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Quantum reflection and transmission in ring systems with double Y-junctions: occurrence of perfect reflection

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Abstract

We consider the scattering problems of a quantum particle in a system with a single Y-junction and in ring systems with double Y-junctions. We provide new formalism for such quantum mechanical problems. Based on a path integral approach, we find compact formulas for probability amplitudes in the ring systems. We also discuss quantum reflection and transmission in the ring systems under scale-invariant junction conditions. It is remarkable that perfect reflection can occur in an anti-symmetric ring system, in contrast with the one-dimensional quantum systems having singular nodes of degree 2.

Keywords: one-dimensional quantum wires, Y-junctions, quantum transmission, quantum reflection

1. Introduction

It is truly interesting that the variety of junction conditions appears when the number of space dimension is reduced in quantum mechanics. This feature becomes apparent in one-dimensional quantum systems. For example, when we consider a node (i.e. a point interaction) on a one-dimensional quantum wire, the two adjacent one-dimensional spaces are completely separated by the node. This situation leads to various non-trivial junction conditions. The characteristics of the non-trivial junction conditions in one-dimensional quantum systems have comprehensively been studied by mathematical works [1–4].

We shed light on quantum problems in a system of a Y-junction. The Y-junction is composed of three one-dimensional quantum wires. These wires intersect at one point. The intersection of degree 3 has a point interaction parametrized by U(3) [1–4]. Such a system was originally investigated by pioneer works (e.g. [5, 6]) with a simplified S-matrix, which was inspired by realized nano-rings. Some mathematical features of transmission of a quantum particle in the system with a Y-junction were investigated in [7] and also in the context of a star graph etc [8–11]. However, the thorough investigation of the system has not yet been completed.

In the present work, we provide new formalism for the quantum mechanical problems in the system with a single Y-junction and ring systems with double Y-junctions. In particular, by focusing on scale-invariant junction conditions, we investigate quantum reflection and transmission in such systems. This paper is organized as follows. In section 2, we formulate the system with a single Y-junction. We also formulate the ring systems with double Y-junctions in section 3. In section 4, we discuss quantum reflection and transmission on the ring systems under the scale-invariant junction conditions. Finally, we provide a summary in section 5.

2. Formulation of a system with a single Y-junction

2.1. The Schrödinger equation

We consider a quantum system with a single Y-junction, in which three one-dimensional quantum wires intersect at one point. Let us assume that the three axes are given by x_1, x_2 and x_3 and directed to the node as shown in figure 1(a) (inward axes). We also assume that the node locates at $x_i = \xi$ $(i = 1, 2, 3)$. Note that the angle between any two axes has no effect on the physical states. On each wire $(x_i < \xi)$, a quantum particle with mass m obeys the Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\Phi_i\left(t, x_i\right) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2}\Phi_i\left(t, x_i\right), \tag{1}$$

where Φ_i denotes the wave function on the x_i-axis. Thus we assume a free particle on the wire.

2.2. Junction conditions

Let us discuss the expression of a junction condition. The junction condition at the node is provided by the conservation of the probability current

$$j_1(t,\xi) + j_2(t,\xi) + j_3(t,\xi) = 0, \tag{2}$$

where the probability current $j_i(t, x_i)$ on the x_i-axis is given by

$$j_i(t,x_i) := -\frac{\mathrm{i}\hbar}{2m}\left\{\Phi_i^*(t,x_i)\Phi_i'(t,x_i) - \Phi_i^{*\prime}(t,x_i)\Phi_i(t,x_i)\right\}, \tag{3}$$

and we implicitly assume the limit

$$j_i(t,\xi) := \lim_{x_i\to\xi} j_i(t,x_i). \tag{4}$$

Here the prime $(')$ denotes the partial differentiation with respect to the spatial coordinate. Equation (2) can be expressed by

$$\Psi'^{\dagger}\Psi - \Psi^{\dagger}\Psi' = 0, \tag{5}$$

where

$$\Psi := \begin{pmatrix}\Phi_1(t,\xi)\\ \Phi_2(t,\xi)\\ \Phi_3(t,\xi)\end{pmatrix}, \quad \Psi' := \begin{pmatrix}\Phi_1'(t,\xi)\\ \Phi_2'(t,\xi)\\ \Phi_3'(t,\xi)\end{pmatrix}. \tag{6}$$

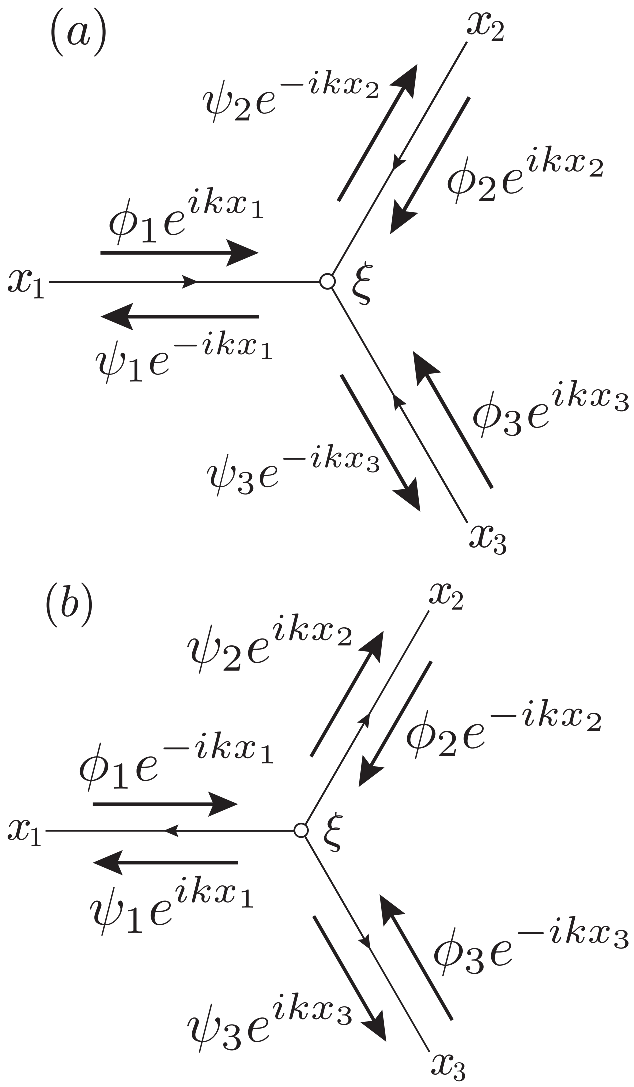

Figure 1. A quantum system with a Y-junction. The three axes are labeled by x_1, x_2 and x_3. The node is given by $x_i = \xi$ $(i = 1, 2, 3)$. (a) All axes are directed to the node. (b) The axes are taken in the opposite direction in comparison with (a).

