IOP Publishing European Journal of Physics

Eur. J. Phys. **41** (2020) 035001 (7pp) https://doi.org/10.1088/1361-6404/ab6c2c

Impact of a ball with a rigid rod

Rod Cross

Physics Department, Sydney University, Australia

E-mail: rodney.cross@sydney.edu.au

Received 27 November 2019, revised 3 January 2020
Accepted for publication 15 January 2020
Published 24 February 2020

Abstract

The impact of a ball with a rigid rod is a standard textbook problem with several surprises. We consider a hinged rod, as well as a free rod. Intuition suggests that the rod will rotate fastest when struck at one end. However, an end impact generates maximum rotation speed only if the mass of the free rod is at least double the mass of the ball, or the mass of a hinged rod is at least three times the mass of the ball. These results can be explained in terms of the reduced effective mass of a rod near its free end, and a consequent reduction in the impulsive force for impacts near a free end.

Keywords: effective mass, collisions, coefficient of restitution

(Some figures may appear in colour only in the online journal)

1. Introduction

A few years ago, Lemos [1] analysed a textbook problem concerning the impact of a ball with a rigid rod, arriving at an intuitively unexpected result. The rod is assumed to be at rest on a frictionless table and is struck at right angles by a ball of mass m. Assuming that the rod has mass M and is free to rotate, and the collision is perfectly elastic, Lemos found that the rod will rotate at maximum angular velocity if it is struck at one end, as expected, but only if $M > 2m$. If $M < 2m$ then the rod rotates at maximum speed when the impact point lies between the middle and the end of the rod. Lemos attributed the latter result to the fact that the rotation axis of the rod is not fixed, unlike the more familiar case of a hinged door. For a free rod, and a given impact speed, the impulsive force on the rod varies with the impact point and is smallest for an impact at the tip. Consequently, the impulsive torque is not necessarily a maximum at the tip of the rod.

However, a similar result is obtained if the rod is hinged or pivoted at one end, in which case the rotation axis of the rod is fixed, like a hinged door. It is shown below that a hinged rod rotates at maximum speed if it is struck at the free end, but only if $M > 3m$. Otherwise, maximum rotation speed arises if the rod is struck at a point remote from the far end. In the

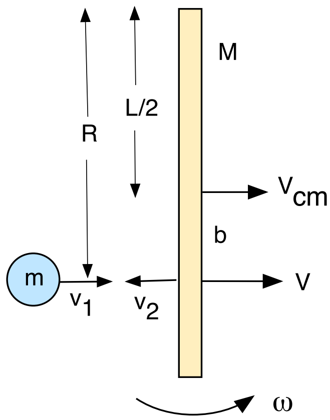

Figure 1. Impact of a ball of mass m with a uniform rod of mass M and length L.

latter case, the impact point that generates maximum rotation speed corresponds to the point where the rebound speed of the ball is zero. We consider both perfectly elastic and inelastic collisions.

2. Theoretical collision models

Consider the situation shown in figure 1 where a ball of mass m is incident at right angles on a uniform rod of mass M and length L, initially at rest. The centre of mass of the rod is located at a distance $L/2$ from each end, and the ball impacts at a distance b from the centre of mass. The ball is incident at speed v_1 and rebounds at speed v_2. Immediately after the collision, the rod rotates at angular velocity ω, the speed of the centre of mass is V_{cm} and the speed of the impact point on the rod is $V = V_{cm} + b\omega$. We consider two separate cases where the rod is either free to rotate and translate, or is hinged at its upper end.

2.1. Free rod

If F is the impact force exerted by the ball on the rod at the impact point then the equations of motion describing the collision are $F = -mdv/dt = MdV_{cm}/dt$ and $Fb = I_{cm}\, d\omega/dt$ where I_{cm} is the moment of inertia of the rod for rotation about its centre of mass. Integration of the equations of motion over the impact duration gives

$$\int F\, dt = m(v_1 + v_2) = MV_{cm} \tag{1}$$

and

$$I_{cm}\, \omega = mb(v_1 + v_2). \tag{2}$$

Since there are three unknowns (v_2, V_{cm} and ω), a third equation is required to solve the problem, describing energy loss during the collision. The latter quantity can be expressed in terms of the coefficient of restitution, e, which can vary from zero to unity depending on the energy loss. Two such coefficients are needed in general (normal and tangential coefficients) to describe oblique collisions, but we consider only normal collisions in this paper. For simplicity it is assumed that e is independent of the impact point, although in practice e

usually varies with the impact point, mainly due to vibration energy losses in the rod. By assuming that the rod is rigid, vibrational energy losses can be ignored.

