IOP Publishing European Journal of Physics

Eur. J. Phys. **41** (2020) 039401 (4pp) https://doi.org/10.1088/1361-6404/ab6301

Addendum

Addendum to 'On the least uncomfortable journey from A to B' (2019 *Eur. J. Phys.* 40 055802)

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Received 28 October 2019
Accepted for publication 17 December 2019
Published 24 February 2020

Introduction

In [1] the solution to the problem of the least uncomfortable journey between two locations on a straight line was obtained by solving the Euler–Lagrange equation with the appropriate boundary conditions and isoperimetric constraint. However, it is a fact that satisfying the Euler–Lagrange equation is only a necessary condition for a function to minimise a functional defined by an integral. In several classic variational problems the physical or geometric meaning of the functional makes it intuitively clear that the solution of the Euler–Lagrange equation with the pertinent constraints and boundary conditions actually minimises the functional. But, even in such circumstances, it is much more satisfying to be in possession of a proof. The trouble is, if one wishes to prove rigorously that the functional attains at least a local (relative) minimum, one usually has to check that Jacobi's equation has a solution that never vanishes on the closed interval of integration and that the Weierstrass excess function is positive [2]. In most cases this is not easy to do. It is worth pointing out, nevertheless, that for some simple problems in mechanics an easy direct proof can be given that the action is a global (absolute) minimum for the physical path [3, 4].

The variational problem of determining the least uncomfortable way to travel from point A to point B on a straight line, with both the travel time and the distance between the two points fixed, was originally proposed in [5]. For the discomfort measured by the acceleration as well the discomfort measured by the jerk the optimal velocity, which minimises the discomfort, was found in [1] without proof that the minimum discomfort is actually attained. Here we provide a direct and elementary proof that in each of the two cases the optimal velocity obtained in [1] does indeed yield the global minimum discomfort.

Discomfort measured by the acceleration

On a straight road, a vehicle departs from point A at $t = 0$ and arrives at point B when $t = T$. Let the coordinate system be so chosen that the departure point A corresponds to $x = 0$ and the arrival point B corresponds to $x = D$. The travel time T is fixed. The problem consists in finding the velocity $v(t)$ that minimises the discomfort functional defined by

$$J[v] = \int_0^T \dot{v}^2 dt \tag{1}$$

with the boundary conditions

$$v(0) = 0, \quad v(T) = 0, \tag{2}$$

and under the isoperimetric constraint

$$\int_0^T v dt = D. \tag{3}$$

As shown in [1], the minimiser $v(t)$ must satisfy

$$2\ddot{v} - \lambda = 0 \tag{4}$$

where the Lagrange multiplier λ is a constant. The solution to this differential equation for $v(t)$ that satisfies the boundary conditions (2) and the isoperimetric constraint (3) is given in [1] as

$$v(t) = \frac{6D}{T}\left(\frac{t}{T} - \frac{t^2}{T^2}\right). \tag{5}$$

Let

$$\bar{v}(t) = v(t) + \eta(t) \tag{6}$$

be any other admissible velocity; that is, $\bar{v}(t)$ satisfies conditions (2) and (3). This, in turn, requires that $\eta(t)$ satisfy

$$\eta(0) = 0, \quad \eta(T) = 0, \tag{7}$$

as well as

$$\int_0^T \eta(t) dt = 0. \tag{8}$$

The discomfort brought about by $\bar{v}(t)$ is

$$\begin{aligned} J[\bar{v}] = \int_0^T \dot{\bar{v}}^2 dt = \int_0^T (\dot{v} + \dot{\eta})^2 dt = \int_0^T (\dot{v}^2 + 2\dot{v}\dot{\eta} + \dot{\eta}^2) dt = J[v] \\ + 2\int_0^T \dot{v}\dot{\eta} dt + \int_0^T \dot{\eta}^2 dt. \end{aligned} \tag{9}$$

An integration by parts gives

$$\int_0^T \dot{v}\dot{\eta} dt = \dot{v}(t)\eta(t)|_0^T - \int_0^T \ddot{v}\eta dt = -\int_0^T \ddot{v}\eta dt \tag{10}$$

where we have used (7). Taking equations (4) and (8) into account, we are led to

$$\int_0^T \dot{v}\dot{\eta} dt = -\frac{\lambda}{2}\int_0^T \eta(t) dt = 0. \tag{11}$$

It follows that equation (9) reduces to

$$J[\bar{v}] = J[v] + \int_0^T \dot{\eta}^2 dt. \tag{12}$$

Since the integral of $\dot{\eta}^2$ is positive, we conclude that $J[\bar{v}] > J[v]$. This proves that the least discomfort is provided by the optimal velocity (5).

