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Photon velocity, power spectrum in Unruh effect with modified dispersion relation

ARNAB MUKHERJEE[1(a)], SUNANDAN GANGOPADHYAY[2(b)] and MANJARI DUTTA[2(c)]

[1] *Department of Physics, Jadavpur University - Kolkata 700032, India*
[2] *Department of Theoretical Sciences, S.N. Bose National Centre for Basic Sciences - JD Block, Sector III, Salt Lake, Kolkata 700106, India*

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Abstract – In this paper we propose a new form of generalized uncertainty principle which involves both a linear and a quadratic term in the momentum. From this we have obtained the corresponding modified dispersion relation which is compared with the corresponding relation in rainbow gravity. The new form of the generalized uncertainty principle reduces to the known forms in appropriate limits. We then calculate the modified velocity of photons and we find that it is energy-dependent, allowing therefore for a superluminal propagation. We then derive the $(1 + 1)$-dimensional Klein-Gordon equation taking into account the effects of the modified dispersion relation. The positive frequency mode solution of this equation is then used to calculate the power spectrum arising due to the Unruh effect. The result shows that the power spectrum depends on the energy of the particle owing its origin to the presence of the generalized uncertainty principle. Our results capture the effects of both the simplest form as well as the linear form of the generalized uncertainty principle and also points out an error in the result of the power spectrum up to first order in the generalized uncertainty principle parameter existing in the literature.

Introduction. – In 1927 Heisenberg proposed the *Uncertainty Principle* [1] from his famous thought experiment. At that time physicists could not believe that the measurement of one physical observable could hamper the measurement of another observable simultaneously. In this principle there is a fundamental limit for the measurement accuracy with which certain pairs of physical observables such as position and momentum or energy and time can be measured simultaneously. Around the mid sixties, the concept of fundamental minimum measurable length was introduced [2]. The existence of this minimum measurable length would mean that Heisenberg uncertainty principle has to be modified. It is for this reason that the generalized uncertainty principle (GUP) was introduced in the literature [3–8]. It is observed that the GUP not only has a minimum measurable length but also a maximum measurable momentum [9]. It is well known that the fundamental length scale introduced by these models breaks Lorentz invariance [10,11]. Although there is no direct evidence of Lorentz symmetry violation at high energies, the beauty of these theories lies in the fact that quantum field theories based on these Lorentz invariance violating models can be formulated in a consistent way. There has been a lot of investigations thereafter incorporating the effects of the GUP in black hole thermodynamics [12–14], quantum systems such as particle in a box, Landau levels, simple harmonic oscillator [15–17]. Thereafter, studies to calculate Planck scale corrections due to modified dispersion relations to the response of the Unruh-DeWitt detector have also been carried out in [18]. Tunneling of fermions with half-integer spin from a higher-dimensional charged anti-de Sitter (AdS) black hole in massive gravity incorporating the effect of modified dispersion relation has also been investigated recently in [19]. Further, the current accuracy of precision measurement of Lamb shift gives an upper bound on the quadratic GUP parameter [15], which although turns out to be weaker than that set by the electroweak scale, is not inconsistent with it. It is expected that with improvements in the accuracy of measurements, the bound will go down by several orders of magnitude.

In this paper we consider a new form of the GUP which is a combination of both the linear and the simplest forms

(a) E-mail: a.m.official.1123@gmail.com
(b) E-mails: sunandan.gangopadhyay@bose.res.in, sunandan.gangopadhyay@gmail.com
(c) E-mail: manjaridutta@boson.bose.res.in

of the GUP exisiting in the literature. Taking this new form of GUP, we derive the corresponding dispersion relation. From this we obtain the velocity of photons up to second order in the linear GUP parameter and first order in the quadratic GUP parameter. We then derive the Klein-Gordon equation in $(1+1)$ dimensions taking into account the effects of the modified dispersion relation. This equation is then solved in an iterative approach and we work with the positive frequency mode solution only. The solution is then written down for a uniformly accelerating observer using the Rindler coordinate transformations. Using this solution for the uniformly accelerating observer, we derive the emission spectrum arising from the Unruh effect. Such an exercise was carried out earlier in [20] taking into account the linear form of the GUP. However, our result capture the effects of both the linear GUP as well as the quadratic GUP parameters in the power spectrum. It also differs from the result in [20] up to first order in the linear GUP parameter as there seems to be an error in the result on dimensional grounds.

