $\lambda_2 = \frac{\gamma - \sqrt{\gamma^2 - 4}}{2}$. Let us note the corresponding eigenvectors by $|+\rangle$ and $|-\rangle$, respectively.

By using equations (31), (33) and (35) the action of J , A and A^\dagger is given by

$$\begin{cases} J|+\rangle = \frac{\gamma + \sqrt{\gamma^2 - 4}}{2}|+\rangle, \\ J|-\rangle = \frac{\gamma - \sqrt{\gamma^2 - 4}}{2}|-\rangle, \end{cases} \quad (36)$$

$$\begin{cases} A|+\rangle = \sqrt{\frac{\gamma + \sqrt{\gamma^2 - 4}}{2}}e^{i\theta_\lambda}|-\rangle, \\ A|-\rangle = \sqrt{\frac{\gamma - \sqrt{\gamma^2 - 4}}{2}}e^{-i\theta_\lambda}|+\rangle, \end{cases} \quad (37)$$

and

$$\begin{cases} A^\dagger|+\rangle = \sqrt{\frac{\gamma - \sqrt{\gamma^2 - 4}}{2}}e^{i\theta_\lambda}|-\rangle, \\ A^\dagger|-\rangle = \sqrt{\frac{\gamma + \sqrt{\gamma^2 - 4}}{2}}e^{-i\theta_\lambda}|+\rangle. \end{cases} \quad (38)$$

If we redefine $|-\rangle = e^{i\theta_\lambda}|-\rangle$, all the phases defined in equations (36)–(38) can be removed. Finally, the only representation of the algebra with $f(J) = AA^\dagger = \gamma - J$, $J = A^\dagger A$ and $A^3 = A$ can be given as

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \gamma - \lambda_1 \end{pmatrix}, \quad A^\dagger = \begin{pmatrix} 0 & \sqrt{\lambda_1} \\ \sqrt{\gamma - \lambda_1} & 0 \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & \sqrt{\gamma - \lambda_1} \\ \sqrt{\lambda_1} & 0 \end{pmatrix}, \quad (39)$$

where $0 \leq \lambda_1 = \frac{\gamma + \sqrt{\gamma^2 - 4}}{2} \leq \gamma$.

Example 2 let us consider a characteristic function of the form $f(J) = \frac{1}{J}$. A possible representation of GHA with $f(J) = AA^\dagger = \frac{1}{J}$, $J = A^\dagger A$ and $A^3 = A$ can be given as

$$J = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \quad A^\dagger = \begin{pmatrix} 0 & x \\ \frac{1}{x} & 0 \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & \frac{1}{x^*} \\ x^* & 0 \end{pmatrix}, \quad (40)$$

where x is a complex number and x^* is its conjugate. The relations $AJ = \frac{1}{J}A$ and $JA^\dagger = A^\dagger \frac{1}{J}$ are satisfied. From the relation $J = A^\dagger A$ it follows that $x = \sqrt{\lambda}e^{i\mu}$ where $\mu \in \mathbb{R}$.

Example 3 Let us consider another example of functions describing a periodic GHA. For $f(J) = J^x = AA^\dagger$ and $J = A^\dagger A$, where x is a complex. for an integer n we have $f^n(J) = J^{(x)^n}$, then $f^n(J) = f(J)$ if and only if $x^n = x$. It follows then that f describes a periodic GHA if $x^n = x$. A matrix representation of A , A^\dagger and J can be deduced for $n = 3$ as given for the examples above. The results in the three examples can be generalized to an arbitrary n .

4. Periodic deformed GHA and Morse potential

In this section, we give the condition on the characteristic function of periodic deformed GHA and give a deformed GHA which contains the Morse potential.