Discomfort measured by the jerk

The problem is the same as before except that now the discomfort functional to be minimised is

$$J[v] = \int_0^T \ddot{v}^2 dt \tag{13}$$

under the same isoperimetric constraint

$$\int_0^T v dt = D \tag{14}$$

and, as argued in [1], the boundary conditions

$$v(0) = 0, \quad v(T) = 0, \quad \ddot{v}(0) = 0, \quad \ddot{v}(T) = 0. \tag{15}$$

As shown in [1], the minimiser $v(t)$ must satisfy

$$2\frac{d^4 v}{dt^4} + \lambda = 0. \tag{16}$$

The solution to this differential equation that satisfies the isoperimetric constraint (14) and the boundary conditions (15) is given in [1] as

$$v(t) = \frac{5D}{T}\left(\frac{t}{T} - 2\frac{t^3}{T^3} + \frac{t^4}{T^4}\right). \tag{17}$$

As before, we take any other admissible velocity $\bar{v}(t)$ as given by equation (6), where now $\eta(t)$ must satisfy

$$\eta(0) = 0, \quad \eta(T) = 0, \quad \ddot{\eta}(0) = 0, \quad \ddot{\eta}(T) = 0 \tag{18}$$

as well as equation (8).

The discomfort associated with $\bar{v}(t)$ is

$$J[\bar{v}] = \int_0^T \ddot{\bar{v}}^2 dt = \int_0^T (\ddot{v} + \ddot{\eta})^2 dt = \int_0^T (\ddot{v}^2 + 2\ddot{v}\ddot{\eta} + \ddot{\eta}^2) dt = J[v] + 2\int_0^T \ddot{v}\ddot{\eta} dt + \int_0^T \ddot{\eta}^2 dt. \tag{19}$$

Two successive integrations by parts give

$$\int_0^T \ddot{v}\ddot{\eta} dt = \ddot{v}(t)\dot{\eta}(t)|_0^T - \frac{d^3 v(t)}{dt^3}\eta(t)\Bigg|_0^T + \int_0^T \frac{d^4 v(t)}{dt^4}\eta(t) dt = \int_0^T \frac{d^4 v(t)}{dt^4}\eta(t) dt \tag{20}$$

where we have used the boundary conditions (15) and (18). Taking equations (16) and (8) into account, we are led to

$$\int_0^T \ddot{v}\ddot{\eta} dt = -\frac{\lambda}{2}\int_0^T \eta(t) dt = 0. \tag{21}$$

It follows that equation (19) reduces to

$$J[\bar{v}] = J[v] + \int_0^T \ddot{\eta}^2 dt. \tag{22}$$

Since the integral of $\ddot{\eta}^2$ is positive, we conclude that $J[\bar{v}] > J[v]$. This proves that the minimum discomfort, as measured by the jerk, is furnished by the optimal velocity (17).

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IOP Publishing European Journal of Physics

Eur. J. Phys. **40** (2019) 055802 (10pp) https://doi.org/10.1088/1361-6404/ab28b4

On the least uncomfortable journey from *A* to *B*

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Received 26 March 2019, revised 21 May 2019
Accepted for publication 11 June 2019
Published 19 August 2019

Abstract

The problem of the 'least uncomfortable journey' between two locations on a straight line, originally discussed by Anderson *et al* (2016 *Am. J. Phys.* **84** 690–5), is revisited. When the integral of the square of the acceleration is used as a measure of the discomfort, the problem is shown to be easily solvable by taking the time, instead of the position, as the independent variable. The solution is quite simple and avoids not only complicated differential equations and the computation of cumbersome integrals, but also the inversion of functions by solving cubic equations. Next, the same problem, but now with the integral of the square of the jerk as a measure of the discomfort, is also exactly solved with time as the independent variable and the appropriate boundary conditions, which are derived. It is argued that the boundary conditions imposed on the velocity in Anderson *et al* (2016 *Am. J. Phys.* **84** 690–5) are inappropriate not only because they are not always physically realisable but also because they do not lead to the minimum discomfort possible.