Equation (5) is equivalently expressed as [4]

$$|\Psi - \mathrm{i}L_0\Psi'| = |\Psi + \mathrm{i}L_0\Psi'|, \tag{7}$$

where L_0 $(\in \mathbb{R})$ is an arbitrary non-vanishing constant with dimension of length (see appendix A for the role of L_0). Hence $\Psi - \mathrm{i}L_0\Psi'$ is connected to $\Psi + \mathrm{i}L_0\Psi'$ via a unitary transformation. Thus, we obtain the junction condition [4]

$$(U - I_3)\,\Psi + \mathrm{i}L_0\,(U + I_3)\,\Psi' = 0, \tag{8}$$

where I_3 is the 3×3 identity matrix, and U is a 3×3 unitary matrix, i.e. $U \in \mathrm{U}(3)$. Therefore the junction condition is characterized by the unitary matrix U.

Next, we discuss parametrization of the unitary matrix U. For this discussion, it is useful to recall that any unitary matrix U can be diagonalized by a unitary matrix W as

$$W^\dagger U W = D := \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \tag{9}$$

where $\theta_i \in \mathbb{R}$ $(i = 1, 2, 3)$. Since $\mathrm{U}(3) = \mathrm{U}(1) \times \mathrm{SU}(3)$ holds locally [12], the unitary matrix W can always be expressed by an element $V \in \mathrm{SU}(3)$ multiplied by a complex factor $e^{i\eta}$, i.e. $W = e^{i\eta} V$, where $\eta \in \mathbb{R}$. When we adopt the Euler angle parametrization [13] for V, we have

$$W = e^{i\eta} e^{i\alpha\lambda_3} e^{i\beta\lambda_2} e^{i\gamma\lambda_3} e^{i\delta\lambda_5} e^{ia\lambda_3} e^{ib\lambda_2} e^{ic\lambda_3} e^{id\lambda_8} \tag{10}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_8$ are the Gell-Mann Matrices (see appendix B for the definition), and $\alpha, \beta, \gamma, \delta, a, b, c, d \in \mathbb{R}$. Then we derive

$$U = W D W^\dagger = \mathcal{V} D \mathcal{V}^\dagger, \tag{11}$$

where

$$\mathcal{V} = e^{i\alpha\lambda_3} e^{i\beta\lambda_2} e^{i\gamma\lambda_3} e^{i\delta\lambda_5} e^{ia\lambda_3} e^{ib\lambda_2}. \tag{12}$$

Thus the unitary matrix U, which provides the junction condition, is specified by the nine real parameters: θ_1, θ_2, θ_3, α, β, γ, δ, a and b.

2.3. The scattering matrices

Let us consider the scattering of a single mode with wave number k by a Y-junction. We assume that incoming waves and outgoing waves are provided by $\phi_i e^{ikx_i}$ and $\psi_i e^{-ikx_i}$ $(i = 1, 2, 3)$, respectively, as shown in figure 1(a). Here $\phi_i, \psi_i \in \mathbb{C}$. Then, we have

$$\Phi_i\left(t, x_i\right) = e^{-i\frac{\mathcal{E}}{\hbar}t}\left(\phi_i e^{ikx_i} + \psi_i e^{-ikx_i}\right), \tag{13}$$

and

$$\Psi = e^{-i\frac{\mathcal{E}}{\hbar}t}\left\{ e^{ik\xi}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} + e^{-ik\xi}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}\right\}, \tag{14}$$

$$\Psi' = e^{-i\frac{\mathcal{E}}{\hbar}t}\left\{ ik e^{ik\xi}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} - ik e^{-ik\xi}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}\right\}, \tag{15}$$

where $\mathcal{E} := \hbar^2 k^2/2m$. Substituting equations (14) and (15) into equation (8) and using equation (11), we can rewrite the expression (8) into the form

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = S(\xi)\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}. \tag{16}$$

Based on the inward axes as shown in figure 1(a), we derive the S-matrix as

$$S(\xi) = S^{(\mathrm{in})}(\xi) := e^{2ik\xi}\mathcal{V} S_0^{(\mathrm{in})} \mathcal{V}^\dagger, \tag{17}$$

where

$$S_0^{(\mathrm{in})} := \begin{pmatrix} \frac{\mathrm{i}kL_1+1}{\mathrm{i}kL_1-1} & 0 & 0 \\ 0 & \frac{\mathrm{i}kL_2+1}{\mathrm{i}kL_2-1} & 0 \\ 0 & 0 & \frac{\mathrm{i}kL_3+1}{\mathrm{i}kL_3-1} \end{pmatrix}, \tag{18}$$

and

$$L_i := L_0 \cot \frac{\theta_i}{2}. \tag{19}$$

Here the superscript '(in)' denotes the result derived on the inward axes. Note that the diagonal component S_{ii} $(i = 1, 2, 3)$ represents the probability amplitude for the reflection from x_i-axis to x_i-axis, while the non-diagonal component S_{ij} $(i \neq j)$ represents the probability amplitude for the transmission from x_j-axis to x_i-axis. The S-matrix (17) is not symmetric in general, because the junction conditions do not necessarily satisfy the time-reversal symmetry (see appendix C for the condition of the time-reversal symmetry).

It should also be noted that when we take the axes in the opposite direction as shown in figure 1(b) (outward axes), we have to replace the wave number k with $-k$. In this case, we derive the S-matrix in the form

$$S(\xi) = S^{(\mathrm{out})}(\xi) := \mathrm{e}^{-2\mathrm{i}k\xi} \mathcal{V} S_0^{(\mathrm{out})} \mathcal{V}^{\dagger}, \tag{20}$$

$$S_0^{(\mathrm{out})} := \begin{pmatrix} \frac{\mathrm{i}kL_1-1}{\mathrm{i}kL_1+1} & 0 & 0 \\ 0 & \frac{\mathrm{i}kL_2-1}{\mathrm{i}kL_2+1} & 0 \\ 0 & 0 & \frac{\mathrm{i}kL_3-1}{\mathrm{i}kL_3+1} \end{pmatrix}. \tag{21}$$

Here the superscript '(out)' denotes the result derived on the outward axes. This expression is also useful to discuss a ring system with double Y-junctions in the next section.

