On total H-irregularity strength of graphs

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Abstract. Let G be a graph with vertex set V and edge set E. Total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., l\}$ called a total l-labeling of a graph G. The total l-labeling is a total H-irregular l-labeling of graph G if for $H \subseteq G$, the total H-weights $wt_f(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ are distinct. The irregularity strength $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. The total $H-$irregular $a-$labeling from the minimum where the graph G is called the total $H-$irregularity strength of G, is denoted by $tHs(G)$. In this paper, we have obtained tHs from linegrid, buttrefly, hexagonal and diamond graphs. To obtain the tHs, we begin to study the total irregularity strength of graph G with subgraph H.

1. Introduction

The graphs that we discussed in this paper are mainly plane graphs which are simple, connected and limited graphs. Let G be a graph that has a set of vertex $V(G)$ and a set of edge $E(G)$. The assignment of integration into vertices or edges, or both is subject to certain conditions called graph labeling [4]. Map that contain vertices and edges until positive integers to the number l are called total labeling [11]. Graph G contains a H-cover if each subgraph of Hj is isomorphic, which conditions each vertex $E(G)$ is included in at least one of the subgraphs $Hj, j = 1, 2, ...s$ [2] For each of the two different edge l_1 and l_2, it holds $w(l_1) \neq$w(l$_2$) the total a labeling is said to be the irregular edge of total a-labeling on graph G. Total edge $H-$irregularity strength G, symbolized by the $tes(G)$ is a minimum where graph G has an irregular total edge labeling [3]. The total a-labeling is said to be a total H-irregular a-labeling of the graph G if for HG, the total H-weights $W(H) = \sum_{v \in V(H)} \varphi(v) + \sum_{e \in E(H)} \varphi(e)$, are distinct.

If the domain is the vertex-set or the edge-set, the labelings are called respectively vertex labelings or edge labelings. If the domain is $V(G)E(G)$ then we call the labeling total labeling. The most complete recent survey of graph labelings is [12]. The assignment of positive label integers $1, 2, ..., l$ for both of vertices and edges is called a label l number of irregular so that the weight is calculated on a different vertices [1].

The edge irregular total $k-$labelling of the graph G is the total $k-$labelling for every two different edges e and f of G there is $wt(e)wt(f)$. And vertex irregular total $k-$labelling of G is the total $k-$labelling if for every two distinct vertices x and y of G there is $wt(x)wt(y)$.

The irregular assignments and the irregularity strength of graphs is the definition of the total edge irregularity strength introduced by Chartrand et.al.[5]. The $k-$labeling of the edges

: : $E1, 2, ..., k$ such that the sum of the labels of edges incident with a vertex is different for all the vertices of G and the smallest k for which there is the an irregular assignment, an irregular assignments is the irregularity strength, $s(G)$ [7].

The irregularity strength was introduced in [5] by Chartrand et al. . The irregularity strength of regular graphs was considered by Faudree and Lehel in [9]. The total edge irregularity strength of G, $tes(G)$ is the minimum k for which the graph G has an edge irregular total $k-$labeling.

The total H-irregular a-labeling from the minimum where the graph G is called the total H-irregularity strength of G, is denoted by $tHs(G)$. The minimum k where G has an edge labeled k is an irregular total k is the definition of the total irregularity, $tes(G)$. The irregularity strength of the edge G, denoted by $es(G)$ is a minimum K where graph G has an irregular labeled k [12]. The minimum where graph G has an irregular total labeling subgraph is the total Irregularity strength of $G, tHs(G)$.

The minimum positive integer k for which graph G has an edge-irregular k-labelling is called the irregularity strength of the graph G, denoted by $s(G)$. An edge-irregular $k$labelling of G is the edge labelling : : $E(G)1, 2, ..., k$ if every two distinct x and y in $V(G)$ satisfy $wt(x)wt(y)$. The total $l-$labeling (φ) which has different weights in every two disparate subgraphs is called total H-irregular labeling, where $H1$ and $H2$ are isomorphic to H there is $wt\varphi(H1) = wt\varphi(H2)$.

Slamin et al. [11] prove that the strength of the total vertex irregularity of the merge is separate from the sun's graph Rajasingh et al. [8].

The total l-labeling $V(G) \cup E(G) \to \{1, 2, \ldots, l\}$ to the total irregular l-label is the labeling of the irregular l-total edge on the graph G if each two different edges have different weights [12]. Rajasingh et al. [8].