Keywords: calculus of variations, higher-derivative variational problem, free endpoints and boundary conditions, least uncomfortable journey

(Some figures may appear in colour only in the online journal)

1. Introduction

The calculus of variations is one of the most important mathematical tools of theoretical physics. The design of simple yet unorthodox variational problems of physical interest is welcome since they help to acquaint advanced undergraduate or beginning graduate students with several useful techniques of the calculus of variations. Recently, Anderson, Desaix, and Nyqvist (ADN) [1] discussed the very interesting problem of determining the least uncomfortable way to travel from point A to point B on a straight line, with both the travel time and the distance between the two points fixed. ADN first minimised the integral of the square of the acceleration, taken as a measure of the discomfort felt during the journey. Next they minimised the integral of the square of the jerk as another possible measure of the discomfort. Less than a year later, a relativistic generalisation of the first version of the problem was published [2].

Frequent acceleration and deceleration make a trip uncomfortable, and the discomfort can be reasonably quantified by the integral of the square of the acceleration taken over the duration of the journey. In the case in which the discomfort is measured by the integral of the square of the acceleration, ADN chose the position x as the independent variable and found the velocity as a function of the position, $v(x)$, that minimises the discomfort. Their approach is perfectly valid but leads to a rather complicated differential equation, equation (9) of ADN, whose solution gives directly x as a function of v. The inversion of their equation (10) to get $v(x)$ requires the trigonometric solution of a cubic equation. A daunting integral gives t as a function of x, whose inversion to yield $x(t)$ is not straightforward either. In section 2 we show that the problem is easily solvable by taking the time, instead of the position, as the independent variable. The solution is very easy and avoids not only complicated differential equations and the computation of intimidating integrals, but also the inversion of functions by solving cubic equations.

A high constant acceleration (quite a few g-factors) is unpleasant and even harmful to the human body [3]. An abrupt change of a constant acceleration is also a source of distress. Since changes in acceleration may possibly cause more discomfort than changes in velocity, it makes sense to define the discomfort in terms of the time rate of change of the acceleration, usually known as jerk [3, 4]. In the case in which the discomfort is measured by the integral of the square of the jerk one has to cope with a higher-derivative variational problem of a kind rarely encountered in mechanics. By taking again the position x as the independent variable, ADN were led to an extremely complicated higher-derivative functional expressed in terms of $V(x) = v(x)^2$, where $v(x)$ is the velocity as a function of the position. Since the ensuing Euler–Lagrange equation for $V(x)$, their equation (19), is too intricate to be exactly solved, ADN resorted to the Rayleigh–Ritz and moment methods in order to find approximate solutions.

In variational problems, especially those involving higher-order derivatives of the unknown function, the determination of the appropriate boundary conditions is crucial. According to ADN, the boundary conditions must be such that 'the initial acceleration/deceleration must be zero in order to have finite jerk at the beginning and end of the journey.' Consequently, their trial velocity functions were chosen in such a way that the acceleration vanishes at the beginning and at the end of the journey. However, since the journey starts from rest and ends at rest, if the initial acceleration is zero then the initial force is also zero and, in most physical cases, the velocity will remain equal to zero for all time[1]. If the force depends on position alone, there can only be a nontrivial motion with both initial velocity and initial acceleration equal to zero if the force is singular, in which case the uniqueness theorem for the solutions of Newton's equations of motion fails [5]. Therefore, the realisation of the boundary conditions assumed by ADN might require pathological, unphysical forces. Also, the specification of the initial acceleration is at odds with the basic principles of Newtonian mechanics. These observations motivated us to search for boundary conditions which are more natural to the problem and free from such physical objections.

