Communications in Theoretical Physics
Commun. Theor. Phys. 72 (2020) 035004 (8pp)
https://doi.org/10.1088/1572-9494/ab690b

Blow-up criteria and periodic peakons for a two-component extension of the μ-version modified Camassa–Holm equation*

Zhigang Li (李志刚)[1] and Zhonglong Zhao (赵忠龙)[2,3]

[1] Department of Mathematics, China University of Mining and Technology, Beijing 100083, China
[2] Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, China

E-mail: lzgcumtb@163.com and zhaozlhit@163.com

Received 13 October 2019, revised 4 January 2020
Accepted for publication 6 January 2020
Published 25 February 2020

Abstract

Some two-component extensions of the modified μ-Camassa–Holm equation are proposed. We show that these systems admit Lax pairs and bi-Hamiltonian structures. Furthermore, we consider the blow-up phenomena for one of these extensions (2μmCH), and the periodic peakons of this system are derived.

Keywords: modified μ-Camassa–Holm equation, bi-Hamiltonian structures, blow-up, peakons

1. Introduction

Over the last two decades, more and more scholars have focused on the Camassa–Holm equation (CH) [1],

$$m_t + 2u_x m + u m_x = 0, \quad m = u - u_{xx}.$$

This famous equation describes a unidirectional propagation of shallow water waves, which was derived by R. Camassa and D. Holm in 1993 by means of an asymptotic expansion of Euler's equations. However, it first appeared from the investigation of a recursion operator to a bi-Hamiltonian equation with infinitely many conservation laws in the article of Fokas and Fuchssteiner in 1986 [2]. One of the remarkable properties of the CH equation is that it possesses peaked solutions, i.e. peakons. A peakon is a continuously weak form traveling wave solution, whose profile remains bounded. Meanwhile, it has a sharp angle at the central elevation, and the slope at the critical point changes rapidly. The other property of the CH equation is wave-breaking phenomena [3–5], which means the solution remains bounded, and the slope of the solution becomes unbounded in finite time.

Another famous (1+1) dimension CH-type equation is the Fokas–Olver–Resenau–Qiao equation (FORQ), or modified CH equation (mCH) [6–8]

$$m_t + (u^2 - u_x^2)m)_x = 0, \quad m = u - u_{xx},$$

which was derived by the authors, respectively. It should be noted that the nonlinear term in the mCH equation is cubic; this is different to the CH equation. Similarly, the mCH equation also admits peakons and wave-breaking phenomena [9]. A natural thought is to generalize such an integrable equation into multi-component sense. Some two-component extensions of the mCH equation have been derived. Song, Qu and Qiao proposed a two-component modified CH system (SQQ) [10]

$$\begin{aligned} m_t + [m(u + u_x)(v - v_x)]_x = 0, \\ n_t + [n(u + u_x)(v - v_x)]_x = 0, \end{aligned}$$

through a 2×2 spectral problem, and the x-part is

$$\phi_x = \begin{pmatrix} \frac{1}{2} & \lambda m \\ \lambda n & -\frac{1}{2} \end{pmatrix} \phi.$$

Li and Li considered other two-component cases by extending the spectral problem into a 4×4 matrix [11]. At the same time, they obtained three two-component generalizations of the

* Supported by the National Nature Science Foundation of China (Grant Nos. 11871471, 11931017), Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi (Grant No. 2019L0531), Shanxi Province Science Foundation for Youths (Grant No. 201901D211274), and the Fund for Shanxi '1331KIRT'.
[3] Author to whom any correspondence should be addressed.

mCH equation by taking reduction. The bi-Hamiltonian structures and reciprocal transformations of these systems have also been considered by the authors.

The μ-version of the mCH equation was constructed by Qu, Fu and Liu [12]

$$m_t + [(2\mu(u)u - u_x^2)m]_x + \gamma u_x = 0, \quad m = \mu(u) - u_{xx},$$

where γ is the coefficient of the lower-order term. This equation can be regarded as a Euclidean-invariant version of the μCH equation.

All the equations mentioned above are integrable, which means they all admit Lax representations and bi-Hamiltonian structures. To the best of our knowledge, there are less multi-component cases of the μmCH equation. Thus, the aim of this paper is to propose some two-component generalizations of the μmCH equation. The outline of this paper is as follows: in section 2, we consider an extended spectral problem and three two-component reductions. It is proved that both reductions admit bi-Hamiltonian structures in section 3. As an example, we derive two blow-up criteria for the 2μmCH system in section 4, and the periodic peakons are derived in section 5. Finally, some conclusions are given in the last section.

2. Extension of Lax pair

In this section, we derive some two-component generalizations of the μmCH equation. Recall the Lax pair

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = P(m, \lambda)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_t = Q(m, u, \lambda, \gamma)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{1}$$

where

$$P(m, \lambda) = \begin{pmatrix} -\sqrt{-\frac{\gamma}{2}}\lambda & \lambda m \\ -\lambda m & \sqrt{-\frac{\gamma}{2}}\lambda \end{pmatrix}, \tag{2}$$

$$Q(m, u, \lambda, \gamma) = \begin{pmatrix} \sqrt{-\frac{\gamma}{2}}\left(\frac{1}{2}\lambda^{-1} + \lambda B(u)\right) & -L(\lambda, u) + \sqrt{-\frac{\gamma}{2}}u_x \\ L(\lambda, u) + \sqrt{-\frac{\gamma}{2}}u_x & -\sqrt{-\frac{\gamma}{2}}\left(\frac{1}{2}\lambda^{-1} + \lambda B(u)\right) \end{pmatrix}, \tag{3}$$

in which $B(u) = 2\mu(u)u - u_x^2$ and $L(\lambda, u) = \frac{1}{2}\lambda^{-1}\mu(u) + \lambda B(u)m$. It is easy to check the μmCH equation arises as a zero-curvature equation $P_t - Q_x + [P, Q] = 0$, which is the compatibility condition of a linear system (1). Now we extend the spectral problem of the μmCH to a 4×4 matrix sense,

$$P(M, \lambda) = \begin{pmatrix} -\sqrt{-\frac{\gamma}{2}}\lambda I & \lambda M \\ -\lambda M & \sqrt{-\frac{\gamma}{2}}\lambda I \end{pmatrix}, \tag{4}$$

$$Q(M, U, \lambda, \gamma) = \begin{pmatrix} \sqrt{-\frac{\gamma}{2}}\left(\frac{1}{2}\lambda^{-1}I + \lambda B(U)\right) & -L(\lambda, U) + \sqrt{-\frac{\gamma}{2}}U_x \\ L(\lambda, U) + \sqrt{-\frac{\gamma}{2}}U_x & -\sqrt{-\frac{\gamma}{2}}\left(\frac{1}{2}\lambda^{-1}I + \lambda B(U)\right) \end{pmatrix}, \tag{5}$$

where ϕ_1 and ϕ_2 are two-dimension column vectors, M, U, $\mu(U)$ and I are 2×2 matrices and I is an identity matrix. Taking P and Q into the zero-curvature equation, we have

$$\begin{aligned} &MU_x + U_xM = B(U)_x,\\ &M\mu(U) = \mu(U)M, \quad MB(U) = B(U)M,\\ &M_t + (B(U)M)_x + \gamma U_x = 0,\\ &M = \mu(U) - U_{xx} = AU. \end{aligned} \tag{6}$$

