

# Analytical and numerical solutions for the current and voltage model on an electrical transmission line with time and distance

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## Abstract

This research paper employs three different techniques on the fractional nonlinear space-time telegraph equation to get the solitary traveling wave solutions, semi-analytical wave solution, and numerical solutions. We implement a modified Khater method, Adomian decomposition method, and B-spline techniques (cubic, quantic, and septic) on the fractional telegraph equation. This model is one of the fundamental equations in an electrical transmission and electromagnetic waves that describes the current and voltage on an electrical transmission line with time and distance. It derived by Oliver Heaviside in the 1880s and used to discuss the mirror phenomena of the electromagnetic waves and wave patterns through along line. New structure forms of solitary traveling wave solutions are obtained, and the comparison between the three kinds of solutions is given. The obtained solutions verified with Maple 16 & Mathematica 12 by placing them back into the original equations. The performance of these methods shows the power and effectiveness of them for applying to many different forms of the nonlinear partial differential equation with integer order and fractional order.

**Keywords:** fractional nonlinear space-time telegraph equation, conformable fractional derivative, semi-analytical and approximate solutions, B-spline techniques

(Some figures may appear in colour only in the online journal)

## 1. Introduction

The Fractional nonlinear partial differential equation is a natural extension of a nonlinear partial differential equation with integer order. During the last three decades, fractional calculus has represented in various fields. The essential property in fractional calculus does not appear in the nonlinear partial differential equations with integer order where it only contains a local estate. Many researchers tried to apply many kinds of derivatives on it to transform it into integer order, such as the Caputo fractional derivatives, Riemann–Liouville fractional derivatives, modified Riemann–Liouville derivative, conformable fractional derivative [1–10]. The

main propose of these kinds of derivatives is transformed the fractional nonlinear partial differential equations into integer order nonlinear partial differential equations, which are considered as the first step of solving that kind of equations. While the second step is using the analytical and numerical methods to obtain exact and approximate solutions of these models [11–20]. In general, it challenging to find the exact solutions of fractional nonlinear partial differential equations without using above mentioned fractional derivatives.

In this paper, we study one of the fractional nonlinear partial differential equations which are fractional telegraph equation to show the nonlocal property that is not in the integer order of the same model. Telegraph equation is

one of hyperbolic models in nonlinear partial differential equations [21–25]. This model represents the current and voltage on an electrical transmission line with time and distance. It has many applications like telegraph wires, radio frequency conductors, and so forth. In the general ward, the telegraph equation has fundamental applications in electromagnetic radiation. So, this equation can be applied for high-frequency transmission lines, designing high-voltage transmission lines and electromagnetic radiation. Electromagnetic radiation is a type of radiation encompass visible light, x-rays, radio waves, microwaves, infrared, (visible) light, ultraviolet, and gamma rays. It also refers to the waves of the electromagnetic field radiating through space-time carrying electromagnetic radiant energy. This branch of science began by Maxwell when derived a waveform of the electric and magnetic equations [26, 27]

$$\nabla \cdot E = \begin{cases} 0, \\ -\frac{\partial B}{\partial t}, \end{cases} \quad (1.1)$$

$$\nabla \cdot B = \begin{cases} 0, \\ \mu_0 \epsilon_0 \frac{\partial E}{\partial t}, \end{cases} \quad (1.2)$$

where  $E$ ,  $B$ ,  $\nabla$ ,  $\epsilon_0$ ,  $\mu_0$  represent an electric field, the magnetic field, a vector differential operator, a relative permeability, and the phase velocity, respectively. Equations (1.1), (1.2) show the relationship between the electric and magnetic fields which are related together by Lorentz force equation and the constitutive relations. This radiation has many properties, we will mention some of them in the following lines:

Electromagnetic waves can polarize, reflect, refract, diffract, or interfere with each other, and also, it obeys the properties of superposition. For paradigm, in optics two or more coherent light-waves can react and by deductive or noxious involvement yield a consequent radiance oblique from the sum of the ingredient radiance of the solitary light-waves.

This research treats with one of the models that used to characterize the electromagnetic waves that called the fractional nonlinear space-time telegraph equation. The telegraph equations with an integer order are given by [28–30]

$$\begin{cases} V_x = -L I_t - R I, \\ I_x = -C V_t - G V. \end{cases} \quad (1.3)$$

It also can be written in the following form

$$\begin{cases} V_{xx} = L C V_{tt} + (R C + G L) V_t + G R V, \\ I_{xx} = L C I_{tt} + (R C + G L) I_t + G R I. \end{cases} \quad (1.4)$$

In other hand, the fractional nonlinear space-time telegraph equation can be written is given by

$$D_{tt}^{\alpha} v - D_{xx}^{\alpha} v + D_t^{\alpha} v + b v + d v^3 = 0. \quad (1.5)$$

According to the diversity in applications and versatility in many branches of science, many computational mathematical methods were applied to this model in its both cases (equations (1.5), (1.4))

The strategy of this paper is summed up as follows: section 2, apples the modified auxiliary equation method, the A domain decomposition method, and B-spline techniques to the fractional nonlinear space-time telegraph equation. Section 3, discusses our solutions and making the discussion between our results and that obtained by using different techniques. Section 4, gives the conclusions.

## 2. Application

Using the following definition of the conformable fractional derivative and its properties on equation (1.5)

**Definition 2.1.** The conformable fractional derivative of order  $\vartheta$  [31–35]

$$D_{\vartheta} f(x) = \lim_{\tau \rightarrow 0} \frac{f(t + \tau t^{1-\vartheta}) - f(t)}{\tau} = t^{1-\vartheta} f'(t), \quad (2.1)$$

where  $(0 < \vartheta < 1)$ . Some fundamental features for the conformable fractional derivative as follows:

$$D_t^{\vartheta} t^r = r t^{r-\vartheta}, \quad (2.2)$$

$$D_t^{\vartheta} (f(t)g(t)) = g(t)D_t^{\vartheta} f(t) + f(t)D_t^{\vartheta} g(t), \quad (2.3)$$

$$D_t^{\vartheta} \left( \frac{f(t)}{g(t)} \right) = \frac{g(t)D_t^{\vartheta} f(t) - f(t)D_t^{\vartheta} g(t)}{g^2(t)} \quad (2.4)$$

and the following wave transformation  $\left[ v(x, t) = v(\xi), \xi = \frac{ax^{\vartheta}}{\vartheta} - \frac{ct^{\vartheta}}{\vartheta} \right]$ , yield:

$$(c^2 - a^2)v'' - c v' + b v + d v^3 = 0. \quad (2.5)$$

Applying the balance rule between the highest order derivatives term and nonlinear term to get the value of the balance, leads to  $(v'' \& v^3) \Rightarrow (N = 1)$ .