In section 3 we re-examine the problem of the least uncomfortable journey with the discomfort measured by the integral of the square of the jerk. Time is again taken as the

[1] There are exceptions, particularly if the force depends only on time. For instance, if $F(t) = 6t$ the equation of motion $\ddot{x} = F(t)$ with $x(0) = 0$, $\dot{x}(0) = 0$, $\ddot{x}(0) = 0$ is solved by $x(t) = t^3$, and it follows that $v(t) = 3t^2$.

independent variable and, by a combination of physical and mathematical arguments, we derive the physically appropriate boundary conditions. Then the exact solution is straightforwardly found and it is shown that it yields for the minimum discomfort a value which is much smaller than the one implied by the ADN boundary conditions.

Section 4 is dedicated to some final remarks.

2. Discomfort measured by the acceleration

When the integral of the square of the acceleration is taken as the measure of the discomfort, let us show that the problem of the least uncomfortable journey between two locations on a straight line becomes much simpler if one takes the time as the independent variable, instead of the position. Let the coordinate system be so chosen that the departure point A corresponds to $x = 0$ and the arrival point B corresponds to $x = D$. The vehicle departs from A at $t = 0$ and arrives at B when $t = T$. The travel time T is fixed. With the time t as the independent variable, the discomfort functional to be minimised is

$$J[v] = \int_0^T \dot{v}^2 \mathrm{d}t \tag{1}$$

with the boundary conditions

$$v(0) = 0, \quad v(T) = 0, \tag{2}$$

and under the isoperimetric constraint

$$\int_0^T v \mathrm{d}t = D. \tag{3}$$

According to Euler's rule [6, section 5.3], in order to find the minimiser $v(t)$ one must set up the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{\partial L}{\partial v} = 0 \tag{4}$$

with the 'Lagrangian'

$$L = \dot{v}^2 + \lambda v, \tag{5}$$

where the Lagrange multiplier λ is a constant. If v were the position of a particle, (5) would be the Lagrangian for a particle with mass $m = 2$ subject to a constant force λ.

With the 'Lagrangian' (5) the Euler–Lagrange equation (4) becomes

$$2\ddot{v} - \lambda = 0. \tag{6}$$

By two successive integrations the general solution to this equation is immediately found:

$$v = A + Bt + \frac{\lambda}{4}t^2, \tag{7}$$

where A and B are constants. The boundary conditions (2) yield $A = 0$ and $\lambda = -4B/T$. Therefore

$$v = BT\left(\frac{t}{T} - \frac{t^2}{T^2}\right). \tag{8}$$

Finally, the isoperimetric constraint (3) determines the remaining constant as $B = 6D/T^2$. Thus, the least uncomfortable journey is achieved by the velocity as the following function of time:

$$v = \frac{6D}{T}\left(\frac{t}{T} - \frac{t^2}{T^2}\right). \tag{9}$$

The optimal velocity grows steadily from zero when $t = 0$ to its maximum value when $t = T/2$, then decreases monotonically to zero when $t = T$, being a symmetric function about $t = T/2$. The maximum value of the velocity is

$$v_{\max} = \frac{3D}{2T}, \tag{10}$$

which is 50% larger than the average velocity $v_{\text{ave}} = D/T$.

Inasmuch as $x(0) = 0$, the position as a function of time is given by

$$x(t) = \int_0^t v(t')\mathrm{d}t' = \frac{6D}{T}\int_0^t \left(\frac{t'}{T} - \frac{t'^2}{T^2}\right)\mathrm{d}t', \tag{11}$$

whence

$$x = D\left(3\frac{t^2}{T^2} - 2\frac{t^3}{T^3}\right), \tag{12}$$

which is equivalent to equation (13) of ADN because they picked the coordinate system in such a way that $x(0) = -D$ and $x(T) = D$.

It is also worth noting that, in the present case, the least uncomfortable journey takes place with constant jerk: $j = \ddot{v} = -12D/T^3$.