The total l-labeling (φ) which has different weights in every two disparate subgraphs is called total H-irregular labeling, where H_1 and H_2 are isomorphic to H there is $wt_\varphi(H_1) \neq wt_\varphi(H_2)$. We determine H-weight as

$$wt_\varphi(H) = \sum_{v \in V(H)} \varphi(v) + \sum_{e \in E(H)} \varphi(e),$$

for the subgraph $H \subseteq G$ under the total l-labeling (φ).

The total Irregularity strength of the graph G $(tHs(G, H))$. To find the total $H-$irregularity strength $(tHs(G, H))$ We use plane graphs such as grid graph and triangular ladder graph [10]. The smallest value of l that is owned by graph G has the total l-labeling H-irregular is the total H-irregularity strength graph G $(tHs(G, H))$. We use theorem 1 as the lower bound of the total H-irregularity strength.

Theorem 1. [2]Let G be a graph that accepts the H-covering given by t isomorphic subgraphs to H. Then

$$tHs(G, H) \geq \left\lceil \frac{t-1}{|V(H)| + |E(H)|} \right\rceil$$

In this paper, we study about the total $H-$irregularity strength of several graphs, namely the line grid graph, the butterfly graph, the hexagonal graph and the diamond graph. From this paper, we study the $tHs(G)$ with H-covering irregularity strength $(tHs(G, H))$ of each of these different graphs. Line grid graphs is a denoted by GLn.

2. Results

In this segment, we present the results of the total H-irregularity strength in our graph research field, as follows.

Theorem 2. Let $GLn(3,n)$, $n \geq 2$, be a line grid graph admitting a C_4-covering. The total $H-$irregularity strength of $GLn(3,n)$ is$\left\lceil \frac{n+7}{11} \right\rceil$.

Proof. Let $GLn(3,n)$, $n \geq 2$, be a line grid graph with the vertex set $V(L(Gn(3,n))) = \{x^i_j; 1 \leq i \leq n-1, 1 \ \leq i \leq n, i = \text{odd}\} \cup x^i_j; 1 \leq i \leq n, 1 \ \leq i \leq n, i = \text{even}\}$and the cardinality is $|V(L(Gn(3,n)))| = 3(n-1) - 2(n)$. The set of edges is $E(L(Gn)) = \{x^i_j x^{i+1}_j, ; 1 \leq i \leq n, 1 \leq j \leq n \cup \{x^i_j x^{i+1}_{j+1}, ; 1 \leq i \leq n, i \text{ odd}, 1 \leq j \leq n-1\} \cup \{x^i_j x^{i+1}_{j-1}; 1 \leq i \leq n, i \text{ even}, 2 \leq j \leq n\}$ and the cardinality is $|E(L(Gn(3,n)))| = 8n - n$. The line grid graph $G(3,n)$n ≥ 2, contains a C_4-covering with exactly $2n-2$ cycles C_4. The lower bound that we get from the theorem 2, $tHs(L(Gn), C_4) \geq \left\lceil \frac{n+7}{8} \right\rceil$. Put l $tHs(L(Gn), C_4) \geq \left\lceil \frac{n+7}{8} \right\rceil$. We define C_4-irregular total l-labeling $\varphi_4 : V(L(Gn(3,n))) \cup E(L(Gn(3,n))) \rightarrow \{1, 2, \ldots, l\}$ with aim the indicating tof that l is the upper bound for total C_4-irregular strength $L(Gn)$.

A C_4-irregular total l-labeling $\varphi_4 : V(L(Gn)) \cup E(L(Gn)) \rightarrow \{1, 2, \ldots, l\}$ is as follows: for $j = 1, 2, \ldots, l$,

$$\text{f}(\text{x}^1_i) = \begin{cases} 1 & ; i = 1 \\ 3t-1 & ; 8t-6 \leq i \leq 8t-4 \\ 3t & ; 8t-3 \leq i \leq 8t-1 \\ 3t+1 & ; 8t \leq i \leq 8t+1 \end{cases} \qquad \text{f}(\text{x}^2_i) = \begin{cases} 1 & ; i = 1 \\ 3t-2 & ; i = 8t-6 \\ 3t-1 & ; 8t-5 \leq i \leq 8t-4 \\ 3t & ; 8t-3 \leq i \leq 8t-1 \\ 3t+1 & ; 8t \leq i \leq 8t+1 \end{cases}$$