### 2.1. Analytical solution

Applying the modified modified auxiliary equation method to equation (2.5) enables putting the general form of solution of the telegraph equation in the next form

$$v(\xi) = \sum_{i=1}^n a_i K^{if(\xi)} + \sum_{i=1}^n b_i K^{-if(\xi)} + a_0, \quad (2.6)$$

where  $a_i$ ,  $b_i$ ,  $K$ ,  $(i = 1, 2, 3, \dots)$  are arbitrary constant and  $f(\xi)$  is a solution function of the following auxiliary equation

$$f'(\xi) = \frac{\beta + \alpha K^{-f(\xi)} + \sigma K^{f(\xi)}}{\ln(K)}, \quad (2.7)$$

where  $\beta$ ,  $\alpha$ ,  $\sigma$  are arbitrary constants will be determine later. Substituting equation (2.6) along (2.7) into equation (2.5) and collecting all coefficients of the same power of  $K^{f(\xi)}$ , give a system of algebraic equations. Solving this system of equation, yields

Family I:

$$a_0 = \frac{\sqrt{b}(\beta(\sqrt{\beta^2 - 4\alpha\sigma} + \beta) - 4\alpha\sigma)}{2\sqrt{\beta^2 - 4\alpha\sigma}\sqrt{-d(\beta^2 - 4\alpha\sigma)}}, a_1 = 0,$$

$$b_1 = \frac{\alpha\sqrt{b}}{\sqrt{4\alpha d\sigma - \beta^2 d}}, c = \frac{3b}{2\sqrt{\beta^2 - 4\alpha\sigma}}, a = \frac{\sqrt{2b - 9b^2}}{2\sqrt{4\alpha\sigma - \beta^2}},$$

where  $d < 0$ ,  $b \in ]\frac{2}{9}, \infty[$ .

When,  $[\sigma \neq 0]$ :

$$\nu(x, t) = \frac{\sqrt{b}}{2\sqrt{-d(\beta^2 - 4\alpha\sigma)}} \left[ \frac{\beta^2}{\sqrt{\beta^2 - 4\alpha\sigma}} + \alpha\sigma \left( -\frac{4}{\sqrt{\beta^2 - 4\alpha\sigma}} \right. \right. \\ \left. \left. - \frac{4}{\sqrt{\beta^2 - 4\alpha\sigma} \tanh \left( \frac{\frac{\sqrt{(2-9b)b}x^\vartheta \sqrt{\beta^2 - 4\alpha\sigma} - 3bt^\vartheta}}{\frac{\sqrt{4\alpha\sigma - \beta^2}}{4\vartheta}} \right) + \beta} \right) + \beta \right], \quad (2.8)$$

$$\nu(x, t) = \frac{\sqrt{b}}{2\sqrt{-d(\beta^2 - 4\alpha\sigma)}} \left[ \frac{\beta^2}{\sqrt{\beta^2 - 4\alpha\sigma}} + \alpha\sigma \left( -\frac{4}{\sqrt{\beta^2 - 4\alpha\sigma}} \right. \right. \\ \left. \left. - \frac{4}{\sqrt{\beta^2 - 4\alpha\sigma} \coth \left( \frac{\frac{\sqrt{(2-9b)b}x^\vartheta \sqrt{\beta^2 - 4\alpha\sigma} - 3bt^\vartheta}}{\frac{\sqrt{4\alpha\sigma - \beta^2}}{4\vartheta}} \right) + \beta} \right) + \beta \right]. \quad (2.9)$$

When,  $[\alpha = -\sigma]$ :

$$\nu(x, t) = \frac{\sqrt{b}}{2\sqrt{-d(\beta^2 - 4\alpha\sigma)}} \left[ \frac{\beta^2}{\sqrt{\beta^2 - 4\alpha\sigma}} + 4\alpha \left( \frac{\alpha}{\sqrt{\beta^2 - 4\alpha\sigma}} \right. \right. \\ \left. \left. + \frac{\alpha}{\sqrt{4\alpha^2 + \beta^2} \tanh \left( \frac{\sqrt{4\alpha^2 + \beta^2} \left( \frac{\sqrt{(2-9b)b}x^\vartheta \sqrt{\beta^2 - 4\alpha\sigma} - 3bt^\vartheta}}{\frac{\sqrt{4\alpha\sigma - \beta^2}}{4\vartheta}} - \frac{3bt^\vartheta}{\sqrt{\beta^2 - 4\alpha\sigma}} \right) \right) + \beta} \right) + \beta \right], \quad (2.10)$$

$$\nu(x, t) = \frac{\sqrt{b}}{2\sqrt{-d(\beta^2 - 4\alpha\sigma)}} \left[ \frac{\beta^2}{\sqrt{\beta^2 - 4\alpha\sigma}} + 4\alpha \left( \frac{\alpha}{\sqrt{\beta^2 - 4\alpha\sigma}} \right. \right. \\ \left. \left. + \frac{\alpha}{\sqrt{4\alpha^2 + \beta^2} \coth \left( \frac{\sqrt{4\alpha^2 + \beta^2} \left( \frac{\sqrt{(2-9b)b}x^\vartheta}{\sqrt{4\alpha\sigma - \beta^2}} - \frac{3bt^\vartheta}{\sqrt{\beta^2 - 4\alpha\sigma}} \right) \right) + \beta} \right) + \beta \right] \quad (2.11)$$

