

Semi-analytical solution for one-dimensional unsteady sediment transport model in open channel with concentration-dependent settling velocity

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Abstract

This paper aims to solve the one-dimensional unsteady suspended sediment transport equation in open channels through a semi-analytical approach. The presence of a large amount of particles in flow change the settling velocity of a particle, and this phenomenon, commonly known as hindered settling effect, must be taken into account to deal with high concentrated flows. Inclusion of this mechanism makes the governing equation nonlinear, and together with this nonlinear governing equation, a generalized bottom boundary condition is taken in terms of deposition velocity and equilibrium bottom concentration. An explicit series solution is presented using the method of lines based homotopy analysis method, and the convergence of the series solution is gained through a convergence control parameter. The solution is validated by comparing it with the existing solution as well as a numerical one. Apart from that, the solution has also been validated under a steady-state condition with available experimental data. Results are interpreted both graphically and physically. It is found that the hindered settling effect is dominant in the main flow region only, for sediment free inlet for all types of turbulent diffusion coefficients. On the other hand, in the case of uniform sediment concentration at the inlet, hindered settling affects the concentration in the top portion of the channel too for linear and parabolic profiles of turbulent diffusion coefficients.

Keywords: open channel flow, diffusion equation, homotopy analysis method, hindered settling mechanism

(Some figures may appear in colour only in the online journal)

Nomenclature

c	= volumetric sediment concentration (—)	c_*, γ	= depth independent parameters (—)
c_a	= reference sediment concentration (—)	u_*	= shear velocity (m s^{-1})
a	= reference level above the bed (m)	ω_s	= settling velocity of sediment particle (m s^{-1})
h	= channel depth (m)	ω_0	= settling velocity in clear fluid (m s^{-1})
		κ	= von Karman constant (—)
		ϵ_s, ϵ_t	= sediment and turbulent diffusivity, respectively ($\text{m}^2 \text{s}^{-1}$)

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β	= ratio between turbulent diffusivity to sediment diffusivity (–)
t	= time (s)
x, y, z	= longitudinal, transverse and vertical coordinates, respectively (m)
u, v, w	= velocities along x, y and z directions, respectively (m s^{-1})
ϵ_m	= molecular diffusion coefficient ($\text{m}^2 \text{s}^{-1}$)
$\epsilon_{sx}, \epsilon_{sy}, \epsilon_{sz}$	= sediment diffusion coefficients along x, y and z directions, respectively ($\text{m}^2 \text{s}^{-1}$)
n_H	= exponent reduction constant (–)
α_1, α_2	= constant (–)
$C, C_*, Z, T, A,$	= dimensionless parameters (–)
B, V_0	
N	= number of interior node points (–)

1. Introduction

The study of transportation of sediment in turbulent flow through an open channel is an important topic of concern to the researchers as it resembles river flow in a practical situation. Numerous models have been developed to describe the distribution of sediment in the suspension. However, the majority of the models deal with one-dimensional (vertical) steady concentration only [1–10], which may not be the case practically. The steady 1D models are governed by the ordinary differential equations (ODE) and hence easy to handle both numerically and analytically. But when the variation of concentration is considered in vertical as well as in the main flow direction, or vertical concentration is considered to vary with time, the governing equation becomes a partial differential equation (PDE), which is not always easy to tackle. Hjelmfelt and Lenau [11] studied the non-equilibrium transport of suspended sediment analytically, and the formulated problem was solved by the method of separating variables. However, the boundary conditions employed in their study had constraints as they did not consider any deposition or entrainment flux. Much early to this work, Monin [12] and Calder [13] proposed a near-bed boundary condition where the deposition was considered, and entrainment was ignored. Dobbins [14] considered both entrainment and deposition, but in his work, entrainment rate and deposition flux both were constant and related to settling velocity, which was an unnecessary constraint on the near-bed boundary condition. Cheng [15] generalized the bottom boundary condition for non-equilibrium transport of sediment considering both entrainment and deposition by introducing a parameter γ . With this boundary condition, Liu and Laymatullah [16] solved the one-dimensional unsteady transport equation with arbitrary eddy viscosity using general integral transform technique. Later, Liu [17] studied the two-dimensional transport of sediment with the same boundary condition and methodology.

All the works mentioned above considered the governing equation as a linear PDE and neglected an important physical phenomenon called hindered settling mechanism inclusion of which would have made the governing equation nonlinear. Long back, it has been proved by Richardson and Zaki [18] that the presence of sediment particles in flow affects the particle settling velocity, and it must be taken into account when dealing flow with high concentration. Recently, Jing *et al* [19] considered this effect in their two-dimensional concentration equation, but they did not provide any analytical solution to the problem. To tackle a mathematical problem, two approaches can be adopted: (i) numerical and (ii) analytical/semi-analytical. Both of them have their own merits and demerits. The numerical method is a trial and error procedure that produces an approximation to the true solution (s), whereas the analytical/semi-analytical solution provides answers to a whole set of problems. Also, numerical approaches do not give deep insight into the problem, but analytic methods can. Checking a numerical solution is difficult, and most of the time, the source code is not error-free. On the other hand, analytical solutions can contribute to proofs of new ideas. Both approaches contribute holistically to the field of science and mathematics, and both are equally important. Looking at the merits of analytical/semi-analytical approach, the present study aims to solve the problem by a semi-analytical approach. Liao [20] originated a method known as the homotopy analysis method (HAM), which can be successfully applied for solving nonlinear ordinary/partial differential equations analytically. The method has been widely applied to solve nonlinear differential equations in different areas of science and engineering [21–26], but its applicability to open channel flow in general and sediment transport in particular remains unexplored except a few works [10, 27, 28]. So the present work aims to derive an approximate analytical solution of the one-dimensional unsteady advection-diffusion equation considering the generalized bottom boundary condition given by Cheng [15] together with the effect of hindered settling which makes the governing equation nonlinear. As the direct solution of the PDE using HAM creates some mathematical difficulty which will be discussed in the paper, the PDE is solved by the method of lines (MOLs) based HAM, where the PDE is first converted into a system of ODEs through a semi-discretization technique, and then the HAM is used to solve the system. The obtained results have been compared with the existing similar type of work, and physical justification has been provided.