3. Formulations of a ring system with double Y-junctions

3.1. Preliminary

We consider a ring system made of double Y-junctions as shown in figure 2. In this system, the Y-junction on the left has the inward axes of x_1, x_2 and x_3, while the Y-junction on the right has the outward axes of x_2, x_3 and x_4. The nodes are given by $x_1 = x_2 = x_3 = \xi_1$ and $x_2 = x_3 = x_4 = \xi_2$, where $\xi_1 > \xi_2$. We also assume that the node on the Y-junction $(x_1 x_2 x_3)$ has the same parameters as the node on the Y-junction $(x_4 x_2 x_3)$ symmetrically, as shown in figure 3(a). We call this type of a ring *a symmetric ring*. (We also discuss *an anti-symmetric ring* below.) When we consider a scattering problem, we take the wave functions on the wires as

$$\Phi_1(t, x_1) = \mathrm{e}^{-\mathrm{i}\frac{\mathcal{E}}{\hbar}t} \left(\mathrm{e}^{\mathrm{i}kx_1} + A\mathrm{e}^{-\mathrm{i}kx_1}\right), \tag{22}$$

$$\Phi_2(t, x_2) = \mathrm{e}^{-\mathrm{i}\frac{\mathcal{E}}{\hbar}t} \left(B\mathrm{e}^{-\mathrm{i}kx_2} + C\mathrm{e}^{\mathrm{i}kx_2}\right), \tag{23}$$

$$\Phi_3(t, x_3) = \mathrm{e}^{-\mathrm{i}\frac{\mathcal{E}}{\hbar}t} \left(D\mathrm{e}^{-\mathrm{i}kx_3} + E\mathrm{e}^{\mathrm{i}kx_3}\right), \tag{24}$$

$$\Phi_4(t, x_4) = \mathrm{e}^{-\mathrm{i}\frac{\mathcal{E}}{\hbar}t} F\mathrm{e}^{\mathrm{i}kx_4}, \tag{25}$$

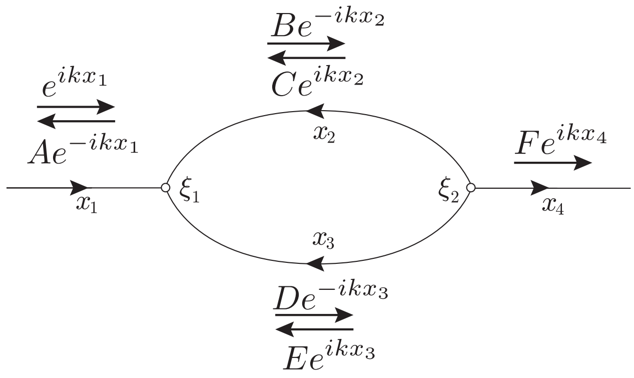

Figure 2. A ring system with double Y-junctions.

where $A, B, C, D, E, F \in \mathbb{C}$. Applying these expressions to equation (16), we have the relations

$$\begin{pmatrix} A \\ B \\ D \end{pmatrix} = S_1^{(\text{in})}(\xi_1) \begin{pmatrix} 1 \\ C \\ E \end{pmatrix}, \tag{26}$$

$$\begin{pmatrix} F \\ C \\ E \end{pmatrix} = S_2^{(\text{out})}(\xi_2) \begin{pmatrix} 0 \\ B \\ D \end{pmatrix}, \tag{27}$$

where S_1 denotes the S-matrix on the Y-junction $(x_1x_2x_3)$, and S_2 denotes the S-matrix on the Y-junction $(x_4x_2x_3)$. By solving equations (26) and (27), we can derive the coefficients A, B, C, D, E and F as functions of the components of $S_1^{(\text{in})}(\xi_1)$ and $S_2^{(\text{out})}(\xi_2)$ (see appendix D).

3.2. Path integral description

Let us derive the solution for the coefficients A, B, C, D, E and F in equations (22)–(25) based on a path integral approach. Our approach is similar to the method of a path decomposition expansion developed in [14] (see [10, 15, 16] for general treatments). The path integral makes physical interpretation clear.

First, we obtain the solution for F. We now consider all possible paths from x_1 to x_4. Each path can be characterized by the number of scattering at the nodes. Let us assume that F_n denotes the contribution from the path having n times scattering to F. Here $n = 2\ell_1$ $(\ell_1 = 1, 2, 3, \cdots)$ is only permitted. Thus, we have

$$F = F_2 + F_4 + F_6 + \cdots. \tag{28}$$

For example, F_2 is composed of two paths: $(x_1 \to x_2 \to x_4)$ and $(x_1 \to x_3 \to x_4)$. Hence, when we write the components of $S^{(\text{in})}(\xi_1)$ and $S^{(\text{out})}(\xi_2)$ as

$$S_1^{(\text{in})}(\xi_1) := \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \tag{29}$$

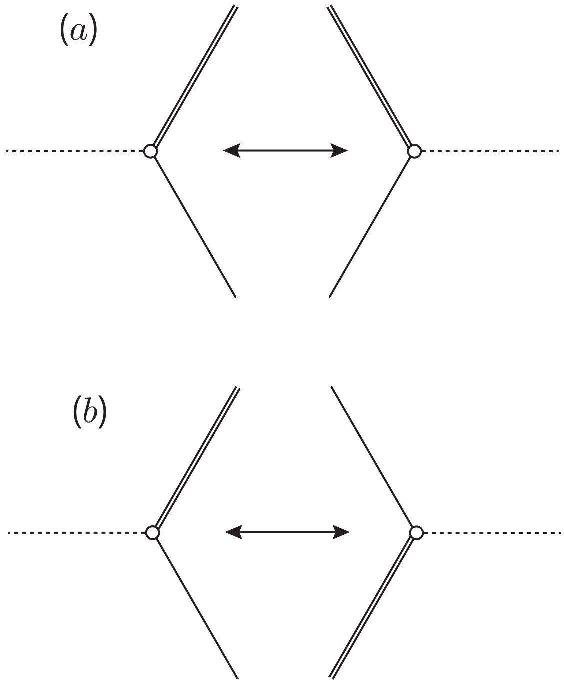

Figure 3. (a) A symmetric ring system. (b) An anti-symmetric ring system.