The reader should compare the extreme simplicity of our solution with the difficult steps that led to the same result in ADN [1], whose approach, based on taking x as the independent variable, requires the integration of a complicated differential equation, the trigonometric solution of cubic equations and the computation of nontrivial integrals.

This is a good example of how the choice of independent variable can sometimes make a variational problem much more tractable.

3. Discomfort measured by the jerk

We presently show that this second version of the problem can also be explicitly solved in a very simple way once time is adopted as the independent variable and the physically appropriate boundary conditions are taken into consideration. Once again we choose the coordinate system in such a way that point A corresponds to $x = 0$ and point B corresponds to $x = D$. As before, the vehicle departs from A at $t = 0$ and arrives at B when $t = T$. The travel time T is fixed. With the time t as the independent variable, the discomfort functional to be minimised is

$$J[v] = \int_0^T \ddot{v}^2\mathrm{d}t \tag{13}$$

with the boundary conditions

$$v(0) = 0, \quad v(T) = 0, \tag{14}$$

and under the isoperimetric constraint

$$\int_0^T v\mathrm{d}t = D. \tag{15}$$

The process of minimising the functional (13) under the constraint (15) must take into account that, on physical grounds, no *a priori* boundary conditions other than (14) should be imposed on $v(t)$.

In the standard treatment of the problem of minimising a functional that depends on the second derivative of the unknown function, one is naturally led to require that both the values of the unknown function and its derivative be prescribed at the endpoints [6, section 4.1]. In the present case, however, $\dot{v}(0)$ and $\dot{v}(T)$ should not be prescribed for the physical reason given in the Introduction, namely, since the initial and final accelerations are determined by the force acting on the vehicle at the beginning and at the end of the journey, as a general rule their values cannot be specified *a priori*. In fact, finding the values of $\dot{v}(0)$ and $\dot{v}(T)$ is an integral part of the variational problem. This means that $\dot{v}(0)$ and $\dot{v}(T)$ must be treated as arbitrary and, as far as $\dot{v}(t)$ is concerned, we have to deal with a variational problem with variable endpoints. Let us briefly review how to tackle this kind of problem [6, section 9.1].

3.1. Variational considerations and boundary conditions

For a functional of the form

$$S[v] = \int_0^T L(v,\, \dot{v},\, \ddot{v})\mathrm{d}t, \tag{16}$$

the variation

$$v(t) \;\longrightarrow\; \bar{v}(t) = v(t) + \eta(t) \tag{17}$$

induces the first variation of the functional, defined by

$$\delta S = \int_0^T \left(\frac{\partial L}{\partial v}\eta + \frac{\partial L}{\partial \dot{v}}\dot{\eta} + \frac{\partial L}{\partial \ddot{v}}\ddot{\eta}\right)\mathrm{d}t. \tag{18}$$

Two successive integrations by parts give

$$\begin{aligned}\delta S = &\int_0^T \left[\frac{\partial L}{\partial v} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{v}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{v}}\right)\right]\eta\mathrm{d}t \\ &+ \left[\left(\frac{\partial L}{\partial \dot{v}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{v}}\right)\eta\right]_0^T + \frac{\partial L}{\partial \ddot{v}}\dot{\eta}\,\Big|_0^T .\end{aligned} \tag{19}$$

If the admissible variations are restricted only by the requirement that they satisfy

$$\eta(0) = \eta(T) = 0, \tag{20}$$

the variation (19) reduces to

$$\delta S = \int_0^T \left[\frac{\partial L}{\partial v} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{v}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{v}}\right)\right]\eta\mathrm{d}t + \frac{\partial L}{\partial \ddot{v}}\dot{\eta}\,\Big|_0^T . \tag{21}$$

The first variation of the functional must vanish for *all* allowed variations $\eta(t)$, no matter what the values of $\dot{\eta}(0)$ and $\dot{\eta}(T)$ may be. Therefore, it must also vanish for those variations such that $\dot{\eta}(0) = \dot{\eta}(T) = 0$, for which the boundary term on the right-hand side of (21) is zero. In this case, because η is otherwise arbitrary, the fundamental lemma of the calculus of variations [6, p 39] establishes that the condition $\delta S = 0$, which is necessary for a minimum, implies that the coefficient of η in the integral on the right-hand side of (21) vanishes. Consequently, the minimiser $v(t)$ must satisfy the generalised Euler–Lagrange equation

$$\frac{\partial L}{\partial v} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{v}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{v}}\right) = 0. \tag{22}$$