$$\text{f}(\text{x}^3_i) = \begin{cases} 1 & ; i = 1 \\ 3t-1 & ; 8t-6 \leq i \leq 8t-4 \\ 3t & ; 8t-3 \leq i \leq 8t-2 \\ 3t+1 & ; 8t-1 \leq i \leq 8t+1 \end{cases} \qquad \text{f}(\text{x}^4_i) = \begin{cases} 1 & ; i = 1 \\ 3t-2 & ; i = 8t-7 \\ 3t-1 & ; 8t-6 \leq i \leq 8t-4 \\ 3t & ; 8t-3 \leq i \leq 8t-1 \\ 3t+1 & ; 8t-4 \leq i \leq 8t+1 \end{cases}$$

$$\text{f}(\text{x}^5_i) = \begin{cases} 1 & ; i = 1 \\ 3t-1 & ; 8t-6 \leq i \leq 8t-5 \\ 3t & ; 8t-4 \leq i \leq 8t-2 \\ 3t+1 & ; 8t-1 \leq i \leq 8t+1 \end{cases} \qquad \text{f}(\text{x}^1_i x^2_i) = \begin{cases} 1 & ; i = 1 \\ 3t-2 & ; 8t-6 \leq i \leq 8t-5 \\ 3t-1 & ; 8t-4 \leq i \leq 8t-2 \\ 3t & ; 8t-1 \leq i \leq 8t \\ 3t+1 & ; i = 8t+1 \end{cases}$$

$$\text{f}(\text{x}^1_i x^2_{i+1}) = \begin{cases} 13t-2 & ; 8t-7 \leq i \leq 8t-6 \\ 3t-1 & ; 8t-5 \leq i \leq 8t-3 \\ 3t & ; 8t-2 \leq i \leq 8t \\ 3t+1 & ; i = 8t+1 \end{cases} \qquad \text{f}(\text{x}^2_i x^3_i) = \begin{cases} 1 & ; i = 1 \\ 3t-2 & ; 8t-7 \leq i \leq 8t-5 \\ 3t-1 & ; 8t-4 \leq i \leq 8t-3 \\ 3t & ; 8t-2 \leq i \leq 8t \\ 3t+1 & ; i = 8t+1 \leq i \leq 8t+2 \end{cases}$$

$$\text{f}(\text{x}^2_{i+1} x^3_i) = \begin{cases} 1 & ; i = 1 \\ 3t-2 & ; i = 8t-6 \\ 3t-1 & ; 8t-5 \leq i \leq 8t-3 \\ 3t & ; 8t-2 \leq i \leq 8t-1 \\ 3t+1 & ; 8t \leq i \leq 8t+1 \end{cases} \qquad \text{f}(\text{x}^3_i x^4_i) = \begin{cases} 1 & ; i = 1 \\ 3t-2 & ; 8t-7 \leq i \leq 8t-6 \\ 3t-1 & ; 8t-5 \leq i \leq 8t-3 \\ 3t & ; 8t-2 \leq i \leq 8t \\ 3t+1 & ; i = 8t+1 \end{cases}$$

$$\text{f}(\text{x}^3_i x^4_{i+1}) = \begin{cases} 1 & ; i = 1 \\ 3t-2 & ; 8t-7 \leq i \leq 8t-6 \\ 3t-1 & ; 8t-5 \leq i \leq 8t-3 \\ 3t & ; 8t-2 \leq i \leq 8t \\ 3t+1 & ; i = 8t+1 \end{cases} \qquad \text{f}(\text{x}^4_i x^5_i) = \begin{cases} 1 & ; i = 1 \\ 3t-2 & ; 8t-7 \leq i \leq 8t-6 \\ 3t-1 & ; 8t-4 \leq i \leq 8t-3 \\ 3t & ; 8t-2 \leq i \leq 8t-1 \\ 3t+1 & ; 8t \leq i \leq 8t+2 \end{cases}$$

$$f(x_{i+1}^4 x_i^5) = \begin{cases} 1 & ; i = 1 \\ 3t-2 & ; 8t-7 \le i \le 8t-6 \\ 3t-1 & ; 8t-5 \le i \le 8t-4 \\ 3t & ; 8t-3 \le i \le 8t-1 \\ 3t+1 & ; 8t \le i \le 8t+1 \end{cases}$$