In general, solutions of these equations are algebraically complicated, especially when considering inelastic collisions. However the algebra and the physics are both simplified if we define two new quantities, namely $e_A = v_2/v_1$ and an effective mass, M_e, given by $F = M_e dV/dt$. The quantity e_A is an apparent coefficient of restitution, ignoring recoil of the rod. The actual coefficient of restitution, e, is given by

$$e = \frac{v_2 + V}{v_1}. \tag{3}$$

For a perfectly elastic collision, $e = 1$, and for an inelastic collision, $0 < e < 1$. The apparent coefficient of restitution is more easily measured and is generally a more useful quantity when describing the impact of a striking implement swung at an incoming ball [2–4].

The effective mass of the rod varies along its length and can be regarded as a point mass M_e that recoils at speed V when subject to the impact force F. The collision between the ball and the rod is then simplified, both algebraically and conceptually, since it is equivalent to one between a point mass m and another point mass M_e. Since $V > V_{cm}$, M_e is less than the actual mass, M, of the rod. Since $V = V_{cm} + b\omega$,

$$\frac{dV}{dt} = \frac{dV_{cm}}{dt} + b\frac{d\omega}{dt}, \tag{4}$$

so

$$\frac{1}{M_e} = \frac{1}{M} + \frac{b^2}{I_{cm}}. \tag{5}$$

From equations (1), (2) and (5) it is easy to show that

$$V = V_{cm} + b\omega = (1 + e_A)\frac{m}{M_e}v_1. \tag{6}$$

Alternatively, equation (6) follows directly from conservation of momentum since $mv_1 = M_e V - mv_2$. Substitution of (6) in (3) then gives

$$e_A = \frac{v_2}{v_1} = \frac{eM_e - m}{M_e + m}, \tag{7}$$

while the angular velocity of the rod is given from (1) to (2) by

$$\omega = \frac{b\int F\, dt}{I_{cm}} = \frac{(1 + e_A)bmv_1}{I_{cm}}. \tag{8}$$

For a uniform rod with $I_{cm} = ML^2/12$, substitution of equations (5) and (7) in (8) gives

$$\omega = \frac{(1 + e)v_1 x}{L[x^2 + (1 + M/m)/12]}, \tag{9}$$

where $x = b/L$. In that case, ω is a maximum when $x^2 = (1 + M/m)/12$. For example ω is a maximum at $x = 1/2$ if $M = 2m$. Since x cannot be larger than $1/2$, ω is a maximum for an impact at the tip of the rod provided that $M > 2m$. Otherwise, ω is a maximum when $x < 1/2$. Since ω is proportional to $1 + e$, the same conclusions can be drawn regardless of the value of e.

2.2. Hinged rod

If the rod is hinged at its upper point in figure 1 then $F = -mdv/dt = M_e dV/dt$ and $FR = I_A d\omega/dt$, where $R = L/2 + b$ and $I_A = ML^2/3$ is the moment of inertia of the rod for rotation about a fixed axis at one end. The net force on the rod includes a reaction force at the hinged end which can be ignored for now. Integration over the impact duration gives

$$I_A\,\omega = mR(v_1 + v_2) = M_e RV. \tag{10}$$

Since $V = R\omega$, $dV/dt = Rd\omega/dt$ so

$$M_e = \frac{F}{dV/dt} = \frac{I_A}{R^2} \tag{11}$$

and

$$\omega = \frac{V}{R} = \frac{(1 + e_A)mv_1}{RM_e}. \tag{12}$$

From equations (3) and (10) it is easy to show that e_A is given by equation (7) for a hinged rod, although the expressions for M_e are different for free and hinged rods. Equation (7) applies to both free and hinged rods because it is a standard result that applies to the head-on collision between any two point masses m and M_e. For a uniform, hinged rod where $I_A = ML^2/3$ and $M_e = ML^2/3R^2$, equations (7) and (12) give

$$\omega = \frac{(1 + e)v_1 x}{L(x^2 + M/3m)}, \tag{13}$$

where $x = R/L$. In that case, ω is a maximum when $x^2 = M/3m$, meaning that ω is a maximum for an impact at the tip of the rod when $M > 3m$ but is a maximum at $R < L$ when $M < 3m$, regardless of the values of e, v_1 and L.