When,  $[\alpha = \sigma]$ :

$$\nu(x, t) = \frac{\sqrt{b}}{2\sqrt{-d(\beta^2 - 4\alpha^2)}} \left[ -\frac{4\alpha^2}{\sqrt{\beta^2 - 4\alpha^2}} + \frac{\beta^2}{\sqrt{\beta^2 - 4\alpha^2}} \right. \\ \left. - \frac{4\alpha^2}{\sqrt{\beta^2 - 4\alpha^2} \tanh \left( \frac{\sqrt{\beta^2 - 4\alpha^2} \left( \frac{\sqrt{(2-9b)b}x^\vartheta}{\sqrt{4\alpha^2 - \beta^2}} - \frac{3bt^\vartheta}{\sqrt{\beta^2 - 4\alpha^2}} \right) \right) + \beta} \right] + \beta, \quad (2.12)$$

$$\nu(x, t) = \frac{\sqrt{b}}{2\sqrt{-d(\beta^2 - 4\alpha^2)}} \left[ -\frac{4\alpha^2}{\sqrt{\beta^2 - 4\alpha^2}} + \frac{\beta^2}{\sqrt{\beta^2 - 4\alpha^2}} \right. \\ \left. - \frac{4\alpha^2}{\sqrt{\beta^2 - 4\alpha^2} \coth \left( \frac{\sqrt{\beta^2 - 4\alpha^2} \left( \frac{\sqrt{(2-9b)b}x^\vartheta}{\sqrt{4\alpha^2 - \beta^2}} - \frac{3bt^\vartheta}{\sqrt{\beta^2 - 4\alpha^2}} \right) \right) + \beta} \right] + \beta. \quad (2.13)$$

*Family II*

$$a_0 = \frac{\sqrt{b}(\beta(\sqrt{\beta^2 - 4\alpha\sigma} + \beta) - 4\alpha\sigma)}{2\sqrt{\beta^2 - 4\alpha\sigma}\sqrt{-d(\beta^2 - 4\alpha\sigma)}}, a_1 = \frac{\sqrt{b}\sigma}{\sqrt{4\alpha d\sigma - \beta^2 d}}, \\ b_1 = 0, c = -\frac{3b}{2\sqrt{\beta^2 - 4\alpha\sigma}}, a = -\frac{\sqrt{2b - 9b^2}}{2\sqrt{4\alpha\sigma - \beta^2}},$$

where  $d < 0$ ,  $b \in ]\frac{2}{9}, \infty[$ .

When,  $[\sigma \neq 0]$ :

$$\nu(x, t) = -\frac{\sqrt{b}\sqrt{\beta^2 - 4\alpha\sigma}}{2\sqrt{-d(\beta^2 - 4\alpha\sigma)}} \left[ \tanh\left(\frac{3bt^\vartheta + \frac{\sqrt{(2-9b)b}x^\vartheta\sqrt{4\alpha\sigma - \beta^2}}{\sqrt{\beta^2 - 4\alpha\sigma}}}{4\vartheta}\right) - 1 \right], \quad (2.14)$$

$$\nu(x, t) = -\frac{\sqrt{b}\sqrt{\beta^2 - 4\alpha\sigma}}{2\sqrt{-d(\beta^2 - 4\alpha\sigma)}} \left[ \coth\left(\frac{3bt^\vartheta + \frac{\sqrt{(2-9b)b}x^\vartheta\sqrt{4\alpha\sigma - \beta^2}}{\sqrt{\beta^2 - 4\alpha\sigma}}}{4\vartheta}\right) - 1 \right]. \quad (2.15)$$

When,  $[\alpha = -\sigma]$ :

$$\begin{aligned} \nu(x, t) = & \frac{\sqrt{b}}{2\alpha\sqrt{\beta^2 - 4\alpha\sigma}\sqrt{-d(\beta^2 - 4\alpha\sigma)}} \left[ -4\alpha^2\sigma + \alpha\beta(\sqrt{\beta^2 - 4\alpha\sigma} + \beta) + \beta\sigma\sqrt{\beta^2 - 4\alpha\sigma} \right. \\ & \left. + \sigma\sqrt{4\alpha^2 + \beta^2}\sqrt{\beta^2 - 4\alpha\sigma} \tanh\left(\frac{\sqrt{4\alpha^2 + \beta^2}\left(\frac{3bt^\vartheta}{\sqrt{\beta^2 - 4\alpha\sigma}} - \frac{\sqrt{(2-9b)b}x^\vartheta}{\sqrt{4\alpha\sigma - \beta^2}}\right)}{4\vartheta}\right) \right], \end{aligned} \quad (2.16)$$

$$\begin{aligned} \nu(x, t) = & \frac{\sqrt{b}}{2\alpha\sqrt{\beta^2 - 4\alpha\sigma}\sqrt{-d(\beta^2 - 4\alpha\sigma)}} \left[ -4\alpha^2\sigma + \alpha\beta(\sqrt{\beta^2 - 4\alpha\sigma} + \beta) + \beta\sigma\sqrt{\beta^2 - 4\alpha\sigma} \right. \\ & \left. + \sigma\sqrt{4\alpha^2 + \beta^2}\sqrt{\beta^2 - 4\alpha\sigma} \coth\left(\frac{\sqrt{4\alpha^2 + \beta^2}\left(\frac{3bt^\vartheta}{\sqrt{\beta^2 - 4\alpha\sigma}} - \frac{\sqrt{(2-9b)b}x^\vartheta}{\sqrt{4\alpha\sigma - \beta^2}}\right)}{4\vartheta}\right) \right]. \end{aligned} \quad (2.17)$$