Insertion of this equation into (21) further reduces the first variation of the functional S to

$$\delta S = \frac{\partial L}{\partial \ddot{v}} \dot{\eta} \Big|_0^T . \tag{23}$$

Since the values of $\dot{\eta}(t)$ are arbitrary at $t = 0$ and $t = T$, we can choose the variation $\eta(t)$ such that $\dot{\eta}(0) \neq 0$, $\dot{\eta}(T) = 0$ and vice versa. Therefore, the condition $\delta S = 0$ requires that the following boundary conditions be obeyed:

$$\frac{\partial L}{\partial \ddot{v}} \Big|_{t=0} = 0, \quad \frac{\partial L}{\partial \ddot{v}} \Big|_{t=T} = 0. \tag{24}$$

These are *natural boundary conditions* [6, p 194] for the problem with variable endpoints as regards $\dot{v}(t)$.

The specification of the appropriate boundary conditions, in addition to the equations of motion, is an essential part of a physical problem stated in the form of a variational principle. It is worth noting at this juncture that the boundary conditions (24) are similar to the 'edge conditions' that arise in bosonic string theory [7].

3.2. The least uncomfortable journey

Now back to the problem of the least uncomfortable journey. Since (14) are the only *a priori* boundary conditions that the velocity is demanded to obey, the admissible variations are only required to satisfy (20), and the previous discussion applies. According to Euler's rule [6, section 5.3], in order to find the minimiser $v(t)$ for the functional (13) under the constraint (15), one must set up the generalised Euler–Lagrange equation (22) with the 'Lagrangian'

$$L = \ddot{v}^2 + \lambda v, \tag{25}$$

where the Lagrange multiplier λ is a constant. With this 'Lagrangian' the generalised Euler–Lagrange equation (22) becomes

$$2\frac{d^4 v}{dt^4} + \lambda = 0. \tag{26}$$

From (25) it follows that the boundary conditions (24) become simply

$$\ddot{v}(0) = 0, \quad \ddot{v}(T) = 0. \tag{27}$$

The general solution to equation (26) is immediately written as

$$v = c_0 + c_1 t + c_2 t^2 + c_3 t^3 - \frac{\lambda}{48} t^4, \tag{28}$$

where c_0, c_1, c_2, c_3 are constants. The boundary conditions $v(0) = 0$ and $\ddot{v}(0) = 0$ yield $c_0 = c_2 = 0$. The boundary conditions $v(T) = 0$ and $\ddot{v}(T) = 0$ lead to

$$c_1 T + c_3 T^3 - \frac{\lambda}{48} T^4 = 0, \quad 6c_3 T - \frac{\lambda}{4} T^2 = 0, \tag{29}$$

which are solved by

$$\lambda = 24\frac{c_3}{T}, \quad c_1 = -\frac{c_3 T^2}{2}. \tag{30}$$

It follows that

$$v = c_3 T^3 \left(-\frac{t}{2T} + \frac{t^3}{T^3} - \frac{t^4}{2T^4} \right). \tag{31}$$

Finally, the isoperimetric constraint (15) determines the remaining constant as $c_3 = -10D/T^4$. Thus, the least uncomfortable journey is achieved by the velocity as the following function of time:

$$v = \frac{5D}{T}\left(\frac{t}{T} - 2\frac{t^3}{T^3} + \frac{t^4}{T^4}\right). \tag{32}$$

This optimal velocity yields for the discomfort its minimum value:

$$J_{\min} = \int_0^T \ddot{v}^2 \mathrm{d}t = \frac{3600D^2}{T^6}\int_0^T \left(\frac{t}{T} - \frac{t^2}{T^2}\right)^2 \mathrm{d}t = 120\frac{D^2}{T^5}. \tag{33}$$

