ICOPAMBS 2019 IOP Publishing
Journal of Physics: Conference Series **1465** (2020) 012029 doi:10.1088/1742-6596/1465/1/012029

On rainbow antimagic coloring of some graphs

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Abstract. Let $G(V, E)$ be a connected and simple graphs with vertex set V and edge set E. Define a coloring $c : E(G) \longrightarrow \{1, 2, 3, ..., k\}$, $k \in N$ as the edges of G, where adjacent edges may be colored the same. If there are no two edges of path P are colored the same then a path P is a rainbow path. The graph G is rainbow connected if every two vertices in G has a rainbow path. A graph G is called antimagic if the vertex sum (i.e., sum of the labels assigned to edges incident to a vertex) has a different color. Since the vertex sum induce a coloring of their edges and there always exists a rainbow path between every pair of two vertices, we have a rainbow antimagic coloring. The rainbow antimagic connection number, denoted by $rc_A(G)$ is the smallest number of colors that are needed in order to make G rainbow connected under the assignment of vertex sum for every edge. We have found the exact value of the rainbow antimagic connection number of ladder graph, triangular ladder, and diamond.

1. Introduction

The most fundamental graph-theoretical subject is connectivity. Graph theory have found many powerful and elegant results on connectivity [11]. For example Chartrand, et al. [3] introduced rainbow connection in 2008.

If there are no two edges of path in G are colored the same, then the path is called rainbow path. If every pair of two vertices connected by at least one rainbow path the graph G is rainbow connected [10]. The smallest number of color needed to color its edges is he rainbow connection number of G denoted by $rc(G)$. The distance between two vertices in graph G (the shortest path of two vertices in G) is $d(u, v)$. The shortest path from farthest u to v in G is $diam(G)$. The degree of the vertex with the greatest number of edge incident to it is the maximum degree of G denoted by $\triangle(G)$. A shortest cycle that contains the vertices u and v is $C(u, v)$. Furthermore, if G is a nontrivial connected graph of size m, then

$$diam(G) \leq rc(G) \leq src(G) \leq m$$

Define a function $g : E(G) \longrightarrow \{1, 2, \ldots, |E|\}$ in G such that the vertex sum (i.e., sum of the labels assigned to edges incident to a vertex) for distinct vertices are different is called antimagic [10]. When the edge weight $w(e) = g(u) + g(v)$ induce a coloring of their edges and there always exists a rainbow path between every pair of two vertices, we have a rainbow antimagic coloring. The rainbow antimagic connection number, denoted by $rc_A(G)$ is the smallest number of colors that are needed in order to make G rainbow connected under the assignment of edge weight $w(e) = g(u) + g(v)$ of every edge.

We will present some research related to rainbow connection, such as rainbow coloring of shadows graphs by Arputharnarya and Mercy [2], on the total rainbow connection of the wheel related graphs by Hasan, et al. [8], rainbow connection number of prism and product of two graphs by Darmawan and Dafik [5], on the (strong) rainbow vertex connection of graphs resulting from edge comb product by Dafik, et al. [4], on rainbow k-connection number of special graphs and it's sharp lower bound by Agustin, et al. [1].

Next we have some research related to antimagic labeling, such as on anti-magic labeling for graph products by Wang, et al. [13], on the local vertex antimagic total coloring of some families tree by Putri, et al. [12], antimagic labeling of regular graphs by Feihuang, et al. [6].

2. Result

We determine the rainbow connection number $rc(G)$ and rainbow antimagic connection number $rc_A(G)$ of ladder L_n, triangular ladder TL_n, and diamond D_n.

Theorem 1. *Let G be any connected graph, then $rc_A(G) \geq max\{\triangle(G), rc(G)\}$*

Proof. By property of antimagic labeling of G, we get $rc_A(G) \geq \triangle(G)$. Meanwhile, by property of rainbow coloring of G, then $rc_A(G) \geq rc(G)$. We have $rc_A(G) \geq max\{\triangle(G), rc(G)\}$.