The paper is organized as follows. In the next section, we consider the new form of the GUP and using this we derive the modified dispersion relation and obtain the photon velocity. In the third section, we derive the Klein-Gordon equation incorporating the modified dispersion relation and solve this equation. In the fourth section, we obtain the power spectrum of the Unruh effect using the solution in the third section. We conclude in the fifth section.

Modified dispersion relation and photon velocity. – In this section our goal is to derive the GUP corrected dispersion relation. Let p^A be the modified four momentum and k^A be the usual four momentum. Now we introduce a general form of the GUP given by the expression

$$[x_i, p_j] = i\hbar\left[\delta_{ij} - \alpha\left(\delta_{ij}p + \frac{p_i p_j}{p}\right) + \beta\left(\delta_{ij}p^2 + 2p_i p_j\right) - \alpha^2(\delta_{ij}p^2 + p_i p_j)\right], \qquad (1)$$

where $p^2 \equiv |\vec{p}|^2 = \eta_{ij}p^i p^j$; $i, j = 1, 2, 3$. The above commutator ensures by the Jacobi identity that

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0. \qquad (2)$$

The commutation relation between the position operator and the usual three-momentum operator is the standard Heisenberg algebra given by

$$[x_i, k_j] = i\hbar\delta_{ij}. \qquad (3)$$

The relations between the modified and the usual momenta which give the commutation relation (1) read

$$p^0 = k^0, \qquad (4)$$

$$p^i = k^i(1 - \alpha k + \beta k^2), \qquad (5)$$

where $\alpha = \alpha_0/(M_{Pl}c)$ and $\beta = \beta_0/(M_{Pl}c)^2$ are small parameters, M_{Pl} is the Planck mass, $k^2 \equiv |\vec{k}|^2 = \eta_{ij}k^i k^j$. It is reassuring to note that in the limit $\alpha \to 0$, eq. (1) reduces to the simplest form of the GUP proposed in the literature [7]

$$[x_i, p_j] = i\hbar\left[\delta_{ij} + \beta\left(\delta_{ij}p^2 + 2p_i p_j\right)\right]. \qquad (6)$$

Further, setting $\beta = 2\alpha^2$ in eq. (1) yields the expression [15]

$$[x_i, p_j] = i\hbar\left[\delta_{ij} - \alpha\left(\delta_{ij}p + \frac{p_i p_j}{p}\right) + \alpha^2\left(\delta_{ij}p^2 + 3p_i p_j\right)\right]. \qquad (7)$$

Setting $i = j$, eq. (1) leads to the following modified uncertainty relation:

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}\left(1 - \alpha\left\langle p + \frac{p_i p_i}{p}\right\rangle - (\alpha^2 - \beta)((\Delta p)^2 + \langle p\rangle^2) - (\alpha^2 - 2\beta)((\Delta p_i)^2 + \langle p_i\rangle^2)\right). \qquad (8)$$

We now consider the background spacetime metric η_{AB}, $(A, B = 0, 1, 2, 3)$ in (3+1) dimensions to be the Minkowski spacetime with signature $(-,+,+,+)$, that is

$$\mathrm{d}s^2 = \eta_{AB}\mathrm{d}x^A\mathrm{d}x^B = -c^2\mathrm{d}t^2 + \eta_{ij}\mathrm{d}x^i\mathrm{d}x^j \qquad (9)$$

with $\eta_{00} = -1$, $\eta_{ij} = \delta_{ij}$. Hence the square of the four momentum in this background becomes

$$\begin{aligned} p^A p_A &= \eta_{AB}p^A p^B \\ &= -(p^0)^2 + \eta_{ij}p^i p^j \\ &= -(k^0)^2 + \eta_{ij}k^i k^j(1 - \alpha k + \beta k^2)^2 \\ &= -(k^0)^2 + k^2(1 - \alpha k + \beta k^2)^2, \end{aligned} \qquad (10)$$

where eqs. (4), (5) have been used in the third line of the equality.