With the change of variable $\tau = t - T/2$, the right-hand side of (32) does not contain odd powers of τ. Therefore, the velocity that minimises the discomfort is a symmetric function about $t = T/2$ that grows steadily from zero at $t = 0$ to a maximum at $t = T/2$, then decreases monotonically from the maximum at $t = T/2$ to zero at $t = T$. The maximum value of the velocity is

$$v_{\max} = \frac{25D}{16T}, \tag{34}$$

which is about 56% larger than the average velocity $v_{\mathrm{ave}} = D/T$. This means that, when the discomfort is measured by the acceleration, the least uncomfortable journey is slightly smoother than when the discomfort is measured by the jerk, as illustrated in figure 1.

By the way, it should be noted that, by approximate methods based on velocity trial functions enforcing physically doubtful boundary conditions, ADN found[2]

$$v_{\max}^{\mathrm{ADN}} = 1.84\frac{D}{T}, \tag{35}$$

which is more than 17% above the exact value (34). The discrepancy in the predicted minimum value for the discomfort is much more pronounced, as we proceed to show. Starting from (28), the implementation of the isoperimetric constraint (15) and of the ADN boundary conditions, namely $v(0) = v(T) = 0$ as well as $\dot{v}(0) = \dot{v}(T) = 0$, leads to the supposedly optimal velocity

$$v^{\mathrm{ADN}} = \frac{30D}{T}\left(\frac{t^2}{T^2} - 2\frac{t^3}{T^3} + \frac{t^4}{T^4}\right). \tag{36}$$

The discomfort associated with this velocity is

$$J_{\min}^{\mathrm{ADN}} = \int_0^T (\ddot{v}^{\mathrm{ADN}})^2 \mathrm{d}t = \frac{3600D^2}{T^6}\int_0^T \left(1 - 6\frac{t}{T} + 6\frac{t^2}{T^2}\right)^2 \mathrm{d}t = 720\frac{D^2}{T^5}. \tag{37}$$

This is six times as large as the true minimum (33).

In figure 2 the optimal velocity given by equation (32) is compared with the velocity (36) derived from the inappropriate ADN boundary conditions. The ADN boundary conditions lead not only to a much larger discomfort but also to a maximum velocity which is 20% above the exact value.

[2] See the line right below equation (36) in [1], where $2D$ is taken for the distance between points A and B.

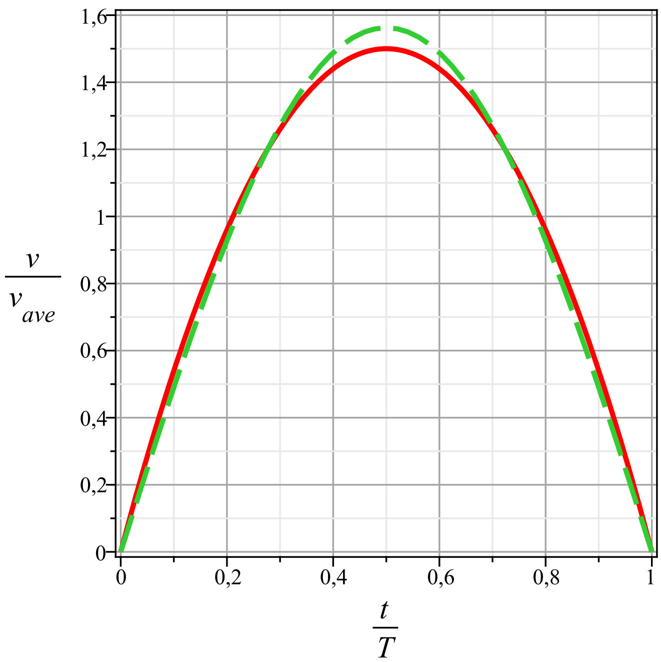

Figure 1. The solid line represents the velocity given by equation (9), whereas the dashed line stands for the velocity given by equation (32), with $v_{\mathrm{ave}} = D/T$.