Theorem 2. *Let P_n be a path of order n with $n \geq 3$. The rainbow antimagic connection number of P_n is $n-1$.*

Proof. We define a function $g : V(P_n) \longrightarrow \{1, 2, \ldots, n\}$ as follows:

$$g(x_a) = a$$

It is clear that the edge-weights of P_n under g are as follows

$$w(x_a x_{a+1}) = 2a + 1, \text{ for } 1 \leq a \leq n$$

Thus, function g induces a proper edge $(n-1)$-coloring of P_n where the color of $x_a x_{a+1}$ is $w(x_a x_{a+1})$. Next, consider any two distinct vertices $u, v \in V(P_n)$. Let $u = x_a$ and $v = x_b$ for each $a \in \{1, 2, \ldots, n\}$ and let $a < b$, then there exist a rainbow $u - v$ path, namely $x_a, x_{a+1}, \ldots, x_b$. Therefore, $rc_A \leq n-1$. Next we prove $rc_A(P_n) \geq n-1$. We know $\triangle(P_n) = 2$ and $rc(P_n) = n-1$, based on Theorem 1 we get $rc_A(P_n) \geq max\{2, n-1\}$. That's mean $rc_A(P_n) \geq n-1$.

Theorem 3. *Let L_n be a ladder graph with $n \geq 4$, the rainbow connection number of L_n is n.*

Proof. The ladder is a connected graph with $V(L_n) = \{x_a y_a, 1 \leq a \leq n\}$ and $E(L_n) = \{x_a x_{a+1}, 1 \leq a \leq n-1\} \cup \{y_a y_{a+1}, 1 \leq a \leq n-1\} \cup \{x_a y_a, 1 \leq a \leq n\}$. The cardinality of L_n are $|V(L_n)| = 2n$ and $|E(L_n)| = 3n-2$.

We define a function $c : E \longrightarrow \{1, 2, \ldots, |E|\}$ as follows:

$$c(e) = \begin{cases} c(x_a x_{a+1}) = a, & \text{for } 1 \leq a \leq n-1 \\ c(y_a y_{a+1}) = a, & \text{for } 1 \leq a \leq n-1 \\ c(x_a y_a) = n, & \text{for } 1 \leq a \leq n \end{cases}$$

Since the edge function $e = x_a y_a$ reach a maximum value, thus $rc(L_n) \leq n$. Next, we show that $rc(L_n) \geq n$. We take two vertices namely $u = y_a$ and $v = x_n$, the vertex u and v is farthest vertex in L_n. From that fact we get $diam(L_n) = n$. Chartrand, et al. [3] said $diam(G) \leq rc(G)$ and we know $diam(L_n) = n$. From that fact we get $rc(L_n) \geq n$.

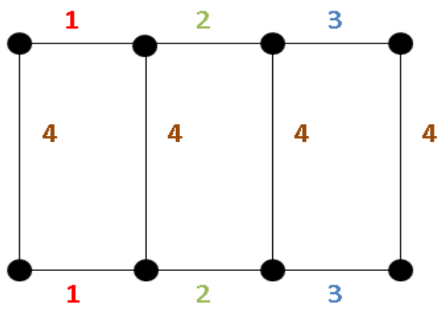

Figure 1. Ladder with 4 rainbow connection

Figure 1 shows that an illustration of ladder graph with $n = 4$. The colored number is color of the edge. So from the Figure 1 we know that $rc(L_4) = 4$.

Theorem 4. *Let L_n be a ladder graph with $n \geq 4$. The rainbow antimagic connection number of L_n are $n \leq rc_A \leq$*

$$rc_A(L_n) = \begin{cases} n, & \textit{if n=1 mod 2} \\ n+1, & \textit{if n=0 mod 2} \end{cases}$$

Proof. The ladder is a connected graph with $V(L_n) = \{x_a y_a, 1 \leq a \leq n\}$ and $E(L_n) = \{x_a x_{a+1}, 1 \leq a \leq n-1\} \cup \{y_a y_{a+1}, 1 \leq a \leq n-1\} \cup \{x_a y_a, 1 \leq a \leq n\}$. The cardinality of L_n are $|V(L_n)| = 2n$ and $|E(L_n)| = 3n - 2$.
We define a function $g : V \longrightarrow \{1, 2, \ldots, |V|\}$ as vertex function

For n =0 mod 2,

$$g(v) = \begin{cases} g(x_a) = 2a, & \text{if a=1 } mod \text{ 2} \\ g(x_a) = 2a - 1, & \text{if a=0 } mod \text{ 2} \\ g(y_a) = 2n - 2a + 2, & \text{if a=1 } mod \text{ 2} \\ g(y_a) = 2n - 2a + 1, & \text{if a=0 } mod \text{ 2} \end{cases}$$

