A connection between Rastall-type and $f(R, T)$ gravities

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Abstract – Recently a Lagrangian formulation for Rastall Gravity (RG) has been proposed in
the framework of $f(R, T)$ gravity. In the present work we obtain Lagrangian formulation for the
standard Rastall theory assuming the matter content is a perfect fluid with linear Equation of
State (EoS). We therefore find a relation between the coupling constant of the Lagrangian, the
Rastall gravitational coupling constant and the EoS parameter. We also propose a Lagrangian for
the Generalized Rastall Gravity (GRG). In this case the Rastall parameter which is a constant
is replaced by a variable one. More exactly, it appears as a function of the Ricci scalar and the
trace of the Energy Momentum Tensor (EMT). In both mentioned models, the Lagrangians are
constructed by a linear function of R and T.

Introduction. – In 1972 Rastall questioned the issue
of conservation of EMT in the usual form $\nabla_\mu T^{\mu\nu} = 0$ [1,2].
The basic point in favor of the Rastall proposal is that the
conventional expression for EMT conservation has only
been examined in special relativity and there are no ex-
perimental evidences to prove it in non-flat spacetimes.
The only non-experimental supporter which validates such
an extension is the principle of equivalence [1]. Therefore,
Rastall came to the conclusion that "*the assumptions from
which one derives* $\nabla_\mu T^{\mu\nu} = 0$, *are all questionable, so one
should not accept this question without further investiga-
tion*". He initially proposed the assumption $\nabla_\mu T^{\mu\nu} = a^\nu$,
where the vector field a_ν must vanish in a flat spacetime.
He then chose the simplest curvature-dependent form for
this vector field, *i.e.*, $a_\nu = \lambda \nabla_\nu R$, where R is the Ricci
scalar and λ is called the Rastall coupling parameter.

Since the advent of Rastall gravity, various aspects of
this theory have been widely explored. For example,
in [3–11], the cosmological consequences of the theory have
been investigated, Gödel-type solutions have been inves-
tigated in [12] and the Brans-Dicke scalar field has been
discussed in [13,14]. Static spherically symmetric solutions
have been obtained in [15–20] and the authors of [21] have
shown that the Rastall proposal is compatible with the

Mach's principle. Other features of the Rastall theory can
be found in [22–35].

Although in the original theory, the Rastall parameter
is a constant, some modifications of RG have been pro-
posed, *e.g.*, in a recent work [36], the authors have gener-
alized the theory to include a variable Rastall parameter
(we hereafter call this theory the generalized Rastall grav-
ity). They then investigated cosmological consequences
of the model $\nabla_\mu T^{\mu\nu} = \nabla_\nu(\lambda' R)$, where λ' is now a
function of spacetime coordinates, and predicted a pri-
mary inflationary phase which can exist even in the ab-
sence of matter components in a flat FRW spacetime. In
a similar work [37], the authors have concluded that a
cosmological scenario including pre- and post-inflationary
eras can be rendered in the framework of GRG. An-
other formulation for GRG has been constructed assuming
$a_\nu = \lambda \nabla_\nu f(R)$ [20] where the authors obtained electri-
cally and magnetically neutral regular black hole solution.
Another idea in this context is to consider possible rela-
tions between GRG and modified gravity theories such as
$f(R, T)$ and quadratic gravity [38]. Work along this line
has been also carried out in finding other formulations of
RG whose physical attributes can be investigated in differ-
ent areas [25,39]. In this regard, the issue of establishing
Lagrangians that provide the field equations of RG has be-
come a question of high priority in the literature. Recently,
in the context of modified gravity with curvature-matter

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coupling, a Lagrangian formulation of RG has been proposed [40] and it is shown that RG can be interpreted as a particular case of these theories of gravity. In the present work we obtain a Lagrangian formulation for the original version of RG [1] and GRG [36]. In the latter case we find that the Rastall parameter is a function of the trace of EMT. Also, we introduce a new version of GRG [36] and obtain the corresponding Lagrangian via comparing their field equations and also conservation laws to those of $f(R,T)$ gravity [41][1]. In the case of this new theory we find that both Rastall gravitational coupling and the Rastall parameter are functions of the Ricci scalar and the trace of EMT. For both theory we obtain those $f(R,T)$ Lagrangians that give the field equations and conservation laws. In the framework of Einstein's gravity, the possibility of dependence of gravitational coupling to EMT trace or Ricci scalar without resorting to the least action principle was firstly considered in [60] and, particularly, cosmological consequences of matter-matter coupling, i.e., $\kappa = 8\pi G - \alpha T$ and matter-curvature coupling, i.e., $\kappa = 8\pi G + \alpha R$ were studied in the context of the λCDM model. It is however interesting to note that in our Lagrangian formalism for GRG we have obtained $\kappa = 8\pi G + \partial h(T)/\partial T$, where $h(T)$ is a function which satisfies some conditions. Incidentally, the form of $h(T)$ is similar to those which have been already considered in a previous work [50].