To obtain two-component generalizations of the μmCH equation, we must restrict the choice of matrix U. It is sufficient to choose $U = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ by taking

$$a = d, \quad bc_x = b_xc.$$

Under this assumption, all the matrices M, U and $\mu(U)$ are commutable. Here are three possible choices for U, which can be written as

$$U = \begin{pmatrix} u & v \\ v & u\end{pmatrix}, \quad U = \begin{pmatrix} u & v \\ -v & u\end{pmatrix}, \quad U = \begin{pmatrix} u & v \\ 0 & u\end{pmatrix}, \tag{7}$$

and the corresponding expressions of M and $\mu(U)$ are

$$M = \begin{pmatrix} m & n \\ n & m\end{pmatrix}, \quad M = \begin{pmatrix} m & n \\ -n & m\end{pmatrix}, \quad M = \begin{pmatrix} m & n \\ 0 & m\end{pmatrix}, \tag{8}$$

and

$$\mu(U) = \begin{pmatrix} \mu(u) & \mu(v) \\ \mu(v) & \mu(u)\end{pmatrix}, \quad \mu(U) = \begin{pmatrix} \mu(u) & \mu(v) \\ -\mu(v) & \mu(u)\end{pmatrix},$$

$$\mu(U) = \begin{pmatrix} \mu(u) & \mu(v) \\ 0 & \mu(u)\end{pmatrix}. \tag{9}$$

Under the assumptions of (7)–(9), we finally obtain three two-component generalizations of the μmCH equation (for simplicity, we only consider $\gamma = 0$)

$$\begin{aligned} &m_t + [(2\mu(u)u + 2\mu(v)v - (u_x^2 + v_x^2))m\\ &\quad + 2(\mu(u)v + \mu(v)u - u_xv_x)n]_x = 0,\\ &n_t + [(2\mu(u)u + 2\mu(v)v - (u_x^2 + v_x^2))n\\ &\quad + 2(\mu(u)v + \mu(v)u - u_xv_x)m]_x = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} &m_t + [(2\mu(u)u - 2\mu(v)v - (u_x^2 - v_x^2))m\\ &\quad - 2(\mu(u)v + \mu(v)u - u_xv_x)n]_x = 0,\\ &n_t + [(2\mu(u)u - 2\mu(v)v - (u_x^2 - v_x^2))n\\ &\quad + 2(\mu(u)v + \mu(v)u - u_xv_x)m]_x = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &m_t + [(2\mu(u)u - u_x^2)m]_x = 0,\\ &n_t + [(2\mu(u)u - u_x^2)n\\ &\quad + 2(\mu(u)v + \mu(v)u - u_xv_x)m]_x = 0. \end{aligned} \tag{12}$$

3. Bi-Hamiltonian structures of reductions

In this section, we show that systems (10), (11) and (12) admit bi-Hamiltonian structures. This property is very important for integrable systems. It is already known that the

μmCH equation admits a bi-Hamiltonian structure

$$m_t = K\frac{\delta H_2}{\delta m} = J\frac{\delta H_1}{\delta m},$$

in which the Hamiltonian functionals are

$$\begin{aligned} H_1 &= -\int_{\mathbb{S}} mu\mathrm{d}x,\\ H_2 &= -\int_{\mathbb{S}} \left(\mu^2(u)u^2 + \mu(u)uu_x^2 - \frac{1}{12}u_x^4\right)\mathrm{d}x, \end{aligned} \tag{13}$$

and the associated Hamiltonian operators are

$$J = \partial m\partial^{-1}m\partial, \quad K = \partial A = -\partial^3, \tag{14}$$

where $\partial = \partial_x$. To derive the Hamiltonian structure of (10) and (11), inspired by Li [11], we denote $\bar{u} = u + v, \bar{v} = u - v$ and $\bar{u} = u + \mathrm{i}v, \bar{v} = u - \mathrm{i}v$, then (10) and (11) can be rewritten as two decoupled μmCH equations

$$\begin{aligned} &\bar{m}_t + [(2\mu(\bar{u})\bar{u} - \bar{u}_x^2)\bar{m}]_x = 0,\\ &\bar{n}_t + [(2\mu(\bar{v})\bar{v} - \bar{v}_x^2)\bar{n}]_x = 0. \end{aligned} \tag{15}$$

Thus, the Hamiltonian functionals are

$$\begin{aligned} \bar{H}_1 &= \frac{1}{2}\int_{\mathbb{S}} (\bar{m}\bar{u} + \bar{n}\bar{v})\mathrm{d}x,\\ \bar{H}_2 &= \int_{\mathbb{S}} \Big(4\mu^2(\bar{u})\bar{u}^2 + 4\mu(\bar{u})\bar{u}\bar{u}_x^2 - \frac{1}{3}\bar{u}_x^4)\\ &\quad - (4\mu^2(\bar{v})\bar{v}^2 + 4\mu(\bar{v})\bar{v}\bar{v}_x^2 - \frac{1}{3}\bar{v}_x^4\Big)\mathrm{d}x. \end{aligned} \tag{16}$$

