

# Invariance analysis, exact solutions and conservation laws of (2+1)-dimensional dispersive long wave equations

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## Abstract

In this article, some new exact explicit solutions of (2+1)-dimensional dispersive long wave (DLW) equations are obtained by using the similarity transformation method under some restrictions imposed on the infinitesimals. This method reduces the dimension of PDEs by one after applying once. By choosing the suitable values of arbitrary functions involved in the expressions of infinitesimals, the system of PDEs is converted into the system of ODEs with the help of similarity variables. Under the suitable choice of arbitrary constants, the graphical representation of the obtained solutions are shown in order to highlight the importance of the study. The adjoint table, conservation laws and optimal system of DLW system are also obtained.

Keywords: dispersive long wave (DLW) equations, similarity transformation method, infinitesimal generator, adjoint equations, conservation laws, similarity solutions

(Some figures may appear in colour only in the online journal)

## 1. Introduction

Theory of nonlinear evolution equations attracts the attention of physicists and mathematicians because of their applications in real life problems. Many researchers are trying to find the exact solutions of these complex phenomena in many field of science and engineering such as plasma physics, biology, chemistry, nonlinear optics, fluid dynamics etc. The complex phenomena are modeled in the form of nonlinear partial differential equations (PDEs). There are so many powerful and efficient methods that have been derived to solve these PDEs like Hyperbolic tangent method [1], Homogeneous balance method [2], Inverse scattering transformation [3] and Lie symmetry method [4–7] etc. The Lie symmetry method, which was first introduced by Lie [8] in 1881, is one of the powerful and reliable method to solve PDEs.

In this article, the Lie symmetry technique is used to find the explicit solution expressions of the (2+1)-dimensional

dispersive long wave (DLW) equations which is of the form [9]:

$$\begin{aligned} u_{yt} + v_{xx} + (uu_x)_y &= 0, \\ v_t + u_x + (uv)_x + u_{xy} &= 0. \end{aligned} \quad (1.1)$$

### 1.1. Formation of the DLW system

The integrable DLW equations are given as [10]:

$$\begin{aligned} u_t &= -v_x - \frac{(u^2)_x}{2}, \\ v_t &= -(uv + u + u_{xx})_x, \end{aligned} \quad (1.2)$$

where  $u(x, t)$  is the horizontal velocity and  $v(x, t)$  is the deviation height of the surface wave, which propagates along  $x$ -axis. This system of equations has been derived in context of water waves which propagates in infinitely long channels having finite constant depth. Then, to cover the situation of open seas or wide channels, two different extensions of this

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system have been provided. Out of which, one is the physical (2+1)-dimensional long wave equation, obtained by approximation from some basic equations of hydrodynamics, is of the form:

$$\begin{aligned} u_t &= -v_x - \frac{(u^2)_x}{2}, \\ v_{tx} &= -(uv + u + u_{xx})_{xx} - u_{yy}. \end{aligned} \quad (1.3)$$

The second extension also involves an additional space variable  $y$  and the equations are obtained by Boiti *et al* [11] as a compatibility condition for a weak Lax pair. They defined weak Lax pair as a pair of linear operators, that commute on the null-space of one of the operators rather than everywhere. Assuming that  $T_1$  and  $T_2$  are the operators then

$$[T_1, T_2]\phi = 0 \text{ for } T_2\phi = 0. \quad (1.4)$$

For detailed information, we refer readers to the original article [11]. Then, the derived equations are as follows:

$$\begin{aligned} u_{ty} &= -v_{xx} - \frac{(u^2)_{xy}}{2}, \\ v_t &= -(uv + u + u_{xy})_x. \end{aligned} \quad (1.5)$$

### 1.2. Related works

The DLW system of equations was first determined by Boiti *et al* [11], as a compatibility condition for a weak Lax pair. The symmetry algebra of these equations was described by Paquin and Winternitz, as infinite dimensional [10]. The similarity solutions of DLW system in two space dimensions are obtained by Lou [12].

Ma [13] explained the diversity of exact solutions to a restricted BLP DLW system. Zheng *et al* [14] used extended mapping approach, to obtain new variable separation excitations of (2+1)-dimensional dispersive long-water wave system. Ma and Hu [9] defined solitons, chaos and fractals in the (2+1)-dimensional DLW equation. Fucui You and Xia [15] explained the Hamiltonian structures of DLW system of equation hierarchy. Eslami *et al* [16] obtained the Explicit solutions of nonlinear (2+1)-dimensional DLW equation. Wazwaz [17] found the rational solutions and multi-soliton solutions of dispersive long water-wave solution by using Painleve–Backlund transformation. Kumar *et al* [18] have been used the lie group analysis to obtain the exact solutions of (3+1)-dimensional generalized  $B$ -type  $KP$  type equation.

### 1.3. Outline

The paper is organized as follows: in section 2, we have presented general procedure of symmetry method. In section 3, the infinitesimals are determined for the system (1.1) using the Lie symmetry technique. With the help of infinitesimals, the similarity variables are obtained which convert the system (1.1) into the system of (1+1)-dimension of ODEs. The same process is continued till the system of ODEs is obtained. Then, by back substitution, the solutions for the original system are obtained. The graphical

representation of the obtained solutions is also given to better understand the physical meaning of the solutions. Also, the commutator and adjoint table are presented in section 3. We have also obtained the adjoint table and optimal system [19] for DLW equations in section 3. The classification of group invariant solutions of differential equations by means of the optimal systems is one of the main applications of Lie group analysis to differential equations. We can always construct a family of group invariant solutions corresponding to a subgroup of a symmetry group admitted by a given differential equation. Since there is an infinite number of such subgroups, it is not possible to list all of the group invariant solutions. An effective and systematic way of classifying these group invariant solutions is to obtain optimal systems of subalgebras of the Lie symmetry algebra. The obtained results are discussed physically in section 4. In section 5, conservation laws for DLWs are determined. As conservation laws are mathematical expressions of the physical laws, such as conservation of energy, mass, and momentum. They are of great significance in the solution process and reduction of PDEs. Conservation laws have been widely used in studying the stability of solutions of nonlinear PDEs. For detail information one can follow the research paper [20]. In section 6, a brief conclusion is given about the whole study.

## 2. General procedure of symmetry method

Sometimes Lie group method is also called symmetry analysis. Now, to study symmetry method, we consider a general system  $v$  of nonlinear PDEs of order  $n$  having  $\alpha$  dependent and  $\beta$  independent variables which is defined as follows:

$$\Delta_v(x, u^{(n)}) = 0, \quad v = 1, 2, \dots, a. \quad (2.1)$$

where  $x = (x^1, \dots, x^\beta)$ ,  $u = (u^1, \dots, u^\alpha)$  and  $u^{(n)}$  defines the  $n$ th derivative of  $u$ .

A symmetry group of system of nonlinear PDEs transforms the solutions of system to another solutions. We can directly use the properties of group and construct new solutions from the known solutions.