When,  $[\alpha = \sigma]$ :

$$\begin{aligned} \nu(x, t) = & \frac{\sqrt{b}}{2\alpha\sqrt{\beta^2 - 4\alpha^2}\sqrt{-d(\beta^2 - 4\alpha^2)}} \left[ -4\alpha^3 + \alpha\beta(\sqrt{\beta^2 - 4\alpha^2} + \beta) - \alpha\beta\sqrt{\beta^2 - 4\alpha^2} \right. \\ & \left. - \alpha(\beta^2 - 4\alpha^2) \tanh\left(\frac{\sqrt{\beta^2 - 4\alpha^2}\left(\frac{3bt^\vartheta}{\sqrt{\beta^2 - 4\alpha^2}} - \frac{\sqrt{(2-9b)b}x^\vartheta}{\sqrt{4\alpha^2 - \beta^2}}\right)}{4\vartheta}\right) \right], \end{aligned} \quad (2.18)$$

$$\begin{aligned} \nu(x, t) = & \frac{\sqrt{b}}{2\alpha\sqrt{\beta^2 - 4\alpha^2}\sqrt{-d(\beta^2 - 4\alpha^2)}} \left[ -4\alpha^3 + \alpha\beta(\sqrt{\beta^2 - 4\alpha^2} + \beta) - \alpha\beta\sqrt{\beta^2 - 4\alpha^2} \right. \\ & \left. - \alpha(\beta^2 - 4\alpha^2) \coth\left(\frac{\sqrt{\beta^2 - 4\alpha^2}\left(\frac{3bt^\vartheta}{\sqrt{\beta^2 - 4\alpha^2}} - \frac{\sqrt{(2-9b)b}x^\vartheta}{\sqrt{4\alpha^2 - \beta^2}}\right)}{4\vartheta}\right) \right]. \end{aligned} \quad (2.19)$$

## 2.2. Semi-analytical solution

Implement of the Adomian decomposition method enables rewriting equation (2.5) to be in the following form:

$$L \nu(\xi) + R \nu(\xi) + N \nu(\xi) = 0, \quad (2.20)$$

where  $L$ ,  $R$ ,  $N$  represent a differential operator, a linear operator and nonlinear term, respectively. Using the inverse operator  $L^{-1}$  on (2.20), we get

$$\begin{aligned} \sum_{i=0}^{\infty} \nu_i(\xi) = \nu(0) + \nu'(0)\xi - \frac{c}{c^2 - a^2} L^{-1} \left( \sum_{i=0}^{\infty} (\nu_i)' \right) - \frac{b}{c^2 - a^2} L^{-1} \left( \sum_{i=0}^{\infty} \nu_i \right) \\ - \frac{d}{c^2 - a^2} L^{-1} \left( \sum_{i=0}^{\infty} A_i \right). \end{aligned} \quad (2.21)$$

Under the following condition [ $\alpha = 2$ ,  $\beta = 3$ ,  $b = 4$ ,  $d = -4$ ,  $\sigma = 1$ ] on equation (2.8), we get:

$$\nu_{\text{exact}} = \frac{2e^\xi}{2e^\xi + 1}. \quad (2.22)$$

So that, we obtain:

$$\nu_0 = \frac{2\xi}{9} + \frac{2}{3}, \quad (2.23)$$

$$\nu_1 = -\frac{4\xi^5}{3645} - \frac{4\xi^4}{243} - \frac{2\xi^3}{81} + \frac{19\xi^2}{27} - \frac{4\xi}{9} - \frac{4}{15}, \quad (2.24)$$

$$\begin{aligned} \nu_2 = \frac{32\xi^{10}}{39858075} + \frac{64\xi^9}{2657205} + \frac{164\xi^8}{688905} - \frac{44\xi^7}{688905} - \frac{398\xi^6}{32805} \\ - \frac{1069\xi^5}{18225} + \frac{268\xi^4}{3645} + \frac{931\xi^3}{1215} - \frac{94\xi^2}{135}. \end{aligned} \quad (2.25)$$

$$\begin{aligned} \nu_3 = -\frac{16\xi^{13}}{1554464925} - \frac{512\xi^{12}}{1315316475} - \frac{224\xi^{11}}{48715425} + \frac{92\xi^{10}}{10333575} + \frac{3239\xi^9}{6200145} + \frac{1919\xi^8}{1148175} \\ - \frac{8377\xi^7}{382725} - \frac{289\xi^6}{4050} + \frac{2459\xi^5}{12150} + \frac{1981\xi^4}{4860} - \frac{1202\xi^3}{2025} + \frac{16\xi^2}{75}. \end{aligned} \quad (2.26)$$

According equations (2.23)–(2.26), we get an approximate solution of equation (2.5) in the next formula

$$\begin{aligned} \nu_{\text{approximate}} = -\frac{16\xi^{13}}{1554464925} - \frac{512\xi^{12}}{1315316475} - \frac{224\xi^{11}}{48715425} + \frac{2708\xi^{10}}{279006525} + \frac{2033\xi^9}{3720087} + \frac{6577\xi^8}{3444525} \\ - \frac{75613\xi^7}{3444525} - \frac{27389\xi^6}{328050} + \frac{1733\xi^5}{12150} + \frac{1355\xi^4}{2916} + \frac{899\xi^3}{6075} + \frac{149\xi^2}{675} - \frac{2\xi}{9} + \frac{2}{5}. \end{aligned} \quad (2.27)$$

## 2.3. Numerical solutions

This section study the numerical solutions of the fractional nonlinear space-time telegraph equation via applying the B-spline techniques to which one of them is considered as the most suitable method of this model. Using the exact solution equation (2.22) with the following initial conditions  $\left[ \nu(0) = \frac{2}{3}, \nu(1) = \frac{2e}{1+2e}, \nu'(0) = \frac{2}{9}, \nu'(1) = \frac{2e}{1+2e} - \frac{4e^2}{(1+2e)^2}, \nu''(0) = -\frac{2}{27}, \nu''(1) = 2e \left( \frac{8e^2}{(1+2e)^3} - \frac{2e}{(1+2e)^2} \right) + \frac{2e}{1+2e} - \frac{8e^2}{(1+2e)^2} \right]$ , allows applying the following B-spline schemes, as follows:

**2.3.1. Cubic-spline.** Using this kind of techniques allows putting the approximate solution of equation (2.5) in the next formula

$$v(\xi) = \sum_{i=-1}^{n+1} c_i B_i, \quad (2.28)$$

where  $c_i$ ,  $B_i$  satisfy the collection condition, respectively:

$$L v(\xi) = f(\xi_i, v(\xi_i)) \text{ where } (i = 0, 1, \dots, n)$$

and

$$B_i(\xi) = \frac{1}{6h^3} \begin{cases} (\xi - \xi_{i-2})^3, & \xi \in [\xi_{i-2}, \xi_{i-1}], \\ -3(\xi - \xi_{i-1})^3 + 3h(\xi - \xi_{i-1})^2 + 3h^2(\xi - \xi_{i-1}) + h^3, & \xi \in [\xi_{i-1}, \xi_i], \\ -3(\xi_{i+1} - \xi)^3 + 3h(\xi_{i+1} - \xi)^2 + 3h^2(\xi_{i+1} - \xi) + h^3, & \xi \in [\xi_i, \xi_{i+1}], \\ (\xi_{i+2} - \xi)^3, & \xi \in [\xi_{i+1}, \xi_{i+2}], \\ 0, & \text{otherwise.} \end{cases} \quad (2.29)$$

where  $i \in [-2, n+2]$ , we obtain

$$v_i(\xi) = c_{i-1} + 4c_i + c_{i+1}. \quad (2.30)$$

Substituting equation (2.30) and its derivatives into equation (2.5), we get  $(n+3)$  of equations. Solving this system of equations to get the value of  $c_i$ , we obtain

**2.3.2. Quantic-spline.** Using this kind of techniques allows putting the approximate solution of equation (2.5) in the next formula

$$v(\xi) = \sum_{i=-1}^{n+1} c_i B_i, \quad (2.31)$$

where  $c_i$ ,  $B_i$  satisfy the collection condition, respectively:

$$L v(\xi) = f(\xi_i, v(\xi_i)) \text{ where } (i = 0, 1, \dots, n)$$

and

$$B_i(\xi) = \frac{1}{h^5} \begin{cases} (\xi - \xi_{i-3})^5, & \xi \in [\xi_{i-3}, \xi_{i-2}], \\ (\xi - \xi_{i-3})^5 - 6(\xi - \xi_{i-2})^5, & \xi \in [\xi_{i-2}, \xi_{i-1}], \\ (\xi - \xi_{i-3})^5 - 6(\xi - \xi_{i-2})^5 + 15(\xi - \xi_{i-1})^5, & \xi \in [\xi_{i-1}, \xi_i], \\ (\xi_{i+3} - \xi)^5 - 6(\xi_{i+2} - \xi)^5 + 15(\xi_{i+1} - \xi)^5, & \xi \in [\xi_i, \xi_{i+1}], \\ (\xi_{i+3} - \xi)^5 - 6(\xi_{i+2} - \xi)^5, & \xi \in [\xi_{i+1}, \xi_{i+2}], \\ (\xi_{i+3} - \xi)^5, & x \in [\xi_{i+2}, \xi_{i+3}], \\ 0, & \text{otherwise.} \end{cases} \quad (2.32)$$

where  $i \in [-2, n+2]$ , we obtain

$$v_i(\xi) = c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}. \quad (2.33)$$

Substituting equation (2.33) and its derivatives into equation (2.5), we get  $(n+5)$  of equations. Solving this system of equations to get the value of  $c_i$ , we obtain

**2.3.3. Septic-spline.** Using the septic spline method allows putting the approximate solution of equation (2.5) in the next formula

$$v(\xi) = \sum_{i=-1}^{n+1} c_i B_i, \quad (2.34)$$

**Table 1.** Analytical, semi-analytical, and absolute error between them.

Value of $\xi$	Approximate	Exact	Absolute value of error
0.01	0.397 800 004 496 347	0.668 885 172 890 993	0.271 085 168 394 646
0.02	0.395 645 110 519 643	0.671 096 198 355 410	0.275 451 087 835 766
0.03	0.393 536 375 349 101	0.673 299 670 844 592	0.279 763 295 495 491
0.04	0.391 474 971 077 693	0.675 495 519 388 437	0.284 020 548 310 744
0.05	0.389 462 186 169 816	0.677 683 674 266 821	0.288 221 488 097 004
0.06	0.387 499 426 955 551	0.679 864 067 013 955	0.292 364 640 058 404
0.07	0.385 588 219 060 458	0.682 036 630 422 358	0.296 448 411 361 900
0.08	0.383 730 208 769 824	0.684 201 298 546 419	0.300 471 089 776 594
0.09	0.381 927 164 326 310	0.686 358 006 705 584	0.304 430 842 379 273
0.1	0.380 180 977 159 943	0.688 506 691 487 145	0.308 325 714 327 201

**Table 2.** Analytical, approximate solution, and absolute value of error between them that obtained by using cubic-spline.

Value of $\xi$	Approximate	Exact	Absolute value of error
0	0.222 222 222 222 222	0.666 666 666 666 666	0.444 444 444 444 444
0.1	0.239 657 614 223 278	0.688 506 691 487 145	0.448 849 077 263 866
0.2	0.257 930 634 646 522	0.709 539 212 929 809	0.451 608 578 283 286
0.3	0.276 987 415 392 828	0.729 709 101 066 375	0.452 721 685 673 547
0.4	0.296 753 111 678 516	0.748 973 892 837 003	0.452 220 781 158 487
0.5	0.317 128 456 252 841	0.767 303 462 381 101	0.450 175 006 128 259
0.6	0.337 985 549 279 125	0.784 679 405 758 260	0.446 693 856 479 134
0.7	0.359 162 517 509 953	0.801 094 197 327 335	0.441 931 679 817 381
0.8	0.380 456 501 851 322	0.816 550 177 334 318	0.436 093 675 482 995
0.9	0.401 614 191 831 015	0.831 058 428 707 847	0.429 444 236 876 831
1	0.422 318 798 251 518	0.844 637 596 503 036	0.422 319 000 000 000