For n =1 mod 2,

$$g(v) = \begin{cases} g(x_a) = 2a, & \text{if a=1 } mod \text{ 2} \\ g(x_a) = 2i - 1, & \text{if a=0 } mod \text{ 2} \\ g(y_a) = 2n - 2a + 1, & \text{if a=1 } mod \text{ 2} \\ g(y_a) = 2n - 2a + 2, & \text{if a=0 } mod \text{ 2} \end{cases}$$

Clearly, that the edge weight of L_n as follows
For n =0 mod 2,

$$w(e) = \begin{cases} w(x_a x_{a+1}) = 4a + 1, & \text{for } 1 \leq a \leq n-1 \\ w(y_a y_{a+1}) = 4n - 4a + 1, & \text{for } 1 \leq a \leq n-1 \\ w(x_a y_a) = 2n + 4, & \text{for } 1 \leq a \leq n-1\text{, and a=1 } mod \text{ 2} \\ w(x_a y_a) = 2n, & \text{for } 1 \leq a \leq n-1\text{, and a=0 } mod \text{ 2} \end{cases}$$

For n =1 mod 2,

$$w(e) = \begin{cases} w(x_a x_{a+1}) = 4a + 1, & \text{for } 1 \leq a \leq n-1 \\ w(y_a y_{a+1}) = 4n - 4a + 1, & \text{for } 1 \leq a \leq n-1 \\ w(x_a y_a) = 2n + 1, & \text{for } 1 \leq a \leq n-1 \end{cases}$$

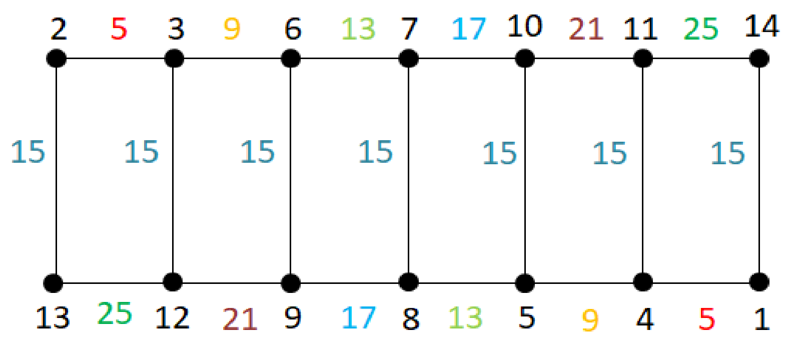

Figure 2. Ladder with 7 rainbow antimagic connection

We prove that for n =1 mod 2, $rc_A(L_n) \leq n$. Assume that $rc_A(L_n) \leq n-1$. We will proof in three cases.
Case 1. For path $x_1 - x_2 - x_3 - \ldots - x_n$
We take any two vertices in upper section of L_n, namely x_1 and x_n. We choose x_1 and x_n because x_1 and x_n is the farthest vertex in upper section of L_n. We know from x_1 and x_n certainly forming a path namely $x_1 - x_{x2} - x_3 - \ldots - x_n$ and it has $diam(x_1 - x_{x2} - x_3 - \ldots - x_n) = n-1$, so the minimum color is $n-1$.
Case 2. For path $y_1 - y_2 - y_3 - \ldots - y_n$
We take any two vertices in bottom section of L_n, namely y_1 and y_n. We choose y_1 and y_n because y_1 and y_n is the farthest vertex in bottom section of L_n. We know from y_1 and y_n certainly forming a path namely $y_1 - y_2 - y_3 - \ldots - y_n$ and it has $diam(y_1 - y_2 - y_3 - \ldots - y_n) = n-1$, so the minimum color is $n-1$.
Case 3. For path $x_1 - y_1 - y - 2 - \ldots - y_n$
We take two vertices from bottom section and upper section of L_n, namely x_1 and y_n. We choose x_1 and y_n because x_1 and y_n isthe farthest vertex of L_n. We know from x_1 and y_n certainly forming a path namely $x_1-y_1-y-2-\ldots-y_n$ and it has $diam(x_1-y_1-y-2-\ldots-y_n) = n-1+1$, so the minimum color is n. This is contradiction with the assumption $rc_A(L_n) \leq n-1$, so $rc_A(L_n) \leq n$.