4.1. Morse potential system

The vibrations inside a diatomic molecule can be described by the Morse potential [17] whose model is given by the following Schrödinger equation

$$H\psi(x) = \left(\frac{\hat{p}^2}{2m_r} + V(x) \right) \psi(x) = \left(\frac{\hat{p}^2}{2m_r} + V_0(e^{-2\beta x} - 2e^{-\beta x}) \right) \psi(x) = E\psi(x), \quad (41)$$

where x is the displacement of the two atoms from their equilibrium positions. V_0 is the depth of the potential well at the equilibrium $x = 0$. While, β is the width of the potential and m_r is the reduced mass of the oscillating system composed by two atoms of masses m_1 and m_2 .

The energy spectrum is finite and given by

$$\epsilon_n = -\frac{\hbar^2 \beta^2}{2m_r} (p - n)^2, \quad (42)$$

where

$$p = \frac{\nu - 1}{2}, \quad \nu = \sqrt{\frac{8m_r V_0}{\beta^2 \hbar^2}}, \quad (43)$$

where $n = \{0, 1, 2, \dots, [p]\}$ with $[p]$ is the integer part of p .

The energy eigenfunctions are given by

$$\psi_n^\nu(y) = \mathcal{N}_n e^{-\frac{y}{2}} y^s L_n^{2s}(y), \quad (44)$$

where we have used the change of variable $y = \nu e^{-\beta x}$, and $L_n^{2s}(y)$ are the associated Laguerre functions with $2s = \nu - 2n - 1$ and \mathcal{N}_n is the normalization constant given by

$$\mathcal{N}_n = \sqrt{\frac{\beta(\nu - 2n - 1)\Gamma(n + 1)}{\Gamma(\nu - n)}}. \quad (45)$$

Now, let us recall the action of GHA generators on an eigenvector $|n\rangle$ of the dimensionless Hamiltonian J [5], the set $\{\psi_n^\nu(y)\}$ is a basis of the vector space of Morse potential wave functions, let us note it by \mathcal{H}^M . In the following we will note $\psi_n^\nu(y) \equiv |n\rangle$.

4.2. Action of A , A^\dagger and J

Let $J = H / \left(\frac{\hbar^2 \beta^2}{2m_r} \right)$ whose eigenvalues are $E_n = \epsilon_n / \left(\frac{\hbar^2 \beta^2}{2m_r} \right)$. From equation (42), we have for $n = \{0, 1, 2, \dots, [p] - 1\}$

$$E_{n+1} - E_0 = (n + 1)(2p - n - 1). \quad (46)$$

Then, it follows that for $n = \{0, 1, 2, \dots, [p]\}$

$$A|n\rangle = \sqrt{n(2p - n)} |n - 1\rangle, \quad (47)$$

and

$$A^\dagger |n\rangle = \sqrt{(n + 1)(2p - n - 1)} |n + 1\rangle, \quad (48)$$

for $n = \{0, 1, 2, \dots, [p] - 1\}$. From equation (47) we have the vacuum state condition $A|0\rangle = 0$, and since $A = (A^\dagger)^\dagger$, the action of A^\dagger on $|n_{\max}\rangle$ is

$$A^\dagger |n_{\max}\rangle = 0 \text{ where } n_{\max} = [p]. \quad (49)$$

Then, for any $|n\rangle$ we have $(A^\dagger)^{n_{\max}+1}|n\rangle = 0$ [5]. Then, A^\dagger is a nilpotent operator, otherwise we lose the vacuum state condition. The GHA, in this case, is then a nilpotent algebra and $n_{\max} + 1$ is the index of the nilpotency.

The operator J in terms of A and A^\dagger is given by

$$J = A^\dagger A - p^2. \quad (50)$$

4.3. The characteristic function

Now, we present the characteristic function of GHA which describes the Morse potential. From equation (42) we have for any $n < n_{\max}$

$$E_{n+1} = -(p - (n + 1))^2, \quad (51)$$

this implies that

$$E_{n+1} = E_n + 2\sqrt{-E_n} - 1. \quad (52)$$

Therefore, we can conclude that the characteristic function can be written as follows

$$f(x) = x + 2\sqrt{-x} - 1. \quad (53)$$

Then, the GHA commutation relations defined in equations (1)–(3) become

$$\begin{aligned} [J, A^\dagger] &= 2A^\dagger \sqrt{-J} - A^\dagger, \quad [J, A] = -2\sqrt{-J}A + A, \\ [A, A^\dagger] &= 2\sqrt{-J} - \mathbb{1}, \end{aligned} \quad (54)$$

where $\mathbb{1}$ is the identity operator acting on \mathcal{H}^M .