2. Problem formulation

2.1. Governing equation

The transport process of suspended sediment is governed by the advection-diffusion equation. The most generalized form of the three-dimensional advection-diffusion equation of

suspended sediment motion is given as follows [29]:

$$\begin{aligned} \frac{\partial c}{\partial t} + \frac{\partial(uc)}{\partial x} + \frac{\partial(vc)}{\partial y} + \frac{\partial(wc)}{\partial z} \\ = \frac{\partial}{\partial x} \left([\epsilon_m + \epsilon_{sx}] \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial y} \left([\epsilon_m + \epsilon_{sy}] \frac{\partial c}{\partial y} \right) \\ + \frac{\partial}{\partial z} \left([\epsilon_m + \epsilon_{sz}] \frac{\partial c}{\partial z} \right), \end{aligned} \quad (1)$$

where t denotes time, c is the volumetric sediment concentration, x , y , and z are the longitudinal, transverse, and vertical directions, respectively; u , v , w represent the time-averaged velocity components in three directions, ϵ_{sx} , ϵ_{sy} , ϵ_{sz} are the sediment diffusion coefficients in x , y , z directions, respectively and ϵ_m is the molecular diffusion coefficient. In turbulent flow, the molecular diffusion coefficient ϵ_m is negligible as compared to the sediment diffusion, i.e. $\epsilon_m \approx 0$. Replacing the vertical velocity component w by the downward settling velocity $-\omega_s$ of sediment, the governing equation for one-dimensional unsteady, uniform flow in a wide channel becomes

$$\frac{\partial c}{\partial t} - \frac{\partial}{\partial z} (\omega_s c) = \frac{\partial}{\partial z} \left(\epsilon_{sz} \frac{\partial c}{\partial z} \right). \quad (2)$$

In literature, it is observed that the magnitude of the settling velocity ω_s of sediment particles is less than that in clear fluid due to the presence of the surrounding particles. The relationship between ω_s and c , according to Richardson and Zaki [18], is given as follows:

$$\omega_s = \omega_0 (1 - c)^{n_H}, \quad (3)$$

where n_H is the exponent of reduction in settling velocity, and ω_0 is the settling velocity in clear fluid. The exponent n_H depends on the particle Reynolds number and varies between 2.3 and 4.9 [18]. However, for mathematical convenience, one can consider an average value, $n_H = 4$ [30]. The present study also assumes the same. Therefore, substituting equation (3) into equation (2) and denoting the sediment diffusivity ϵ_{sz} in the vertical direction z by ϵ_s , the governing equation for the suspended sediment concentration becomes:

$$\frac{\partial c}{\partial t} - \omega_0 \frac{\partial}{\partial z} [c(1 - c)^4] = \frac{\partial}{\partial z} \left(\epsilon_s \frac{\partial c}{\partial z} \right). \quad (4)$$

2.2. Boundary conditions

The suspension of sediment particles occurs from a particular reference level $z = a$ to the free surface $z = h$. At the free surface, $z = h$, no mass transfer takes place, and hence zero flux can be specified thereat. The boundary condition at the free surface is given as

$$\epsilon_s \frac{\partial c}{\partial z} + \omega_s c = 0 \text{ at } z = h. \quad (5)$$

Different kinds of boundary conditions can be prescribed at the reference level $z = a$ [11, 31, 32]. Considering the flux to be equal to the rate of placing the sediment into suspension,

Cheng [15] proposed a generalized boundary condition which reads as follows

$$\epsilon_s \frac{\partial c}{\partial z} + \omega_s c = \gamma(c - c_*) \text{ at } z = a. \quad (6)$$

All the other boundary conditions are special cases of equation (6). Here, γ is a parameter whose value depends on the bottom boundary condition, and c_* is the equilibrium concentration at the bottom surface. From a physical point of view, γ is considered to be a reflectivity coefficient of the bottom surface, e.g. $\gamma = 0$ represents a perfectly reflective surface, and $\gamma = \infty$ corresponds to a perfectly absorbing surface [15, 17]. An ideal condition lies in between these values.

At the inlet, the concentration profile can be taken arbitrary and has the following form

$$c(t = 0, z) = c_0(z). \quad (7)$$

The governing equation (4) is non-dimensionalized as follows:

$$\begin{aligned} C = \frac{c}{c_a}, \quad C_* = \frac{c_*}{c_a}, \quad Z = \frac{z}{h}, \quad T = \frac{tu_*}{h} \\ A = \frac{a}{h}, \quad K(Z) = \frac{\epsilon_t}{\beta u_* h}, \quad B = \frac{\gamma}{u_*}, \quad V_0 = \frac{\omega_0}{u_*}, \end{aligned}$$

where c_a is the reference concentration at reference height $z = a$, u_* is the shear velocity, h is the maximum flow depth, and β is the ratio of turbulent diffusivity ϵ_t to sediment diffusivity ϵ_s . Accordingly, the dimensionless form of equation (4) becomes:

$$\frac{\partial C}{\partial T} - V_0 \frac{\partial}{\partial Z} [C(1 - C)^4] = \frac{\partial}{\partial Z} \left(K(Z) \frac{\partial C}{\partial Z} \right). \quad (8)$$

Researchers [9, 33] have shown that the effect of the hindered settling mechanism is dominant only in the main flow region and is negligible at the water surface or near the channel bottom. Therefore, considering $\omega_s = \omega_0$ at the bottom boundary and performing non-dimensionalization, the boundary and initial conditions become:

$$K(Z) \frac{\partial C}{\partial Z} + V_0 C = 0 \text{ at } Z = 1 \quad (9)$$

$$K(Z) \frac{\partial C}{\partial Z} + (V_0 - B)C = -BC_* \text{ at } Z = A \quad (10)$$

and

$$C(T = 0, Z) = C_0(Z) \quad (11)$$

Two cases can be considered for $C_0(Z)$: (a) the inlet is clear water with no sediment, i.e. $C_0(Z) = 0$ and (b) there is uniform sediment concentration at the inlet, i.e. $C_0(Z) = 1$.

It can be noticed that equation (8) is a highly nonlinear PDE together with non-homogeneous BCs equations (9), (10) and the IC equation (11). Also that the PDE has nonlinearity in power with a variable coefficient, and as such, the analytical solution is a challenging task. We attempt to find an approximate analytical solution of the PDE using a unified non-perturbation method, known as the HAM, in combination

with some other mathematical tools. The detailed solution procedure is described below.

3. Solution procedure

In literature, several analytical techniques are available which deal with nonlinear ODE/PDEs, such as classical perturbation method [34], Lyapunov's artificial small parameter method [35], δ -expansion method [36], and Adomian's decomposition method [37]. These techniques essentially convert the original nonlinear equation into an infinite system of linear equations and then approximate the solution considering the sum of solutions of the first few linear equations. However, the region and rate of convergence in these techniques cannot be adjusted properly. Also, few of the methods stop working if small parameters (often called perturbation quantity) are not present in the governing equation or boundary conditions and also are not valid for strongly nonlinear problems. Interestingly, these shortcomings have shown to be overcome using a novel analytical method called the HAM proposed by Liao [20]. The basic idea of the method is first to choose an appropriate set of base functions to represent the solution, and then accordingly, a proper set of a linear operator, initial approximation, and auxiliary function is constructed. The convergence of the series solution obtained by HAM strongly depends on the choice of initial approximation [38]. However, in the present case of PDE equation (8) together with the BCs and IC equations (9)–(11), it is not easy to choose such operators and the initial approximation. The non-homogeneous nature of the boundary condition and difficulty in choosing an appropriate set of base functions make the task difficult. Liao [39] showed that it is easy to apply HAM to the ODE as compared to the partial differential equations. Therefore, first, we aim to convert the PDE into a system of nonlinear ODEs through semi-discretization technique, which is described below.