Next step we prove $rc_A L_n \geq n$. We know $diam(L_n) = n$ and $\triangle(L_n) = 3$, based on Theorem 1 we get $rc_A(L_n) \geq max\{\triangle(L_n), rc(L_n)\}$. From that we know $rc_A(L_n) \geq max\{3, n\}$. It means $rc_A(L_n) \geq n$.

Next we prove for $n \leq rc_A(L_n) \leq n+1$ for n =0 mod 2, based on Theorem 1 we know that $rc_A(G) \geq max\{\triangle(G), rc(G)\}$. We know $\triangle(L_n) = 3$ and $rc(L_n) = n$, then $rc_A(L_n) \geq max\{\triangle(L_n), rc(L_n)\}$ then $rc_A(L_n) \geq max\{3, n\}$. It means $n \leq rc_A \leq n+1$.

Figure 2 shows that an illustration of ladder graph with $n = 7$. The colored number is the weight and then be the color of the edge. So from the Figure 1 we know that $rc_A(L_7) = 7$.

Theorem 5. *Let TL_n be a triangular ladder graph with $n \geq 4$, the rainbow connection number of TL_n is n.*

Proof. The triangular ladder is a connected graph with $V(L_n) = \{x_a y_a, 1 \leq a \leq n\}$ and $E(L_n) = \{x_a x_{a+1}, 1 \leq a \leq n-1\} \cup \{y_a y_{a+1}, 1 \leq a \leq n-1\} \cup \{x_a y_a, 1 \leq a \leq n\} \cup \{x_a y_{a+1}, 1 \leq a \leq n-1\}$. The cardinality of TL_n are $|V(L_n) = 2n$ and $|E(L_n) = 4n-3$. We define a function $c : E \longrightarrow \{1, 2, \ldots, |E|\}$ as follows

$$c(e) = \begin{cases} c(x_a x_{a+1}) = i, & \text{for } 1 \leq a \leq n-1 \\ c(y_a y_{a+1}) = a, & \text{for } 1 \leq a \leq n-1 \\ c(x_a y_a) = n, & \text{for } 1 \leq a \leq n \\ c(x_a y_{a+1}) = 1, & \text{for } 1 \leq a \leq n-1 \end{cases}$$

Since the edge function $e = x_a y_a$ reach a maximum value, thus, $rc(TL_n) \leq n$. Then we prove that $rc(TL_n) \geq n$. We take two vertices namely $u = y_a$ and $v = x_n$, the vertex u and v is farthest vertex in TL_n. From that fact we get $diam(TL_n) = n$. Chartrand, et all. [3] said $diam(G) \leq rc(G)$ and we know $diam(TL_n) = n$. From that fact we get $rc(TL_n) \geq n$.

Theorem 6. *Let TL_n be a triangular ladder graph with $n \geq 4$. The rainbow antimagic connection number of TL_n is $n \leq rc_A(TL_n) \leq n+2$.*

Proof. The triangular ladder is a connected graph with $V(L_n) = \{x_a y_a, 1 \leq a \leq n\}$ and $E(L_n) = \{x_a x_{a+1}, 1 \leq a \leq n-1\} \cup \{y_a y_{a+1}, 1 \leq a \leq n-1\} \cup \{x_a y_a, 1 \leq a \leq n\} \cup \{x_a y_{a+1}, 1 \leq a \leq n-1\}$. The cardinality of TL_n are $|V(L_n) = 2n$ and $|E(L_n) = 4n-3$.
We define a function $g : V \longrightarrow \{1, 2, \ldots, |V|\}$ as vertex function

For n =0 mod 2,

$$g(v) = \begin{cases} g(x_a) = 2a, & \text{if a=1 } mod \text{ 2} \\ g(x_a) = 2a-1, & \text{if a=0 } mod \text{ 2} \\ g(y_a) = 2n-2a+2, & \text{if a=1 } mod \text{ 2} \\ g(y_a) = 2n-2a+1, & \text{if a=0 } mod \text{ 2} \end{cases}$$

For n =1 mod 2,

$$g(v) = \begin{cases} g(x_a) = 2a, & \text{if a=1 } mod \text{ 2} \\ g(x_a) = 2a-1, & \text{if a=0 } mod \text{ 2} \\ g(y_a) = 2n-2a+1, & \text{if a=1 } mod \text{ 2} \\ g(y_a) = 2n-2a+2, & \text{if a=0 } mod \text{ 2} \end{cases}$$