$$S_2^{(\mathrm{out})}(\xi_2) = \begin{pmatrix} \tilde{s}_{11} & \tilde{s}_{12} & \tilde{s}_{13} \\ \tilde{s}_{21} & \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{s}_{31} & \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix}, \tag{30}$$

we have

$$F_2 = \tilde{s}_{12}s_{21} + \tilde{s}_{13}s_{31} = \sum_{i=2,3} \tilde{s}_{1i}s_{i1}. \tag{31}$$

Similarly, we derive

$$F_4 = \sum_{i_n=2,3} \tilde{s}_{1i_1}s_{i_1i_2}\tilde{s}_{i_2i_3}s_{i_31}, \tag{32}$$

$$F_6 = \sum_{i_n=2,3} \tilde{s}_{1i_1}s_{i_1i_2}\tilde{s}_{i_2i_3}s_{i_3i_4}\tilde{s}_{i_4i_5}s_{i_51}, \tag{33}$$

$$\vdots$$

$$F_{2\ell_1} = \sum_{i_n=2,3} \tilde{s}_{1i_1}s_{i_1i_2}\tilde{s}_{i_2i_3}s_{i_3i_4}\tilde{s}_{i_4i_5}\cdots s_{i_{2\ell_1-1}1}. \tag{34}$$

When we define 2×2 matrices

$$s := \begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix}, \tag{35}$$

$$\tilde{s} := \begin{pmatrix} \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix}, \tag{36}$$

then we derive

$$F_2 = \begin{pmatrix} \tilde{s}_{12} & \tilde{s}_{13} \end{pmatrix} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{37}$$

$$F_4 = \begin{pmatrix} \tilde{s}_{12} & \tilde{s}_{13} \end{pmatrix} s\tilde{s} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{38}$$

$$F_6 = \begin{pmatrix} \tilde{s}_{12} & \tilde{s}_{13} \end{pmatrix} (s\tilde{s})^2 \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \\ \vdots \tag{39}$$

$$F_{2\ell_1} = \begin{pmatrix} \tilde{s}_{12} & \tilde{s}_{13} \end{pmatrix} (s\tilde{s})^{\ell_1 - 1} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}. \tag{40}$$

Therefore, we obtain the amplitude F as

$$F = \begin{pmatrix} \tilde{s}_{12} & \tilde{s}_{13} \end{pmatrix} \left\{ I_2 + s\tilde{s} + (s\tilde{s})^2 + \cdots \right\} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{41}$$

where I_2 is the 2×2 identity matrix.

Next, we obtain the solution for A. We consider all possible paths from x_1 to x_1. Let us assume that A_n denotes the contribution from the path having n times scattering to A. Here $n = 2\ell_2 + 1$ $(\ell_2 = 0, 1, 2, \cdots)$. Then we have

$$A = A_1 + A_3 + A_5 + \cdots. \tag{42}$$

For example, A_1 is composed of the one path: $(x_1 \to x_1)$, and A_3 is composed of four paths: $(x_1 \to x_2 \to x_2 \to x_1), (x_1 \to x_2 \to x_3 \to x_1), (x_1 \to x_3 \to x_2 \to x_1)$ and $(x_1 \to x_3 \to x_3 \to x_1)$. Thus we derive

$$A_1 = s_{11} \tag{43}$$

$$A_3 = \begin{pmatrix} s_{12} & s_{13} \end{pmatrix} \tilde{s} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{44}$$

$$A_5 = \begin{pmatrix} s_{12} & s_{13} \end{pmatrix} \tilde{s}s\tilde{s} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \\ \vdots \tag{45}$$

$$A_{2\ell_2+1} = \begin{pmatrix} s_{12} & s_{13} \end{pmatrix} \tilde{s} \, (s\tilde{s})^{\ell_2 - 1} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix} \quad (\ell_2 \geqslant 1). \tag{46}$$

Therefore we obtain

$$A = s_{11} + \begin{pmatrix} s_{12} & s_{13} \end{pmatrix} \tilde{s} \left\{ I_2 + s\tilde{s} + (s\tilde{s})^2 + \cdots \right\} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}. \tag{47}$$

The coefficient A is expressed by the sum of a series of $s\tilde{s}$ in the same manner as F.

Finally, in a similar way, we obtain the other coefficients B, C, D and E as

$$B = s_{21} + \begin{pmatrix} s_{22} & s_{23} \end{pmatrix} \tilde{s} \left\{ I_2 + s\tilde{s} + (s\tilde{s})^2 + \cdots \right\} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{48}$$

$$C = \begin{pmatrix} \tilde{s}_{22} & \tilde{s}_{23} \end{pmatrix} \left\{ I_2 + s\tilde{s} + (s\tilde{s})^2 + \cdots \right\} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{49}$$

$$D = s_{31} + \begin{pmatrix} s_{32} & s_{33} \end{pmatrix} \tilde{s} \left\{ I_2 + s\tilde{s} + (s\tilde{s})^2 + \cdots \right\} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{50}$$

$$E = \begin{pmatrix} \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix} \left\{ I_2 + s\tilde{s} + (s\tilde{s})^2 + \cdots \right\} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}. \tag{51}$$

Thus the coefficients B, C, D and E are also expressed by the sum of a series of $s\tilde{s}$.