3. Comparison of free and hinged rods

Solutions of the above equations are shown in figures 2 and 3 for a ball of mass $m = 0.1$ kg impacting a uniform rod of mass $M = 0.2$ kg and length $L = 0.5$ m, as functions of the impact parameter, b. The solutions are given for a perfectly elastic collision with $e = 1$ but results for other values of e are easily determined since ω is proportional to $1 + e$. For this rod, $I_{cm} = 0.004\,17$ kg m^2 and $I_A = 0.016\,67$ kg m^2.

The effective mass of the free and hinged rods is shown in figure 2. For a free rod, $M_e = M$ when $b = 0$ and $M_e = M/4$ when $b = L/2$. For the hinged rod, $M_e = 4M/3$ when $b = 0$ and $M_e = M/3$ when $b = L/2$. The effective mass of the rod is independent of the mass of the ball and is independent of e, but it decreases as b increases since the rod rotates at relatively high speed for impacts near the tip. That is, $M_e = F/(dV/dt)$ decreases as b increases since V increases as b increases.

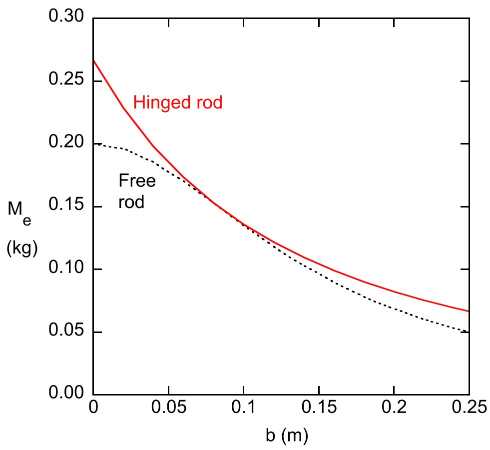

Figure 2. Effective mass, M_e, of the free and hinged rods versus the impact parameter, b.

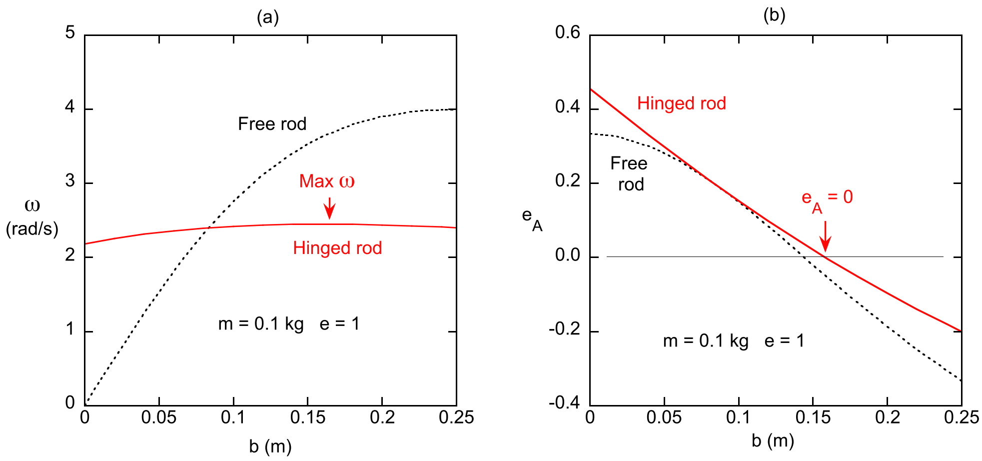

Figure 3. (a) Angular velocity, ω, and (b) e_A for the free and hinged rods versus the parameter, b.

The value of M_e for the free rod is the same as that for the hinged rod if the impact occurs at $b = L/6$ since the impact point then coincides with the centre of percussion. That is, the free rod rotates about an axis through the far end, so a free rod then behaves in the same manner as a hinged rod. The location of the centre of percussion can be calculated from equations (6) and (8) and from the relation $V_e = V - (L/2 + b)\omega = 0$ where V_e is the velocity of the far end of the free rod.