With a direct calculation, we find (10) can be reformulated as a bi-Hamiltonian system

$$\begin{pmatrix} m_t\\ n_t \end{pmatrix} = K\begin{pmatrix} \delta H_2/\delta m\\ \delta H_2/\delta n \end{pmatrix} = J\begin{pmatrix} \delta H_1/\delta m\\ \delta H_1/\delta n \end{pmatrix}, \tag{17}$$

by using Hamiltonian functionals

$$\begin{aligned} H_1 &= -\int_{\mathbb{S}} (mu + nv)\mathrm{d}x,\\ H_2 &= -\int_{\mathbb{S}} \Big(2[\mu(u)u + \mu(v)v][\mu(u)v + \mu(v)u + u_x v_x]\\ &\quad + [\mu(u)v + \mu(v)u - \frac{1}{3}u_x v_x](u_x^2 + v_x^2)\Big)\mathrm{d}x, \end{aligned} \tag{18}$$

and associated Hamiltonian operators

$$\begin{aligned} J &= \begin{pmatrix} \partial m\partial^{-1}m\partial + \partial n\partial^{-1}n\partial & \partial m\partial^{-1}n\partial + \partial n\partial^{-1}m\partial\\ \partial m\partial^{-1}n\partial + \partial n\partial^{-1}m\partial & \partial m\partial^{-1}m\partial + \partial n\partial^{-1}n\partial \end{pmatrix},\\ K &= \begin{pmatrix} 0 & \partial A\\ \partial A & 0 \end{pmatrix}. \end{aligned} \tag{19}$$

Similarly, (11) can also be reformulated as a bi-Hamiltonian system. The Hamiltonian functionals are

$$\begin{aligned} H_1 &= -\int_{\mathbb{S}} (mu - nv)\mathrm{d}x,\\ H_2 &= -\int_{\mathbb{S}} \Big(2[\mu(u)u - \mu(v)v][\mu(u)v + \mu(v)u + u_x v_x]\\ &\quad + [\mu(u)v + \mu(v)u - \frac{1}{3}u_x v_x](u_x^2 - v_x^2)\Big)\mathrm{d}x. \end{aligned} \tag{20}$$

The Hamiltonian operator K is defined in (19), and J is

$$J = \begin{pmatrix} \partial m\partial^{-1}m\partial - \partial n\partial^{-1}n\partial & \partial m\partial^{-1}n\partial + \partial n\partial^{-1}m\partial\\ \partial m\partial^{-1}n\partial + \partial n\partial^{-1}m\partial & -\partial m\partial^{-1}m\partial + \partial n\partial^{-1}n\partial \end{pmatrix}. \tag{21}$$

It is not difficult to check that (12) admits the following two conserved quantities

$$\begin{aligned} H_1 &= -\int_{\mathbb{S}} (mv + nu)\mathrm{d}x,\\ H_2 &= -\int_{\mathbb{S}} \Big(2\mu(u)u[\mu(u)v + \mu(v)u + u_x v_x]\\ &\quad + [\mu(u)v + \mu(v)u - \frac{1}{3}u_x v_x]u_x^2\Big)\mathrm{d}x. \end{aligned} \tag{22}$$

Thus, we can also rewrite (12) into the bi-Hamiltonian form, the associated Hamiltonian operator K is the same as (10) and (11), and J is

$$J = \begin{pmatrix} 0 & \partial m\partial^{-1}m\partial\\ \partial m\partial^{-1}m\partial & \partial m\partial^{-1}n\partial + \partial n\partial^{-1}m\partial \end{pmatrix}. \tag{23}$$

4. Blow-up criteria

In this section, we provide a criterion for the blow up of solutions to the Cauchy problem of system (10) (2μmCH)

$$\begin{aligned} &m_t + [(2\mu(u)u + 2\mu(v)v - (u_x^2 + v_x^2)m\\ &\quad + 2(\mu(u)v + \mu(v)u - u_x v_x)n]_x = 0,\\ &n_t + [(2\mu(u)u + 2\mu(v)v - (u_x^2 + v_x^2)n\\ &\quad + 2(\mu(u)v + \mu(v)u - u_x v_x)m]_x = 0,\\ &u(0, x) = u_0(x), \quad u(t, x + 1) = u(t, x),\\ &\quad m = \mu(u) - u_{xx},\\ &v(0, x) = u_0(x), \quad v(t, x + 1) = u(t, x),\\ &\quad n = \mu(v) - v_{xx}. \end{aligned} \tag{24}$$

Firstly, we offer the local well-posedness of (24) in Sobolev space. The process of the proof is similar to the paper [8]. Therefore, we only show the result here.

Theorem 1. *Let* $(u_0, v_0) \in (H^s(\mathbb{S}))^2$ *with* $s > \frac{5}{2}$. *Then a time* $T > 0$ *exists such that the initial-value problem (24) has a unique strong solution* $(u, v) \in (C([0, T]; H^s)\bigcap C^1([0, T]; H^{s-1})^2$, *and the map $(u_0, v_0) \to (u, v)$ is continuous from a neighborhood of (u_0, v_0) in $(H^s)^2$ into $(C([0, T]; H^s) \bigcap C^1([0, T]; H^{s-1})^2$.*

Now we introduce the following transport estimation lemma, which is crucial to the proof of the blow-up criterion for (24).

Lemma 1 (10). *Consider the one-dimensional linear transport equation*

$$\partial_t f + v\partial_x f = g, \quad f|_{t=0} = f_0. \tag{25}$$

Let $0 \leqslant \sigma < 1$, and suppose that

$$f_0 \in H^\sigma, \quad g \in L^1([0, T]; H^\sigma),$$
$$v_x \in L^1([0, T]; L^\infty), \quad f \in L^\infty([0, T]; H^\sigma) \cap C([0, T]; \mathcal{S}').$$

Then $f \in C([0, T]; H^\sigma)$. More precisely, there exists a constant C depending only on σ such that, for every $0 < t \leqslant T$,

$$\|f(t)\|_{H^\sigma} \leqslant \|f_0\|_{H^\sigma} + C\int_0^t (\|g(\tau)\|_{H^\sigma} + \|f(\tau)\|_{H^\sigma} V'(\tau)\mathrm{d}\tau,$$

and hence

$$\|f(t)\|_{H^\sigma} \leqslant \mathrm{e}^{CV(t)}\left(\|f_0\|_{H^\sigma} + C\int_0^t \|g(\tau)\|_{H^\sigma}\mathrm{d}\tau\right)$$
$$\text{with} \quad V(t) = \int_0^t \|\partial_x v(\tau)\|_{L^\infty}\mathrm{d}\tau.$$

The blow-up criterion for (24) is as follows.