Keeping terms up to $\mathcal{O}(\alpha^2, \beta)$ in the above expression, we obtain

$$\begin{aligned} p^A p_A &= -(k^0)^2 + k^2[1 - 2\alpha k + \alpha^2 k^2 + 2\beta k^2] \\ &= k^A k_A + k^2[-2\alpha k + \alpha^2 k^2 + 2\beta k^2]. \end{aligned} \qquad (11)$$

Using the usual dispersion relation

$$k^A k_A = -m^2 c^2, \qquad (12)$$

eq. (11) takes the form

$$p^A p_A = -m^2 c^2 + k^2[-2\alpha k + (\alpha^2 + 2\beta)k^2]. \qquad (13)$$

Setting $\beta = 2\alpha^2$, the above relation reduces to [20]

$$p^A p_A = -m^2 c^2 + k^2[-2\alpha k + 5\alpha^2 k^2]. \qquad (14)$$

To express the usual momentum k in terms of the modified momentum p up to $\mathcal{O}(\alpha^2, \beta)$, we choose an ansatz of the form

$$k = a_0 + \alpha a_1 + \beta a_2 + \alpha^2 a_3. \qquad (15)$$

Now taking the magnitude of both sides of eq. (5), we get

$$p = k(1 - \alpha k + \beta k^2), \quad p = |\vec{p}|. \tag{16}$$

Substituting eq. (15) in eq. (16), we get

$$p = a_0 + (a_1 - a_0^2)\alpha + (a_3 - 2a_0 a_1)\alpha^2 + (a_0^3 + a_2)\beta. \tag{17}$$

Comparing coefficients of α^0, α, α^2, β on both sides of the above equation, we have

$$a_0 = p, \tag{18}$$
$$a_1 = a_0^2 = p^2, \tag{19}$$
$$a_3 = 2a_0 a_1 = 2p^3, \tag{20}$$
$$a_2 = -a_0^3 = -p^3. \tag{21}$$

This gives the following relation between the usual three-momentum k^i in terms of the modified three-momentum p^i:

$$k^i = p^i[1 + \alpha p + (2\alpha^2 - \beta)p^2]. \tag{22}$$

Substituting the above expression for the usual three-momentum k^i in eq. (13) and retaining terms up to $\mathcal{O}(\alpha^2, \beta)$, the modified dispersion relation corresponding to the form of the GUP considered here takes the form

$$p^A p_A = -m^2c^2 + p^2[-2\alpha p - (5\alpha^2 - 2\beta)p^2]. \tag{23}$$

Writing the left-hand side of the above equation as

$$p^A p_A = -\mathcal{M}^2 c^2 \tag{24}$$

yields

$$\mathcal{M} = \sqrt{m^2 + \frac{p^2[\,2\alpha p + (5\alpha^2 - 2\beta)p^2]}{c^2}}\,. \tag{25}$$

The above result indicates that $\mathcal{M}$ is an effective mass of the particle generating solely due to the GUP.

From eq. (23), the time component of the four-momentum squared, up to $\mathcal{O}(\alpha^2, \beta)$ can be written as

$$(p^0)^2 = m^2c^2 + p^2[1 + 2\alpha p + (5\alpha^2 - 2\beta)p^2]. \tag{26}$$

Hence the energy of the particle up to $\mathcal{O}(\alpha^2, \beta)$ is given by

$$E^2 = m^2c^4 + p^2c^2[1 + 2\alpha p + (5\alpha^2 - 2\beta)p^2]. \tag{27}$$

The above relation is the most general modified dispersion relation. It is interesting to compare this result with the rainbow gravity generalization of the modified dispersion relations in doubly special relativity to curved spacetime. These modified dispersion relations are given by [21,22]