The present work is arranged as follows. In the following section we give a brief review on $f(R,T)$ gravity and derive the field equations which are required for our discussions. The third section is devoted to a concise review on the Rastall model and GRG. In the fourth section we obtain a Lagrangian from $f(R,T)$ gravity whose metric variation gives the field equation of RG. GRG and a new version of it are discussed in the fifth section and, finally, our conclusion is drawn in the final section.

Basic equations for $f(R,T)$ gravity. – In the present section, we review the field equations of $f(R,T)$ gravity which were initially introduced in [41]. Let us consider the following action:

$$S = \int \sqrt{-g}\mathrm{d}^4x \left[\frac{1}{2\kappa} f(R,T) + L^{(\mathrm{m})} \right], \quad (1)$$

where G is the gravitational coupling constant and $T \equiv g^{\mu\nu}T_{\mu\nu}$, R, $L^{(\mathrm{m})}$ are the trace of EMT, the Ricci curvature scalar and the Lagrangian of matter component, respectively. The determinant of the spacetime metric is denoted by g, and the units have been set so that $c = 1$. The EMT for matter fields reads

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta \left[\sqrt{-g}L^{(\mathrm{m})} \right]}{\delta g^{\mu\nu}}. \quad (2)$$

The metric variation of action (1) gives the following field equation [41]:

$$F(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)F(R,T) =$$
$$(\kappa - \mathcal{F}(R,T))T_{\mu\nu} - \mathcal{F}(R,T)\Theta_{\mu\nu}, \quad (3)$$

where

$$\Theta_{\mu\nu} \equiv g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = -2T_{\alpha\beta} + g_{\alpha\beta}L^{(\mathrm{m})} - 2g^{\alpha\beta}\frac{\partial^2 L^{(\mathrm{m})}}{\partial g^{\alpha\beta}\partial g^{\mu\nu}}, \quad (4)$$

and we have used the following definitions for the sake of brevity

$$\mathcal{F}(R,T) \equiv \frac{\partial f(R,T)}{\partial T} \quad \text{and} \quad F(R,T) \equiv \frac{\partial f(R,T)}{\partial R}. \quad (5)$$

The field equation (3) is applicable when the matter Lagrangian is introduced. Considering then $L^{(\mathrm{m})} = p$ for a perfect fluid in expression (4) along with choosing the signature of metric as $(-,+,+,+)$, we obtain

$$\Theta_{\mu\nu} = -2T_{\alpha\beta} + pg_{\alpha\beta}, \quad (6)$$

where p is the pressure of the fluid. Substituting (6) into (3) we get

$$F(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)F(R,T) =$$
$$(\kappa + \mathcal{F}(R,T))T_{\mu\nu} - \mathcal{F}(R,T)pg_{\mu\nu}. \quad (7)$$

One can obtain the following covariant equation by applying the Bianchi identity to the field equation (7):

$$(\kappa+\mathcal{F})\nabla^\mu T_{\mu\nu} + \frac{1}{2}\mathcal{F}\nabla_\mu T + T_{\mu\nu}\nabla^\mu\mathcal{F} - \nabla_\nu(p\mathcal{F}) = 0, \quad (8)$$

where we dropped the argument of $\mathcal{F}(R,T)$ for abbreviation. It is suitable to rewrite eq. (8) in a convenient form for latter purposes, i.e., as

$$\nabla^\mu T_{\mu\nu} = \frac{-\frac{1}{2}\mathcal{F}\nabla_\mu T - T_{\mu\nu}\nabla^\mu\mathcal{F} + \nabla_\nu(p\mathcal{F})}{\kappa + \mathcal{F}}. \quad (9)$$

As is obvious, in $f(R,T)$ gravity, EMT is not conserved automatically and this leads to an irreversible particle creation in cosmology [44]. In [44] it is discussed that such particle creation is the result of the energy flow from the gravitational field to the matter. Note that the only class of solutions for which EMT is conserved is of the form $f(R,T) = R + CT^{1/2}$ for arbitrary constant C (see footnote [2]).