Theorem 2. *Let $(u_0, v_0) \in (H^s)^2$ be as in theorem 1 with $s > \frac{5}{2}$. Let (u,v) be the corresponding solution to (24). Assume that $T^* > 0$ is the maximum time of existence. If T^* is finite, then we have*

$$\int_0^{T^*} (\|m(\tau)\|_{L^\infty} + \|n(\tau)\|_{L^\infty})^2 \mathrm{d}\tau = \infty. \tag{26}$$

Proof. In order to apply lemma 1 to prove theorem 1, we consider the equivalent form of (24) as we have just mentioned in (15)

$$\begin{aligned} &\bar{m}_t + [(2\mu(\bar{u})\bar{u} - \bar{u}_x^2)\bar{m}]_x = 0,\\ &\bar{n}_t + [(2\mu(\bar{v})\bar{v} - \bar{v}_x^2)\bar{n}]_x = 0,\\ &\bar{m}(0, x) = \bar{m}_0, \quad \bar{n}(0, x) = \bar{n}_0, \end{aligned} \tag{27}$$

where $\bar{m} = m + n$ and $\bar{n} = m - n$. Now we break the proof into three steps.

Step 1. For $s \in \left(\frac{1}{2}, 1\right)$, applying lemma 1 to (15), we have

$$\begin{aligned} \|\bar{m}\|_{H^s} &\leqslant \|\bar{m}_0\|_{H^s} + C\int_0^t (\|(2\mu(\bar{u})\bar{u} - \bar{u}_x^2)_x(\tau)\\ &\quad \times \|_{L^\infty}\|\bar{m}(\tau)\|_{H^s} + \|(\bar{u}_x\bar{m}^2(\tau)\|_{H^s})\mathrm{d}\tau,\\ \|\bar{n}\|_{H^s} &\leqslant \|\bar{n}_0\|_{H^s} + C\int_0^t (\|(2\mu(\bar{v})\bar{v} - \bar{v}_x^2)_x(\tau)\\ &\quad \times \|_{L^\infty}\|\bar{n}(\tau)\|_{H^s} + \|(\bar{v}_x\bar{n}^2(\tau)\|_{H^s})\mathrm{d}\tau, \end{aligned} \tag{28}$$

for all $0 < t < T^*$. Recall the 1-D Moser-type estimates [13], for $s \geqslant 0$, the following estimates hold:

$$\|fg\|_{H^s} \leqslant C(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}),$$
$$\|f\partial_x g\|_{H^s} \leqslant C(\|f\|_{H^{s+1}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\partial_x g\|_{H^s}),$$

where the constant C is independent of f and g. Owing to the first Moser-type estimate, we have

$$\begin{aligned} \|\bar{u}_x\bar{m}^2\|_{H^s} &\leqslant C(\|\bar{u}_x\|_{H^s}\|\bar{m}^2\|_{L^\infty} + \|\bar{u}_x\|_{L^\infty}\|\bar{m}^2\|_{H^s})\\ &\leqslant C(\|\bar{u}_x\|_{H^s}\|\bar{m}\|_{L^\infty}^2 + \|\bar{u}_x\|_{L^\infty}\|\bar{m}\|_{L^\infty}\|\bar{m}\|_{H^s}). \end{aligned} \tag{29}$$

As A is invertible, the inverse $A^{-1}m = u$ is given explicitly by

$$\begin{aligned} u(x) = &\left[\frac{1}{2}\left(x - \frac{1}{2}\right)^2 + \frac{23}{24}\right]\mu(m)\\ &+ \left(x - \frac{1}{2}\right)\int_{\mathbb{S}}\int_0^y m(s)\mathrm{d}s\mathrm{d}y\\ &- \int_0^x\int_0^y m(s)\mathrm{d}s\mathrm{d}y + \int_{\mathbb{S}}\int_0^y\int_0^s m(r)\mathrm{d}r\mathrm{d}s\mathrm{d}y, \end{aligned}$$

and the Green function of the operator A^{-1}, $u = A^{-1}m = G^*m$ is given by

$$G(x) = \frac{1}{2}\left(x - [x] - \frac{1}{2}\right)^2 + \frac{23}{24}.$$

With an application of Young's inequality, we obtain

$$\|\bar{u}_x\|_{L^\infty} \leqslant \|G_x\|_{L^1}\|\bar{m}\|_{L^\infty} \leqslant \|\bar{m}\|_{L^\infty}. \tag{30}$$

Combining the fact $\|\bar{u}_x\|_{H^s} \leqslant C\|\bar{m}\|_{H^s}$ with (29) gives rise to

$$\|\bar{u}_x\bar{m}^2\|_{H^s} \leqslant C\|\bar{m}\|_{H^s}\|\bar{m}\|_{L^\infty}^2, \tag{31}$$

and

$$\begin{aligned} &\|(2\mu(\bar{u})\bar{u} - \bar{u}_x^2)_x\|_{L^\infty}\\ &= 2\|\bar{m}\bar{u}_x\|_{L^\infty} \leqslant C\|\bar{m}\|_{L^\infty}\|\bar{u}_x\|_{L^\infty} \leqslant C\|\bar{m}\|_{L^\infty}^2. \end{aligned} \tag{32}$$

Plugging (31) and (32) into the first equation of (28), we have

$$\|\bar{m}(t)\|_{H^s} \leqslant \|\bar{m}_0\|_{H^s} + C\int_0^t \|\bar{m}(\tau)\|_{L^\infty}^2\|\bar{m}(\tau)\|_{H^s}\mathrm{d}\tau, \tag{33}$$

and taking advantage of Gronwall's inequality yields

$$\|\bar{m}(t)\|_{H^s} \leqslant \|\bar{m}_0\|_{H^s}\mathrm{e}^{C\int_0^t \|\bar{m}(\tau)\|_{L^\infty}^2\mathrm{d}\tau}. \tag{34}$$

With a similar computation, we have

$$\|\bar{n}(t)\|_{H^s} \leqslant \|\bar{n}_0\|_{H^s}\mathrm{e}^{C\int_0^t \|\bar{n}(\tau)\|_{L^\infty}^2\mathrm{d}\tau}. \tag{35}$$

Squaring both sides of (34) and (35), and by the definition of the Sobolev norm, we have

$$2(\|m(t)\|_{H^s}^2+\|n(t)\|_{H^s}^2)\leqslant 2(\|m_0\|_{H^s}^2+\|n_0\|_{H^s}^2)\mathrm{e}^{2C\int_0^t(\|m(\tau)\|_{L^\infty}+\|n(\tau)\|_{L^\infty})^2\mathrm{d}\tau},$$

which means

$$\|m(t)\|_{H^s}+\|n(t)\|_{H^s}\leqslant\sqrt{2(\|m_0\|_{H^s}^2+\|n_0\|_{H^s}^2)}\,\mathrm{e}^{C\int_0^t(\|m(\tau)\|_{L^\infty}+\|n(\tau)\|_{L^\infty})^2\mathrm{d}\tau}. \tag{36}$$

Thus, if the maximal existence time $T^*<\infty$ satisfies

$$\int_0^{T^*}(\|m(\tau)\|_{L^\infty}+\|n(\tau)\|_{L^\infty})^2\mathrm{d}\tau<\infty,$$

throw inequality (36), we have

$$\limsup_{t\to T^*}(\|m(t)\|_{H^s}+\|n(t)\|_{H^s})<\infty, \tag{37}$$

which contradicts the assumption that the maximal existence time T^* is finite.