3.1. MOLs-HAM

In the MOLs approach, the boundary value derivatives (also known as spatial derivatives in space-time problem) present in the PDE are discretized by algebraic approximations using finite difference or finite volume or finite element methods and hence the given PDE reduces into a system of ODEs. This method is a well-known semi-analytical approach for solving complex PDEs [40]. So, one can convert a PDE into a coupled system of ODEs by MOL using finite difference approximations for the independent boundary value variable. Using the second-order central finite difference approximation for the derivatives with respect to Z , i.e. $\frac{\partial C_i}{\partial Z} = \frac{C_{i+1} - C_{i-1}}{2\Delta Z}$ and $\frac{\partial^2 C_i}{\partial Z^2} = \frac{C_{i+1} - 2C_i + C_{i-1}}{\Delta Z^2}$, where $C_i = C_i$

(T) = $C(Z = Z_i, T)$, equation (8) is converted to:

$$\begin{aligned} \frac{dC_i}{dT} - V_0((1 - 5c_a C_i)(1 - c_a C_i)^3) \left(\frac{C_{i+1} - C_{i-1}}{2\Delta Z} \right) \\ = \frac{d}{dZ}(K(Z)) \Big|_{Z=Z_i} \left(\frac{C_{i+1} - C_{i-1}}{2\Delta Z} \right) + K(Z_i) \\ \times \left(\frac{C_{i+1} - 2C_i + C_{i-1}}{\Delta Z^2} \right), \end{aligned} \quad (12)$$

where $i (=1, 2, \dots, N)$ is the node point, $\Delta Z = \frac{L}{N+1}$ is the distance between two node points, N is the total number of interior node points, and L is the length of the domain. Clearly, $Z = A$ and $Z = 1$ correspond to the nodes $i = 0$ and $i = N + 1$, respectively, and $Z_i = A + i\Delta Z$. Using the second-order backward finite difference approximation for Eq. (9) and second-order forward finite difference approximation for Eq. (10), the boundary conditions reduce to:

$$K(Z_{N+1}) \left(\frac{3C_{N+1} - 4C_N + C_{N-1}}{2\Delta Z} \right) + V_0 C_{N+1} = 0, \quad (13)$$

$$K(Z_0) \left(\frac{-3C_0 + 4C_1 - C_2}{2\Delta Z} \right) + (V_0 - B)C_0 = -BC_* \quad (14)$$

One can obtain the values of C_{N+1} and C_0 from equations (13) and (14), respectively, such as $C_{N+1} = \frac{K(Z_{N+1})}{3K(Z_{N+1}) + 2\Delta Z V_0} (4C_N - C_{N-1})$ and $C_0 = \frac{1}{\frac{-3K(Z_0)}{2\Delta Z} + (V_0 - B)} (-BC_* + \frac{K(Z_0)}{2\Delta Z} (C_2 - 4C_1))$. Now, substituting them in equation (12), a closed system of N ODEs with N number of unknowns C_1, C_2, \dots, C_N is obtained as follows:

$$\begin{aligned} \frac{dC_1}{dT} - V_0((1 - 5c_a C_1)(1 - c_a C_1)^3) \left(\frac{C_2 - C_0}{2\Delta Z} \right) \\ = \frac{d}{dZ}(K(Z)) \Big|_{Z=Z_1} \left(\frac{C_2 - C_0}{2\Delta Z} \right) + K(Z_1) \\ \times \left(\frac{C_2 - 2C_1 + C_0}{\Delta Z^2} \right) \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{dC_i}{dT} - V_0((1 - 5c_a C_i)(1 - c_a C_i)^3) \left(\frac{C_{i+1} - C_{i-1}}{2\Delta Z} \right) \\ = \frac{d}{dZ}(K(Z)) \Big|_{Z=Z_i} \left(\frac{C_{i+1} - C_{i-1}}{2\Delta Z} \right) + K(Z_i) \\ \times \left(\frac{C_{i+1} - 2C_i + C_{i-1}}{\Delta Z^2} \right) \\ \text{for } i = 2, 3, \dots, N-1 \end{aligned} \quad (16)$$

and

$$\begin{aligned} \frac{dC_N}{dT} - V_0((1 - 5c_a C_N)(1 - c_a C_N)^3) \left(\frac{C_{N+1} - C_{N-1}}{2\Delta Z} \right) \\ = \frac{d}{dZ}(K(Z)) \Big|_{Z=Z_N} \left(\frac{C_{N+1} - C_{N-1}}{2\Delta Z} \right) + K(Z_N) \\ \times \left(\frac{C_{N+1} - 2C_N + C_{N-1}}{\Delta Z^2} \right), \end{aligned} \quad (17)$$

where C_0 and C_{N+1} can be estimated using the expressions mentioned earlier. Finally, one has a coupled system of N

ODEs given by equations (15)–(17) subject to the initial conditions $C_i(T=0) = C_0(Z)$.

To apply HAM, one can write the system of nonlinear ODEs given by equations (15)–(17) in the following form

$$C_i' = f_i(T, C_1, C_2, \dots, C_N) \quad (18)$$

subject to

$$C_i(0) = C_0(Z), \quad (19)$$

where $C_i' = \frac{dC_i}{dT}$, for $i = 1, 2, \dots, N$. Now, using the embedding parameter $q \in [0, 1]$, define a family of equations as

$$(1 - q)\mathcal{L}_i[\Phi_i(T; q) - C_{i,0}(T)] - q\tilde{h}_i H_i(T)\mathcal{N}_i[\Phi_i(T; q)] = 0, \quad (20)$$

where \mathcal{L}_i are the linear operators with the property that $\mathcal{L}_i[f] = 0$ when $f = 0$, $H_i(T)$ are the non-zero auxiliary functions, \tilde{h}_i are the non-zero convergence-control parameters, $C_{i,0}(T)$ are the initial approximations of $C_i(T)$, $\Phi_i(T; q)$ are unknown functions and \mathcal{N}_i are the nonlinear operators defined by equation (18). Equation (20) is known as the zeroth-order deformation equation. It is clear from equation (20) that when $q = 0$, $\Phi_i(T; 0) = C_{i,0}(T)$ and when $q = 1$, $\Phi_i(T; 1) = C_i(T)$. So, as the embedding parameter q increases from 0 to 1, $\Phi_i(T; q)$ continuously varies from the initial guess $C_{i,0}(T)$ to the final solution $C_i(T)$. Fortunately, in the framework of HAM, one has great flexibility to choose the initial guess $C_{i,0}(T)$, the auxiliary linear operators \mathcal{L}_i , and the auxiliary function $H_i(T)$. However, this freedom is in accordance with Liao's rule of solution expression and rule of coefficient ergodicity [41]. Now expanding $\Phi_i(T; q)$ by Maclaurin series with respect to the embedding parameter q , we have

$$\Phi_i(T; q) = \Phi_i(T; 0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi_i(T; q)}{\partial q^m} \bigg|_{q=0} q^m \quad (21)$$