For each vertex and edge label under the φ_4-labeling is almost l. We can see and observe that every vertex and edge under φ_4-labeling are almost l. We get the C_4-weight of C_4^l, $l = 1, 2, \ldots, 2m-2$, under the total labeling φ_4, we get

$$wt_{\varphi_4}(C_4^l) = \sum_{v \in V(C_4^l)} \varphi(v) + \sum_{e \in E(C_4^l)} \varphi(e),$$

From the $wt_{\varphi_4}(C_4^l)$ that the amount of label vertexs and edges, we obtain the sequence increases. And it is enough to prove that $wt_{\varphi_4}(C_4^l) < wt_{\varphi_4}(C_4^{l+1})$, $l = 1, 2, \ldots, n$.

The function label of vertex and edge wt_{φ_4} point are variable periodic functions n. For each positive integer $l, l = 1, 2, 3, ...$, the wt_{φ_4} function is used to label the vertex and edge of the graph $G(Ln)$. For each weight $G(Ln)$:

$$\begin{aligned} w_1 &= \varphi_4(x_i^1) + \varphi_4(x_i^2) + \varphi_4(x_i^3) + \varphi_4(x_{i+1}{}^2_1) + \varphi_4(x_i^1)(x_i^2) + \varphi_4(x_i^2)(x_i^3) + \varphi_4(x_{i+1}{}^2_1)(x_i^3) + \\ &\quad \varphi_4(x_i^1)(x_{i+1}{}^2_1) \\ w_2 &= \varphi_4(x_{i+1}{}^2_1) + \varphi_4(x_i^3) + \varphi_4(x_{i+1}{}^4_1) + \varphi_4(x_{i+1}{}^3_1) + \varphi_4(x_i +_+ 1^2)(x_i^3) + \varphi_4(x_i{+}^2_1)(x_{i+1}{}^3_1) + \\ &\quad \varphi_4(x_i^3)(x_{i+1}{}^4_1) + \varphi_4(x_{i+1}{}^3_1)(x_{i+1}{}^4_1) \\ w_3 &= \varphi_4(x_i^3) + \varphi_4(x_i^4) + \varphi_4(x_{i+1}{}^4_1) + \varphi_4(x_i^5) + \varphi_4(x_i^3)(x_i^4) + \varphi_4(x_i^3)(x_{i+1}{}^4_1) + \varphi_4(x_i^4)(x_i^5) + \\ &\quad \varphi_4(x_{i+1}{}^4_1)(x_i^5) \end{aligned}$$

for each value s, weights are obtained

$$\begin{aligned} w_1 &= 8, 11, 14, ..., 3n+5 \\ w_2 &= 10, 13, 16, ..., 3n+7 \\ w_3 &= 9, 12, 15, ..., 3n+6 \end{aligned}$$

The based on the equation obtained $wt\varphi_{L(Gn)} = 1 + wt\varphi_{L(Gn)}$. total weight of L(Gn) is $wt\varphi_{L(Gn)} = \mathrm{wt}_{\varphi_4}(C_4^l) = \sum_{v \in V(C_4^l)} \varphi(v) + \sum_{e \in E(C_4^l)} \varphi(e) = 8, 9, 10, 11, 12, ..., 3n+7$.

we respect to $wt_{\varphi_3}(C_n^l l) < wt_{\varphi_3}(C_n^{l+1})$, $l = 1, 2, \ldots, n$ then $wt_{\varphi_3}(C_n^{l+1}) = 2 + wt_{\varphi_3}(C_n^l)$ we can observed the illustration of total C_3-irregularity strength of line grid graph on Figure 1.

Theorem 3. Let $Wd(3, n)$, $n \ge 2$, be a butterfly graph recognizing a C_3-covering. The total $H-$irregularity strength of $Wd(3, n)$ is

$$tHs(Wd, C_3) = \left\lceil \frac{n+5}{6} \right\rceil$$

Proof. let $Wd(3, n)$, $n \ge 2$, be a butterfly graph with the vertex set $V(Wd, c3) = \{x_i; 1 \le i \le n+1\} \cup \{y_i; 1 \le i \le n\} \cup \{z_i; 1 \le i \le n+1\}$ and the cardinality is $|V(Wd, c3)| = 3n+2$. The set of edges is $E(Wd, C_3) = \{x_i y_i; 1 \le i \le n\} \cup \{y_i x_{i+1}, ; 1 \le i \le n\} \cup \{x_i z_i; 1 \le i \le n+1\} \cup \{z_i y_i, ; 1 \le i \le n\} \cup \{y_i z_{i+1}, ; 1 \le i \le n\}$ and the cardinality is $|E(Wd, C_3)| = 5n+1$.