Now, we consider a one parameter ( $\epsilon$ ) Lie group of infinitesimal transformations acting on independent variable  $x$  and dependent variable  $u$

$$\begin{aligned} \tilde{x}^a &= x^a + \epsilon \xi^a(x, u) + O(\epsilon^2), \quad a = 1, 2, \dots, \beta, \\ \tilde{u}^b &= u^b + \epsilon \varphi^b(x, u) + O(\epsilon^2), \quad b = 1, 2, \dots, \alpha, \end{aligned} \quad (2.2)$$

where  $\epsilon$  is very small parameter of group and  $\xi^a$ ,  $\varphi^b$  are infinitesimals for independent and dependent variables, respectively. Then, the infinitesimal generator of the above group of transformation is

$$V = \sum_{a=1}^{\beta} \xi^a(x, u) \partial_{x^a} + \sum_{b=1}^{\alpha} \varphi^b(x, u) \partial_{u^b}. \quad (2.3)$$

After the infinitesimal transformation, the invariance condition of the group is as follows:

$$\begin{aligned} Pr^{(n)} \Delta_v(x, u^{(n)}) &= 0, \quad v = 1, 2, \dots, a, \\ \text{whenever } \Delta_v(x, u^{(n)}) &= 0, \end{aligned} \quad (2.4)$$

where  $Pr^{(n)}$  is  $n$ th order prolongation of infinitesimal generator  $V$ , given as:

$$Pr^{(n)}V = V + \sum_{k=1}^{\alpha} \sum_B \varphi_k^B(x, u^{(n)}) \partial u_B^k. \quad (2.5)$$

This prolonged space is called **Jet space** having all independent and dependent variables, and all order partial derivatives of dependent variables as its coordinates. Its dimension is given by

$$\dim V^n = \left[ \beta + \alpha \binom{\beta + n}{n} \right], \quad (2.6)$$

where  $B = (b_1, \dots, b_s)$ ,  $1 \leq b_s \leq \beta$ ,  $1 \leq s \leq n$ , and the summation runs over  $B$ . If  $B = s$ , then the coefficients  $\varphi_k^B$  of  $\partial u_B^k$  will depend on  $s$ th and derivatives of  $u$  as follows:

$$\varphi_k^B(x, u^{(n)}) = D_B(\varphi_k - \sum_{a=1}^{\beta} \xi^a u_a^k) + \sum_{a=1}^{\beta} \xi^a u_{B,a}^k, \quad (2.7)$$

where  $u_a^k = \frac{\partial u^k}{\partial x^a}$ ,  $u_{B,a}^k = \frac{\partial u_B^k}{\partial x^a}$ .

### 3. Lie symmetry analysis and similarity solution of DLW system

We consider a one-parameter ( $\epsilon$ ) Lie group of transformations for the dependent and independent variables as follows:

$$\begin{aligned} \tilde{x} &= x + \epsilon \xi^{(1)}(x, y, t, u, v) + O(\epsilon^2), \\ \tilde{y} &= y + \epsilon \xi^{(2)}(x, y, t, u, v) + O(\epsilon^2), \\ \tilde{t} &= t + \epsilon \tau(x, y, t, u, v) + O(\epsilon^2), \\ \tilde{u} &= u + \epsilon \eta^{(1)}(x, y, t, u, v) + O(\epsilon^2), \\ \tilde{v} &= v + \epsilon \eta^{(2)}(x, y, t, u, v) + O(\epsilon^2), \end{aligned}$$

where  $\xi^{(1)}$ ,  $\xi^{(2)}$ ,  $\tau$ ,  $\eta^{(1)}$  and  $\eta^{(2)}$  are the infinitesimal for the variables  $x$ ,  $y$ ,  $t$ ,  $u$  and  $v$ , respectively

Then, the vector field, which generates the Lie symmetry, is

$$V = \xi^{(1)} \partial_x + \xi^{(2)} \partial_y + \tau \partial_t + \eta^{(1)} \partial_u + \eta^{(2)} \partial_v. \quad (3.1)$$

Here, subscripts denote the partial derivatives with respect to the respective variables. For DLW equation, we consider  $n = 5$ , thus the infinitesimal criteria for the invariance of (1.1) would be

$$Pr^{(5)} \Delta_v(x, u^{(5)}) = 0, \text{ whenever } \Delta_v(x, u^{(5)}) = 0. \quad (3.2)$$

Now, applying the fifth order prolongation  $Pr^{(5)}$  of  $V$  to (3.2), the invariant conditions are formulated as:

$$\begin{aligned} &[\eta_{yt}^{(1)}] + [\eta_{xx}^{(2)}] + u_x [\eta_y^{(1)}] + u_y [\eta_x^{(1)}] \\ &+ u [\eta_{xy}^{(1)}] + u_{xy} \eta^{(1)} = 0, \\ &[\eta_t^{(2)}] + [\eta_x^{(1)}] + u_x \eta^{(2)} + v [\eta_x^{(1)}] + u [\eta_x^{(2)}] \\ &+ v_x \eta^{(1)} + [\eta_{xy}^{(1)}] = 0. \end{aligned} \quad (3.3)$$

Thus, this is the system of PDEs containing partial derivatives of  $\eta$ , where  $[\eta_y^{(1)}]$  is partial derivative of  $u$  with respect to  $y$ . Other terms can be understood in similar way. Substituting the values of  $[\eta_i^{(j)}]$  from [4] in above equations, we obtain the system of partial derivatives of infinitesimals. Equating to zero the coefficients of different monomials of obtained equations, the following values of the infinitesimals are obtained as

$$\begin{aligned} \xi^{(1)} &= \frac{1}{2} F_1'(t)x + F_2(t), \\ \xi^{(2)} &= H_1(y), \\ \tau &= F_1(t), \\ \eta^{(1)} &= \frac{-1}{2} F_1'(t)u + \frac{1}{2} F_1''(t)x + F_2'(t), \\ \eta^{(2)} &= \frac{-1}{2} [F_1'(t) + 2H_1'(y)](v+1), \end{aligned}$$

where  $F_1(t)$ ,  $F_2(t)$  and  $H_1(y)$  are the arbitrary functions and prime denotes the derivative of the arbitrary functions.

#### Symmetry reduction and invariant solutions of DLW system

We construct similarity variables to reduce the number of independent variables of PDEs. Characteristic equation for the obtained set of infinitesimals is

$$\begin{aligned} \frac{dx}{\frac{1}{2} F_1'(t)x + F_2(t)} &= \frac{dy}{H_1(y)} = \frac{dt}{F_1(t)} \\ &= \frac{du}{\frac{-1}{2} F_1'(t)u + \frac{1}{2} F_1''(t)x + F_2'(t)} \\ &= \frac{dv}{\frac{-1}{2} [F_1'(t) + 2H_1'(y)](v+1)}. \end{aligned} \quad (3.4)$$

From first and third equation we have,

$$\frac{dx}{dt} - \frac{F_1'(t)}{2F_1(t)} = \frac{F_2(t)}{F_1(t)}. \quad (3.5)$$

Here, the solutions of these integrals are obtained by assuming some meaningful values of the arbitrary functions  $F_1(t)$ ,  $F_2(t)$  and  $H_1(y)$ . As far as integrability of similarity reductions is concerned can be asserted by Painlevé conjecture [21]. According to which their general solutions should not have any movable singularities other than poles whereas according to Weiss *et al* [22], PDEs can be tested directly without recourse to the reduction to ODEs. The choice of the functions may provide the rich physical structure to the solutions of system (1.1). For physically meaningful solution of the problem, we have taken  $aF_1'(t) = F_2(t)$  so that the solutions can be explained meaningfully in the graphical representation. The symmetry of system (1.1) can be written as follows:

$$V = V_1(F_1) + V_2(F_2) + V_3(H_1), \quad (3.6)$$

where

$$\begin{aligned} V_1(F_1) &= \frac{x}{2} F_1'(t) \frac{\partial}{\partial x} + F_1(t) \frac{\partial}{\partial t} - \frac{1}{2} (F_1'(t)u \\ &\quad - F_1''(t)x) \frac{\partial}{\partial u} - \frac{1}{2} (v+1) F_1'(t) \frac{\partial}{\partial v}, \\ V_2(F_2) &= F_2(t) \frac{\partial}{\partial x} + F_2'(t) \frac{\partial}{\partial u}, \\ V_3(H_1) &= H_1(y) \frac{\partial}{\partial y} - H_1'(y)(v+1) \frac{\partial}{\partial v}. \end{aligned}$$