A brief review on Rastall-type gravities. – In this section we briefly review various versions of Rastall gravity. RG is a relatively simple theory in which both the Rastall parameter and Rastall gravitational coupling are

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[1]$f(R,T)$ gravity has been widely studied in the literature, see, e.g., [42–59] for some recent works.

[2]For a comprehensive discussion on the cosmological consequences of these types of models, see, e.g., [42,43].

constants. However, one can also investigate other generalizations (GRG) which include one or both mentioned parameter(s) as variable(s). In this sense, within the present work, we investigate both cases. The main motivation for such generalizations is to seek for theories with better consistency with the observational data. For example, recent cosmological discoveries have indicated that there are two acceleration phases in the history of evolution of the universe and one of which is absent in the ΛCDM model. Work along this line has been pursued in [36,37] where it is shown that GRG leads to inflationary eras in the early and late times of evolution of the universe.

The original version of this theory was introduced in [1] via modification of the EMT conservation as follows:

$$\nabla_\mu T^{\mu\nu} = \lambda\nabla^\nu R, \qquad (10)$$

where λ is an arbitrary constant. This modification leads to the following field equation:

$$G_{\mu\nu} + \lambda\kappa' R g_{\mu\nu} = \kappa' T_{\mu\nu}, \qquad (11)$$

where κ' is the Rastall gravitational constant. Here, because of different gravitational rule we have different gravitational coupling. Taking the trace of (11) and substituting the result into the field equation (11) together with using the conservation rule (10) leads to

$$G_{\mu\nu} = \kappa' T_{\mu\nu} - \frac{\lambda\kappa'^2}{4\kappa'\lambda - 1}T g_{\mu\nu}, \qquad (12)$$

$$\nabla_\mu T^{\mu\nu} = \frac{\lambda\kappa'}{4\kappa'\lambda - 1}\nabla^\nu T. \qquad (13)$$

Equations (12) and (13) have been written in a suitable form which we shall discuss in the next section.

A modified version of the Rastall theory, *i.e.*, GRG, has been proposed, through using a coordinate-dependent Rastall parameter [36], as

$$\nabla_\mu T^{\mu\nu} = \nabla^\nu(\lambda' R), \qquad (14)$$
$$G_{\mu\nu} + \lambda'\kappa'' R g_{\mu\nu} = \kappa'' T_{\mu\nu}, \qquad (15)$$

where we have used a different gravitational coupling. The trace of the field equation (15) leaves us with a relation between three variables T, R and λ', as

$$(4\kappa''\lambda' - 1)R = \kappa'' T, \qquad (16)$$

with the help of which eq. (14) gives

$$\nabla_\mu T^{\mu\nu} = \nabla^\nu\left(\frac{\kappa''\lambda' T}{4\kappa''\lambda' - 1}\right). \qquad (17)$$

A suggestion for Lagrangian of Rastall gravities from $f(R,T)$ theory. – In this section we try to find an acceptable Lagrangian from $f(R,T)$ gravity for Rastall-type gravities. We begin with the original Rastall model [1]. Since in the Rastall field equation (12) there

appear no derivatives of the Ricci scalar, a comparison to eq. (7) suggests a function of the type

$$f(R,T) = R + \alpha T, \qquad (18)$$

where α is some coupling constant. Substituting this function into (7) gives

$$G_{\mu\nu} = (\kappa + \alpha)T_{\mu\nu} + \alpha\left(\frac{T}{2} - p\right)g_{\mu\nu}. \qquad (19)$$

Using the perfect fluid EoS, $p = w\rho$, leads to the relation $T = -\rho + 3p = (3w - 1)\rho$ from which we have

$$G_{\mu\nu} = (\kappa + \alpha)T_{\mu\nu} + \frac{\alpha(w-1)}{2(3w-1)}T g_{\mu\nu}. \qquad (20)$$

Therefore, eq. (20) gives (12) provided that the following equalities hold:

$$\kappa' = \kappa + \alpha \quad \text{and} \quad -\frac{\lambda\kappa'^2}{4\kappa'\lambda - 1} = \frac{\alpha(w-1)}{2(3w-1)}, \qquad (21)$$

from which we get the coupling constant in Lagrangian (18) as

$$\alpha = \frac{2(1-3w)\lambda\kappa'^2}{(w-1)(4\kappa'\lambda - 1)}. \qquad (22)$$

It is easy to check that for solution (22) and function (18), the two eqs. (9) and (13) become identical. In this case for choice (18) we obtain

$$\nabla_\mu T^{\mu\nu} = \frac{\alpha}{2(\kappa+\alpha)}\frac{w-1}{1-3w}\nabla^\nu T = \frac{\lambda\kappa'}{4\kappa'\lambda - 1}\nabla^\nu T. \qquad (23)$$

Therefore, introducing a Lagrangian formulation from $f(R,T)$ requires a change in the Rastall gravitational constant, κ'. As a result, we can say RG can be understood from the $f(R,T)$ Lagrangian provided that results (21) are satisfied and also function (18) is utilized.