and define

$$C_{i,m}(T) = \frac{1}{m!} \frac{\partial^m \Phi_i(T; q)}{\partial q^m} \bigg|_{q=0}, \quad (22)$$

where $\frac{\partial^m \Phi_i(T; q)}{\partial q^m}$ are known as m th order deformation derivatives. Suppose that \mathcal{L}_i , \tilde{h}_i , $H_i(T)$ and $C_{i,0}(T)$ are chosen in such a way that the series equation (21) converges at $q = 1$; then the series becomes:

$$C_i(T) = C_{i,0}(T) + \sum_{m=1}^{\infty} C_{i,m}(T). \quad (23)$$

Differentiating zeroth-order deformation equation m times with respect to q , and setting $q = 0$, and then dividing by $m!$, the higher order approximations $C_{i,m}(T)$ for $m \geq 1$ can be obtained as follows:

$$\mathcal{L}_i[C_{i,m}(T) - \chi_m C_{i,m-1}(T)] = \tilde{h}_i H_i(T) R_{i,m}(\vec{C}_{i,m-1}) \quad (24)$$

subject to

$$C_{i,m}(0) = 0, \quad (25)$$

where

$$R_{i,m}(\vec{C}_{i,m-1}(T)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}_i[\Phi_i(T; q)]}{\partial q^{m-1}} \bigg|_{q=0} \quad (26)$$

and

$$\chi_m = \begin{cases} 0 & \text{when } m = 1, \\ 1 & \text{otherwise} \end{cases} \quad (27)$$

Equation (24) is known as the m th order deformation equation. The approximate solution of equation (18) can be obtained by considering up to a finite number of terms of equation (23) as follows:

$$C_i(T) \approx C_{i,0}(T) + \sum_{m=1}^k C_{i,m}(T) \quad (28)$$

which is known as k th order approximation of HAM.

In the present work, one has a system of N coupled nonlinear ODEs given by equations (15)–(17) subject to $C_i(0) = C_0(Z)$, $i = 1, \dots, N$. To solve the system equations (15)–(17), let us consider the set of base functions as the polynomials $\{T^n; n = 0, 1, \dots\}$ to represent the solutions $C_i(T)$ in the form of

$$C_i(T) = \sum_{n=0}^{\infty} \alpha_n T^n, \quad (29)$$

where α_n 's are the coefficients of the series. Keeping the initial condition and the set of base function in mind, one can simply choose the initial approximations as $C_{i,0}(T) = C_0(Z)$ for $i = 1, 2, \dots, N$ and the single term auxiliary linear operator as

$$\mathcal{L}_i[\Phi_i(T; q)] = \frac{\partial \Phi_i(T; q)}{\partial T}. \quad (30)$$

To avoid computational difficulty, the auxiliary functions are simply chosen as $H_i(T) = 1$ for $i = 1, 2, \dots, N$ [42]. Accordingly, higher order terms are estimated as

$$C_{i,m}(T) = \chi_m C_{i,m-1}(T) + \tilde{h}_i \int_0^T R_{i,m}(\vec{C}_{i,m-1}) dT. \quad (31)$$

Therefore, the k th order solution can be obtained from equation (28) where the higher order terms are given by equation (31).

4. Results and discussion

In this section, first, we define the input parameters which are required to obtain the solution of equation (8) and then the derived approximate analytical solution is compared with the analytical solution provided by Cheng [15] taking constant turbulent diffusion coefficient $K(Z)$ and also with the numerical solution of the problem. After that, transient suspended sediment concentration profiles with different eddy viscosity distributions are plotted followed by important discussion on the hindered settling mechanism and the distribution of bottom sediment concentration.

4.1. Input function and parameters

To assess the solution of the one-dimensional unsteady suspended sediment transport model, expression for $K(Z)$ is needed. In literature [43], three types of expressions are available for turbulent diffusion coefficient $K(Z)$, namely constant, linear, and parabolic which are given below:

(i) Constant

$$K(Z) = \frac{\kappa}{\alpha_1} \quad (32)$$

(ii) Linear

$$K(Z) = \frac{\kappa}{\alpha_2} Z \quad (33)$$

and (iii) Parabolic

$$K(Z) = \kappa Z(1 - Z), \quad (34)$$

where κ is the von-Karman constant whose value is 0.41. The details of the model parameters α_1 and α_2 can be found in van Rijn [43]. Accordingly, the values $\alpha_1 = 6$ and $\alpha_2 = 3$ are used in this work.

4.2. Validation of the HAM-based solution

The derived HAM based semi-analytical solution is validated here with the analytical solution of Cheng [15]. Cheng [15] solved the one-dimensional, unsteady linear diffusion equation together with the generalized boundary conditions and constant turbulent diffusion coefficient. Taking $V_0 = 1$ and neglecting the hindered settling effect, i.e. $n_H = 0$, the model considered in this study reduces to the problem solved by Cheng [15]. It can be seen from equation (31) that the series solution is dependent on the auxiliary parameter $\hbar_i = \hbar$ whose value needs to be determined to evaluate the solution. According to Liao [41], an appropriate set of values for the auxiliary parameters ensures the convergency of the series solution over the domain. For that purpose, the squared residual error (Δ_m) at the m th order approximation can be calculated as follows:

$$\Delta_m = \int_{T \in \Omega} \sum_{i=1}^N (\mathcal{N}_i(C_1, C_2, \dots, C_N))^2 dX. \quad (35)$$

Sometimes it is not possible to compute the integration given in equation (35); so to avoid the difficulty one can proceed with the discrete form of integration as follows:

$$\Delta_m = \frac{1}{l+1} \sum_{k=0}^l \left(\sum_{i=1}^N (\mathcal{N}_i(C_1(T_k), C_2(T_k), \dots, C_N(T_k)))^2 \right), \quad (36)$$

where $l+1$ is number of equally distributed node points. It is shown that as the squared residual error (Δ_m) tends to 0, the series solution converges to the exact solution of the problem. So it is sufficient to deal with the squared residual error for the convergence of the solution. For that purpose, choosing a relevant set of parameters and taking $N = 9$, squared residual error (Δ_m) with different order of approximation m is plotted in figure 1 and the value of optimal convergence control parameter is found as $\hbar = -0.060464$. It can be seen from figure 1 that as the order of approximation is increasing, the

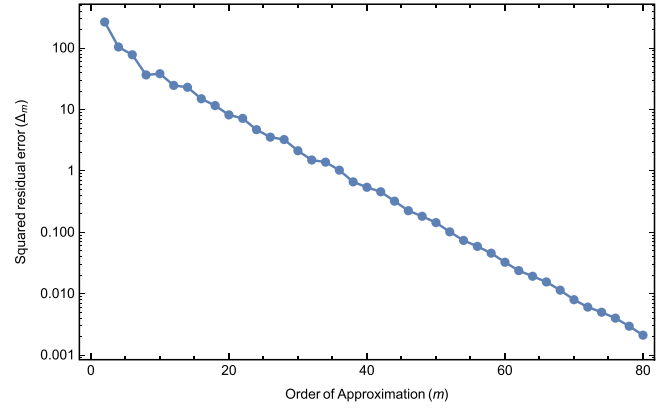


Figure 1. Squared residual error equation (36) with order of approximation m .