The set of infinitesimal symmetries of the system of PDEs forms Lie algebra under Lie bracket  $[V_i, V_j] = V_i V_j - V_j V_i$ , which satisfies skew symmetry with diagonal entries zero. The commutator table of Lie algebra for DLW system is given as:

$[V_i, V_j]$	$V_1$	$V_2$	$V_3$
$V_1$	0	$V_2(F_1 F_2' - \frac{1}{2} F_2 F_1')$	0
$V_2$	$-V_2(F_1 F_2' - \frac{1}{2} F_2 F_1')$	0	0
$V_3$	0	0	0

Now, the adjoint representation of Lie group on its Lie algebra can be defined as:

$$\text{ad}(v)(w) := \frac{d}{d\epsilon} \text{Ad}(\exp(\epsilon v))(w) = [w, v], \quad (3.7)$$

with Lie series [23]

$$\begin{aligned} \text{Ad}(\exp(\epsilon v))(w) &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} (\text{ad } v)^n(w) \\ &= w - \epsilon [v, w] + \frac{\epsilon^2}{2} [v, [v, w]] - \dots \end{aligned} \quad (3.8)$$

Thus, the adjoint table is constructed as:

Ad	$V_1$	$V_2$	$V_3$
$V_1$	$V_1$	$V_2 - \epsilon V_2$	$V_3$
$V_2$	$V_1 - \epsilon V_2$	$V_2$	$V_3$
$V_3$	$V_1$	$V_2$	$V_3$

### Optimal system for (2+1)-dimensional DLW equations

We consider the Symmetry algebra of equation (1.1), whose adjoint representation was determined in the above table. Let  $V = r_1 V_1 + r_2 V_2 + r_3 V_3$  be a non zero vector. Now, all the coefficients have to be simplified by using the convenient adjoint maps. For this purpose, first we assume that  $r_3 \neq 0$ . Let  $r_3 = 1$ , then we act on a  $V$  by  $\text{Ad}(\exp(r_2 V_2))$ , to make the coefficient of  $V_2$  vanish:

$$V' = \text{Ad}(\exp(r_2 V_2))V = r_1' V_1 + V_3. \quad (3.9)$$

Here,  $r_1'$  depends on  $r_1$ ,  $r_2$  and  $r_3$ . Now, we act  $V'$  by  $\text{Ad}(\exp(r_1' V_1))$ , to make the coefficient of  $V_1$  vanish, so that  $V$  is equivalent to  $V_3$  under the adjoint representation. In other words, every one-dimensional sub-algebra generated by  $V$  with  $r_3 \neq 0$  is equal to the subalgebra spanned by  $V_3$ . The

remaining one-dimensional sub-algebras are spanned by vectors of the above form with  $r_3 = 0$ . Let  $r_2 = 1$ , then we get  $V'' = r_1 V_1 + V_2$ . We can further act on  $V''$  by the group generated by  $V_1$ , this has the net effect of scaling the coefficients of  $V_2$  and  $V_3$ :

$$V''' = \text{Ad}(\exp(\epsilon V_1))V'' = r_1'' \epsilon V_1 + V_2. \quad (3.10)$$

So, it depends on the sign of  $r_1''$ . We consider the values of  $r_1''$  as  $+1$ ,  $-1$  and  $0$ . Thus, any one-dimensional subalgebra spanned by  $V$  with  $r_3 = 0$  and  $r_2 \neq 0$ , is equivalent to one spanned by either  $V_2$ ,  $V_2 + V_1$  or  $V_2 - V_1$ . The further simplifications are not possible. Then, an optimal system of the (2+1)-dimensional DLW equation is given by

$$V_3, V_2, V_2 + V_1, V_2 - V_1. \quad (3.11)$$

**Case 1:** Considering  $F_1(t) = 2t$ ,  $H_1(y) = y$ , the similarity reduction provides the following set of solution:

$$u = \frac{J(X, Y)}{\sqrt{t}}, \quad v = \frac{K(X, Y)}{t} - 1, \quad (3.12)$$

where  $X$  and  $Y$  are the similarity variables, while  $J(X, Y)$  and  $K(X, Y)$  are the similarity functions given as:

$$X = \frac{(x + 2a)}{\sqrt{t}}, \quad Y = \frac{y}{\sqrt{t}}. \quad (3.13)$$

After substituting these values in the system of equations (1.1), we obtain

$$\begin{aligned} JJ_{XY} + J_X J_Y - J_Y - \frac{1}{2} [X J_{YY} + Y J_{YY}] + K_{XX} &= 0, \\ JK_X + K J_X + J_{XX} - K - \frac{1}{2} [X K_X + Y J_Y] &= 0. \end{aligned} \quad (3.14)$$

Again, we consider new infinitesimals  $\xi^{(X)}$ ,  $\xi^{(Y)}$ ,  $\eta^{(J)}$  and  $\eta^{(K)}$  for variables  $X, Y, J$  and  $K$ . The values of new infinitesimals after applying the similarity transformation method (STM) are

$$\xi^{(X)} = \frac{A_2 X}{2} + A_3, \quad \xi^{(Y)} = [\log Y A_2 + A_1] Y, \quad (3.15)$$

$$\begin{aligned} \eta^{(J)} &= \frac{(X - J) A_2}{2} + \frac{A_3}{2}, \\ \eta^{(K)} &= - \left[ \log Y A_2 + A_1 + \frac{3}{2} A_2 \right] K, \end{aligned} \quad (3.16)$$

where  $A_1$ ,  $A_2$  and  $A_3$  are the arbitrary constants. So, the Lie algebra of infinitesimal symmetry is spanned by

$$\begin{aligned} \tilde{V}_1 &= Y \frac{\partial}{\partial Y} - K \frac{\partial}{\partial K}, \\ \tilde{V}_2 &= \frac{X}{2} \frac{\partial}{\partial X} + Y \log Y \frac{\partial}{\partial Y} + \frac{(X - J)}{2} \frac{\partial}{\partial J} \\ &\quad - \left( \log Y + \frac{3}{2} \right) K \frac{\partial}{\partial K}, \\ \tilde{V}_3 &= \frac{\partial}{\partial X} + \frac{1}{2} \frac{\partial}{\partial J}. \end{aligned}$$

For simplicity, assuming  $A_1 = A_2 = 0$ ,  $A_3 = 1$ , the Lagrange's form is

$$\frac{dX}{1} = \frac{dY}{0} = \frac{dJ}{\frac{1}{2}} = \frac{dK}{0}, \quad (3.17)$$

which gives  $Y = Y_1$ ,  $J = \frac{1}{2}X + j(Y_1)$  and  $K = k(Y_1)$ , where  $Y_1$  is an arbitrary constant.