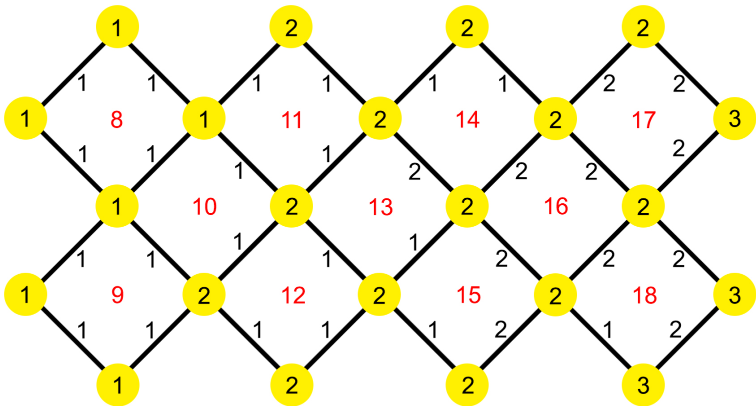

Figure 1. Illustration of line grid graph $L(Gn)(3,4)$

The butterfly graph (Wd, C_3), contains a C_3-covering with exactly $2n-2$ cycles C_3. The lower bound that we get from the theorem 3, $tHs(Wd, C_3) \geq \left\lceil \frac{n+5}{6} \right\rceil$. Put l $tHs(Wd, C_3) \geq \left\lceil \frac{n+5}{6} \right\rceil$. We specify a C_3-irregular total l-labeling $\varphi_3 : V(Wd, C_3) \cup E(Wd, C_3) \to \{1, 2, \ldots, l\}$ is prove that α as an upper bound for the total Wd-irregularity strength of Wd.

A C_3-irregular total l-labeling $\varphi_3 : V(L(Gn)) \cup E(L(Gn)) \to \{1, 2, \ldots, l\}$ is as follows:

$$\begin{aligned}
f(x_i) &= \left\lceil \tfrac{i+2}{3} \right\rceil & f(y_i) &= \left\lceil \frac{i+2}{3} \right\rceil \\
f(z_i) &= \left\lceil \tfrac{i+1}{3} \right\rceil & f(y_i z_i) &= \left\lceil \frac{i}{3} \right\rceil \\
f(x_i y_i) &= \left\lceil \tfrac{i+1}{3} \right\rceil & f(x_{i+1} y_i) &= \left\lceil \frac{i+1}{3} \right\rceil \\
f(x_i z_i) &= \left\lceil \tfrac{i}{3} \right\rceil & f(y_i z_{i+1}) &= \left\lceil \frac{i}{3} \right\rceil
\end{aligned}$$

We get the upper bound from the function of C_3-irregular total $Wd(3,n)$-labelling. We get to present to the upper bound of the graph in the theorem 3, $tHs((Wdn), C_3) \leq \left\lceil \frac{i+2}{3} \right\rceil$

Based on the labeling above, we can show the all weights are different by the following equation:

$$\begin{aligned}
wt\varphi_n(Wd_n^{j+2}) - wt\varphi_n(Wd_n^{j}) &= \varphi_3(x_{i+1}) + \varphi_3(y_{i+1}) + \varphi_3(z_{i+1}) + \varphi_3(x_i z_{i+1}) + \varphi_3(x_i y_{i+1}) + \\
&\quad \varphi_3(z_i y_{i+1}) - \varphi_3(x_i) - \varphi_3(y_i) - \varphi_3(z_i) - \varphi_3(x_i z_i) - \varphi_3(x_i y_i) - \\
&\quad \varphi_3(z_i y_i) \\
&= 2
\end{aligned}$$

for every w odd

$$\begin{aligned}
wt\varphi_n(Wd_n^{j+3}) - wt\varphi_n(Wd_n^{j+1}) &= \varphi_3(x_{i+2}) + \varphi_3(y_i) + \varphi_3(z_{i+2}) + \varphi_3(x_{i+1} y_{i+1}) + \varphi_3(z_{i+1} y_{i+1}) \\
&\quad + \varphi_3(x_i z_{i+2}) - \varphi_3(x_{i+1}) - \varphi_3(y_i) - \varphi_3(z_{i+1}) - \varphi_3(x_{i+1} y_i) -
\end{aligned}$$