Step 2. For $s\in[1,2)$, differentiating (27) with respect to x, we have

$$\begin{aligned}\bar m_{xt}+(2\mu(\bar u)\bar u-\bar u_x^2)\bar m_{xx}&=-2\bar u_{xx}\bar m^2-3\bar u_x(\bar m^2)_x,\\ \bar n_{xt}+(2\mu(\bar v)\bar v-\bar v_x^2)\bar n_{xx}&=-2\bar v_{xx}\bar n^2-3\bar v_x(\bar n^2)_x.\end{aligned} \tag{38}$$

Applying lemma 1 to the first equation of (38) yields

$$\begin{aligned}\|\bar m_x\|_{H^{s-1}}\leqslant&\|\bar m_{0x}\|_{H^{s-1}}\\&+C\int_0^t(\|(2\mu(\bar u)\bar u-\bar u_x^2)_x\|_{L^\infty}\|\bar m_x\|_{H^{s-1}}\\&+\|\bar u_x(\bar m^2)_x\|_{H^{s-1}}+\|\bar u_{xx}\bar m^2\|_{H^{s-1}})\mathrm{d}\tau.\end{aligned} \tag{39}$$

With a similar computation to (31), by the Moser-type estimates and the fact

$$\begin{aligned}\|\bar u_{xx}\|_{L^\infty}&=\|\mu(\bar m)-\bar m\|_{L^\infty}\\&\leqslant C(\|\bar m\|_{L^\infty}+\|G^*\bar m\|_{L^\infty})\leqslant C\|\bar m\|_{L^\infty},\end{aligned}$$

we have

$$\begin{aligned}\|\bar u_x(\bar m^2)_x\|_{H^{s-1}}&\leqslant C\|\bar m\|_{L^\infty}^2\|\bar m\|_{H^s},\\ \|\bar u_{xx}\bar m^2\|_{H^{s-1}}&\leqslant C\|\bar m\|_{L^\infty}^2\|\bar m\|_{H^{s-1}}.\end{aligned} \tag{40}$$

Plugging (32) and (40) into (39), we see (33) still holds for $1\leqslant s<2$. Following the same argument in step 1 from (34) to (37), we see that theorem 1 still holds for $1\leqslant s<2$.

Step 3. For $s\in[2,\infty)$, assume $2\leqslant k\in\mathbb{N}$ is a positive integer. We prove that by induction, if (26) holds when $k-1\leqslant s<k$, then it still holds for $k\leqslant s<k+1$. Differentiate the first equation of (27) k times with respect to x, we have

$$\begin{aligned}&(\partial_x^k\bar m)_t+(2\mu(\bar u)\bar u-\bar u_x^2)(\partial_x^k\bar m)_x\\&\quad=-\textstyle\sum_{j=0}^{k-1}C_k^j\partial_x^{k-j}(2\mu(\bar u)\bar u-\bar u_x^2)\partial_x^{j+1}\bar m-2\partial_x^k(\bar u_x\bar m^2),\\&(\partial_x^k\bar n)_t+(2\mu(\bar v)\bar v-\bar v_x^2)(\partial_x^k\bar n)_x\\&\quad=-\textstyle\sum_{j=0}^{k-1}C_k^j\partial_x^{k-j}(2\mu(\bar v)\bar v-\bar v_x^2)\partial_x^{j+1}\bar n-2\partial_x^k(\bar v_x\bar n^2).\end{aligned} \tag{41}$$

Taking advantage of lemma 1, consider the first equation of (41), we have

$$\begin{aligned}\|\partial_x^k\bar m(t)\|_{H^{s-k}}\leqslant&\|\partial_x^k\bar m_0\|_{H^{s-k}}\\&+C\int(\|\partial_x^k\bar m(\tau)\|_{H^{s-k}}\|(\bar m\bar u_x)(\tau)\|_{L^\infty}\\&+\Big\|\big(\textstyle\sum_{j=0}^{k-1}C_k^j\partial_x^{k-j}(2\mu(\bar u)\bar u-\bar u_x^2)\partial_x^{j+1}\bar m\\&+2\partial_x^k(\bar u_x\bar m^2)\big)(\tau)\Big\|_{H^{s-k}})\mathrm{d}\tau.\end{aligned} \tag{42}$$

Using Moser-type estimates and the Sobolev embedding inequality, it is easy to check that

$$\begin{aligned}&\|\partial_x^k(\bar u_x\bar m^2)\|_{H^{s-k}}\leqslant C\|\bar m\|_{L^\infty}^2\|\bar m\|_{H^s},\\&\Big\|\textstyle\sum_{j=0}^{k-1}C_k^j\partial_x^{k-j}(2\mu(\bar u)\bar u-\bar u_x^2)\partial_x^{j+1}\bar m\Big\|_{H^{s-k}}\\&\quad\leqslant C\|\bar m\|_{H^s}\|\bar m\|_{H^{k-\frac12+\epsilon}}^2,\end{aligned} \tag{43}$$

where ϵ is a sufficiently small positive number such that $H^{\frac12+\epsilon}\hookrightarrow L^\infty$ holds. Taking (43) into (42) and by Sobolev embedding $H^{k-\frac12+\epsilon}\hookrightarrow L^\infty$ with $k\geqslant 2$, we have

$$\|\bar m(t)\|_{H^s}\leqslant\|\bar m_0\|_{H^s}+C\int_0^t\|\bar m(\tau)\|_{H^{k-\frac12+\epsilon}}^2\|\bar m(\tau)\|_{H^s}\mathrm{d}\tau, \tag{44}$$

and applying Gronwall's inequality, we obtain

$$\|\bar m(t)\|_{H^s}\leqslant\|\bar m_0\|_{H^s}\mathrm{e}^{C\int_0^t\|\bar m(\tau)\|_{H^{k-\frac12+\epsilon}}^2\mathrm{d}\tau}. \tag{45}$$