$$E^2 f^2(E/E_p) - p^2c^2 g^2(E/E_p) = m^2c^4, \tag{28}$$

where E_p is the Planck energy and the functions $f(E/E_p)$ and $g(E/E_p)$ are called rainbow functions. Specific forms of the rainbow functions read [23]

$$f(E/E_p) = 1, \quad g(E/E_p) = \sqrt{1 - \eta\left(\frac{E}{E_p}\right)^n}, \tag{29}$$

where η is the rainbow parameter. In [24], it was argued from the universality of logarithmic corrections to black hole entropy that n gets restricted to $n = 1, 2$. Setting $n = 2$, eq. (28) gives

$$E^2 = \frac{m^2c^4 + p^2c^2}{1 + \eta\frac{p^2c^2}{E_p^2}}. \tag{30}$$

Keeping terms up to linear order in $\eta p^2c^2/E_p^2$ in the above relation yields

$$E^2 = m^2c^4 + p^2c^2\left[1 - \eta\frac{m^2c^4}{E_p^2} - \eta\frac{p^2c^2}{E_p^2}\right]. \tag{31}$$

The above relation has a very similar structure to the one derived in eq. (27).

Setting $\beta = 2\alpha^2$ in eq. (27), we get

$$E^2 = m^2c^4 + p^2c^2[1 + 2\alpha p + \alpha^2 p^2]. \tag{32}$$

It is reassuring to note that in the absence of quantum gravity corrections, that is, $\alpha = 0$ and $\beta = 0$, we have $p^i = k^i$, and hence one gets back the standard dispersion

$$E^2 = m^2c^4 + k^2c^2. \tag{33}$$

We shall now proceed to investigate how the velocity of photon gets affected by the modified dispersion relation (eq. (27)). In the usual case, the velocity of photon $c = E/\mathrm{k}$. However, since the momentum is modified here due to the GUP, it is expected that the velocity of photon will also get modified. In Minkowskian background, the velocity of photon can be calculated as

$$u = \frac{\partial E}{\partial p}\,. \tag{34}$$

Setting $m = 0$ in eq. (27), we get

$$E^2 = p^2c^2[1 + 2\alpha p + (5\alpha^2 - 2\beta)p^2]. \tag{35}$$

Substituting eq. (35) in eq. (34) and keeping terms up to $\mathcal{O}(\alpha^2, \beta)$, we get

$$u = \frac{\partial E}{\partial p} = c[1 + 2\alpha p + 3(2\alpha^2 - \beta)p^2]. \tag{36}$$

Now we proceed to invert eq. (35) to express p in terms of energy E. This is required to rewrite eq. (36) in terms of energy E. To do this we take p to be of the form

$$p = a + \alpha b + \beta e + \alpha^2 d + \mathcal{O}(\alpha^3, \beta^2, \alpha^2\beta). \tag{37}$$

Substituting eq. (37) in eq. (35) and comparing the coefficients of α^0, α, β, α^2 on both sides of the equation, we get

$$a^2c^2 = E^2, \tag{38}$$
$$abc^2 + c^2a^3 = 0, \tag{39}$$
$$ac^2e - a^4c^2 = 0, \tag{40}$$
$$b^2c^2 + 2adc^2 + 6c^2a^2b + 5c^2a^4 = 0. \tag{41}$$

Solving the above equations, we obtain

$$a = \frac{E}{c}, \tag{42}$$

$$b = -a^2 = -\frac{E^2}{c^2}, \tag{43}$$

$$d = 0, \tag{44}$$

$$e = a^3 = \frac{E^3}{c^3}. \tag{45}$$

Substituing the above values in eq. (37), we have

$$p = \frac{E}{c}\left[1 - \alpha\frac{E}{c} + \beta\frac{E^2}{c^2}\right] + \mathcal{O}(\alpha^3, \beta^2, \alpha^2\beta). \tag{46}$$

Substituting eq. (46) into eq. (36) and keeping terms up to $\mathcal{O}(\alpha^2,\,\beta)$, the modified photon velocity is obtained to be

$$u = c\left[1 + \frac{2\alpha E}{c} + \frac{(4\alpha^2 - 3\beta)E^2}{c^2}\right]. \tag{47}$$

The above result for the velocity of photon captures the effect of the GUP for both the linear and the quadratic terms in momentum in the GUP (8).