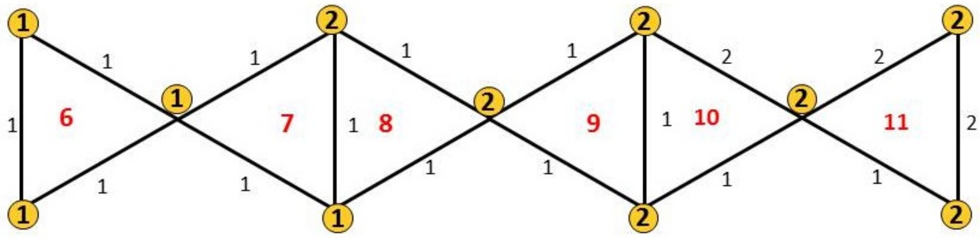

Figure 2. Illustration of butterfly graph (Wd)

$$\begin{aligned} & \varphi_3(z_{i+1}y_i) - \varphi_3(x_i z_{i+1}) \\ = \;& 2 \end{aligned}$$

We respect to $wt_{\varphi_3}(C_n^l) < wt_{\varphi_3}(C_n^{l+1})$, $l = 1, 2, \ldots, n$ then $wt_{\varphi_3}(C_n^{l+1}) = 2 + wt_{\varphi_3}(C_n^l)$ The all H-weights are distinct. This matter concludes that $tHs(Wd_n) = \left\lceil \frac{n+5}{6} \right\rceil$. The example of total C_3-irregularity of butterfly graph labeling, we can see on Figure 2, and we get $tHs(Wd, C_3)$ $= 2$

Theorem 4. Let $Hn(3, n)$, $n \geq 2$, be a Hexagonal graph recognizing a C_6-covering. The total $H-$irregularity strength of $Hn(n)$ is

$$tHs(Hn, C_6) = \left\lceil \frac{n+11}{12} \right\rceil$$

Proof. let $Hn(6, n)$, $n \geq 2$, be a cycle graph with the vertex set $V(C_6) = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n+1\} \cup \{z_i; 1 \leq i \leq n+1\} \cup \{k_i; 1 \leq i \leq n\}$ and the cardinality is $|V(C_6)| = 4n+2$. The set of edges is $E(C_6) = \{x_i y_i; 1 \leq i \leq n\} \cup \{x_i y_{i+1}, ; 1 \leq i \leq n\} \cup \{y_i z_i; 1 \leq i \leq n+1\} \cup \{z_i k_i, ; 1 \leq i \leq n\} \cup \{z_{i+1} k_1, ; 1 \leq i \leq n\}$ and the cardinality is $|E(Hn, C_6)| = 5n+1$. The cycle graph (Hn, C_6), contains a C_6-covering with exactly $2n-2$ cycles C_6. The lower bound that we get from the theorem 4, $tHs(Hn) \geq \left\lceil \frac{n+11}{12} \right\rceil$. Put l $tHs(Hn) \geq \left\lceil \frac{n+11}{12} \right\rceil$. We specify a C_6-irregular total l-labeling $\varphi_6 : V(Hn, C_6) \cup E(C_6) \to \{1, 2, \ldots, l\}$ is prove that α as an upper bound for the total Hn-irregularity strength of Hn.