The angular velocity of the rods is shown in figure 3(a). For the free rod, ω is zero for an impact in the middle of the rod since the torque about the centre of mass is then zero. The angular velocity increases as b increases, since the torque Fb increases with b, and is a maximum at the tip of the rod. For the hinged rod, ω is finite for an impact in the middle of the rod since the torque about the hinged end is finite. Even though R increases as b increases, ω is almost independent of b, with a weak maximum at $b = 0.16$ m. The explanation is that the impulse $\int F\,dt$ decreases by a factor of about two when R is doubled since the ball rebounds at lower speed when bouncing off a smaller mass. As a result, the impulsive torque about the hinged end depends only weakly on b.

The bounce speed to incident speed ratio, $e_A = v_2/v_1$, is shown in figure 3(b). The incident ball bounces best when M_e is large but it comes to a complete stop with $e_A = 0$ if $eM_e = m$, as indicated by equation (7) and as shown in figure 3(b). When $e = 1$, the ball comes to a stop if $M_e = m$, in the same way that a billiard comes to a stop if it strikes an identical, stationary ball. For impacts closer to the tip, where $M_e < m$, the ball does not bounce at all. Instead, the ball keeps moving in the same direction, at reduced speed. At least, that is the case in figure 3. Lighter balls, with $m < eM_e$ at the tip of the rod, will bounce backwards off the tip.

A surprising result is that ω is a maximum when $e_A = 0$ for the hinged rod, but not for the free rod. The hinged rod result is easily explained, since maximum energy is transferred to the rod if the ball comes to rest and since the kinetic energy of the rod is $\frac{1}{2}I_A\omega^2$. The kinetic energy of the free rod is $\frac{1}{2}MV_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$, which is maximised when $e_A = 0$, but the energy is partly translational and partly rotational.

4. Discussion

The intentions of this article are primarily to rephrase the findings of Lemos [1] in simpler terms so that the physics is more transparent, and to extend his findings to include inelastic collisions and a hinged rod. The discussion is greatly simplified by considering only a perfectly rigid rod. Nevertheless, a flexible rod behaves in a similar fashion, as shown in previous studies concerning the impact of a ball and a striking implement [2–4].

There are two main differences between perfectly rigid and flexible (i.e. real) rods. One is that a flexible rod will vibrate if it is struck anywhere other than a vibration node point. Consequently, some of the initial impact energy of the ball is lost in the form of rod vibrations. The loss may be comparable to the energy lost in the ball itself, especially for impacts near the tip of the rod. The effect of vibrational energy loss is that the coefficient of restitution is reduced, an effect that can be modelled simply by reducing the value of e in the above equations, especially near the tip of the rod. Vibrational energy losses can be minimised by using a very stiff rod, which is part of the reason that modern tennis racquets are now made from stiff graphite rather than wood. Interesting results can be observed by dropping a ball vertically on a flat beam. If the ball slows down almost to a stop, multiple impacts can arise due to vibrations of the beam [5–7]. A result is shown in [5] where a ball bounced seven times on a beam before being ejected by the beam.

The other main difference concerns bending wave propagation along the rod. In many cases of interest, the bending wave generated by the impact takes several milliseconds or more to travel to the far end of the rod and back to the impact point. If the ball bounces off the rod before the wave returns, then no information is transmitted back to the ball concerning the end of the rod. In that case, the rebound speed of the ball will be independent of whether the far end of the rod is free to rotate or is hinged or rigidly clamped [2]. The implication in ball sports is that it makes no difference to the exit speed of a ball whether the handle end of the striking implement is gripped firmly or loosely.

5. Conclusions

While the impact of a ball with a rigid rod is a topic suitable for undergraduate physics students, a straightforward mathematical analysis is relatively cumbersome in general and tends to hide the underlying physics. By defining an effective mass of the rod, both the algebra and the physics are simplified considerably since the impact can be treated as one between two point masses. Since the effective mass of a rod is smallest at a free end, the rebound speed of the ball and the impulsive force are both reduced for impacts near the free end. The rebound speed of the ball may even be zero if the effective mass of the rod is comparable to the mass of the ball. Consequently, the impulsive torque is not necessarily a maximum for an impact at the free end, so the angular velocity of the rod is not necessarily a maximum for an impact at the free end. For a free rigid rod, maximum angular velocity occurs for an impact remote from the tip if the actual mass of the rod is less than twice the mass of the ball, regardless of the coefficient of restitution. For a hinged rigid rod, maximum angular velocity occurs for an impact remote from the tip if the mass of the rod is less than three times the mass of the ball.

ORCID iDs

Rod Cross https://orcid.org/0000-0001-9409-2791

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