It is clear that the edge weight of TL_n as follows
We define a function $w : E \longrightarrow \{1, 2, \ldots, |E|\}$ as follows
For n =0 mod 2,

$$w(e) = \begin{cases} w(x_a x_{a+1}) = 4a+1, & \text{for } 1 \leq a \leq n-1 \\ w(y_a y_{a+1}) = 4n-4i+1, & \text{for } 1 \leq a \leq n-1 \\ w(x_a y_{a+1}) = 2n-1, & \text{for } 1 \leq a \leq n-1 \\ w(x_a y_a) = 2n+4, & \text{for } 1 \leq a \leq n-1, \text{ and a=1 } mod \text{ 2} \\ w(x_a y_a) = 2n, & \text{for } 1 \leq a \leq n-1, \text{ and a=0 } mod \text{ 2} \end{cases}$$

For n =1 mod 2,

$$w(e) = \begin{cases} w(x_a x_{a+1}) = 4a+1, & \text{for } 1 \leq a \leq n-1 \\ w(y_a y_{a+1}) = 4n-4a+1, & \text{for } 1 \leq a \leq n-1 \\ w(x_a y_a) = 2n+1, & \text{for } 1 \leq a \leq n-1 \\ w(x_a y_{a+1}) = 2n, & \text{for } 1 \leq a \leq n-1, \text{ and a=1 } mod \text{ 2} \\ w(x_a y_{a+1}) = 2n-2, & \text{for } 1 \leq a \leq n-1, \text{ and a=0 } mod \text{ 2} \end{cases}$$

Now we prove $rc_A(TL_n) \geq n$. We know $\triangle(TL_n) = 4$ and $rc(TL_n) = n$, we know that $rc_A(TL_n) \geq max\{\triangle(TL_n), rc(TL_n)\}$ so $rc_A(TL_n) \geq max\{4, n\}$.

Figure 3 shows that an illustration of triangular ladder graph with $n = 7$. The colored number is the weight and then be the color of the edge. So from the picture we know that $rc_A(TL_7) = 9$.

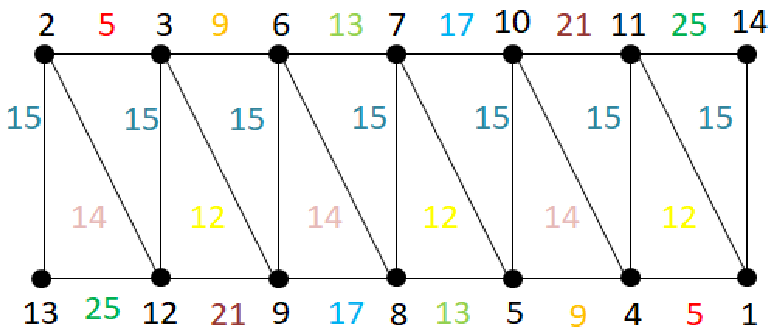

Figure 3. Triangular ladder with 9 rainbow antimagic connection

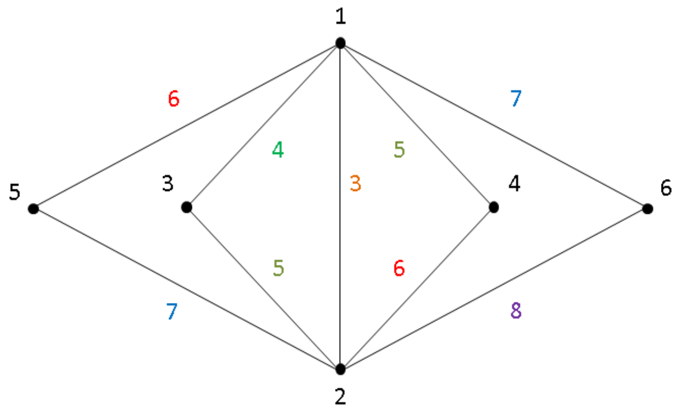

Figure 4. Diamond graph with 6 rainbow antimagic coloring

Theorem 7. *Let D_n be a diamond graph with $n \geq 2$. The rainbow connection number of D_n is $rc(D_n) = 2$.*

Proof. The diamond graph is a connected graph with $V(D_n) = \{p\} \cup \{q\} \cup \{x_a, y_a, 1 \leq a \leq n\}$ and $E(D_n) = \{pq\} \cup \{px_a, 1 \leq a \leq n\} \cup \{qx_a, 1 \leq a \leq n\} \cup \{py_a, 1 \leq a \leq n\} \cup \{qy_a, 1 \leq a \leq n\}$. The cardinality of D_n are $|V(D_n)| = 2n + 2$ and $E(D_n) = 4n + 1$.