Generalization of Rastall gravity and a proposal for its Lagrangian. – In this section we obtain a Lagrangian formulation for GRG and introduce a new version of GRG and propose its corresponding Lagrangian. We first suppose that, in general, the gravitational coupling κ'' in eq. (15) is a variable. This assumption gives a new version of GRG for which the Bianchi identity cannot be satisfied, obviously. In the case of a varying κ'' applying the Bianchi identity to eq. (15) gives

$$\kappa''\nabla_\mu(\lambda' R) + \lambda' R\nabla_\mu\kappa'' = \kappa''\nabla^\nu T_{\mu\nu} + T_{\mu\nu}\nabla^\nu\kappa''. \qquad (24)$$

As can be seen, to reach the original equation (14), there is a constraint which must hold, *i.e.*,

$$\lambda' R\nabla_\mu\kappa'' - T_{\mu\nu}\nabla^\nu\kappa'' = 0. \qquad (25)$$

Now, substituting the Lagrangian

$$f(R,T) = R + h(T) \qquad (26)$$

into the field equation (7) we find

$$G_{\mu\nu} = \left(\kappa + \frac{\mathrm{d}h}{\mathrm{d}T}\right) T_{\mu\nu} + \left(\frac{h}{2} - p\frac{\mathrm{d}h}{\mathrm{d}T}\right) g_{\mu\nu}. \qquad (27)$$

In this case, the equality of the two eqs. (15) (for a varying κ'') and (27) requires that the following relations be satisfied:

$$\kappa'' = \kappa + \frac{\mathrm{d}h}{\mathrm{d}T},$$

$$\lambda' = -\frac{\frac{h}{2} - p\frac{\mathrm{d}h}{\mathrm{d}T}}{R\left(\kappa + \frac{\mathrm{d}h}{\mathrm{d}T}\right)}$$

$$= \frac{\frac{h}{2} - p\frac{\mathrm{d}h}{\mathrm{d}T}}{\left(\frac{\mathrm{d}h}{\mathrm{d}T} + \kappa\right)\left[T\left(\frac{\mathrm{d}h}{\mathrm{d}T} + \kappa\right) + 4\left(\frac{h}{2} - p\frac{\mathrm{d}h}{\mathrm{d}T}\right)\right]}. \qquad (28)$$

Next, we note that the constraint (25) should be considered for relations (28). Substituting then for λ' from (28) into (25) gives

$$\left(\frac{h}{2} - \frac{\mathrm{d}h}{\mathrm{d}T}p\right)\nabla_\mu\kappa'' + \left(\kappa + \frac{\mathrm{d}h}{\mathrm{d}T}\right) T_{\mu\nu}\nabla^\nu\kappa'' = 0. \qquad (29)$$

Equation (29) is a constraint that must be satisfied. Assuming a homogeneous gravitational coupling parameter κ'' along with a perfect fluid for the matter content, eq. (29) reduces to the following equation[3]:

$$\frac{\mathrm{d}h}{\mathrm{d}T}T - \frac{3w-1}{2(w+1)}h + \frac{\kappa}{w+1}T = 0, \qquad (30)$$

for which the solution reads

$$h(T) = \frac{2\kappa}{w-3}T + \beta T^{\frac{3w-1}{2(w+1)}}, \qquad (31)$$

where β is an integration constant[4]. Note that in the case of constraint (29) one gets $\kappa''\lambda' = 1/3(1+w)$. Therefore, it suffices to use function (31) as well as the gravitational coupling κ'' in (28) to obtain GRG equations (14) and (15) from (26). However, in our approach, both the Rastall parameter and the gravitational coupling have to be variables instead of constant ones, contrary to the case of GRG which was originally introduced in [36].