The above expressions can be simplified. Since $|s_{ij}| < 1$ and $|\tilde{s}_{ij}| < 1$ $(i, j = 2, 3)$, we have

$$\lim_{n\to\infty} (s\tilde{s})^n = O_2, \tag{52}$$

where O_2 is the 2×2 zero matrix. Hence the series of $s\tilde{s}$ converges as

$$I_2 + s\tilde{s} + (s\tilde{s})^2 + \cdots = (I_2 - (s\tilde{s}))^{-1}. \tag{53}$$

Therefore we can rewrite the above results as

$$A = s_{11} + \begin{pmatrix} s_{12} & s_{13} \end{pmatrix} \tilde{s} \, (I_2 - (s\tilde{s}))^{-1} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{54}$$

$$B = s_{21} + \begin{pmatrix} s_{22} & s_{23} \end{pmatrix} \tilde{s} \, (I_2 - (s\tilde{s}))^{-1} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{55}$$

$$C = \begin{pmatrix} \tilde{s}_{22} & \tilde{s}_{23} \end{pmatrix} (I_2 - (s\tilde{s}))^{-1} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{56}$$

$$D = s_{31} + \begin{pmatrix} s_{32} & s_{33} \end{pmatrix} \tilde{s} \, (I_2 - (s\tilde{s}))^{-1} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{57}$$

$$E = \begin{pmatrix} \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix} (I_2 - (s\tilde{s}))^{-1} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, \tag{58}$$

$$F = \begin{pmatrix} \tilde{s}_{12} & \tilde{s}_{13} \end{pmatrix} (I_2 - (s\tilde{s}))^{-1} \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}. \tag{59}$$

By straightforward calculations, we can confirm that the above results coincide with the results derived by solving equations (26) and (27) algebraically (see appendix D). Using equations (54) and (59), we can calculate the probability for reflection $|A|^2$ and that for transmission $|F|^2$.

4. Systems under scale-invariant junction conditions

In this section, we restrict our attention to the cases of scale-invariant junction conditions for explicit examples.

4.1. A system with a single Y-junction

We consider a system with a single Y-junction. Under the scale-invariant junction conditions, the eigen values of U take $+1$ or -1, that is, θ_i takes 0 or π (see appendix E for the details). Then, the diagonal matrix D is given by

$$D = D_S := \begin{pmatrix} \epsilon(\theta_1) & 0 & 0 \\ 0 & \epsilon(\theta_2) & 0 \\ 0 & 0 & \epsilon(\theta_3) \end{pmatrix}, \tag{60}$$

where

$$\epsilon(\theta_i) = \begin{cases} +1 & (\theta_i = 0) \\ -1 & (\theta_i = \pi) \end{cases}. \tag{61}$$

Substituting equations (11) and (60) into equation (8), we find that the junction condition is divided into two conditions

$$(U - I_3)\,\Psi = 0, \tag{62}$$

$$(U + I_3)\,\Psi' = 0. \tag{63}$$

Thus the dimensional parameter L_0 is dropped in the junction conditions. Furthermore, from equation (19), we have

$$L_i \to \pm\infty \quad \text{as} \quad \theta_i = 0, \tag{64}$$

$$L_i = 0 \quad \text{as} \quad \theta_i = \pi. \tag{65}$$

Hence we derive

$$S_0^{(\mathrm{in})} = S_0^{(\mathrm{out})} = D_S. \tag{66}$$

It should be noted that the k-dependence of $S_0^{(\mathrm{in})}$ and $S_0^{(\mathrm{out})}$ disappears under the scale-invariant junction conditions. Thus, we obtain the S-matrix

$$S^{(\mathrm{in})}(\xi) = \mathrm{e}^{2ik\xi}\,\mathcal{V} D_S \mathcal{V}^\dagger, \tag{67}$$

$$S^{(\mathrm{out})}(\xi) = \mathrm{e}^{-2ik\xi}\,\mathcal{V} D_S \mathcal{V}^\dagger. \tag{68}$$

Since the three parameters θ_i $(i = 1, 2, 3)$ are fixed, six parameters remain in the S-matrices. From equations (67) and (68), we find the probability for reflection from the x_i-axis to the x_i-axis

$$P\,(i \to i) = \left|\left(\mathcal{V} D_S \mathcal{V}^\dagger\right)_{ii}\right|^2, \tag{69}$$

and the probability for transmission from the x_i-axis to the x_j-axis

$$P\,(i \to j) = \left|\left(\mathcal{V} D_S \mathcal{V}^\dagger\right)_{ji}\right|^2. \tag{70}$$

In these expressions, k does not appear. Therefore, when we consider the scale-invariant junction conditions, the probabilities for reflection and transmission become constant with respect to k.

4.2. A symmetric ring system with double Y-junctions

We consider a symmetric ring system with double Y-junctions described in section 3.1 (see also figure 3(a)). From equations (67) and (68), we have the S matrix at ξ_1 and that at ξ_2 as

$$S_1^{(\mathrm{in})}(\xi_1) = \mathrm{e}^{2\mathrm{i}k\xi_1}\mathcal{V}D_S\mathcal{V}^\dagger, \tag{71}$$

$$S_2^{(\mathrm{out})}(\xi_2) = \mathrm{e}^{-2\mathrm{i}k\xi_2}\mathcal{V}D_S\mathcal{V}^\dagger. \tag{72}$$

We drop the superscripts '(in)' and '(out)' in what follows. From these equations, we find the relations

$$S_1(\xi_1)S_1(\xi_1)^\dagger = S_1(\xi_1)^\dagger S_1(\xi_1) = I_3, \tag{73}$$

$$S_2(\xi_2)S_2(\xi_2)^\dagger = S_2(\xi_2)^\dagger S_2(\xi_2) = I_3, \tag{74}$$

$$S_1(\xi_1)S_1(\xi_1) = \mathrm{e}^{4\mathrm{i}k\xi_1}, \tag{75}$$

$$S_2(\xi_2)S_2(\xi_2) = \mathrm{e}^{-4\mathrm{i}k\xi_2}, \tag{76}$$

$$S_2(\xi_2) = \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}S_1(\xi_1)^\dagger. \tag{77}$$

From equations (73) and (77), we can derive

$$\tilde{s} = \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}\begin{pmatrix} s_{22}^* & s_{32}^* \\ s_{23}^* & s_{33}^* \end{pmatrix}, \tag{78}$$

and

$$s\tilde{s} = \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}\begin{pmatrix} |s_{22}|^2+|s_{23}|^2 & -s_{21}s_{31}^* \\ -s_{21}^*s_{31} & |s_{32}|^2+|s_{33}|^2 \end{pmatrix}. \tag{79}$$

Furthermore, we obtain

$$s\tilde{s}\begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix} = \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}|s_{11}|^2\begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}. \tag{80}$$