Similarly, we have the estimation for $\bar n$

$$\|\bar n(t)\|_{H^s}\leqslant\|\bar n_0\|_{H^s}\mathrm{e}^{C\int_0^t\|\bar n(\tau)\|_{H^{k-\frac12+\epsilon}}^2\mathrm{d}\tau}. \tag{46}$$

By the definition of the Sobolev norm, and squaring both sides of (45) and (46), finally we get

$$\begin{aligned}&\|m(t)\|_{H^s}+\|n(t)\|_{H^s}\\&\leqslant\sqrt{2(\|m_0\|_{H^s}^2+\|n_0\|_{H^s}^2)}\,\mathrm{e}^{C\int_0^t(\|m(\tau)\|_{H^{k-\frac12+\epsilon}}+\|n(\tau)\|_{H^{k-\frac12+\epsilon}})^2\mathrm{d}\tau}.\end{aligned} \tag{47}$$

If the maximal existence time T^* is finite, and

$$\int_0^{T^*}(\|m(\tau)\|_{L^\infty}+\|n(\tau)\|_{L^\infty})^2\mathrm{d}\tau<\infty,$$

we find that $\|m(\tau)\|_{H^{k-\frac12+\epsilon}}+\|n(\tau)\|_{H^{k-\frac12+\epsilon}}$ is uniformly bounded in $(0,T^*)$ by the induction assumption. Throw (47), and we get that (37) still holds for $s\geqslant 2$, which contradicts the assumption that the maximal existence time T^* is finite. Thus we complete the proof of theorem 2.

Remark 1. In theorem 2, we have already proved that if $m(t)$ or $n(t)$ blow up in finite time, we have

$$\int_0^{T^*}(\|m(\tau)\|_{L^\infty}+\|n(\tau)\|_{L^\infty})^2\mathrm{d}\tau=\infty,$$

which means

$$\limsup_{t\to T^*}\|m(t)\|_{L^\infty}=\infty,\quad \text{or} \limsup_{t\to T^*}\|n(t)\|_{L^\infty}=\infty. \tag{48}$$

On the other hand, if (48) holds, then by the Sobolev embedding theorem $\|f\|_\infty \leqslant C\|f\|_{H^s}$, we have

$$\limsup_{t\to T^*}\|m(t)\|_{H^s}=\infty,\quad or \quad \limsup_{t\to T^*}\|n(t)\|_{H^s}=\infty.$$

Thus we have the following corollary.

Corollary 1. *Let $(u_0, v_0)\in(H^s)^2$ be as in theorem 1 with $s>\frac{5}{2}$, and (u, v) be the corresponding solution to (24). Then (u, v) blow up in finite time if and only if (48) holds.*

The following theorem shows that the wave-breaking only depends on the infimum of mu_x, mv_x, nu_x and nv_x.

Theorem 3. *Let $(u_0, v_0)\in(H^s)^2$ be as in theorem 1 with $s>\frac{5}{2}$. Then the solution (u,v) blow ups in finite time $T^*>0$ if and only if*

$$\lim\inf_{t\to T^*}[\inf_{x\in\mathbb{S}}(ab(t, x)]=-\infty, \tag{49}$$

where $a\in\{m, n\}$ and $b\in\{u_x, v_x\}$.

Proof. Arguing by density, it suffices to prove the case $s=3$. Multiplying the first equation of (24) by $2m$ and the second equation by $2n$, integrating with respect to x over $\mathbb{S}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}}m^2\mathrm{d}x=&-\int_{\mathbb{S}}2m[(2\mu(u)u+2\mu(v)v\\&-(u_x^2+v_x^2)m\\&+2(\mu(u)v+\mu(v)u-u_xv_x)n]_x\mathrm{d}x,\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}}n^2\mathrm{d}x=&-\int_{\mathbb{S}}2n[(2\mu(u)u+2\mu(v)v\\&-(u_x^2+v_x^2)n\\&+2(\mu(u)v+\mu(v)u-u_xv_x)m]_x\mathrm{d}x.\end{aligned} \tag{50}$$

Plugging the two equations of (50) and integration by parts, yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}}(m^2+n^2)\mathrm{d}x\\&=-\int_{\mathbb{S}}(m^2+n^2)(2mu_x+2nv_x)-2mn(2mv_x+2nu_x)\mathrm{d}x.\end{aligned} \tag{51}$$

Differentiating (24) with respect to x, we have

$$\begin{aligned}&m_{xt}+[(2\mu(u)u+2\mu(v)v-(u_x^2+v_x^2)m\\&\quad+2(\mu(u)v+\mu(v)u-u_xv_x)n]_{xx}=0,\\&n_{xt}+[(2\mu(u)u+2\mu(v)v-(u_x^2+v_x^2)n\\&\quad+2(\mu(u)v+\mu(v)u-u_xv_x)m]_{xx}=0.\end{aligned} \tag{52}$$

Multiplying the first equation of (52) by $2m_x$ and the second equation by $2n_x$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}}m_x^2\mathrm{d}x=&-\int_{\mathbb{S}}2m_x[(2\mu(u)u+2\mu(v)v-(u_x^2+v_x^2)m\\&+2(\mu(u)v+\mu(v)u-u_xv_x)n]_{xx}\mathrm{d}x,\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}}n_x^2\mathrm{d}x=&-\int_{\mathbb{S}}2n_x[(2\mu(u)u+2\mu(v)v-(u_x^2+v_x^2)n\\&+2(\mu(u)v+\mu(v)u-u_xv_x)m]_{xx}\mathrm{d}x.\end{aligned} \tag{53}$$

Plugging the two equations of (53) and integration by parts, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}}(m_x^2+n_x^2)\mathrm{d}x\\&=-\int_{\mathbb{S}}(8(m_x^2+n_x^2)(mu_x+nv_x)+16m_xn_x(mv_x+nu_x)\mathrm{d}x.\end{aligned} \tag{54}$$

By virtue of (51) and (54), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|m(t)\|_{H^1}^2+\|n(t)\|_{H^1}^2)\\&\quad=-\int_{\mathbb{S}}(m^2+n^2)(2mu_x+2nv_x)-2mn(2mv_x+2nu_x)\\&\qquad+8(m_x^2+n_x^2)(mu_x+nv_x)+16m_xn_x(mv_x+nu_x)\mathrm{d}x.\end{aligned} \tag{55}$$