Setting $\beta = 2\alpha^2$ in the above relation, we get

$$u = c\left[1 + \frac{2\alpha E}{c} - \frac{2\alpha^2 E^2}{c^2}\right]. \tag{48}$$

From the above relation, we observe that the velocity of photon is energy-dependent. This energy dependence is due to the modified dispersion relation arising from the GUP. The other important point to note is that the photon velocity is larger than the speed of light c which indicates that quantum gravity effects allow a superluminal photon propagation.

Klein-Gordon equation and modified dispersion relation. – In this section we are going to write down the modified Klein-Gordon equation in (1+1)-dimensional Minkowski spacetime

$$\mathrm{d}s^2 = -c^2\mathrm{d}T^2 + \mathrm{d}X^2. \tag{49}$$

To obtain the modified Klein-Gordon equation, we first recast eq. (11) in the form

$$p_A p^A = -(k^0)^2 + k^2 + k^2[-2\alpha k + \alpha^2 k^2 + 2\beta k^2]. \tag{50}$$

Elevating k^0 and k to operators and using their standard representations

$$k^0 = \frac{i\hbar}{c}\partial_T, \quad k^1 = -i\hbar\partial_X \tag{51}$$

and keeping terms up to $\mathcal{O}(\alpha^2,\,\beta)$, we get the modified Klein-Gordon equation in $(1+1)$ dimensions for massless particles to be

$$\begin{aligned} p_A p^A\ \Phi(T,X) = \hbar^2 \bigg[&\frac{1}{c^2}\partial_T^2 - \partial_X^2 - 2i\alpha\hbar\partial_X^3 \\ &+(\alpha^2 + 2\beta)\hbar^2\partial_X^4\bigg]\,\Phi(T,X) = 0. \end{aligned} \tag{52}$$

The third and fourth terms in the right-hand side of the above equation are the ones that have emerged due to the GUP. We now take a solution of the above equation in the form

$$\Phi(T,X) = \exp(-i\omega T)\Psi(X), \tag{53}$$

where $\hbar\omega$ is the energy of the particle.

Substituting this in the above equation, we get

$$\left[\partial_X^2 + 2i\alpha\hbar\partial_X^3 - (\alpha^2 + 2\beta)\hbar^2\partial_X^4 + \frac{\omega^2}{c^2}\right]\Psi(X) = 0. \tag{54}$$

The above equation reduces to that derived in [20] up to $\mathcal{O}(\alpha, \beta^0)$.

Setting

$$\Psi(X) = \exp(nX) \tag{55}$$

we obtain

$$n^2 + 2i\hbar\alpha n^3 - (\alpha^2 + 2\beta)\hbar^2 n^4 + \frac{\omega^2}{c^2} = 0\,.$$

To solve this equation, we choose an ansatz

$$n = n_0 + \alpha n_1 + \beta n_2 + \alpha^2 n_3, \tag{56}$$

where n_0, n_1, n_2, n_3 are to be determined. Substituting this ansatz in the above equation and comparing the coeeficients of α^0, α, β and α^2 on both sides of the equation, we get

$$\begin{aligned} \frac{\omega^2}{c^2} + n_0^2 &= 0, \\ 2n_0 n_1 + 2i\hbar n_0^3 &= 0, \\ n_1^2 + 2n_0 n_3 + 6i\hbar n_0^2 n_1 - \hbar^2 n_0^4 &= 0, \\ -2\hbar^2 n_0^4 + 2n_0 n_2 &= 0. \end{aligned}$$