4.4. Morse potential system from deformed GHA

Following the same approach as in equation (22), the restriction on f defined in equations (11)–(13) for which the operators a and b satisfy $a^n = a$ and $b^n = b$ (with $n \geq 0$) is

$$f^n(h) = f(h). \quad (55)$$

Let $\phi_n \equiv |n\rangle$, where $|n\rangle$ is the eigenvector of the dimensionless Hamiltonian J of Morse potential, the set $\{|n\rangle\}$ form a basis of the Hilbert space \mathcal{H}^M . Let us consider that $b\phi_n = \sqrt{(n + 1)(2p - n - 1)} \phi_{n+1}$ for $n = \{0, 1, \dots, n_{\max}-1\}$ and $b\phi_{n_{\max}} = \phi_0$ and $a\phi_n = \sqrt{n(2p - n)} \phi_{n-1}$ for $n = \{0, 1, \dots, n_{\max}\}$ where $n_{\max} = [p]$. Let us note $N_n = \sqrt{n(2p - n)}$. Then, the matrix representation of b and a in this case can be given as

follow

$$b = \begin{pmatrix} 0 & 0 & 0 & \dots & \dots & 1 \\ N_1 & 0 & 0 & \dots & \dots & 0 \\ 0 & N_2 & 0 & \dots & \dots & 0 \\ 0 & 0 & N_3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & N_{n_{\max}} & 0 \end{pmatrix}, \quad (56)$$

$$a = \begin{pmatrix} 0 & N_1 & 0 & \dots & \dots & 0 \\ 0 & 0 & N_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & N_3 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \ddots & N_{n_{\max}} \\ 0 & 0 & 0 & \dots & \dots & 0 \end{pmatrix}. \quad (57)$$

With these deformed generators a and b , the generator h , in this case, is hermitian and is nothing but the Hamiltonian J . However, in general $h \neq h^\dagger$. The representation of h is

$$h = \begin{pmatrix} E_0 & 0 & 0 & \dots & \dots & 0 \\ 0 & E_1 & 0 & 0 & \dots & \vdots \\ 0 & 0 & E_2 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 & E_{n_{\max}} \end{pmatrix}. \quad (58)$$

From equations (56)–(57) we can conclude that

$$h = ba - p^2, \quad (59)$$

and that

$$h = a^\dagger b^\dagger - p^2, \quad (60)$$

we can see also from equations (56) and (57) that $b^\dagger \neq a$.

We can conclude that the Morse potential system can be described by a deformed GHA in addition to the nilpotent GHA already given in [5] and recalled in the section (4.2). The relation $b\phi_n = \phi_0$ implies that $b^{n_{\max}+1}\phi_0 = \phi_0$. Then $f^{n_{\max}+1}(h) = f(h)$. Thus, the characteristic function of the deformed GHA describing Morse potential is

$$f(J) = J + 2\sqrt{-J} - 1, \text{ with } (f(h))^{n_{\max}+1} = f(h). \quad (61)$$

5. Conclusion

In this work, we have given the necessary and sufficient condition on the characteristic function of periodic GHA and periodic deformed GHA. Then, we have analyzed particular examples. The representation of GHA and deformed GHA generators, in these cases, is finite. Moreover, we have given a periodic deformed GHA describing the Morse system by providing an added and necessary condition on the corresponding creation GHA operator. The Morse potential can be then deduced from a periodic deformed GHA as it can be described by a nilpotent GHA.