Here, two new arbitrary functions  $j$  and  $k$  have been introduced. After substituting their values in equations of system (3.14), the following ODEs obtained

$$j' + Y_1 j'' = 0, \quad (3.18)$$

which gives

$$j = C_2 + \frac{\sqrt{t}}{y} C_3, \quad (3.19)$$

and

$$k + Y_1 k' = 0, \quad (3.20)$$

which gives

$$k = \frac{C_1}{Y_1}. \quad (3.21)$$

Here,  $C_2$  and  $C_3$  are the arbitrary constants. Thus, by substituting these values in equations (3.12) and (3.13), we get the values of  $u$  and  $v$  as:

$$u = \frac{1}{2} \frac{(x + 2a)}{t} + \frac{C_2}{\sqrt{t}} + \frac{C_3}{y}, \quad v = \frac{C_1}{y\sqrt{t}} - 1. \quad (3.22)$$

These are the solutions of given problem. These solutions are new as far as we are aware about this. Since the solutions involve many arbitrary constants  $a$ ,  $C_1$ ,  $C_2$  and  $C_3$ , thus the physical structure of the solutions depend on the values of these arbitrary constants. The graphical representation of the solution given by equation (3.22) is shown in figures 1 and 2. Singularity occurs in figures 1 and 2 only at  $y = 0$  in finite time or even simultaneously. This causes the solutions to become very unstable. The singularities occurs when the nonlinear effects dominates the linear effects. Nonlinearity acts to disperse the solution. Due to singularities, solution becomes unstable.

**Case 2:** Considering  $F_1(t) = t^2$ ,  $H_1(y) = 1$ , the similarity reduction provides solutions as:

$$u = X + \frac{J(X, Y)}{t}, \quad v = \frac{K(X, Y)}{t} - 1, \quad (3.23)$$

where  $X$  and  $Y$  are the similarity variables while  $J(X, Y)$  and  $K(X, Y)$  are the similarity functions defined as:

$$X = \frac{(x + 2a)}{t}, \quad Y = y + \frac{1}{t}. \quad (3.24)$$

After substituting these values in system of equations (1.1), we get the reduced form of the system as:

$$\begin{aligned} JJ_{XY} + J_X J_Y - J_{YY} + K_{XX} &= 0, \\ JK_X + K J_X + J_{XXY} - K_Y &= 0. \end{aligned} \quad (3.25)$$

Now, we consider new infinitesimals  $\xi^{(X)}$ ,  $\xi^{(Y)}$ ,  $\eta^{(J)}$  and  $\eta^{(K)}$  for variables  $X$ ,  $Y$ ,  $J$  and  $K$ . The values of new infinitesimals after applying the STM are

$$\xi^{(X)} = \frac{1}{2} A_1 X + A_3, \quad \xi^{(Y)} = A_1 Y + A_2, \quad (3.26)$$

$$\eta^{(J)} = \frac{-1}{2} A_1 J, \quad \eta^{(K)} = \frac{-3}{2} A_1 K, \quad (3.27)$$

where  $A_1$ ,  $A_2$  and  $A_3$  are the arbitrary constants. So, the Lie algebra of the infinitesimal symmetry is spanned by

$$\begin{aligned} \tilde{V}_1 &= \frac{X}{2} \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} - \frac{J}{2} \frac{\partial}{\partial J} - \frac{3K}{2} \frac{\partial}{\partial K}, \\ \tilde{V}_2 &= \frac{\partial}{\partial Y}, \\ \tilde{V}_3 &= \frac{\partial}{\partial X}. \end{aligned}$$

For simplicity, assuming  $A_1 = A_3 = 0$ , the Lagrange form is

$$\frac{dX}{0} = \frac{dY}{A_2} = \frac{dJ}{0} = \frac{dK}{0}. \quad (3.28)$$

which gives  $X = X_1$ ,  $J = j(X_1)$  and  $K = k(X_1)$ , where  $X_1$  is an arbitrary constant. Here, two new arbitrary functions  $j$  and  $k$  have been introduced. After substituting these values in the system of equations (3.25), the following ODEs are obtained:

$$k''(X_1) = 0, \quad (3.29)$$

which gives

$$k = \alpha X + \beta, \quad (3.30)$$

where  $\alpha$  and  $\beta$  are the arbitrary constants and

$$jk' + kj' = 0, \quad (3.31)$$

which gives

$$j = \frac{1}{\alpha X + \beta}. \quad (3.32)$$

Thus, by substituting these values in equations (3.23) and (3.24), the values of  $u$  and  $v$  are obtained as:

$$\begin{aligned} u &= \frac{(x + 2a)}{t} + \frac{1}{\alpha(x + 2a) + \beta t}, \\ v &= \frac{\alpha(x + 2a)}{t^2} + \frac{\beta}{t} - 1. \end{aligned} \quad (3.33)$$

This case introduces new similarity solutions. Here,  $a$ ,  $\alpha$  and  $\beta$  are the arbitrary constants. By assuming the suitable choice of these constants the physical structure of the solutions can be understood. The graphical representation of the solution given by equation (3.33) is shown in figures 3 and 4.

**Case 3:** Considering  $F_1(t) = e^{2t}$ ,  $H_1(y) = 1$ , the similarity reduction provides the solutions as:

$$u = (x + 2a) + \frac{J(X, Y)}{(x + 2a)}, \quad v = K e^{-t} - 1. \quad (3.34)$$

where  $X$  and  $Y$  are the similarity variables while  $J(X, Y)$  and  $K(X, Y)$  are the similarity functions defined as:

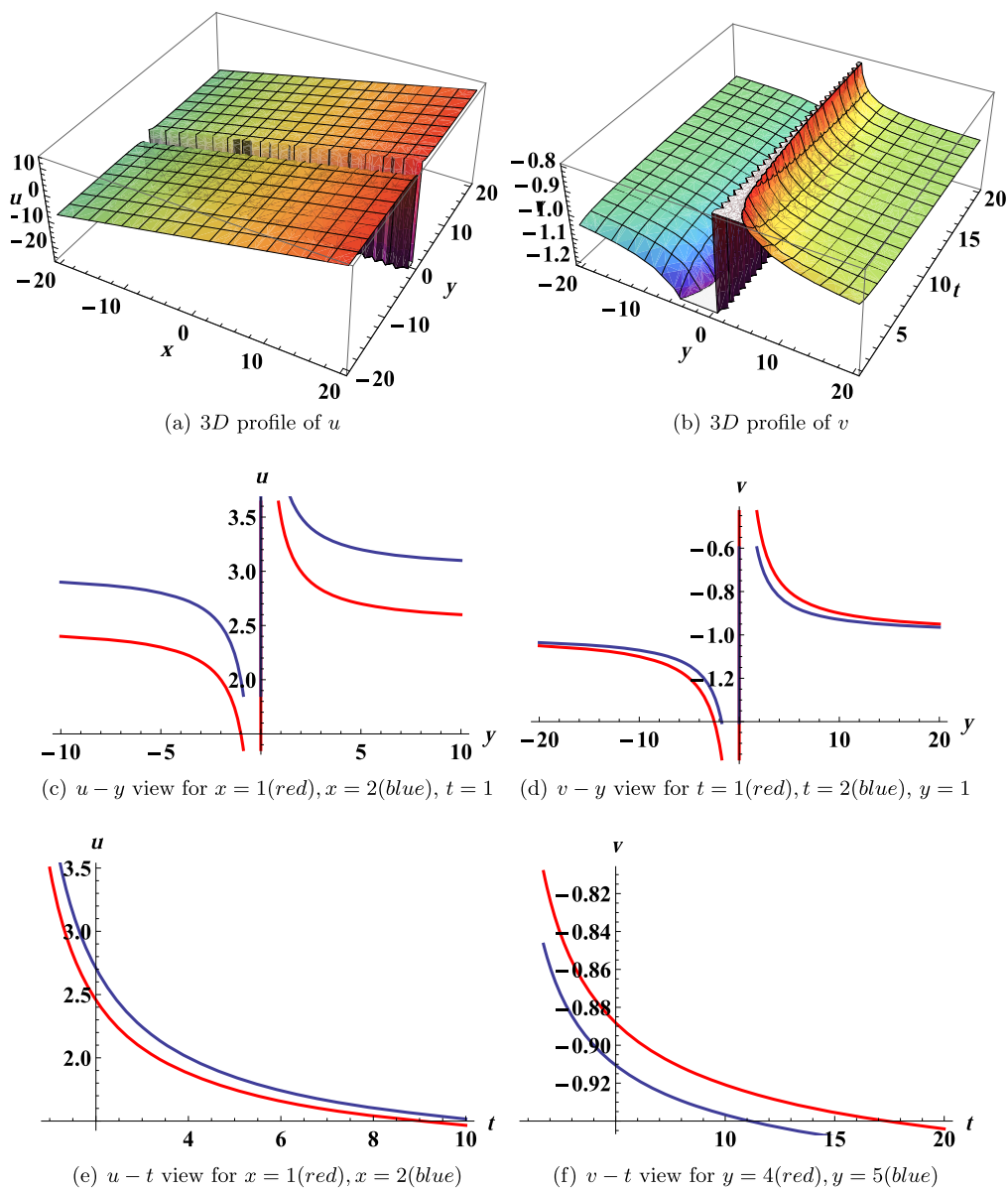
$$X = (x + 2a) e^{-t}, \quad Y = y + \frac{e^{-2t}}{2}. \quad (3.35)$$

After substituting these values in the system of equations (1.1), we obtain

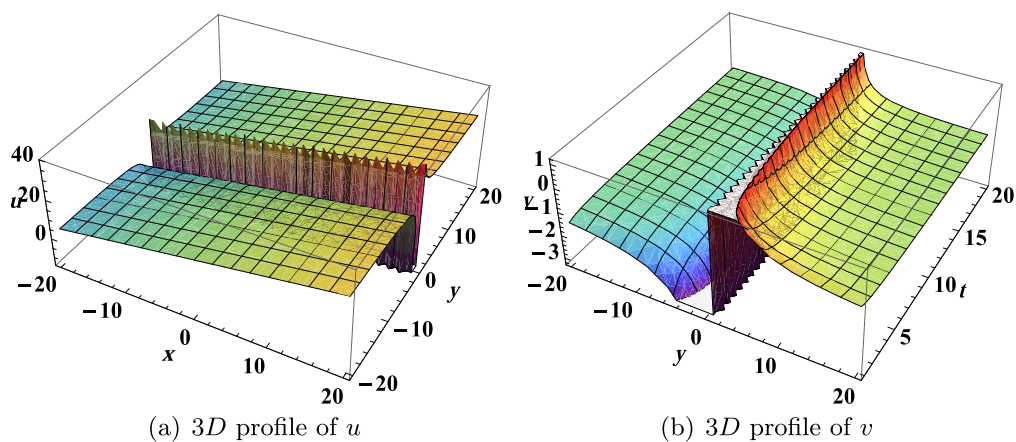
$$\begin{aligned} XJJ_{XY} - 2JJ_Y + XJ_X J_Y - X^2 J_{YY} \\ + X^3 K_{XX} &= 0, \end{aligned} \quad (3.36)$$

$$\begin{aligned} X^2 (JK_X + K J_X) - XJK + 2J_Y - 2XJ_{XY} \\ + X^2 J_{XXY} - X^3 K_Y &= 0. \end{aligned} \quad (3.37)$$