Solving these equations, we get

$$n_0 = \frac{i\omega}{c}, \tag{57}$$

$$n_1 = i\hbar\frac{\omega^2}{c^2}, \tag{58}$$

$$n_2 = -i\hbar^2\frac{\omega^3}{c^3}, \tag{59}$$

$$n_3 = 2i\hbar^2\frac{\omega^3}{c^3}. \tag{60}$$

Note that we have considered only the outgoing mode solutions since they give the radiation spectrum. The positive frequency outgoing solution of the modified Klein-Gordon equation therefore reads

$$\begin{aligned} \Phi(T,X) = \exp\bigg[&-i\omega\left(T - \frac{X}{c}\right) + i\alpha\frac{\hbar\omega^2}{c^2}X \\ &+i(2\alpha^2 - \beta)\frac{\hbar^2\omega^3}{c^3}X\bigg]. \end{aligned} \tag{61}$$

The above solution reduces to that derived in [20] up to $\mathcal{O}(\alpha, \beta^0)$. Interestingly, the above solution also reduces to that derived in [20] for $\beta = 2\alpha^2$. In the next section, we shall use this solution to investigate the power spectrum of a uniformly accelerated observer.

GUP corrected power spectrum in Unruh effect. – In this section we essentially follow the analysis in [25] to compute the power spectrum of a uniformly accelerating observer. We first consider a uniformly accelerated frame known as Rindler frame. The coordinate transformation equations connecting Minkowski and Rindler frames with respect to an observer in the Rindler frame moving along the x-axis read

$$X(\tau) = \frac{c}{\kappa}\cosh(\kappa\tau), \quad T(\tau) = \frac{1}{\kappa}\sinh(\kappa\tau), \tag{62}$$

where τ is the proper time of the uniformly accelerating observer. Hence the wave function (eq. (61)) as seen by the Rindler observer will have the form

$$\begin{aligned}&\phi[T(\tau), X(\tau)] = \\ &\exp\left[\frac{i\omega}{\kappa}e^{-\kappa\tau}\left(1+\frac{\alpha\hbar\omega}{2c}+\frac{(2\alpha^2-\beta)\hbar^2\omega^2}{2c^2}\right)\right] \\ &\times\exp\left[\frac{i\hbar\omega^2}{2c\kappa}e^{\kappa\tau}\left(\alpha+(2\alpha^2-\beta)\frac{\hbar\omega}{c}\right)\right].\end{aligned} \tag{63}$$

The above relation can be obtained by substituting eq. (62) in eq. (61).

The power spectrum is now given by [25]

$$P(\nu) = |f(\nu)|^2, \tag{64}$$

where $f(\nu)$ is the Fourier transform of $\phi(\tau)$,

$$f(\nu) = \int_{-\infty}^{+\infty} \mathrm{d}\tau\, \Phi(\tau)e^{i\nu\tau}. \tag{65}$$

Substituting eq. (63) in the above relation yields

$$\begin{aligned}f(\nu) = \int_{-\infty}^{+\infty}&\mathrm{d}\tau\exp\left[\frac{i\omega}{\kappa}e^{-\kappa\tau}\left(1+\frac{\alpha\hbar\omega}{2c}+\frac{(2\alpha^2-\beta)\hbar^2\omega^2}{2c^2}\right)\right] \\ &\times\exp\left[\frac{i\hbar\omega^2}{2c\kappa}e^{\kappa\tau}\left(\alpha+(2\alpha^2-\beta)\frac{\hbar\omega}{c}\right)\right]e^{i\nu\tau}.\end{aligned} \tag{66}$$