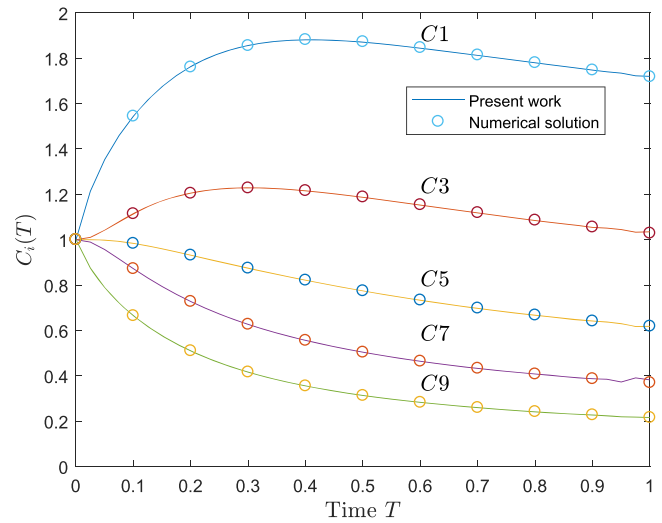


Figure 2. Comparison of 80th order HAM based analytical solution with numerical solution for parameters' values $B = 1.0$, $K = 0.35$, $C_* = 2.0$ and the initial condition $C_0(Z) = 1$.

squared residual error is decreasing and tending to zero, which assures the convergence of the solution.

In figure 2, 80th order HAM based series solutions for C_1 , C_3 , C_5 , C_7 , and C_9 together with the numerical solutions obtained using *dsolve* in Maple are plotted, and it is clear from figure 2 that the series solution matches well with the numerical one.

Now the derived semi-analytical solution for suspended sediment transport in open channels is validated with the analytical solution of Cheng [15] considering the same test cases, i.e. constant eddy viscosity and $n_H = 0$. For that, 80th order HAM based solution for $N = 9$ is computed with different parameter values for B , K , and C_* , and the initial condition $C_0(Z) = 1$. figures 3(a)–(c) plot the vertical sediment concentration distribution at different times. In figure 3(d), bottom sediment concentration as a function of time is plotted for different C_* values. All the required parameter values are mentioned in the figure's caption. It can be observed from figure 3 that the solution agrees well with the solution of Cheng [15].

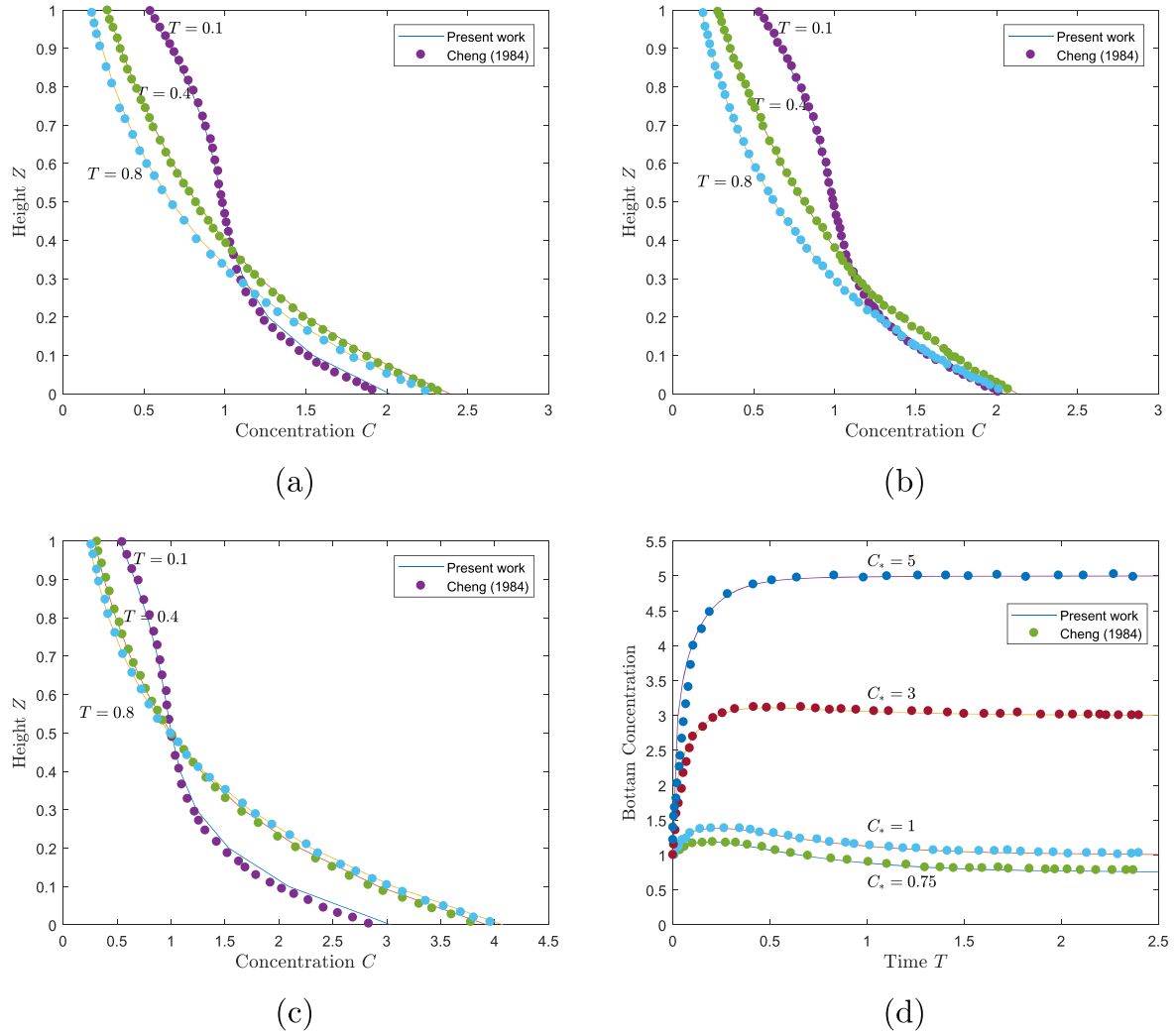


Figure 3. Comparison of the semi-analytical solution with the analytical solution of [15]: (a) $B = 1.0$, $K = 0.35$, $C_* = 2.0$; (b) $B = 4.0$, $K = 0.35$, $C_* = 2.0$; (c) $B = 1.0$, $K = 0.35$, $C_* = 4.0$; (d) bottom sediment concentration with different values of C_* where $B = 2.0$ and $K = 0.35$.