It remains to check whether the equality of eqs. (9) and (14) leads to solution (31). In the case of Lagrangian (26), we can rewrite eq. (9) as

$$\nabla^\mu T_{\mu\nu} =$$
$$\frac{1}{(3w-1)(\kappa+\mathcal{F})}\left[\frac{1-w}{2}\mathcal{F} + (1+w)T\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}T}\right]\nabla_\nu T, \qquad (32)$$

while, for eq. (17), we obtain

$$\nabla^\mu T_{\mu\nu} = \left(\frac{x}{4x-1} - \frac{T}{(4x-1)^2}\frac{\mathrm{d}x}{\mathrm{d}T}\right)\nabla_\nu T, \qquad (33)$$

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[3]Note that in the case of a homogeneous coupling parameter κ'', only the time derivative of κ'' would appear in both terms of (29). Also, we have used $T_t^t = -\rho = T/(1-3w)$ and $p = w\rho = wT/(3w-1)$ to obtain eq. (30).

[4]One can find some interesting features of functions of this form in [50,51].

where, $x \equiv \kappa''\lambda'$. One can easily check that eqs. (32) and (33) lead to the same result provided that eq. (30) holds. More precisely, in this case we have

$$\frac{\mathrm{d}^2 h}{\mathrm{d}T^2}T - \frac{w-3}{2(w+1)}\frac{\mathrm{d}h}{\mathrm{d}T}T + \frac{\kappa}{w+1} = 0, \qquad (34)$$

which is the derivative of eq. (30) with respect to the trace of EMT.

Now, we consider eqs. (25) and (28) for a constant gravitational coupling and again utilize the function (26). Clearly, (25) is automatically satisfied. The upper relation in (28) gives the solution $h(T) = (\kappa'' - \kappa)T + \Lambda$, for arbitrary constant Λ. Substituting this solution into the lower relation in (28) yields

$$\lambda' = \frac{1}{4}\left(\frac{(3w-1)T}{[2(w-1)\kappa + (3-5w)\kappa'']T + 2\Lambda(1-3w)} + \frac{1}{\kappa''}\right). \qquad (35)$$

Next, we must ensure that both conservation equations (9) and (14) provide the same result. Equation (9) for Lagrangian $h(T) = (\kappa'' - \kappa)T + \Lambda$ along with perfect fluid assumption leads to

$$\nabla^\mu T_{\mu\nu} = \frac{1-w}{2(3w-1)}\frac{\mathcal{F}}{\kappa + \mathcal{F}}\nabla_\nu T, \qquad (36)$$

while (14) gives

$$\nabla^\mu T_{\mu\nu} = \kappa''\left(\frac{\lambda'}{4\kappa''\lambda' - 1} - \frac{T\mathrm{d}\lambda'/\mathrm{d}T}{(4\kappa''\lambda' - 1)^2}\right)\nabla_\nu T. \qquad (37)$$

Equality of eqs. (36) and (37) leads to a differential equation for which the solution (35) is recovered provided that the free parameter within the solution of this differential equation is chosen as $-\Lambda/2\kappa''^2$.

Therefore, GRG is equivalent to $f(R,T) = R + (\kappa'' - \kappa)T + \Lambda$ gravity. In this case the Rastall parameter is a function of the trace of EMT and depends on the Rastall gravitational constant κ''.

Concluding remarks. – In this paper we considered a possible connection between RG and its generalizations, and Lagrangians of $f(R,T)$ gravity theories. We first obtained a relation between Rastall field equations and the linear Lagrangian $f(R,T) = R + \alpha T$. In this case one obtains the Rastall field equations as well as the Rastall EMT conservation rule from the mentioned Lagrangian.

In the next step, we investigated the Lagrangian of a version of GRG which was introduced in [36]. It is found that for this version of GRG to be derived from $f(R,T)$ gravity, the Rastall parameter could be running. For this case, we chose the Lagrangian $f(R,T) = R + h(T)$ and compared the field equations derived from this Lagrangian to those of the new version of GRG. We then found the Rastall parameter and the Rastall gravitational coupling parameter in terms of some functions of the Ricci scalar and the trace of EMT. Applying the Bianchi identity on the related field equation of this version of GRG leads to

covariant constraint (29) which must be satisfied. Hence, solving this constraint gives the function (31). Also, it was verified that the EMT conservation equation (9) leads to eq. (14) for solution (31).

Finally, we have shown that GRG is equivalent to $f(R, T) = R + (\kappa'' - \kappa)T + \Lambda$ gravity with a Rastall parameter which is a function of the trace of EMT and a constant Rastall gravitational coupling. It is notable that, recently, similar considerations has been performed for only RG [40]. However, in the present work we also have considered GRG as well as its EMT conservation rule and an extension of it in a more straightforward way.

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