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Addendum

Addendum to 'On the least uncomfortable journey from A to B' (2019 *Eur. J. Phys.* 40 055802)

Nivaldo A Lemos

Instituto de Física—Universidade Federal Fluminense Av. Litorânea, S/N, Boa Viagem, 24210-340, Niterói—Rio de Janeiro, Brazil

E-mail: nivaldo@if.uff.br

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Introduction

In [1] the solution to the problem of the least uncomfortable journey between two locations on a straight line was obtained by solving the Euler–Lagrange equation with the appropriate boundary conditions and isoperimetric constraint. However, it is a fact that satisfying the Euler–Lagrange equation is only a necessary condition for a function to minimise a functional defined by an integral. In several classic variational problems the physical or geometric meaning of the functional makes it intuitively clear that the solution of the Euler–Lagrange equation with the pertinent constraints and boundary conditions actually minimises the functional. But, even in such circumstances, it is much more satisfying to be in possession of a proof. The trouble is, if one wishes to prove rigorously that the functional attains at least a local (relative) minimum, one usually has to check that Jacobi's equation has a solution that never vanishes on the closed interval of integration and that the Weierstrass excess function is positive [2]. In most cases this is not easy to do. It is worth pointing out, nevertheless, that for some simple problems in mechanics an easy direct proof can be given that the action is a global (absolute) minimum for the physical path [3, 4].

The variational problem of determining the least uncomfortable way to travel from point A to point B on a straight line, with both the travel time and the distance between the two points fixed, was originally proposed in [5]. For the discomfort measured by the acceleration as well the discomfort measured by the jerk the optimal velocity, which minimises the discomfort, was found in [1] without proof that the minimum discomfort is actually attained. Here we provide a direct and elementary proof that in each of the two cases the optimal velocity obtained in [1] does indeed yield the global minimum discomfort.

Discomfort measured by the acceleration

On a straight road, a vehicle departs from point A at $t = 0$ and arrives at point B when $t = T$. Let the coordinate system be so chosen that the departure point A corresponds to $x = 0$ and the arrival point B corresponds to $x = D$. The travel time T is fixed. The problem consists in finding the velocity $v(t)$ that minimises the discomfort functional defined by

$$J[v] = \int_0^T \dot{v}^2 dt \tag{1}$$

with the boundary conditions

$$v(0) = 0, \quad v(T) = 0, \tag{2}$$

and under the isoperimetric constraint

$$\int_0^T v dt = D. \tag{3}$$

As shown in [1], the minimiser $v(t)$ must satisfy

$$2\ddot{v} - \lambda = 0 \tag{4}$$

where the Lagrange multiplier λ is a constant. The solution to this differential equation for $v(t)$ that satisfies the boundary conditions (2) and the isoperimetric constraint (3) is given in [1] as

$$v(t) = \frac{6D}{T}\left(\frac{t}{T} - \frac{t^2}{T^2}\right). \tag{5}$$

Let

$$\bar{v}(t) = v(t) + \eta(t) \tag{6}$$

be any other admissible velocity; that is, $\bar{v}(t)$ satisfies conditions (2) and (3). This, in turn, requires that $\eta(t)$ satisfy

$$\eta(0) = 0, \quad \eta(T) = 0, \tag{7}$$

as well as

$$\int_0^T \eta(t) dt = 0. \tag{8}$$

The discomfort brought about by $\bar{v}(t)$ is

$$\begin{aligned} J[\bar{v}] = \int_0^T \dot{\bar{v}}^2 dt = \int_0^T (\dot{v} + \dot{\eta})^2 dt = \int_0^T (\dot{v}^2 + 2\dot{v}\dot{\eta} + \dot{\eta}^2) dt = J[v] \\ + 2\int_0^T \dot{v}\dot{\eta} dt + \int_0^T \dot{\eta}^2 dt. \end{aligned} \tag{9}$$

An integration by parts gives

$$\int_0^T \dot{v}\dot{\eta} dt = \dot{v}(t)\eta(t)|_0^T - \int_0^T \ddot{v}\eta dt = -\int_0^T \ddot{v}\eta dt \tag{10}$$

where we have used (7). Taking equations (4) and (8) into account, we are led to

$$\int_0^T \dot{v}\dot{\eta} dt = -\frac{\lambda}{2}\int_0^T \eta(t) dt = 0. \tag{11}$$

It follows that equation (9) reduces to

$$J[\bar{v}] = J[v] + \int_0^T \dot{\eta}^2 dt. \tag{12}$$

Since the integral of $\dot{\eta}^2$ is positive, we conclude that $J[\bar{v}] > J[v]$. This proves that the least discomfort is provided by the optimal velocity (5).

Discomfort measured by the jerk

The problem is the same as before except that now the discomfort functional to be minimised is

$$J[v] = \int_0^T \ddot{v}^2 dt \tag{13}$$

under the same isoperimetric constraint

$$\int_0^T v dt = D \tag{14}$$

and, as argued in [1], the boundary conditions

$$v(0) = 0, \quad v(T) = 0, \quad \ddot{v}(0) = 0, \quad \ddot{v}(T) = 0. \tag{15}$$

As shown in [1], the minimiser $v(t)$ must satisfy

$$2\frac{d^4 v}{dt^4} + \lambda = 0. \tag{16}$$

The solution to this differential equation that satisfies the isoperimetric constraint (14) and the boundary conditions (15) is given in [1] as

$$v(t) = \frac{5D}{T}\left(\frac{t}{T} - 2\frac{t^3}{T^3} + \frac{t^4}{T^4}\right). \tag{17}$$

As before, we take any other admissible velocity $\bar{v}(t)$ as given by equation (6), where now $\eta(t)$ must satisfy

$$\eta(0) = 0, \quad \eta(T) = 0, \quad \ddot{\eta}(0) = 0, \quad \ddot{\eta}(T) = 0 \tag{18}$$

as well as equation (8).

The discomfort associated with $\bar{v}(t)$ is

$$J[\bar{v}] = \int_0^T \ddot{\bar{v}}^2 dt = \int_0^T (\ddot{v} + \ddot{\eta})^2 dt = \int_0^T (\ddot{v}^2 + 2\ddot{v}\ddot{\eta} + \ddot{\eta}^2) dt = J[v] + 2\int_0^T \ddot{v}\ddot{\eta} dt + \int_0^T \ddot{\eta}^2 dt. \tag{19}$$

Two successive integrations by parts give

$$\int_0^T \ddot{v}\ddot{\eta} dt = \ddot{v}(t)\dot{\eta}(t)|_0^T - \frac{d^3 v(t)}{dt^3}\eta(t)\Bigg|_0^T + \int_0^T \frac{d^4 v(t)}{dt^4}\eta(t) dt = \int_0^T \frac{d^4 v(t)}{dt^4}\eta(t) dt \tag{20}$$

where we have used the boundary conditions (15) and (18). Taking equations (16) and (8) into account, we are led to

$$\int_0^T \ddot{v}\ddot{\eta} dt = -\frac{\lambda}{2}\int_0^T \eta(t) dt = 0. \tag{21}$$

It follows that equation (19) reduces to

$$J[\bar{v}] = J[v] + \int_0^T \ddot{\eta}^2 dt. \tag{22}$$

Since the integral of $\ddot{\eta}^2$ is positive, we conclude that $J[\bar{v}] > J[v]$. This proves that the minimum discomfort, as measured by the jerk, is furnished by the optimal velocity (17).

ORCID iDs

Nivaldo A Lemos https://orcid.org/0000-0002-2386-1247

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