Returning to the exact solution (32), the acceleration is given by

$$a = \dot{v} = \frac{5D}{T^2}\left(1 - 6\frac{t^2}{T^2} + 4\frac{t^3}{T^3}\right). \tag{38}$$

As physically expected, the acceleration takes a positive value ($a_0 = 5D/T^2$) at the beginning of the journey ($t = 0$) and decreases monotonically to a negative value ($a_T = -5D/T^2$) at the end of the journey ($t = T$); it goes through zero and changes sign at $t = T/2$, being an odd function with respect to $t = T/2$. Finally, the jerk is given by

$$j = \ddot{v} = -\frac{60D}{T^3}\left(\frac{t}{T} - \frac{t^2}{T^2}\right). \tag{39}$$

The jerk is a negative function, symmetric about $t = T/2$, that vanishes at the beginning and the end of the journey; its minimum value $j_{\mathrm{min}} = -15D/T^3$ is reached at $t = T/2$.

The position is given by

$$x(t) = \int_0^t v(t')\mathrm{d}t' = \frac{5D}{T}\int_0^t\left(\frac{t'}{T} - 2\frac{t'^3}{T^3} + \frac{t'^4}{T^4}\right)\mathrm{d}t', \tag{40}$$

whence

$$x = \frac{D}{2}\left(5\frac{t^2}{T^2} - 5\frac{t^4}{T^4} + 2\frac{t^5}{T^5}\right). \tag{41}$$

It is now clear why the variational problem for $v(x)$ is so difficult. Finding $v(x)$ requires the solution of the fifth-degree algebraic equation (41) for t in order to determine $t(x)$ for its

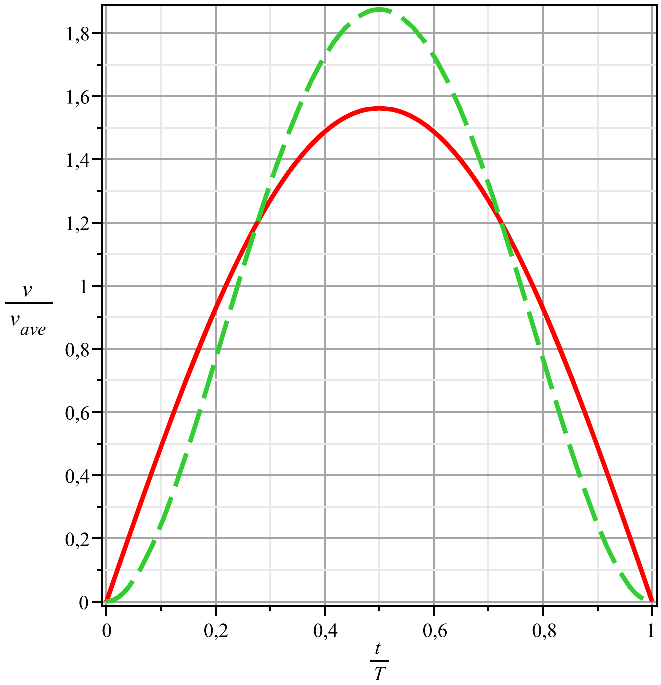

Figure 2. The solid line represents the true optimal velocity given by equation (32), whereas the dashed line stands for the velocity (36) derived from the inappropriate ADN boundary conditions, with $v_{\mathrm{ave}} = D/T$.

substitution into equation (32), which is very hard to do. Expressing the solution for $t(x)$ would require the use of the not-so-well-known Jacobi theta functions [8].

4. Final remarks

The problem of the least uncomfortable journey on a straight line makes it clear that the choice of independent variable may be capable of making a variational problem much more easily solvable. It also provides a nice example of a physical variational problem involving a functional that depends on derivatives of order higher than the first. As we have seen, in such cases the identification of the appropriate boundary conditions calls for a careful combination of physical and mathematical arguments.

The relativistic generalisation of the least uncomfortable journey with the discomfort measured by the acceleration cannot be solved in terms of elementary functions [2]. Although it might be of interest to formulate a relativistic generalisation of the same problem with the discomfort measured by the jerk, there is not much hope that an exact solution in terms of elementary functions can be found.

Acknowledgments

The author is thankful to the anonymous reviewers whose criticisms, corrections and suggestions have contributed to a significant improvement of the original manuscript.

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