Thus an eigen value and an eigen vector of $s\tilde{s}$ are given by $\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}|s_{11}|^2$ and $\begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}$, respectively. Substituting equation (80) into equations (47)–(51) and (41), we derive the simplified results

$$A = \frac{(1-\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)})s_{11}}{1-\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}|s_{11}|^2}, \tag{81}$$

$$B = \frac{s_{21}}{1-\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}|s_{11}|^2}, \tag{82}$$

$$C = \frac{-\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} s_{11} s_{12}^*}{1-\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} |s_{11}|^2}, \tag{83}$$

$$D = \frac{s_{31}}{1-\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} |s_{11}|^2}, \tag{84}$$

$$E = \frac{-\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} s_{11} s_{13}^*}{1-\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} |s_{11}|^2}, \tag{85}$$

$$F = \frac{\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} \left(1-|s_{11}|^2\right)}{1-\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} |s_{11}|^2}. \tag{86}$$

Consequently, from equations (81) and (86), we find that the perfect transmission ($A = 0$) occurs if and only if the condition

$$\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} = 1 \tag{87}$$

holds except the trivial case of $s_{11} = 0$. On the other hand, the perfect reflection ($F = 0$) does not occur except the trivial case of $|s_{11}| = 1$.

4.3. An anti-symmetric ring system with double Y-junctions

We also discuss an anti-symmetric ring system in which the x_2-components in the S-matrix are replaced with the x_3-components and vice versa on the Y-junction $(x_4 x_2 x_3)$ on the right, as shown in figure 3(b). That is, we replace $S_2(\xi_2)$ with $\overline{S}_2(\xi_2)$, where

$$\overline{S}_2(\xi_2) := P S_2(\xi_2) P^{-1}. \tag{88}$$

Here P is defined by

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{89}$$

Under the scale-invariant junction conditions, from equation (77), we derive

$$\begin{aligned} \overline{S}_2(\xi_2) &= \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} P S_1(\xi_1)^\dagger P^{-1} \\ &= \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} \begin{pmatrix} s_{11}^* & s_{31}^* & s_{21}^* \\ s_{13}^* & s_{33}^* & s_{23}^* \\ s_{12}^* & s_{32}^* & s_{22}^* \end{pmatrix}. \end{aligned} \tag{90}$$

Thus, we have

$$\tilde{s} = \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} \begin{pmatrix} s_{33}^* & s_{23}^* \\ s_{32}^* & s_{22}^* \end{pmatrix}, \tag{91}$$

and

$$s\tilde{s} = \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} \begin{pmatrix} s_{22}s_{33}^* + s_{23}s_{32}^* & s_{22}s_{23}^* + s_{23}s_{22}^* \\ s_{32}s_{33}^* + s_{33}s_{32}^* & s_{32}s_{23}^* + s_{33}s_{22}^* \end{pmatrix}. \tag{92}$$

Substituting the last result into equations (54)–(59), we obtain

$$A = \frac{1}{\mathcal{D}}\left[s_{11} + s_{11}\mathrm{e}^{4\mathrm{i}k(\xi_1-\xi_2)} + \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}\Lambda\right] \tag{93}$$

$$\begin{aligned} B = \frac{1}{\mathcal{D}}\Big[s_{21} + \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}\big\{ - s_{21}(s_{32}s_{23}^* + s_{33}s_{22}^*) \\ + s_{31}(s_{22}s_{23}^* + s_{22}^*s_{23})\Big], \end{aligned} \tag{94}$$

$$C = \frac{e^{2\mathrm{i}k(\xi_1-\xi_2)}}{\mathcal{D}}\left[(s_{33}^*s_{21} + s_{23}^*s_{31}) + s_{11}s_{12}^*\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}\right], \tag{95}$$

$$\begin{aligned} D = \frac{1}{\mathcal{D}}\Big[s_{31} + \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}\big\{ - s_{31}(s_{22}s_{33}^* + s_{23}s_{32}^*) \\ + s_{21}(s_{32}s_{33}^* + s_{33}s_{32}^*)\Big], \end{aligned} \tag{96}$$

$$E = \frac{\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}}{\mathcal{D}}\left[(s_{32}^*s_{21} + s_{22}^*s_{31}) + s_{11}s_{13}^*\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}\right], \tag{97}$$

$$F = \frac{\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}}{\mathcal{D}}(s_{31}^*s_{21} + s_{21}^*s_{31})\left(1 - \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}\right), \tag{98}$$

where

$$\begin{aligned} \mathcal{D} := 1 - \mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)}\,(s_{22}s_{33}^* + s_{23}s_{32}^* + s_{32}s_{23}^* + s_{33}s_{22}^*) \\ + |s_{11}|^2\,\mathrm{e}^{4\mathrm{i}k(\xi_1-\xi_2)}, \end{aligned} \tag{99}$$

$$\begin{aligned} \Lambda := -s_{11}(s_{22}s_{33}^* + s_{23}s_{32}^* + s_{32}s_{23}^* + s_{33}s_{22}^*) \\ + s_{12}(s_{33}^*s_{21} + s_{23}^*s_{31}) + s_{13}(s_{32}^*s_{21} + s_{22}^*s_{31}). \end{aligned} \tag{100}$$

From equation (93), we can find a condition for perfect transmission ($A = 0$). From $A = 0$, we derive

$$\cos\left(2k(\xi_1 - \xi_2)\right) = -\frac{\Lambda}{2s_{11}}. \tag{101}$$

Hence, if the condition

$$-\frac{\Lambda}{2s_{11}} \in \mathbb{R} \quad \text{and} \quad \left|\frac{\Lambda}{2s_{11}}\right| \leqslant 1 \tag{102}$$

holds, then the perfect transmission occurs repeatedly as k increases. Furthermore, from equation (98), we find that the perfect reflection ($F = 0$) occurs if and only if the condition

$$\mathrm{e}^{2\mathrm{i}k(\xi_1-\xi_2)} = 1 \tag{103}$$

holds except the trivial cases of $s_{21} = 0$ or $s_{31} = 0$. It should be emphasized that this phenomenon never happen when we consider one-dimensional quantum systems with double point interactions of degree 2 (see [17, 18]).