Thus, if a positive constant $C>0$ exists such that $ab\geqslant -C$ on $[0, T^*)\times\mathbb{S}$, then by the above estimation, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|m(t)\|_{H^1}^2+\|n(t)\|_{H^1}^2)\\&\quad\leqslant 32C\int_{\mathbb{S}}(m^2+m_x^2+n^2+n_x^2)\mathrm{d}x.\end{aligned} \tag{56}$$

Applying Gronwall's inequality finally yields

$$\|m(t)\|_{H^1}^2+\|n(t)\|_{H^1}^2\leqslant \mathrm{e}^{16Ct}(\|m_0\|_{H^1}^2+\|n_0\|_{H^1}^2) \tag{57}$$

for $t\in[0, T^*)$, which contradicts the fact that $T^*<\infty$ is the maximal existence time. On the other hand, if (49) holds, we have

$$\limsup_{t\to T^*}\|m(t)\|_{L^\infty}=\infty,\quad or \quad \limsup_{t\to T^*}\|n(t)\|_{L^\infty}=\infty.$$

Then, by the Sobolev embedding theorem, we claim that the solution of (24) will blow up in finite time via corollary 1.

5. Periodic peakons

A peakon is a continuously weak form traveling wave solution, whose profile remains bounded. A big feature of CH-type equations is that they possess peaked traveling wave solutions. As we have just mentioned that A is invertible and the Green function of operator A^{-1}, $u=A^{-1}m=G^*m$ is given by

$$G(x)=\frac{1}{2}\left(x-[x]-\frac{1}{2}\right)^2+\frac{23}{24}.$$

Thus, similarly to the CH equation, it is natural to assume both (10)–(12) admit periodic peakons with the form

$$\phi_u(x,t) = p(t)G(\xi), \quad \phi_v(x,t) = q(t)G(\xi), \tag{58}$$

where $\xi = x - s(t)$, $p(t)$, $q(t)$ and $s(t)$ are functions of t. As an example, we calculate the peakons for (10). Due to the fact that $G(\xi)$ only admits a first order partial derivative, we have to give the definition of weak solutions for (10) first.

Definition 1. Solutions u and v are called weak for (10), if for any test function $\phi(t,x) \in C^{\infty}_{per}([0,T),\mathbb{S})$, u and v satisfy the following integral equations

$$\begin{aligned}
&\int_0^T\int_{\mathbb{S}}\bigg(u_t\phi_{xx} + \bigg(-\frac{1}{3}u_x^3 - u_xv_x^2 + 2\mu(u)uu_x \\
&+ 2\mu(u)vv_x + 2\mu(v)uv_x + 2\mu(v)u_xv\bigg)\phi_{xx} \\
&+ (2\mu^2(u)u + 2\mu^2(v)u + 4\mu(u)\mu(v)v \\
&+ \mu(u)u_x^2 + \mu(u)v_x^2 \\
&+ 2\mu(v)u_xv_x)\bigg)\phi_x\,\mathrm{d}x\mathrm{d}t = 0, \\
&\int_0^T\int_{\mathbb{S}}\bigg(v_t\phi_{xx} + \bigg(-\frac{1}{3}v_x^3 - u_x^2v_x \\
&+ 2\mu(v)uu_x + 2\mu(v)vv_x + 2\mu(u)u_xv + 2\mu(u)uv_x\bigg)\phi_{xx} \\
&+ (2\mu^2(u)v + 2\mu^2(v)v + 4\mu(u)\mu(v)u \\
&+ \mu(v)u_x^2 + \mu(v)v_x^2 + 2\mu(u)u_xv_x)\bigg)\phi_x\,\mathrm{d}x\mathrm{d}t = 0.
\end{aligned} \tag{59}$$

Denoting $\zeta = \xi - [\xi] - \frac{1}{2}$, and taking (58) into the first equation of (59), we have

$$\begin{aligned}
&\int_{\mathbb{S}}\bigg(\bigg[\dot{p}\bigg(\frac{1}{2}\zeta^2 + \frac{23}{24}\bigg) - p\dot{s}\zeta\bigg]\phi_{xx} \\
&+ \bigg[\bigg(\frac{2}{3}p^3 + 2pq^2\bigg)\zeta^3 + \bigg(\frac{23}{12}p^3 + \frac{23}{4}pq^2\bigg)\zeta\bigg]\phi_{xx} \\
&+ \bigg[(2p^3 + 6pq^2)\zeta^2 + \bigg(\frac{23}{12}p^3 + \frac{23}{4}pq^2\bigg)\bigg]\phi_x\bigg)\mathrm{d}x = 0,
\end{aligned} \tag{60}$$

where $\mu(G(x)) = 1$. For convenience, we denote $\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}t}$, and with a direct calculation, we have

$$\begin{aligned}
&\int_{\mathbb{S}}\bigg[\dot{p}\bigg(\frac{1}{2}\zeta^2 + \frac{23}{24}\bigg) - p\dot{s}\zeta\bigg]\phi_{xx}\mathrm{d}x \\
&= \int_0^s\bigg[\frac{\dot{p}}{2}\bigg(x - s + \frac{1}{2}\bigg)^2 - p\dot{s}\bigg(x - s + \frac{1}{2}\bigg)\bigg]\phi_{xx}\mathrm{d}x \\
&+ \int_s^1\bigg[\frac{\dot{p}}{2}\bigg(x - s - \frac{1}{2}\bigg)^2 - p\dot{s}\bigg(x - s - \frac{1}{2}\bigg)\bigg]\phi_{xx}\mathrm{d}x \\
&= \bigg(\bigg[\frac{\dot{p}}{2}\bigg(x - s + \frac{1}{2}\bigg)^2 - p\dot{s}\bigg(x - s + \frac{1}{2}\bigg)\bigg]\phi_x\bigg)\bigg|_0^s \\
&+ \bigg(\bigg[\frac{\dot{p}}{2}\bigg(x - s - \frac{1}{2}\bigg)^2 - p\dot{s}\bigg(x - s - \frac{1}{2}\bigg)\bigg]\phi_x\bigg)\bigg|_s^1 \\
&- \int_0^s\bigg[\dot{p}\bigg(x - s + \frac{1}{2}\bigg) - p\dot{s}\bigg]\phi_x\mathrm{d}x \\
&- \int_s^1\bigg[\dot{p}\bigg(x - s - \frac{1}{2}\bigg) - p\dot{s}\bigg]\phi_x\mathrm{d}x \\
&= -p\dot{s}\phi_x(s) - \dot{p}\phi(s),
\end{aligned} \tag{61}$$