A C_6-irregular total l-labeling $\varphi_6 : V(C6) \cup E(C6) \to \{1, 2, \ldots, l\}$ is as follows:

$$\begin{aligned} f(x_i) &= \left\lceil \frac{i+10}{12} \right\rceil & f(y_i) &= \left\lceil \tfrac{i+7}{12} \right\rceil \\ f(z_i) &= \left\lceil \frac{i+8}{12} \right\rceil & f(k_i) &= \left\lceil \tfrac{i+6}{12} \right\rceil \\ f(x_i y_i) &= \left\lceil \frac{i+5}{12} \right\rceil & f(y_i x_{i+1}) &= \left\lceil \tfrac{i+4}{12} \right\rceil \\ f(x_i z_i) &= \left\lceil \frac{i}{12} \right\rceil & f(z_i k_i) &= \left\lceil \tfrac{i+3}{12} \right\rceil \\ f(k_i z_{i+1}) &= \left\lceil \frac{i+2}{12} \right\rceil \end{aligned}$$

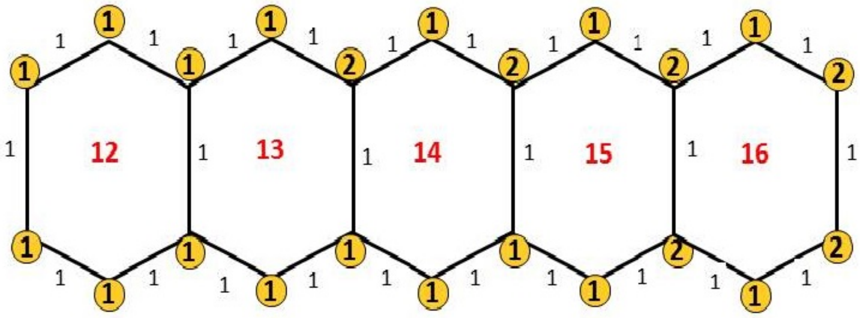

Figure 3. Illustration of Hexagonal graph ($C6$)

We get the upper bound from the function of C_6-irregular total Hn-labelling. We get to present to the upper bound of the graph in the theorem 3, $tHs((Hn), C_6) \leq \left\lceil \frac{i+10}{12} \right\rceil$

Based on the labeling above, we can show the all weights are different by the following equation:

$$\begin{aligned} wt\varphi_n(H_n^{j+1}) - wt\varphi_n(H_n^j) &= \varphi_6(x_i+1) + \varphi_6(y_i+1) + \varphi_6(x_i+2) + f(x_iy_i+1) + \\ &\quad \varphi_6(y_ix_1+1+1) + \varphi_6(z_i+1) - \varphi_6(x_i) - \varphi_6(y_i) - \varphi_6(x_{i+}1) - \\ &\quad \varphi_6(z_i) - \varphi_6(k_i) - \varphi_6(z_{i+}1) - \varphi_6(x_iy_i) - \varphi_6(y_ix_{1+1}) \\ &= 1 \end{aligned}$$

We respet to $wt_{\varphi_6}(C_n^l) < wt_{\varphi_6}(C_n^{l+1})$, $l = 1, 2, \ldots, n$ then $wt_{\varphi_6}(C_n^{l+1}) = 2 + wt_{\varphi_6}(C_n^l)$. The all H-weights are distinct. This matter concludes that $tHs((H_m), H_n) = \left\lceil \frac{n+11}{12} \right\rceil$. The example of total H_n-irregularity of diamond ladder graph labeling, we can see on Figure 3, and we get $tHs(Hn, C_6) = 2$.

Theorem 5. Let $Dn(3, n)$, $n \geq 2$, be a diamond graph recognizing a C_4-covering. The total $H-$irregularity of Dn(n) is

$$tHs(Dn, C_4) = \left\lceil \frac{n+8}{9} \right\rceil$$

Proof. let $Dn(4, n)$, $n \geq 2$, be a diamond graph with the vertex set $V(Dn, C_4) = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n+1\} \cup \{z_i; 1 \leq i \leq n+1\}$ and the cardinality is $|V(Dn, C_4)| = 3n+1$. The set of edges is $E(Dn, C_4) = \{x_iy_i; 1 \leq i \leq n\} \cup \{x_iy_{i+1}, ; 1 \leq i \leq n\} \cup \{y_iz_i; 1 \leq i \leq n\} \cup \{y_iz_{i+1}, ; 1 \leq i \leq n\} \cup \{y_iy_{i+1}, ; 1 \leq i \leq n\}$ and the cardinality is $|E(Dn, C_4)| = 5n$. The diamond graph (Dn, C_4), contains a C_4-covering with exactly $2n-2$ cycles C_4. The lower bound that we get from the theorem 5, $tHs(Dn, C_4) \geq \left\lceil \frac{n+8}{9} \right\rceil$. Put l $tHs(C_4) \geq \left\lceil \frac{n+8}{9} \right\rceil$. We specify a C_4-irregular total l-labeling $\varphi_4 : V(Dn, C_4) \cup E(Dn, C_4) \to \{1, 2, \ldots, l\}$ is prove that α as an upper bound for the total $Dn-$irregularity strength of Dn.