4.3. Transient suspended sediment concentration distributions

Transient suspended sediment concentration profiles with different turbulent diffusion coefficients for times $T = 0.4, 1.0, 2.0$, and ∞ taking initially uniform concentration profile, i.e. $C_0(Z) = 1$ without hindered settling effect are plotted in figure 4. The required parameters are $A = 0.05$, $B = 2.0$, $V_0 = 0.2$, and $C_* = 1$. Figure 4 shows that in the beginning, vertical concentration profiles are almost same except at the top of the channel where the profile corresponding to parabolic diffusivity tends to zero essentially, and the linear diffusivity based profile deviates more at steady state in comparison to the others.

4.4. Effect of hindered settling mechanism on concentration profile

The effect of the hindered settling mechanism is examined in figure 5 for vertical concentration profile with different turbulent diffusion coefficients together with both the initial conditions at fixed time $T = 2$. It is clear from the figure that

for sediment free inlet, the hindered settling effect is dominant in the main flow region only, irrespective of the nature of the turbulent diffusion coefficient. It happens because, near the water surface, the concentration of particles is very low while near the bed region particles do not come in suspension. On the other hand, in the case of uniform sediment concentration at the inlet, i.e. $C(T = 0, Z) = 1$, it can be observed that constant and linear profiles are affected by the hindered settling mechanism in the top portion of the channel too. The reason behind it is that, in case of uniform sediment concentration at the inlet, the free surface of the channel has sufficient particles in the suspension there.

4.5. Bottom sediment concentration profiles

One interesting feature for the 1D suspended sediment transport is that of overshooting. Cheng [15] showed that bottom concentration profile overshoots its equilibrium in the beginning and then slowly decreases to equilibrium. Jobson and Sayre [44] and Celic and Rodi [45] also observed this phenomenon in their experiments, and later, Liu and

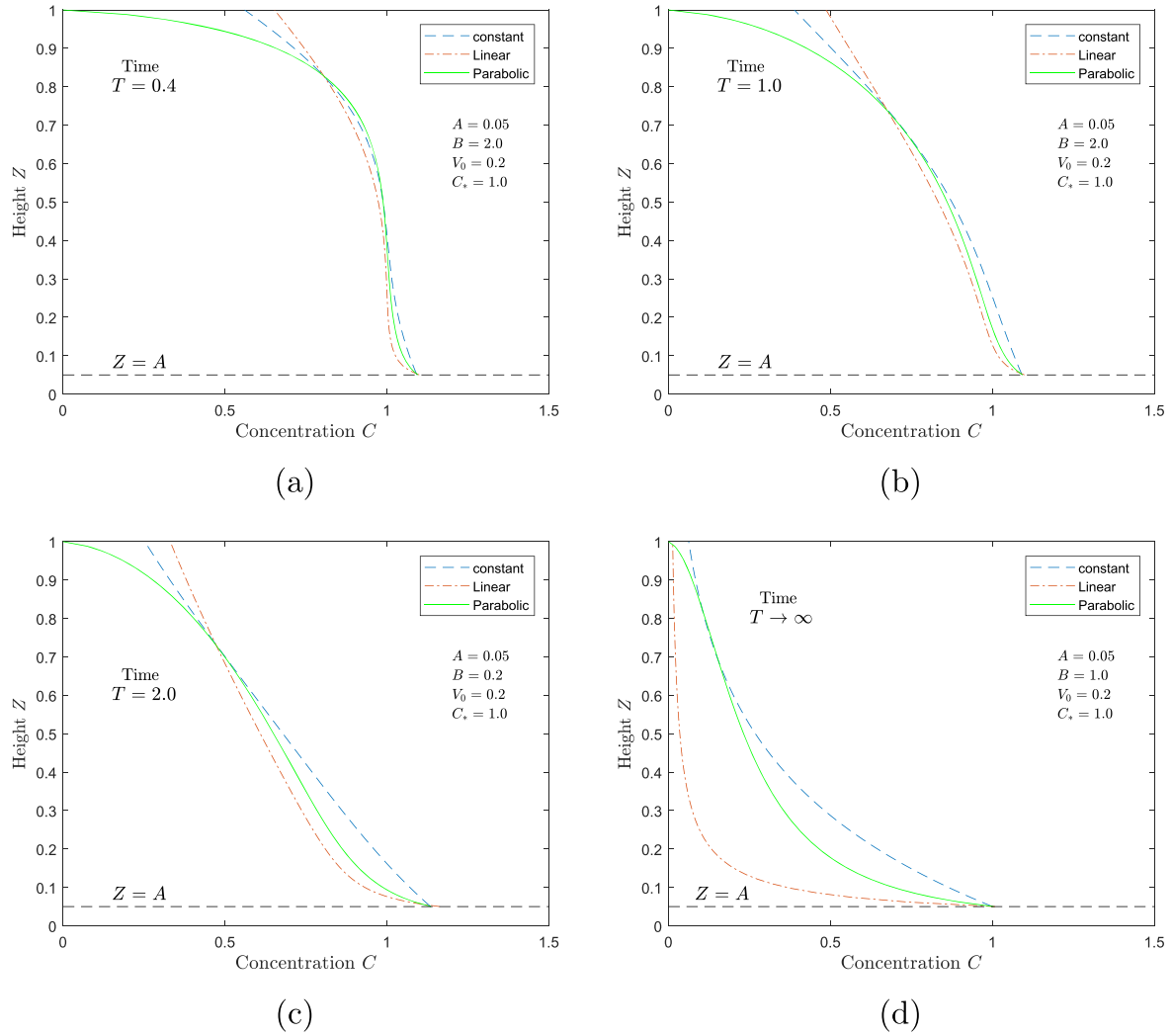


Figure 4. Transient suspended sediment concentration distributions with different turbulent diffusion coefficients with the parameters $A = 0.05$, $B = 1.0$, $V_0 = 0.2$ and $C_* = 1.0$, and initial condition $C(T = 0, Z) = C_0(Z) = 1$; (a) $T = 0.4$; (b) $T = 1.0$; (c) $T = 2.0$; (d) $T \rightarrow \infty$.

Nayamatullah [16] reported the overshooting behavior in their model. To observe this phenomenon, the bottom sediment concentration profiles for both the inlet conditions are plotted in figure 6 with time variation. It can be observed from the figure that the overshooting happens only when the inlet has uniform sediment concentration, i.e. $C_0(Z) = 1$ for all the turbulent diffusion coefficients, while it did not occur when inlet is free of sediment in any case.

4.6. Comparison with experimental data

To the best of the authors' knowledge, no experimental data for one-dimensional unsteady suspended sediment transport problem is available in the literature. However, for vertical distribution of suspended sediment concentration, many experimental data are available. As at large time, unsteady problem behaves like a steady one, we validate our model with the experimental data for vertical distribution of suspended sediment concentration at a large time. To that purpose, two types of data are taken into account: Coleman [46] and Einstein and Chien [47]. Coleman [46] performed

experiments by using a 356 mm wide and 15 m long smooth flume. 40 test cases were performed with three different sand diameters $d = 0.105$, $d = 0.21$ and $d = 0.42$ mm. While Einstein and Chien [47] performed 16 different experiments, named Run S1 to S16 very close to the channel bed with three different particle diameters of 1.3, 0.940 and 0.274 mm. Randomly two data sets Run 13 and Run 23 are taken from Coleman [46] and run S9 and S12 are taken from Einstein and Chien [47]. Figures 7 and 8 show the comparison between these data sets and the proposed model. Here we consider the parabolic diffusion coefficient, and all the values of the parameters required for the calculation are mentioned within the figures. The values of shear velocity, reference label and reference concentration are considered from the respective data sets. It can be seen from the figures that in all the cases, the agreement is quite well.