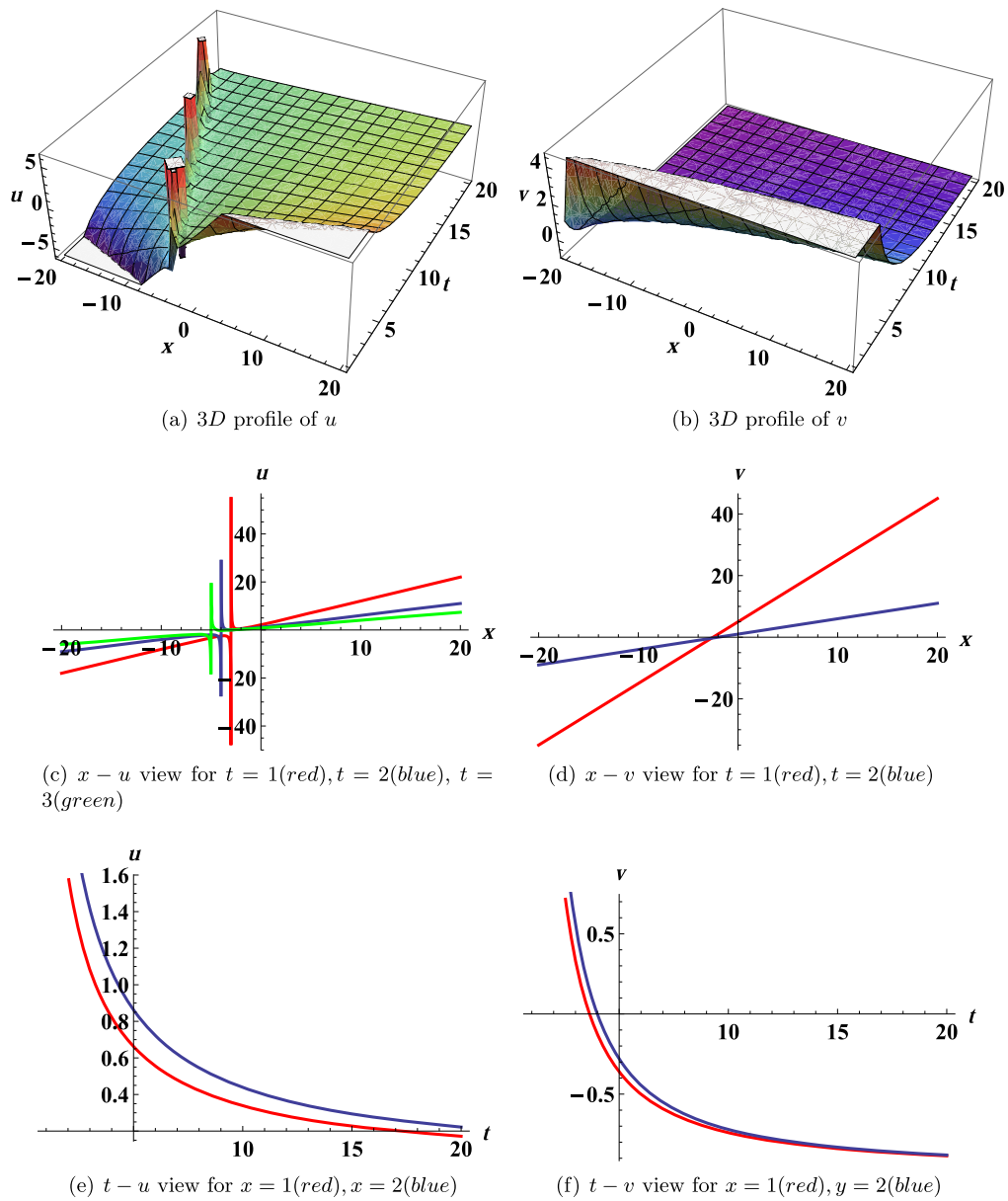




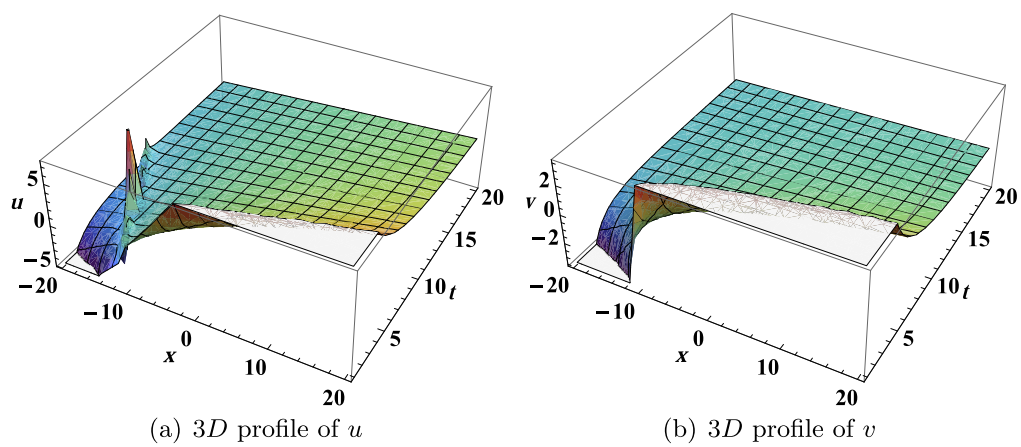
**Figure 1.** Singularity profiles of  $u$  and  $v$  given by equation (3.22) for  $a = 1.0$ ,  $C_1 = 1.0$ ,  $C_2 = 1.0$ ,  $C_3 = 1.0$ .



**Figure 2.** Solution profiles of  $u$  and  $v$  given by equation (3.22) for  $a = 5.0$ ,  $C_1 = 10.0$ ,  $C_2 = 10.0$  and  $C_3 = 10.0$ .



**Figure 3.** Solitary wave profiles of  $u$  and  $v$  given by equation (3.33) for  $a = 1.0$ ,  $\alpha = 2$  and  $\beta = 2$ .



**Figure 4.** Solution profiles of  $u$  and  $v$  given by (3.33) for  $a = 5.0$ ,  $\alpha = 3$  and  $\beta = 3$ .

Again, we consider new infinitesimals  $\xi^{(X)}, \xi^{(Y)}, \eta^{(J)}$  and  $\eta^{(K)}$  for the variables  $X, Y, J$  and  $K$ . The values of the new infinitesimals after applying the STM are

$$\xi^{(X)} = \frac{1}{2}A_1X + A_3, \quad \xi^{(Y)} = A_1Y + A_2, \quad (3.38)$$

$$\eta^{(J)} = \frac{A_3}{2}J, \quad \eta^{(K)} = \frac{3}{2}A_1K, \quad (3.39)$$

where  $A_1, A_2$  and  $A_3$  are the arbitrary constants. So, Lie algebra of the infinitesimal symmetry is spanned by the following vector fields:

$$\tilde{V}_1 = \frac{X}{2} \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} + \frac{3K}{2} \frac{\partial}{\partial K},$$

$$\tilde{V}_2 = \frac{\partial}{\partial Y},$$

$$\tilde{V}_3 = \frac{\partial}{\partial X} + \frac{J}{2} \frac{\partial}{\partial J}.$$

For simplicity, assuming  $A_1 = A_3 = 0$ , the Lagrange form of the set of infinitesimals becomes

$$\frac{dX}{0} = \frac{dY}{A_2} = \frac{dJ}{0} = \frac{dK}{0}, \quad (3.40)$$

which gives  $X = X_1, J = j(X_1)$  and  $K = k(X_1)$ , where  $X_1$  is an arbitrary constant.

Here, two new arbitrary functions  $j$  and  $k$  have been introduced. After substituting these values in system of equations (1.1), the following ODEs are obtained:

$$k''(X_1) = 0 \quad (3.41)$$

which gives

$$k = \gamma(x + 2a)e^{-t} + \delta, \quad (3.42)$$

where  $\gamma$  and  $\delta$  are the arbitrary constants and

$$X[jk' + kj'] - jk = 0, \quad (3.43)$$

which gives

$$j = \frac{X}{\gamma X + \delta}. \quad (3.44)$$

Thus, by substituting these values in equations (3.34) and (3.35), the values of  $u$  and  $v$  are obtained as:

$$u = (x + 2a) + \frac{e^{-t}}{\gamma(x + 2a)e^{-t} + \delta},$$

$$v = \gamma(x + 2a)e^{-2t} + \delta e^{-t} - 1. \quad (3.45)$$

These are another similarity solutions. Here,  $a, \gamma$  and  $\delta$  are the arbitrary constants. By assuming the suitable values of these constants, the physical structure of the solutions can be understood. The graphical representation of the solution given by equation (3.45) is shown in figures 5 and 6 for the different values of the arbitrary constants.

## 4. Results and discussions

This action explains the physical analysis of obtained explicit solutions of the given problem. Figure 1 represents the singularity profiles of  $u$  and  $v$  for  $a = 1.0, C_1 = 1.0, C_2 = 1.0$  and  $C_3 = 1.0$  given by the expressions (3.22). Figures 1(c)

and (d) represents the 2D singular profile of  $u$  and  $v$  given by equation (3.22) respectively, which shows as the value of  $x$  increases  $u$  and  $v$  also increases. The  $t - u$  view and  $t - v$  view of the solutions are represented by figures 1(e) and (f) respectively, which shows as time passes over a bigger range, the value of  $u$  and  $v$  start decrease. Figure 2 represents the singularity profiles of  $u$  and  $v$  for  $a = 5.0, C_1 = 10.0, C_2 = 10.0$  and  $C_3 = 10.0$  at  $t = 1$  given by the expressions (3.22). From figures 1 and 2, it may be concluded that as the values of arbitrary constants increase, the evolutionary profiles of  $u$  and  $v$  increase.

Figure 3 shows the solution profiles of  $u$  and  $v$  obtained in the form of (3.33) for  $a = 1.0, \alpha = 2$  and  $\beta = 2$  at  $t = 1$ . From expression (3.33) it is observed that the continuity of  $u$  depends on the arbitrary constants  $\alpha, a$  and  $\beta$ . The figures 3(c) and (d) represents the  $x - u$  view and  $x - v$  view, respectively. The  $t - u$  view and  $t - v$  view of the solutions are represented by figures 3(e) and (f) respectively, which show that as time increases, value of  $u$  and  $v$  decrease. Figure 4 shows the solution profiles of  $u$  and  $v$  given by the equation (3.33) for  $a = 5.0, \alpha = 3$  and  $\beta = 3$  at  $t = 1$ . We found different behaviors of  $u$  and  $v$  for the different choice of arbitrary constants.