5. Summary

We have newly formulated the system with a single Y-junction and the ring systems with double Y-junctions. For the ring systems, we found the compact formulas for probability amplitudes based on the path integral approach. We have also discussed quantum reflection and transmission on the ring systems. Restricting our attention to the ring systems under the scale-invariant junction conditions, we found the conditions for perfect transmission and/or perfect reflection. It is remarkable that perfect reflection can occur in the case of the anti-symmetric ring system, in contrast to the one-dimensional quantum systems having singular nodes of degree 2. This phenomena might be related to the possible switching of the supercurrent through a ring-shaped Josephson junction, which was investigated by [19]. This probable relation would be investigated elsewhere. General cases beyond the scale-invariant junction conditions should also be investigated. This will be provided in the future works.

Appendix A. The property of L_0 in the junction condition

As shown in [4], the parameter L_0 does not provide any additional freedom for all possible junction conditions characterized by the unitary matrix $U \in U(3)$. To make the present paper self-contained, we explicitly show this fact in the case of $U(3)$, following the discussion of $U(2)$ in [4]. When we use W satisfying equation (9), equation (8) can be written as

$$(D - I_3)\,\tilde{\Psi} + \mathrm{i}L_0\,(D + I_3)\,\tilde{\Psi}' = 0, \tag{A.1}$$

where

$$\tilde{\Psi} = \begin{pmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \\ \tilde{\Phi}_3 \end{pmatrix} := W^\dagger \Psi, \quad \tilde{\Psi}' = \begin{pmatrix} \tilde{\Phi}'_1 \\ \tilde{\Phi}'_2 \\ \tilde{\Phi}'_3 \end{pmatrix} := W^\dagger \Psi'. \tag{A.2}$$

Equation (A.1) is reduced to

$$\tilde{\Phi}_j + L_0 \cot\frac{\theta_j}{2}\tilde{\Phi}'_j = 0 \quad (j = 1, 2, 3). \tag{A.3}$$

We consider an infinitesimal transformation of L_0 as

$$L_0 \longrightarrow \overline{L}_0 = L_0 + \delta L_0. \tag{A.4}$$

Under this transformation, if we consider infinitesimal transformations of θ_j as

$$\theta_j \longrightarrow \overline{\theta}_j = \theta_j + \frac{\delta L_0}{L_0}\sin\theta_j \tag{A.5}$$

then equation (A.3) becomes invariant. Hence, the change of L_0 can always be absorbed by the changes of parameters $(\theta_1, \theta_2, \theta_3)$. In other words, the parameters $(L_0, \theta_1, \theta_2, \theta_3)$ give the same junction condition as that of $(\overline{L}_0, \overline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3)$. Thus, the transformation given by (A.4) and (A.5) can be considered to be a kind of a gauge transformation. Therefore, we can regard the parameter L_0 as a gauge freedom.

Appendix B. Gell-Mann matrices

The Gell-Mann matrices are defined as

$$\lambda_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.1}$$

$$\lambda_2 := \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.2}$$

$$\lambda_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.3}$$

$$\lambda_4 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{B.4}$$

$$\lambda_5 := \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \tag{B.5}$$

$$\lambda_6 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{B.6}$$

$$\lambda_7 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \tag{B.7}$$

$$\lambda_8 := \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{B.8}$$

These eight matrices give bases in SU(3).

Appendix C. The condition of the time-reversal symmetry

We discuss the condition of the time-reversal symmetry for the unitary matrix U. When we consider the time-reversal transformation

$$t \longrightarrow \bar{t} = -t, \tag{C.1}$$

we find the transformation of the wave function from equation (1),

$$\Phi_i(t, x_i) \longrightarrow \overline{\Phi}_i(\bar{t}, x_i) = \Phi_i^*(t, x_i). \tag{C.2}$$

Then we also have

$$\Psi \longrightarrow \overline{\Psi} = \Psi^*, \quad \Psi' \longrightarrow \overline{\Psi'} = \Psi'^*. \tag{C.3}$$

Under the time-reversal transformation, the junction condition (8) is transformed as

$$\begin{aligned} &(U - I_3)\Psi + \mathrm{i}L_0(U + I_3)\Psi' = 0 \\ &\longrightarrow \left(\overline{U} - I_3\right)\Psi^* + \mathrm{i}L_0\left(\overline{U} + I_3\right)\Psi'^* = 0, \end{aligned} \tag{C.4}$$

where we assumed the transformation $U \to \overline{U}$. The complex conjugate of equation (8) multiplied by $(-U^T)$ from the left-hand side, where T denotes transposition, becomes

$$\left(U^T - I_3\right)\Psi^* + \mathrm{i}L_0\left(U^T + I_3\right)\Psi'^* = 0. \tag{C.5}$$

Here we have used the property that U is unitary. Comparing equation (C.4) with (C.5), we derive

$$U \longrightarrow \overline{U} = U^T. \tag{C.6}$$

Thus, the time-reversal symmetry requires the condition

$$U = U^T. \tag{C.7}$$

Therefore, if U is a symmetric matrix, then the boundary condition satisfies the time-reversal symmetry as well as the Schrödinger equation (1) does.

Next, we discuss the parameter space which satisfies the condition (C.7) for the time-reversal symmetry. From equations (11) and (C.7), we derive the condition

$$\mathcal{V} = \mathcal{V}^*, \tag{C.8}$$

where $\mathcal{V}$ is given by equation (12). In this case, the S-matrix also becomes symmetric, i.e. $S = S^T$. Note that while $i\lambda_2$ and $i\lambda_5$ are real, $i\lambda_3$ is imaginary. From the requirements that $\mathrm{e}^{\mathrm{i}\alpha\lambda_3}$, $\mathrm{e}^{\mathrm{i}\gamma\lambda_3}$, and $\mathrm{e}^{\mathrm{i}a\lambda_3}$ should be real, we find the conditions for the time-reversal symmetry,

$$\alpha = 0 \quad \text{or} \quad \pi, \tag{C.9}$$

$$\gamma = 0 \quad \text{or} \quad \pi, \tag{C.10}$$

$$a = 0 \quad \text{or} \quad \pi. \tag{C.11}$$

Therefore, when the above conditions (C.9)–(C.11) hold, the Y-junction satisfies the time-reversal symmetry, in which the six remaining parameters $\beta, \delta, b, \theta_1, \theta_2, \theta_3$ are still free.