A C_4-irregular total l-labeling $\varphi_4 : V(C_4) \cup E(C_4) \to \{1, 2, \ldots, l\}$ is as follows:

$$f(x_i) = \left\lceil \tfrac{i+6}{9} \right\rceil \qquad f(y_i) = \left\lceil \frac{i+7}{9} \right\rceil$$

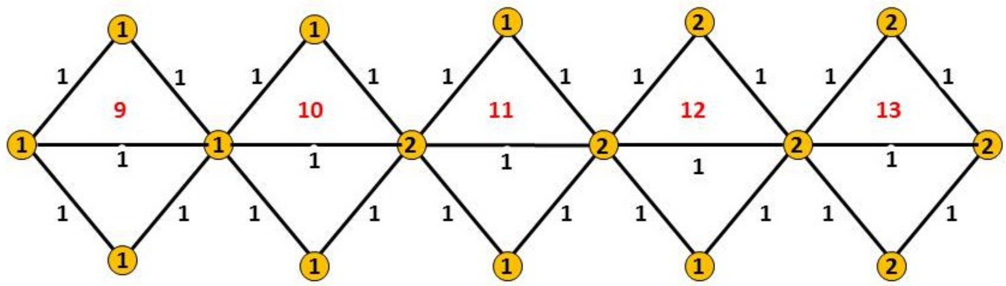

Figure 4. Illustration of Total C_4-Irregularity strength of diamond graph (Dn)

$$f(z_i) = \left\lceil \tfrac{i+5}{9} \right\rceil \qquad f(x_i y_i) = \left\lceil \frac{i+4}{9} \right\rceil$$
$$f(x_i y_{i+1}) = \left\lceil \tfrac{i+3}{9} \right\rceil \quad f(y_i z_1) = \left\lceil \frac{i+2}{9} \right\rceil$$
$$f(y_{i+1} z_i) = \left\lceil \tfrac{i+1}{9} \right\rceil \quad f(y_i y_{i+1}) = \left\lceil \frac{i+2}{9} \right\rceil$$

We get the upper bound from the function of C_4-irregular total Dn-labelling. We get to present to the upper bound of the graph in the theorem 3, $tHs((Dn), C_4) \leq \left\lceil \frac{i+6}{9} \right\rceil$

Based on the labeling above, we can show the all weights are different by the following equation:

$$\begin{aligned} wt\varphi_n(D_n^{j+1}) - wt\varphi_n(D_n^j) &= \varphi_4(x_i+1) + \varphi_4(y_i+1) + \varphi_4(y_i+2) + \varphi_4(z_i+1) + \varphi_4(x_i y_i + 1) + \\ &\quad \varphi_4(x_i y_{i+1} + 1) - \varphi_4(x_i) - \varphi_4(y_i) - \varphi_4(y_i+1) - \varphi_4(z_i) - \varphi_4(x_i y_i) - \\ &\quad \varphi_4(x_i y_{i+1}) - \varphi_4(y_i z_i) - \varphi_4(y_{i+1} z_i) - \varphi_4(y_i y_{i+1}) \\ &= 1 \end{aligned}$$

We respect to $wt_{\varphi_4}(C_n^l) < wt_{\varphi_4}(C_n^{l+1})$, $l = 1, 2, \ldots, n$ then $wt_{\varphi_4}(C_n^{l+1}) = 2 + wt_{\varphi_4}(C_n^l)$. The all H-weights are distinct. This matter concludes that $tHs(D_n) = \left\lceil \frac{n+8}{9} \right\rceil$. The example of total C_4-irregularity of diamond graph labeling, we can see on Figure 4, and we get $tHs(Dn, C_4) = 2$.

3. Conclusion

In this paper, We have given the result of total H-irregularity strength of linegrid graphs, butterfly graphs, hexagonal graphs and diamond graphs. We recognize H-covering on all graphs in this discussion that H is a cycle and fan graph.

Open Problem 1. Find the total $H-$irregularity strength of the graphs with $H \neq C$.

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