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ORCID iDs

Abdessamad Belfakir  <https://orcid.org/0000-0002-8628-7561>

References

- [1] Curado E M F and Rego-Monteiro M A 2000 Thermodynamic properties of a solid exhibiting the energy spectrum given by the logistic map *Phys. Rev. E* **61** 6255
- [2] Curado E M F and Rego-Monteiro M A 2001 Multi-parametric deformed Heisenberg algebras: a route to complexity *J. Phys. A: Math. Gen.* **34** 3253
- [3] Curado E M F, Hassouni Y, Rego-Monteiro M A and Rodrigues L M C S 2008 Generalized Heisenberg algebra and algebraic method: the example of an infinite square-well potential *Phys. Lett. A* **372** 3350–5
- [4] Rego-Monteiro M A, Curado E M F and Rodrigues L M C S 2017 Time evolution of linear and generalized Heisenberg algebra nonlinear Pöschl-Teller coherent states *Phys. Rev. A* **96** 052122
- [5] Hussin V and Marquette I 2011 Generalized Heisenberg Algebras, SUSYQM and degeneracies: infinite well and morse potential *SIGMA* **7** 024
- [6] Gómez C and Sierra G 1993 Quantum harmonic oscillator algebra and link invariants *J. Math. Phys.* **34** 2119
- [7] Arik M and Coon D D 1976 Hilbert spaces of analytic functions and generalized coherent states *J. Math. Phys.* **17** 524
- [8] Bonatsos D and Daskaloyannis C 1999 Quantum groups and their applications in nuclear physics *Prog. Part. Nucl. Phys.* **43** 537
- [9] Klauder J and Skagerstam B 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific) 9971-966-52-2 (<https://doi.org/10.1142/0096>)
- [10] Hassouni Y, Curado E M F and Rego-Monteiro M A 2005 Construction of coherent states for physical algebraic systems *Phys. Rev. A* **71** 022104
- [11] Curado E M F, Rego-Monteiro M A, Rodrigues L M C S and Hassouni Y 2006 Coherent states for a degenerate system: the hydrogen atom *Physica A* **371** 16–9
- [12] Angelova M, Hertz A and Hussin V 2012 Squeezed coherent states and the one-dimensional Morse quantum system *J. Phys. A: Math. Theor.* **45** 244007
- [13] Angelova M and Hussin V 2008 Generalized and Gaussian coherent states for the Morse potential *J. Phys. A: Math. Theor.* **41** 304016
- [14] Berrada K, El Baz M and Hassouni Y 2011 Generalized Heisenberg algebra coherent states for power-law potentials *Phys. Lett. A* **375** 298
- [15] Bagarello F, Curado E M F and Gazeau J P 2018 Generalized Heisenberg algebra and (non linear) pseudo-bosons *J. Phys. A: Math. Theor.* **51** 155201
- [16] Bagarello F 2011 Non linear pseudo-bosons *J. Math. Phys.* **52** 063521
- [17] Morse P M 1929 Diatomic molecules according to the wave mechanics: II. Vibrational levels *Phys. Rev.* **34** 57
- [18] Berrondo M and Palma A 1980 The algebraic approach to the Morse oscillator *J. Phys. A: Math. Gen.* **13** 773

- [19] Balantekin A B 1998 Algebraic approach to shape invariance *Phys. Rev. A* **57** 4188
- [20] Cooper I L 1993 An integrated approach to ladder and shift operators for the Morse oscillator, radial Coulomb and radial oscillator potentials *J. Phys. A: Math. Gen.* **26** 1601
- [21] Dong S, Lemus R and Frank A 2002 Ladder operators for the Morse potential *Int. J. Quantum Chem.* **86** 433–9
- [22] Frank A and Van Isacker P 1994 *Algebraic Methods in Molecular and Nuclear Structure Physics* (Wiley-Interscience publication) (New York: Wiley) 9780471526407
- [23] Iachello F and Levine R D 1995 *Algebraic Theory of Molecules* 99 (Oxford: Oxford University Press) 0-19-508091-2