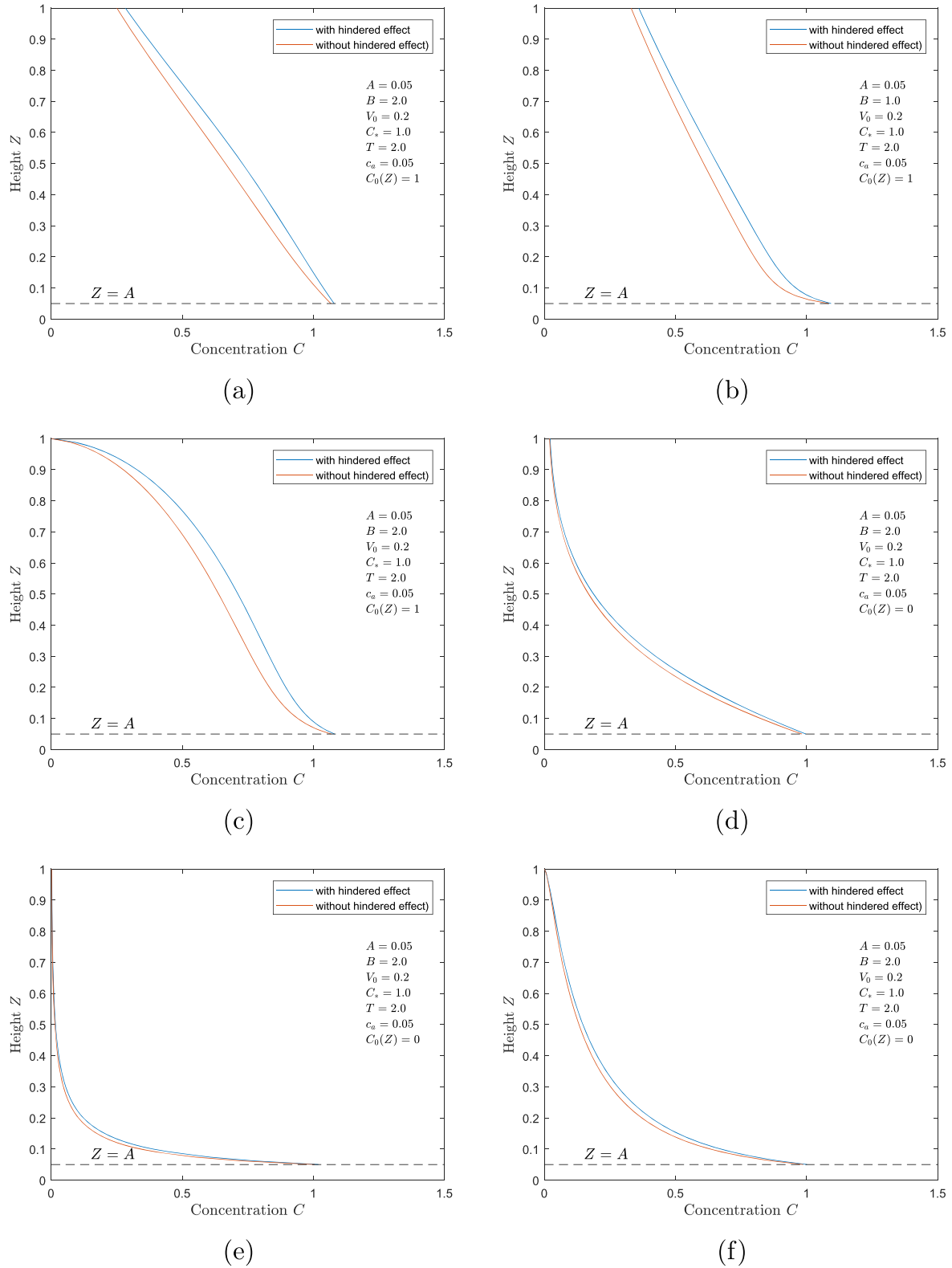


Figure 5. Effect of hindered settling through n_H ; (a)–(c) are constant, linear, and parabolic eddy viscosity profiles, respectively, with initial condition $C_0(Z) = 1$, and (d)–(f) are constant, linear, and parabolic eddy viscosity profiles, respectively, with initial condition $C_0(Z) = 0$.

5. Conclusions

The present work derives a semi-analytical solution for unsteady one-dimensional suspended sediment transport model with arbitrary eddy viscosity profiles together with the

generalized boundary conditions and hindered settling effect. An explicit series solution based on the MOL based HAM is obtained for the governing nonlinear PDE representing the suspended sediment concentration. The proposed methodology is found to be an efficient approach which controls the