Finally, we relate our expression of the S-matrix with the symmetric S-matrix in the previous works [5, 6]. If we adopt the parameters

$$\begin{aligned} &\alpha = 0,\ \beta = \frac{3\pi}{2},\ \gamma = \pi,\ \delta = \frac{\pi}{4}, \\ &a = 0,\ \theta_1 = 0,\ \theta_2 = \theta_3 = \pi, \end{aligned} \tag{C.12}$$

then we obtain

$$S = \begin{pmatrix} -\cos 2b & \frac{1}{\sqrt{2}}\sin 2b & \frac{1}{\sqrt{2}}\sin 2b \\ \frac{1}{\sqrt{2}}\sin 2b & \frac{1}{2}(\cos 2b - 1) & \frac{1}{2}(\cos 2b + 1) \\ \frac{1}{\sqrt{2}}\sin 2b & \frac{1}{2}(\cos 2b + 1) & \frac{1}{2}(\cos 2b - 1) \end{pmatrix}. \tag{C.13}$$

It should be noted that this matrix also corresponds to a scale-invariant junction condition. When $0 \leqslant b \leqslant \frac{\pi}{4}$, we introduce new variables

$$\tilde{\epsilon} := \frac{1}{2}\sin^2 2b, \tag{C.14}$$

$$\tilde{a} := \frac{1}{2}\left(\sqrt{1-2\tilde{\epsilon}}-1\right) = \frac{1}{2}\left(\cos 2b - 1\right), \tag{C.15}$$

$$\tilde{b} := \frac{1}{2}\left(\sqrt{1-2\tilde{\epsilon}}+1\right) = \frac{1}{2}\left(\cos 2b + 1\right). \tag{C.16}$$

Then we find that the S-matrix is reduced to

$$S = \begin{pmatrix} -(\tilde{a}+\tilde{b}) & \tilde{\epsilon}^{1/2} & \tilde{\epsilon}^{1/2} \\ \tilde{\epsilon}^{1/2} & \tilde{a} & \tilde{b} \\ \tilde{\epsilon}^{1/2} & \tilde{b} & \tilde{a} \end{pmatrix}. \tag{C.17}$$

This is the same expression as the S-matrix in [5, 6].

Appendix D. The solution of the amplitude *A*, *B*, *C*, *D*, *E* and *F*

By solving equations (26) and (27) algebraically, we obtain the solution for A, B, C, D, E and F as

$$\begin{aligned} A = s_{11} + \frac{1}{\Delta}\{ & s_{12}\left(A_{12}(1-A_{33}) + A_{13}A_{32}\right) \\ & + s_{13}\left(A_{13}(1-A_{22}) + A_{12}A_{23}\right)\}, \end{aligned} \tag{D.1}$$

$$B = \frac{1}{\Delta}\left\{s_{31}B_{32} + s_{21}(1-B_{33})\right\}, \tag{D.2}$$

$$C = \frac{1}{\Delta}\left\{A_{12}(1-A_{33}) + A_{13}A_{32}\right\}, \tag{D.3}$$

$$D = \frac{1}{\Delta}\left\{s_{21}B_{23} + s_{31}(1-B_{22})\right\}, \tag{D.4}$$

$$E = \frac{1}{\Delta}\left\{A_{13}(1-A_{22}) + A_{12}A_{23}\right\}, \tag{D.5}$$

$$\begin{aligned} F = \frac{1}{\Delta}\{ & \tilde{s}_{12}\left(s_{31}B_{32} + s_{21}(1-B_{33})\right) \\ & + \tilde{s}_{13}\left(s_{21}B_{23} + s_{31}(1-B_{22})\right)\}, \end{aligned} \tag{D.6}$$

where

$$A_{ij} := \sum_{k=2,3} s_{ki}\tilde{s}_{jk}, \tag{D.7}$$

$$B_{ij} := \sum_{k=2,3} \tilde{s}_{ki}s_{jk}, \tag{D.8}$$

$$\begin{aligned}\Delta &:= (1 - A_{22})(1 - A_{33}) - A_{23}A_{32} \\ &= (1 - B_{22})(1 - B_{33}) - B_{23}B_{32}.\end{aligned} \tag{D.9}$$

Appendix E. Scale-invariant junction conditions

We discuss the Weyl scaling transformation of the wave function (see also [4]), which is given by

$$\Phi_i(t, x_i) \longrightarrow \overline{\Phi}_i(t, x_i) = \mathcal{A}\Phi_i(t, \lambda x_i), \tag{E.1}$$

where λ is a positive constant, and $\mathcal{A}$ is a normalization factor. Note that the condition for the normalization of the probability is provided by

$$\sum_{i=1}^{3} \int_{-\infty}^{\xi} |\Phi_i(t, x_i)|^2 \, \mathrm{d}x_i = 1 \tag{E.2}$$

in a system with a single Y-junction (figure 1(a)). This normalization condition becomes, under the Weyl scaling transformation,

$$\begin{aligned}&\sum_{i=1}^{3} \int_{-\infty}^{\xi/\lambda} |\mathcal{A}\Phi_i(t, \lambda x_i)|^2 \, \mathrm{d}x_i \\ &= \sum_{i=1}^{3} \int_{-\infty}^{\xi} |\mathcal{A}\Phi_i(t, y_i)|^2 \, \frac{1}{\lambda}\mathrm{d}y_i = 1.\end{aligned} \tag{E.3}$$

Hence, we derive

$$\Phi_i(t, x_i) \longrightarrow \overline{\Phi}_i(t, x_i) = \lambda^{1/2}\Phi_i(t, \lambda x_i). \tag{E.4}$$

We also have

$$\Psi \longrightarrow \lambda^{1/2}\Psi, \quad \Psi' \longrightarrow \lambda^{3/2}\Psi. \tag{E.5}$$

From these results, we find that the junction condition (8) is transformed, under the Weyl scaling transformation, to

$$\lambda^{1/2}(U - I_3)\Psi + \mathrm{i}L_0\lambda^{3/2}(U + I_3)\Psi' = 0. \tag{E.6}$$

It follows that the junction condition is not invariant in general under the Weyl scaling transformation. If we impose the Weyl scaling invariance, then we obtain

$$(U - I_3)\Psi = 0, \quad (U + I_3)\Psi' = 0. \tag{E.7}$$

The last equation means that the eigen values of U take $+1$ or -1. In this case, we call the junction condition scale invariant.

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