where  $c_i$ ,  $B_i$  satisfy the collection condition, respectively:

$$L v(\xi) = f(\xi_i, v(x_i)) \text{ where } (i = 0, 1, \dots, n)$$

and

$$B_i(\xi) = \frac{1}{h^5} \begin{cases} (\xi - \xi_{i-4})^7, & \xi \in [\xi_{i-4}, \xi_{i-3}], \\ (\xi - \xi_{i-4})^7 - 8(\xi - \xi_{i-3})^7, & \xi \in [\xi_{i-3}, \xi_{i-2}], \\ (\xi - \xi_{i-4})^7 - 8(\xi - \xi_{i-3})^7 + 28(\xi - \xi_{i-2})^7, & \xi \in [\xi_{i-2}, \xi_{i-1}], \\ (\xi - \xi_{i-4})^7 - 8(\xi - \xi_{i-3})^7 + 28(\xi - \xi_{i-2})^7 + 56(\xi - \xi_{i-1})^7, & \xi \in [\xi_{i-1}, \xi_i], \\ (\xi_{i+4} - \xi)^7 - 8(\xi_{i+3} - \xi)^7 + 28(\xi_{i+2} - \xi)^7 + 56(\xi_{i+1} - \xi)^7, & \xi \in [\xi_i, \xi_{i+1}], \\ (\xi_{i+4} - \xi)^7 - 8(\xi_{i+3} - \xi)^7 + 28(\xi_{i+2} - \xi)^7, & \xi \in [\xi_{i+1}, \xi_{i+2}], \\ (\xi_{i+4} - \xi)^7 - 8(\xi_{i+3} - \xi)^7, & \xi \in [\xi_{i+2}, \xi_{i+3}], \\ (\xi_{i+4} - \xi)^7, & \xi \in [\xi_{i+3}, \xi_{i+4}], \\ 0, & \text{otherwise.} \end{cases} \quad (2.35)$$

where  $i \in [-3, n + 3]$ , it leads

$$v_i(\xi) = c_{i-3} + 120 c_{i-2} + 1191 c_{i-1} + 2416 c_i + 1191 c_{i+1} + 120 c_{i+2} + c_{i+3}. \quad (2.36)$$

Substituting equation (2.36), and its derivatives into equation (2.5), we get  $(n + 7)$  of equations. Solving this system of equations to get the value of  $c_i$ , we obtain



**Table 3.** Analytical, approximate solution, and absolute value of error between them that obtained by using quantic-spline.

Value of $\xi$	Approximate	Exact	Absolute value of error
0	0.666 666 666 666 666	0.666 666 666 666 666	$1.110\,223\,024 \times 10^{-16}$
0.1	0.684 588 596 139 129	0.688 506 691 487 145	0.003 918 095 348 016
0.2	0.700 779 615 189 411	0.709 539 212 929 809	0.008 759 597 740 398
0.3	0.718 115 238 373 826	0.729 709 101 066 375	0.011 593 862 692 550
0.4	0.735 708 969 325 661	0.748 973 892 837 003	0.013 264 923 511 343
0.5	0.753 703 201 426 247	0.767 303 462 381 101	0.013 600 260 954 854
0.6	0.771 942 758 777 081	0.784 679 405 758 260	0.012 736 646 981 179
0.7	0.790 405 237 794 915	0.801 094 197 327 335	0.010 688 959 532 419
0.8	0.808 823 580 285 777	0.816 550 177 334 318	0.007 726 597 048 541
0.9	0.827 777 704 985 461	0.831 058 428 707 847	0.003 280 720 000 000
1	0.844 637 596 503 036	0.844 637 596 503 036	0.000 000 000 000 000

**Table 4.** Analytical, approximate solution, and absolute value of error between them that obtained by using septic-spline.

Value of $\xi$	Approximate	Exact	Absolute value of error
0	0.666 666 666 666 666	0.666 666 666 666 666	$1.110\,22 \times 10^{-16}$
0.1	0.688 506 691 486 581	0.688 506 691 487 145	$5.638\,82 \times 10^{-13}$
0.2	0.709 539 212 928 685	0.709 539 212 929 809	$1.124\,21 \times 10^{-12}$
0.3	0.729 709 101 065 317	0.729 709 101 066 375	$1.058\,38 \times 10^{-12}$
0.4	0.748 973 892 836 129	0.748 973 892 837 003	$8.746\,34 \times 10^{-13}$
0.5	0.767 303 462 380 602	0.767 303 462 381 101	$4.988\,23 \times 10^{-13}$
0.6	0.784 679 405 758 176	0.784 679 405 758 260	$8.426\,59 \times 10^{-14}$
0.7	0.801 094 197 327 587	0.801 094 197 327 335	$2.524\,65 \times 10^{-13}$
0.8	0.816 550 177 334 830	0.816 550 177 334 318	$5.120\,35 \times 10^{-13}$
0.9	0.831 058 428 708 228	0.831 058 428 707 847	$3.812\,51 \times 10^{-13}$
1	0.844 637 596 503 036	0.844 637 596 503 036	0

### 3. Results and discussion

This research paper implements an analytical, semi-analytical, and numerical scheme on time-space fractional telegraph equations. It used the modified auxiliary equation method to get solitary wave solutions. We studied the obtained solutions for applying the Adomian decomposition method, and B-spline techniques to show the accuracy of the used analytical method. However, Yildirim used the Adomian decomposition method (ADM) for the same models, but the numerical scheme allows applying many times on the same models. Since every time, we study a new form of solitary wave solutions.

In this part of our paper, we discuss the new and different solutions obtained in our paper than obtained in the previously published papers. In the following steps, we will represent this difference:

1. In [36], Ozkan Gunner and Ahmet Bekir used Jumarie's derivative with the next definition

$$D_{\omega}^{\alpha} \omega^{\gamma} = \frac{\Gamma(1 + \gamma)}{\Gamma(1 + \gamma - \alpha)} \omega^{\gamma - \alpha}, \text{ where } \gamma > 0,$$

while we used conformable fractional derivative with definition (2.1).