Now expanding the second term of the above equation and keeping terms up to $\mathcal{O}(\alpha^2, \beta)$, we obtain

$$\begin{aligned}f(\nu) = &\int_{-\infty}^{\infty}\left[\exp\left[\frac{i\omega}{\kappa}e^{-\kappa\tau}\left(1+\frac{\alpha\hbar\omega}{2c}\right.\right.\right. \\ &\left.\left.\left.+\frac{(2\alpha^2-\beta)\hbar^2\omega^2}{2c^2}\right)\right]+i\nu\tau\right]\mathrm{d}\tau \\ &+\int_{-\infty}^{\infty}\left[\frac{i\hbar\omega^2}{2c\kappa}\left(\alpha+(2\alpha^2-\beta)\frac{\hbar\omega}{c}\right)\exp\left[i\nu\tau+\kappa\tau\right.\right. \\ &\left.\left.+\frac{i\omega}{\kappa}e^{-\kappa\tau}\left(1+\frac{\alpha\hbar\omega}{2c}+\frac{(2\alpha^2-\beta)\hbar^2\omega^2}{2c^2}\right)\right]\right]\mathrm{d}\tau \\ &-\int_{-\infty}^{\infty}\frac{\alpha^2\omega^4\hbar^2}{8c^2\kappa^2}\exp\left[i\nu\tau+2\kappa\tau\right. \\ &\left.+\frac{i\omega}{\kappa}e^{-\kappa\tau}\left(1+\frac{\alpha\hbar\omega}{2c}+\frac{(2\alpha^2-\beta)\hbar^2\omega^2}{2c^2}\right)\right]\mathrm{d}\tau\,.\end{aligned} \tag{67}$$

To compute the above integrals we need to rewrite the integral in a suitable form. For doing that we introduce a new variable $v = e^{\kappa\tau}$. Using this, eq. (67) takes the form

$$\begin{aligned}f(\nu) = &\int_0^{\infty}\frac{1}{\kappa}v^{-\left(1+\frac{i\nu}{\kappa}\right)} \\ &\times\exp\left[\frac{i\omega v}{\kappa}\left(1+\frac{\alpha\hbar\omega}{2c}+\frac{(2\alpha^2-\beta)\hbar^2\omega^2}{2c^2}\right)\right]\mathrm{d}v \\ &+\int_0^{\infty}\frac{i\hbar\omega^2}{2c\kappa^2}\left(\alpha+(2\alpha^2-\beta)\frac{\hbar\omega}{c}\right)v^{-\left(1+\frac{i\nu}{\kappa}\right)-1}\exp\left[\frac{i\omega Av}{\kappa}\right]\mathrm{d}v \\ &-\frac{\alpha^2\omega^4\hbar^2}{8c^2\kappa^3}\int_0^{\infty}v^{-\left(2+\frac{i\nu}{\kappa}\right)-1}\exp\left[\frac{i\omega Av}{\kappa}\right]\mathrm{d}v,\end{aligned} \tag{68}$$

where

$$A = \left(1+\frac{\alpha\hbar\omega}{2c}+\frac{\left(2\alpha^2-\beta\right)\hbar^2\omega^2}{2c^2}\right). \tag{69}$$

To perform the above integrals, we use the standard integral

$$\int_0^{\infty}x^{s-1}\exp(-bx)\mathrm{d}x = \exp(-s\ln b)\Gamma(s). \tag{70}$$

Using this and keeping the terms up to $\mathcal{O}(\alpha^2, \beta)$, we obtain

$$\begin{aligned}f(\nu) = &\frac{1}{\kappa}\left[\frac{wA}{\kappa}\right]^{\frac{i\nu}{\kappa}}\exp\left(\frac{\pi\nu}{2\kappa}\right)\Gamma\left(-\frac{i\nu}{\kappa}\right) \\ &\times\left[1-\frac{\alpha\hbar\omega^3}{2c\kappa^2(1+\frac{i\nu}{\kappa})}+\frac{\beta\hbar^2\omega^4}{2c^2\kappa^2(1+\frac{i\nu}{\kappa})}\right. \\ &\left.-\frac{5\alpha^2\hbar^2\omega^4}{4c^2\kappa^2(1+\frac{i\nu}{\kappa})}+\frac{\hbar^2\alpha^2\omega^6}{8c^2\kappa^4(1+\frac{i\nu}{\kappa})(2+\frac{i\nu}{\kappa})}\right].\end{aligned} \tag{71}$$