$$\begin{aligned}
&\int_{\mathbb{S}}\bigg(\frac{2}{3}p^3 + 2pq^2\bigg)(\zeta^3\phi_x)_x\mathrm{d}x \\
&= \bigg(\frac{2}{3}p^3 + 2pq^2\bigg)\bigg[\bigg(x - s + \frac{1}{2}\bigg)^3\phi_x\bigg|_0^s \\
&+ \bigg(x - s - \frac{1}{2}\bigg)^3\phi_x\bigg|_s^1\bigg] \\
&= \bigg(\frac{1}{6}p^3 + \frac{1}{2}pq^2\bigg)\phi_x(s),
\end{aligned} \tag{62}$$

$$\begin{aligned}
&\int_{\mathbb{S}}\bigg(\frac{23}{12}p^3 + \frac{23}{4}pq^2\bigg)(\zeta\phi_x)_x\mathrm{d}x \\
&= \bigg(\frac{23}{12}p^3 + \frac{23}{4}pq^2\bigg)\bigg[\bigg(x - s + \frac{1}{2}\bigg)\phi_x\bigg|_0^s \\
&+ \bigg(x - s - \frac{1}{2}\bigg)\phi_x\bigg|_s^1\bigg] \\
&= \bigg(\frac{23}{12}p^3 + \frac{23}{4}pq^2\bigg)\phi_x(s).
\end{aligned} \tag{63}$$

Plugging (61)–(63) into (60), we obtain the following ordinary differential equations (ODEs)

$$\begin{cases} \dot{p} = 0, \\ \dot{s} = \frac{25}{12}p^2 + \frac{25}{4}q^2. \end{cases} \tag{64}$$

Similarly, from the second equation of (59), we have

$$\begin{cases} \dot{q} = 0, \\ \dot{s} = \frac{25}{12}q^2 + \frac{25}{4}p^2. \end{cases} \tag{65}$$

Thus $s(t) = ct$, $p^2 = q^2 = \frac{3c}{25}$, $c > 0$,

$$\phi_u(\xi) = \pm\phi_v(\xi) = \pm\frac{\sqrt{3c}}{5}\bigg(\frac{1}{2}\bigg(\xi - [\xi] - \frac{1}{2}\bigg)^2 + \frac{23}{24}\bigg). \tag{66}$$

With a similar technique, one can verify that (11) and (12) also admit periodic peakons with form (58).

6. Conclusions

In this paper, we derived some two-component generalizations of the μmCH equation by extending the Lax pair through a zero-curvature equation, and we established bi-Hamiltonian structures of these systems. As an example, we presented two blow-up criteria and periodic peakons for system (10). The remaining problem is whether the peaked solutions are orbital stable, and how to prove the stability if such an assumption holds by taking advantage of the conserved quantities. In addition, the symmetry analysis method is one of the effective methods used to investigate the properties of partial differential equations [14–17]. Based on the conservation laws, one can construct the nonlocally related systems to search the nonlocal symmetries. One of our authors investigated the conservation laws and nonlocally related systems of the Hunter–Saxton equation [18], which is a short-wave limit of the CH equation. Many new local and nonlocal symmetries were obtained, and the inverse potential system was also constructed. It has been proved that the symmetry analysis method is an effective method for studying the CH-type equation. In the forthcoming days, we will study the symmetry properties of the μmCH equation presented in this paper.

References

[1] Camassa R and Holm D D 1993 An integrable shallow water equation with peaked solitons *Phys. Rev. Lett.* **71** 1661–4
[2] Fuchssteiner B and Fokas A S 1981/1982 Symplectic structures, their Bäcklund transformations and hereditary symmetries *Phys.* D **4** 47–66
[3] Constantin A and Escher J 1998 Global existence and blow-up for a shallow water equation *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **26** 303–28 (http://eudml.org/doc/84330)
[4] Constantin A 2000 Global existence of solutions and breaking waves for a shallow water equation: a geometric approach *Ann. Inst. Fourier (Grenoble)* **50** 321–62
[5] Constantin A and Escher J 1998 Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation *Comm. Pure Appl. Math.* **51** 475–504
[6] Fuchssteiner B 1996 Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation *Physica* D **95** 229–43
[7] Olver P J and Rosenau P 1996 Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support *Phys. Rev.* E **53** 1900–6
[8] Qiao Z and Li X Q 2011 An integrable equation with nonsmooth solitons *Theor. Math. Phys.* **267** 584–9
[9] Gui G L, Liu Y, Olver P J and Qu C Z 2013 Wave-breaking and peakons for a modified Camassa-Holm equation *Commun. Math. Phys.* **319** 731–59
[10] Song J F, Qu C Z and Qiao Z J 2011 A new integrable two-component system with cubic nonlinearity *J. Math. Phys.* **52** 013503
[11] Li H M and Li X Y 2019 On the integrability of some two-component Camassa-Holm-type systems *Pramana J. Phys.* **92** 65
[12] Qu C Z, Fu Y and Liu Y 2014 Well-posedness, wave breaking and peakons for a modified μ-Camassa-Holm equation *J. Funct. Anal.* **266** 433–77
[13] Bahouri H, Chemin J Y and Danchin R 2011 *Fourier Analysis and Nonlinear Partial Differential equations* (Berlin: Springer)
[14] Zhao Z L and Han B 2017 Lie symmetry analysis, Bäcklund transformations, and exact solutions of a (2+1)-dimensional Boiti-Leon-Pempinelli system *J. Math. Phys.* **58** 101514
[15] Zhao Z L 2019 Bäcklund transformations, rational solutions and soliton-cnoidal wave solutions of the modified Kadomtsev-Petviashvili equation *Appl. Math. Lett.* **89** 103–10
[16] Zhao Z L and Han B 2018 Residual symmetry, Bäcklund transformation and CRE solvability of a (2+1)-dimensional nonlinear system *Nonlinear Dyn.* **94** 461–74
[17] Zhao Z L and Han B 2017 Lie symmetry analysis of the Heisenberg equation *Commun. Nonlinear Sci. Numer. Simul.* **45** 220–34
[18] Zhao Z L 2019 Conservation laws and nonlocally related systems of the Hunter-Saxton equation for liquid crystal *Anal. Math. Phys.* **9** 2311–27