2. Equation (2.14) is equal to equation (3.14) in [36] when  $[b = \gamma \ \& \ \beta = -4d]$ .

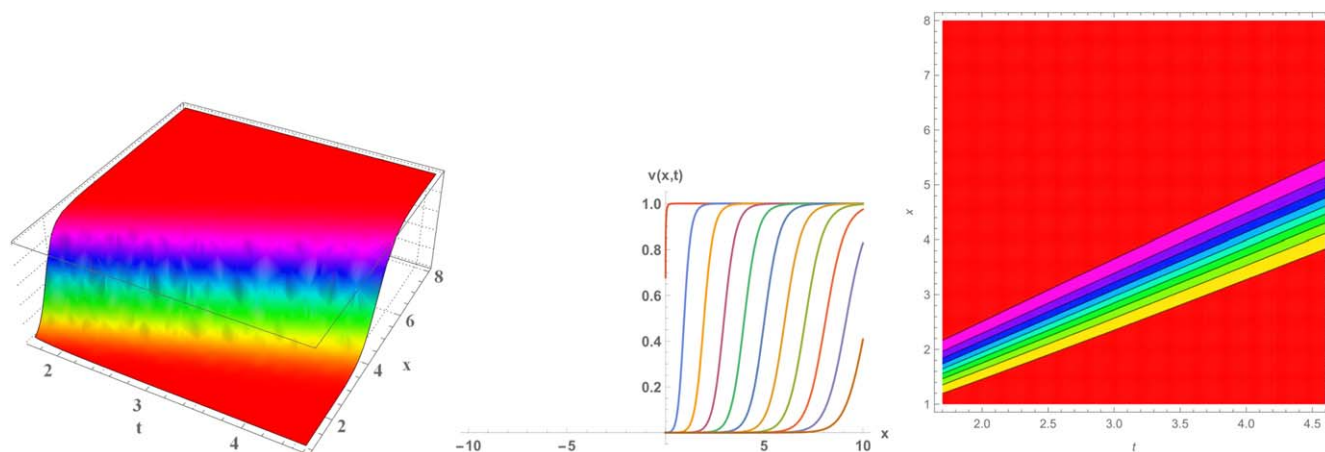
3. All other analytical solutions that obtained in our paper are new and different forms of that obtained in [36].
4. In [37], H Jafari *et al* used a modified Riemann–Liouville derivative with the next definition

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_0^x (x - \xi)^{-\alpha} \times (f(\xi) - f(0)), \text{ where } 0 < \alpha \leq 1,$$

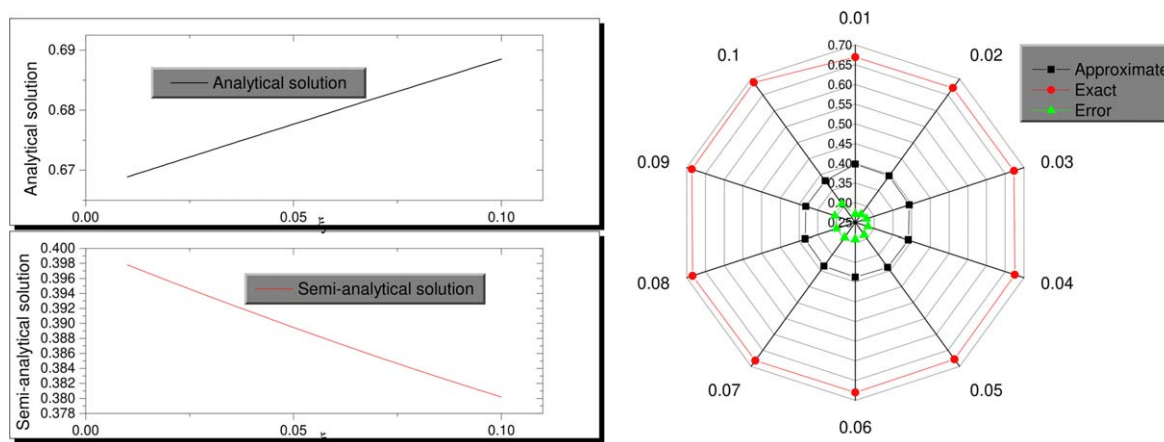
while we used conformable fractional derivative with definition (2.1).

5. In [36], they applied the fractional Fan sub-equation method on three different models where one of them are space-time fractional telegraph equation.
6. All our solutions are new and different from that obtained in [37].
7. In [38], J Biazar and M Eslami studied the integer order of the model under consideration by using differential transform method but also our solutions are different from that obtained in their research even in case of  $(\vartheta = 1)$ .
8. Tables 1–4 show the accuracy of septic B-spline technique on the fractional nonlinear space-time telegraph equation where the numerical value that obtained by this method has the smallest absolute value of error between other methods that are applied in our paper.

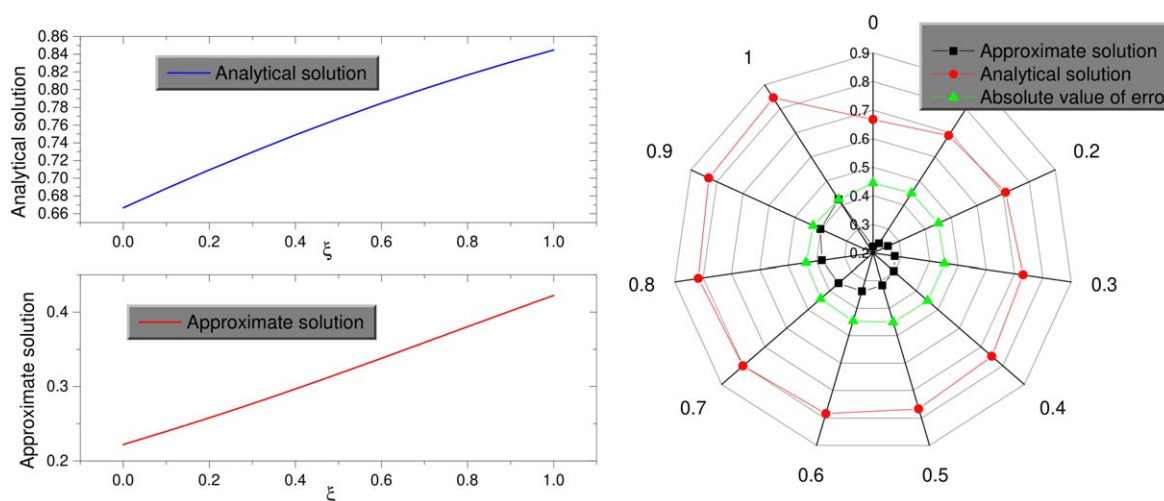
We also discuss our analytical solutions by using Adomian decomposition method to compare the exact solution with