Figure 5 represents the evolutionary profiles of  $u$  and  $v$  which are obtained in the form of equations (3.45). The values of the constants are taken as  $a = 1.0, \alpha = 0.15, \beta = 0.18$  and  $t = 1$ .  $t - u$  view of figure 5(a) represents that a small increment in the value of  $x$  gives a rise in the value of  $u$ , while  $t - v$  view of figure 5(b) represents that there is a small increment in the value of  $v$  for higher values of  $x$ . Figure 6 shows the solution profiles of  $u$  and  $v$  given by the equation (3.45) for  $a = 5.0, \gamma = 0.25$  and  $\delta = 0.35$  at  $t = 1$ .

## 5. Conservation laws for DLW system

One can use conservation laws for mathematical analysis to develop appropriate numerical methods and for stability, uniqueness and existence analysis. The strong indication of its integrability is the existence of large number of conservation laws of a PDE or system. Now, to deduce conservation laws of system of equations (1.1) and to obtain its adjoint equations, we consider a PDE of order  $n$  with  $\beta$  independent variables  $x = (x^1, x^2, \dots, x^\beta)$  and  $\alpha$  dependent variables  $u = (u^1, u^2, \dots, u^\alpha)$  and their respective partial derivatives are

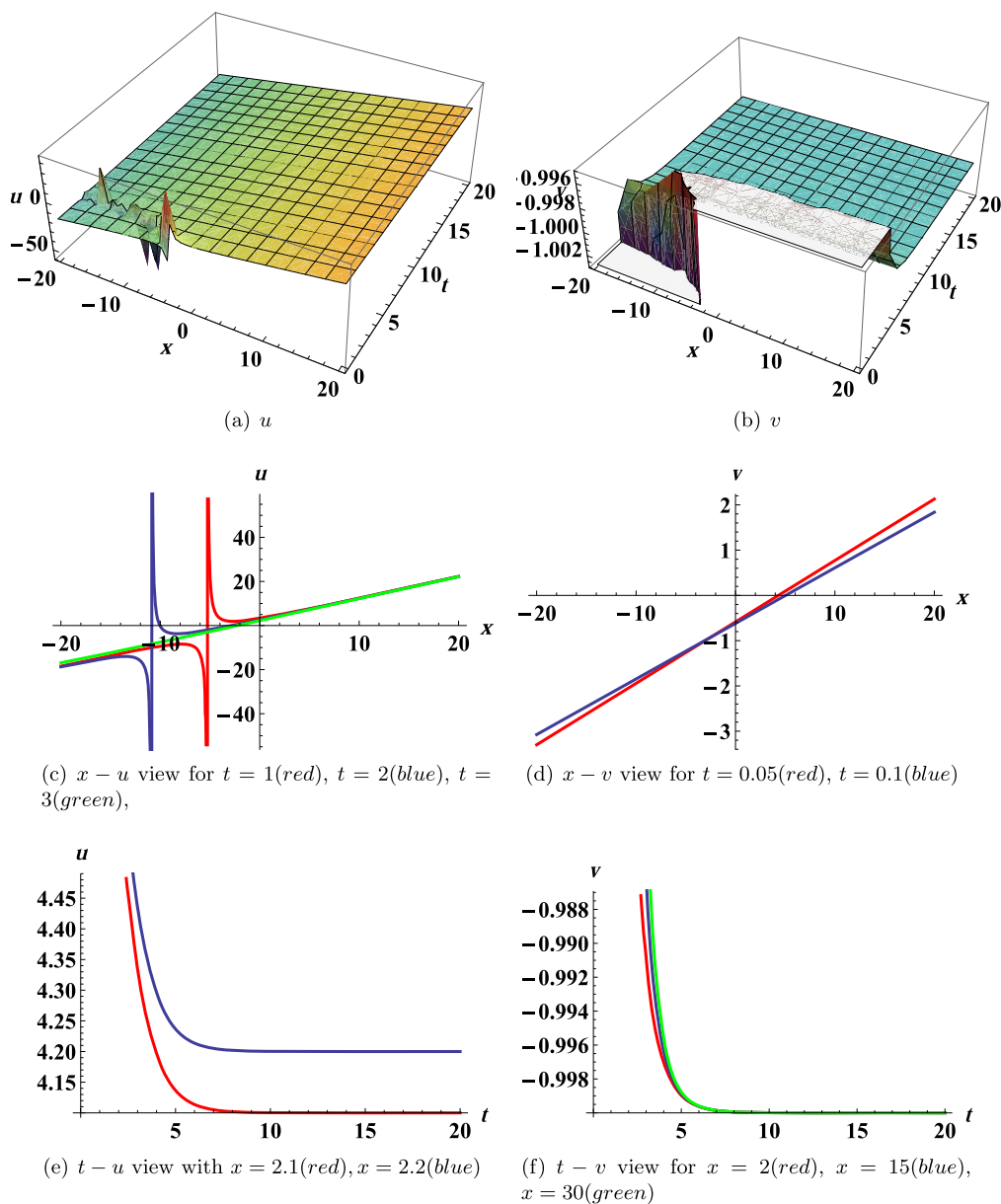
$$u_1 = u_i^m, \quad u_2 = u_{ij}^m, \quad \dots, \quad u_s = u_{i_1 i_2 \dots i_s}^m,$$

where

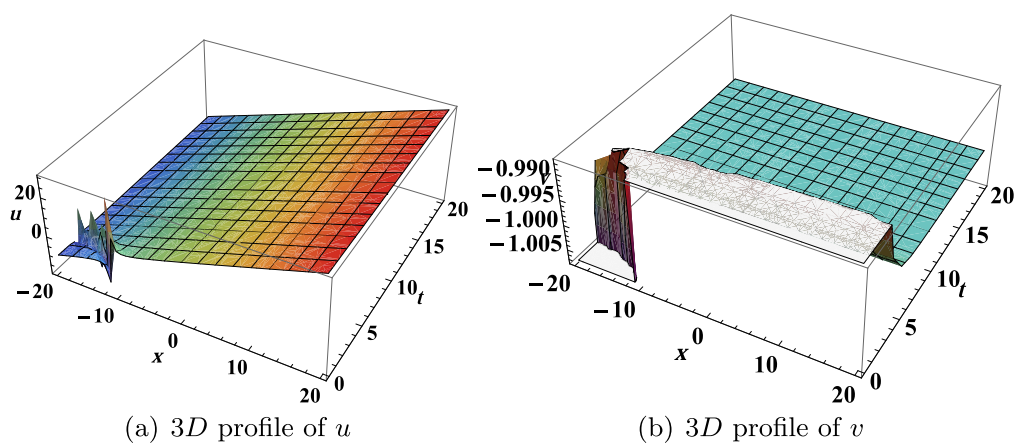
$$u_i^m = D_i(u^m), \quad u_{ij}^m = D_i D_j(u^m),$$

$$u_{i_1 i_2 \dots i_s}^m = D_{i_1 i_2 \dots i_s}(u^m).$$





**Figure 5.** Solitary wave profiles of  $u$  and  $v$  given by (3.45) for  $a = 1.0$ ,  $\gamma = 0.15$  and  $\delta = 0.18$ .