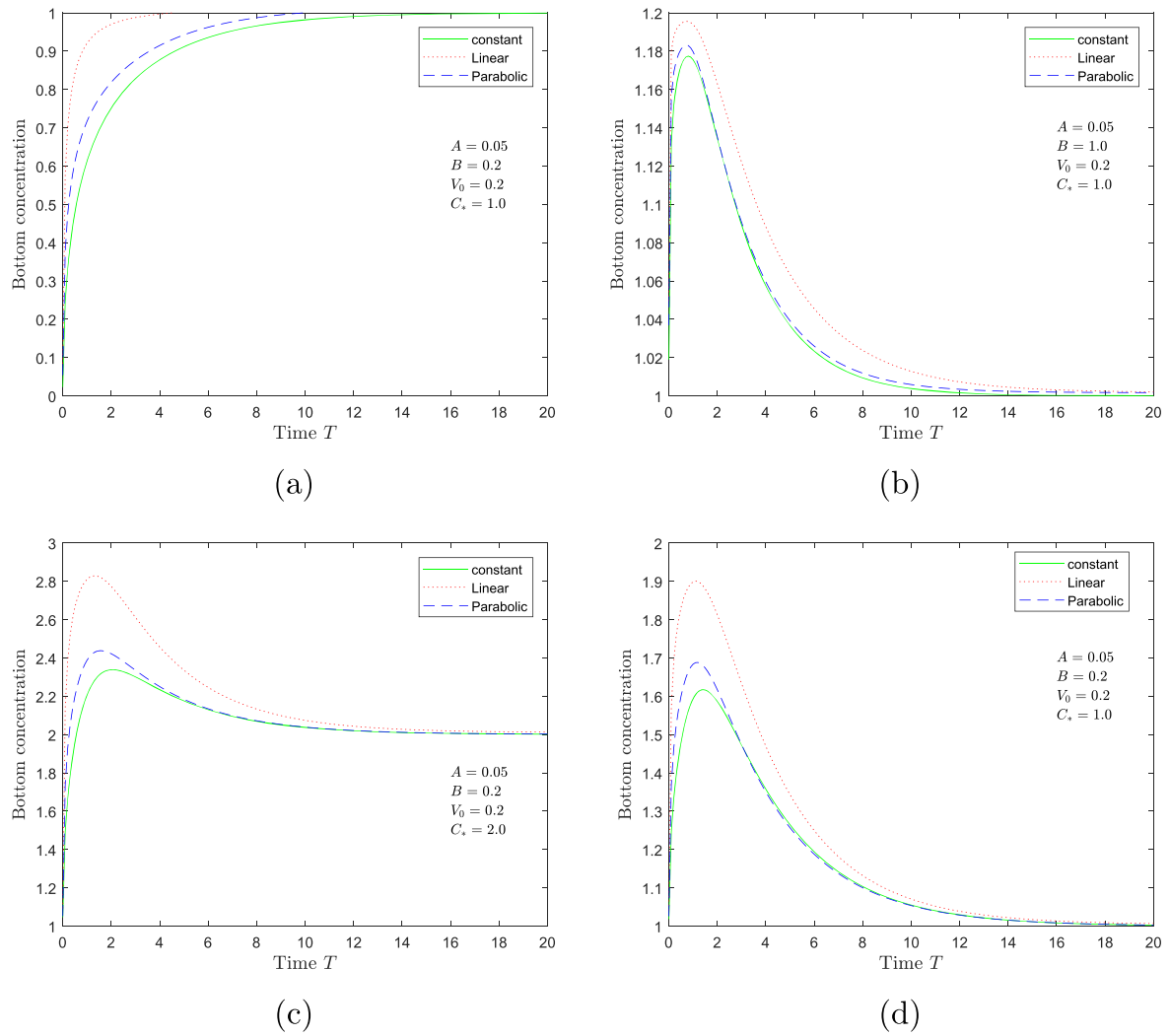


Figure 6. Bottom sediment concentration profiles with: (a) $C_0(Z) = 0$ and (b)–(d) $C_0(Z) = 1$.

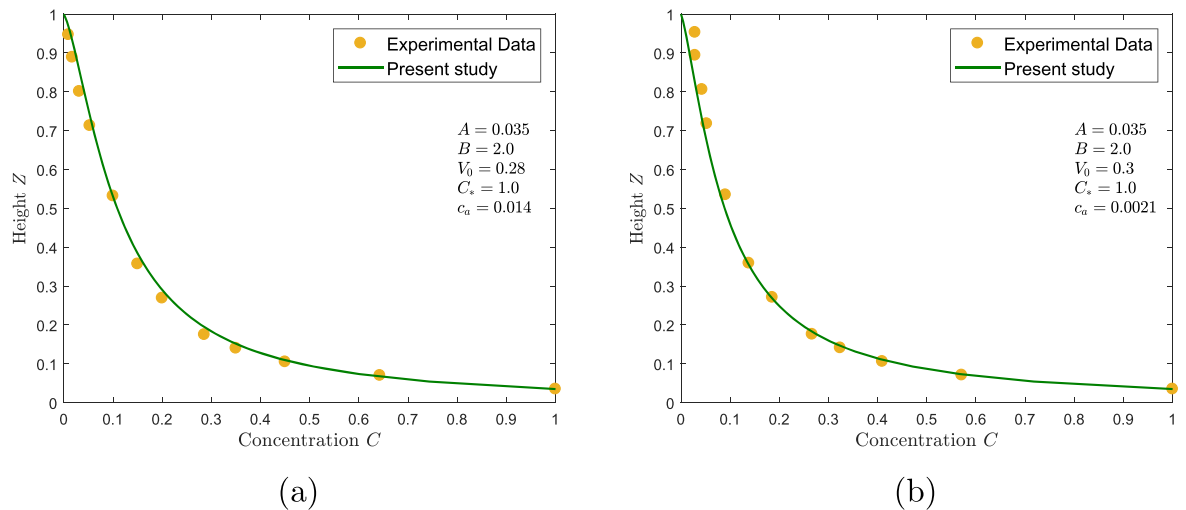


Figure 7. Comparison of present study with Coleman Data [46]: (a) Run-13 and (b) Run-23.

convergence of the series solution through a convergence-control parameter. Moreover, under certain conditions, the derived semi-analytical solution agrees well with the previously obtained analytical solution as well as laboratory data

existing in literature. The behavior of the transient sediment concentration profiles with uniform inlet concentration is depicted. The effect of hindered settling mechanism on the vertical concentration profile is shown for all types of eddy

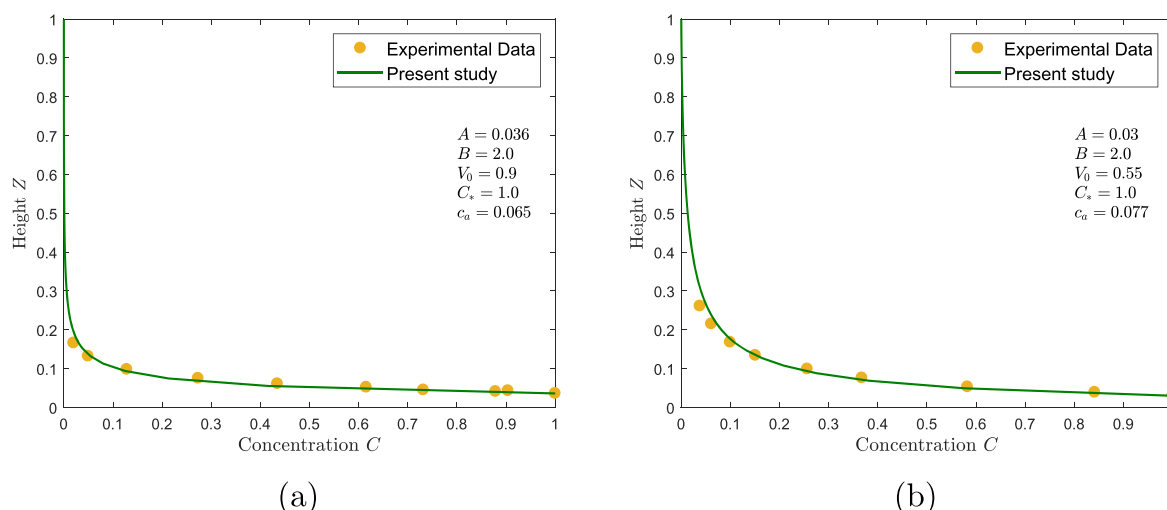


Figure 8. Comparison of present study with Einstein and Chien Data [47]: (a) S-9 and (b) S-12.

viscosity profiles. Also, the overshooting of bottom sediment concentration is observed when the inlet has uniform sediment concentration. Finally, the efficient approach proposed in this study is expected to be extended further for providing the analytical treatment for complicated problems in fluvial hydraulics.

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