**Figure 1.** Kink solitary wave solution of equation (2.8) in three and two-dimensional and contour plot.



**Figure 2.** Analytical and semi-analytical solutions of equation (2.5) in separated and combined forms to show the coincide of each of them.

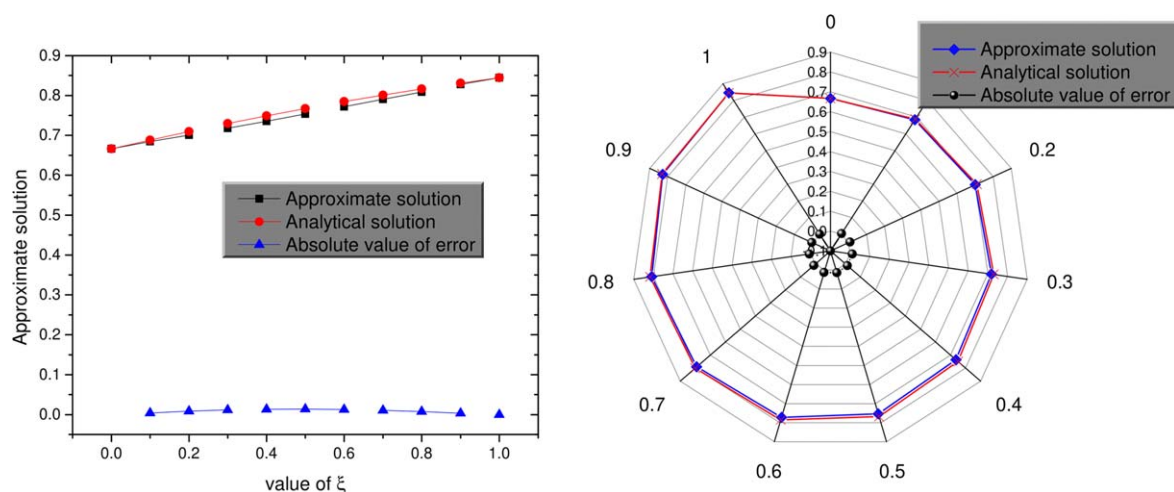


**Figure 3.** Analytical and numerical solutions of equation (2.5) in separated and combined forms to show the coincide of each of them.

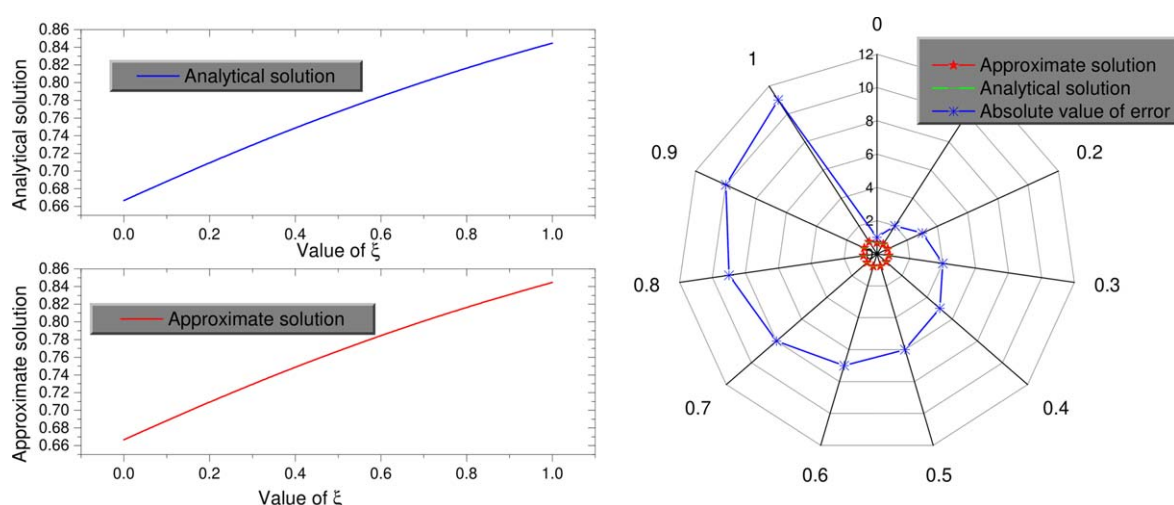
semi-analytical and numerical solutions. That shows the error values between them. We represent both exact and numerical solutions in figures 1–5 in three different techniques (two & three-dimensional, contour, and radar plots) to show how they are closed.

#### 4. Conclusion

In this paper, we successfully employed three different techniques on the space-time fractional telegraph equation. Analytical, semi-analytical, and numerical schemes were used to



**Figure 4.** Analytical and numerical solutions of equation (2.5) in separated and combined forms to show the coincide of each of them.



**Figure 5.** Analytical and numerical solutions of equation (2.5) in separated and combined forms to show the coincide of each of them.

investigate the exact and approximate solutions of this model. New explicit solutions were obtained which is different with that in previous research papers. Some plots are sketched to show more properties of the obtained solutions. The accuracy of obtained solutions was tested by calculating the absolute value of error between exact, semi-analytical, and numerical solutions. The effective and power were shown in the performance of the used analytical method.

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