**Figure 6.** Solution profiles of  $u$  and  $v$  given by (3.45) for  $a = 5.0$ ,  $\alpha = 0.25$  and  $\beta = 0.28$ .

Here, the total differentiation  $D_i$  is given as:

$$D_i = \frac{\partial}{\partial x^i} + u_i^m \frac{\partial}{\partial u^m} + u_{ij}^m \frac{\partial}{\partial u_j^m} + \dots$$

Now, we consider a system having  $\alpha$  differential equations, may be linear or nonlinear which is

$$F_m(x, u, \dots, u_s) = 0, \quad m = 1, 2, \dots, \alpha. \quad (5.1)$$

Then, the adjoint equation of equation (5.1) is defined as:

$$F_m^*(x, u, \dots, u_s, v_s) = 0, \quad m = 1, 2, \dots, \alpha, \quad (5.2)$$

where the adjoint operator is defined as:

$$F_m^*(x, u, \dots, u_s, v_s) = \frac{\delta \tilde{L}}{\delta u^m}. \quad (5.3)$$

Here,  $v = (v^1, \dots, v^\gamma)$  is introduced as dependent variables and  $\tilde{L}$  is Lagrangian function defined as:

$$\tilde{L} = \sum_{\gamma=1}^{\alpha} [v^\gamma F_\gamma(x, u, \dots, u_s)]. \quad (5.4)$$

and the variational derivative is defined as:

$$\frac{\delta}{\delta u^m} = \frac{\partial}{\partial u^m} + \sum_{s=1}^{\infty} (-1)^s D_{i_1 i_2 \dots i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^m}. \quad (5.5)$$

Thus, to determine adjoint, we consider

$$\begin{aligned} F_1 &= u_{yt} + v_{xx} + (uu_x)_y, \\ F_2 &= v_t + u_x + (uv)_x + u_{xy}. \end{aligned} \quad (5.6)$$

Then, we have

$$\tilde{L} = \tilde{u}F_1 + \tilde{v}F_2. \quad (5.7)$$

Now, applying equations (5.3) and (5.5) to equation (5.7), we obtain the following expressions:

$$\begin{aligned} F_1^* &= \frac{\partial \tilde{L}}{\partial u} = -u_y \tilde{u}_x - u_x \tilde{u}_y - \tilde{v}_x \\ &\quad - v \tilde{v}_x + \tilde{u}_{yt} + u \tilde{u}_{xy} - \tilde{v}_{xy}, \\ F_2^* &= \frac{\partial \tilde{L}}{\partial v} = -u \tilde{v}_x + \tilde{u}_{xx} - \tilde{v}_t. \end{aligned} \quad (5.8)$$

Thus,  $F^* = F_1^* + F_2^*$  is the adjoint equation of the system.

For conservation laws, we consider an infinitesimal symmetry

$$V = \sum_{a=1}^{\beta} \xi^a(x, u) \partial_{x^a} + \sum_{m=1}^{\alpha} \varphi^m(x, u) \partial_{u^m}. \quad (5.9)$$

Then, the conservation law  $D_a(C^a) = 0$ , is constructed by

formula [24]

$$\begin{aligned} C^a &= \xi^a \tilde{L} + W^m \left[ \frac{\partial \tilde{L}}{\partial u_a^m} - D_b \left( \frac{\partial \tilde{L}}{\partial u_{ab}^m} \right) \right. \\ &\quad \left. + D_b D_c \left( \frac{\partial \tilde{L}}{\partial u_{abc}^m} \right) - \dots \right] \\ &\quad + D_b (W^m) \left[ \frac{\partial \tilde{L}}{\partial u_{ab}^m} - D_c \left( \frac{\partial \tilde{L}}{\partial u_{abc}^m} \right) + \dots \right] \\ &\quad + D_b D_c (W^m) \left[ \frac{\partial \tilde{L}}{\partial u_{abc}^m} - \dots \right]. \end{aligned} \quad (5.10)$$

Here,  $W$  is defined as  $W^m = \varphi^m - \xi^b u_b^m$ .

**Case 1:** When we consider  $V_2 = \frac{\partial}{\partial x}$ ,  $W = -u_x$ , then infinite conserved vector components are

$$\begin{aligned} C^1 &= -u_x [(u + v + 1)\tilde{v} + u_y \tilde{u} - \tilde{u}_y + \tilde{v}_{xy}] \\ &\quad - \tilde{u} u_{xy} - u_{xx} \tilde{v}_y - u_{xy} \tilde{v} + u_x \tilde{u}_x - u_{xx} \tilde{u}, \end{aligned} \quad (5.11)$$

$$C^2 = -u_x [u_x \tilde{u} - \tilde{u}_t] - u_{xt} \tilde{u}, \quad (5.12)$$

and

$$C^3 = -u_x \tilde{v}. \quad (5.13)$$

**Case 2:** When we consider  $V_2 = \frac{\partial}{\partial y}$ ,  $W = -u_y$ , then infinite conserved vector components are

$$\begin{aligned} C^1 &= -u_y [(u + v + 1)\tilde{v} + u_y \tilde{u} - \tilde{u}_y + \tilde{v}_{xy}] \\ &\quad - \tilde{u} u_{yy} - u_{yx} \tilde{v}_y - u_{xy} \tilde{v} + u_y \tilde{u}_x - u_{yx} \tilde{u}, \end{aligned} \quad (5.14)$$

$$C^2 = -u_y [u_x \tilde{u} - \tilde{u}_t] - u_{yt} \tilde{u}, \quad (5.15)$$

and

$$C^3 = -u_y \tilde{v}. \quad (5.16)$$

**Case 3:** When we consider  $V_2 = \frac{\partial}{\partial t}$ ,  $W = -u_t$ , then infinite conserved vector components are

$$\begin{aligned} C^1 &= -u_t [(u + v + 1)\tilde{v} + u_y \tilde{u} - \tilde{u}_y + \tilde{v}_{xy}] \\ &\quad - \tilde{u} u_{ty} - u_{tx} \tilde{v}_y - u_{tx} \tilde{v} + u_t \tilde{u}_x - u_{tx} \tilde{u}, \end{aligned} \quad (5.17)$$

$$C^2 = -u_t [u_x \tilde{u} - \tilde{u}_t] - u_{tt} \tilde{u}, \quad (5.18)$$

and

$$C^3 = -u_t \tilde{v}. \quad (5.19)$$

## 6. Conclusion

In this article, the exact analytic solutions for the (2+1)-dimensional DLW system are established in equations (3.22), (3.33) and (3.45) by using the Lie symmetry method. The obtained results are new findings as per the authors knowledge. The graphical representation of the solutions are given by choosing the suitable choice of arbitrary constants. The behavior of obtained solutions are physically analyzed. We have considered the three cases on the basis of choice of arbitrary functions involved in the expressions of

infinitesimals consisting polynomial, trigonometric and the exponential functions. The obtained solutions are explicit expressions in  $t$ ,  $x$  and  $y$ , that decay asymptotically to a constant at infinity. These obtained results may be significant to explain the physical phenomena related to the DLW system and validate the precision of many numerical methods. We have also obtained the conservation laws for DLW system, which can be used to develop numerical methods for existence analysis. Our work is very significant as these exact solutions will be helpful in applied sciences areas like as condensed fluid dynamics, nonlinear optics, etc. Thus Lie symmetry method is very powerful tool to establish the exact solutions of PDEs.

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