We define a function $c : E \longrightarrow \{1, 2, \ldots, |E|\}$ as follows

$$c(e) = \begin{cases} c(pq) = 1 & \\ c(px_n) = 2 & \\ c(py_n) = 1 & \\ c(qx_n) = 1 & \\ c(qy_n) = 2 & \\ c(px_a) = 1, & \text{for } 1 \leq a \leq n-1 \\ c(py_a) = 2, & \text{for } 1 \leq a \leq n-1 \\ c(qx_a) = 1, & \text{for } 1 \leq a \leq n \\ c(qy_a) = 2, & \text{for } 1 \leq a \leq n \end{cases}$$

After that we prove for $rc(D_n) \geq 2$. Chartrand, et al. [3] said $diam(G) \leq rc(G)$ and we know $diam(D_n) = 2$, we get $rc(D_n) \geq 2$.

Theorem 8. *Let G be a diamond graph with $n \geq 2$. The rainbow antimagic connection number of D_n is $2n+1 \leq rc_a(D_n) \leq 2n+2$.*

Proof. The diamond graph is a connected graph with $V(D_n) = \{p\} \cup \{q\} \cup \{x_a, y_a, 1 \leq a \leq n\}$ and $E(D_n) = \{pq\} \cup \{px_a, 1 \leq a \leq n\} \cup \{qx_a, 1 \leq a \leq n\} \cup \{py_a, 1 \leq a \leq n\} \cup \{qy_a, 1 \leq a \leq n\}$. The cardinality of D_n are $|V(D_n)| = 2n+2$ and $E(D_n) = 4n+1$. We define a function $g : V \longrightarrow \{1, 2, \ldots, |V|\}$ as follows

$$g(v) = \begin{cases} g(p) = 1 \\ g(q) = 2 \\ g(x_a) = 2a+1 & \text{for } 1 \leq a \leq n \\ g(y_a) = 2a+2, & \text{for } 1 \leq a \leq n \end{cases}$$

We define a function $w : E \longrightarrow \{1, 2, \ldots, |E|\}$ as follows

$$w(e) = \begin{cases} w(px_a) = 2a+2, & \text{for } 1 \leq a \leq n \\ w(qx_i) = 2a+3, & \text{for } 1 \leq a \leq n \\ w(py_a) = 2a+3, & \text{for } 1 \leq a \leq n \\ w(qy_a) = 2a+4, & \text{for } 1 \leq a \leq n \\ w(pq) = 3 \end{cases}$$

Now we prove $rc_A(D_n) \geq 2n+1$. We know $\triangle(D_n) = 2n+1$ and $rc(D_n) = 2n+2$, we know that $rc_A(D_n) \geq max\{\triangle(D_n), rc(D_n)\}$ so $rc_A(D_n) \geq max\{2n+1, 2n+2\}$. It means $rc_A(D_n) \geq 2n+1$.

3. Conclusion

We have initiated to study the rainbow antimagic coloring of related ladder. The rainbow connection number of ladder graph is $rc(L_n) = n$. The rainbow connection number of triangular ladder graph is $rc(TL_n) = n$. The rainbow connection number of diamond graph is $rc(D_n) = 2$. The rainbow antimagic coloring number of ladder graph is $rc_{la}(L_n) = n$ for n =1 mod 2 and $n \leq rc_A(L_n) \leq n+1$ for n =0 mod 2. The rainbow antimagic coloring number of triangular ladder graph is $n \leq rc_A(TL_n) \leq n+2$. The rainbow antimagic coloring number of diamond graph is $2n+1 \leq rc_A \leq 2n+2$. The open problem of this research is:

Open Problem 1. *Let TL_n be a triangular ladder graph with $n \geq 4$, determine the exact value of rainbow antimagic connection number of TL_n.*

Acknowledgement

We gratefully acknowledgement CGANT(Combinatorics, Graph Theory, and Network Topology) for the support and supervision to completing this paper.

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