Hence the power spectrum with a negative frequency is given by

$$\begin{aligned}|f(-\nu)|^2 = &\frac{2\pi}{\nu\kappa}\frac{1}{(e^{\frac{2\pi\nu}{\kappa}}-1)}\left[1-\frac{\alpha\hbar\omega^3}{c\kappa^2(\frac{\nu^2}{\kappa^2}+1)}\right. \\ &\left.+\frac{\beta\hbar^2\omega^4}{c^2\kappa^2(\frac{\nu^2}{\kappa^2}+1)}-\frac{5\,\alpha^2\hbar^2\omega^4}{2c^2\kappa^2(\frac{\nu^2}{\kappa^2}+1)}+\frac{3\alpha^2\hbar^2\omega^6}{2c^2\kappa^4(\frac{\nu^2}{\kappa^2}+1)(\frac{\nu^2}{\kappa^2}+4)}\right].\end{aligned} \tag{72}$$

From this we can obtain the power spectrum per logarithmic band to be

$$\begin{aligned}\nu|f(-\nu)|^2 = &\frac{2\pi}{\kappa}\frac{1}{(e^{\frac{2\pi\nu}{\kappa}}-1)}\left[1-\frac{\alpha\hbar\omega^3}{c\kappa^2(\frac{\nu^2}{\kappa^2}+1)}\right. \\ &\left.+\frac{\beta\hbar^2\omega^4}{c^2\kappa^2(\frac{\nu^2}{\kappa^2}+1)}-\frac{5\,\alpha^2\hbar^2\omega^4}{2c^2\kappa^2(\frac{\nu^2}{\kappa^2}+1)}+\frac{3\alpha^2\hbar^2\omega^6}{2c^2\kappa^4(\frac{\nu^2}{\kappa^2}+1)(\frac{\nu^2}{\kappa^2}+4)}\right].\end{aligned} \tag{73}$$

A few observations are in place now. We would first like to point out that the power spectrum becomes dependent on the frequency ω and hence the energy of the particle due to the presence of the GUP. This result is in contrast to the result in the absence of the GUP where the power spectrum is independent of the energy of the particle. Hence, one can infer that the energy of the particle provides a back reaction effect on the power spectrum due to the generalized uncertainty principle. This phenomenon is similar to the back reaction effects observed in rainbow gravity [26]. We would then like to point out that up to $\mathcal{O}(\alpha)$ our result does not agree with that obtained in [20], and there appears to be an error in the result in [20] in the power of κ.

Further, setting $\beta = 2\alpha^2$ in the above relation gives

$$\nu|f(-\nu)|^2 = \frac{2\pi}{\kappa}\frac{1}{(e^{\frac{2\pi\nu}{\kappa}}-1)}\left[1-\frac{\alpha\hbar\omega^3}{c\kappa^2(\frac{\nu^2}{\kappa^2}+1)} - \frac{\alpha^2\hbar^2\omega^4}{2c^2\kappa^2(\frac{\nu^2}{\kappa^2}+1)} + \frac{3\alpha^2\hbar^2\omega^6}{2c^2\kappa^4(\frac{\nu^2}{\kappa^2}+1)(\frac{\nu^2}{\kappa^2}+4)}\right]. \quad (74)$$

Up to $\mathcal{O}(\alpha)$, the above result reduces to

$$\nu|f(-\nu)|^2 = \frac{2\pi}{\kappa}\frac{1}{(e^{\frac{2\pi\nu}{\kappa}}-1)}\left[1-\frac{\alpha\hbar\omega^3}{c\kappa^2(\frac{\nu^2}{\kappa^2}+1)}\right]. \quad (75)$$

From the non-negativity of the power spectrum, the above result imposes a constraint on the linear GUP parameter α, which reads

$$\frac{\alpha\hbar\omega^3}{c\kappa^2(\frac{\nu^2}{\kappa^2}+1)} < 1. \quad (76)$$

Setting $\alpha = 0$ in eq. (73), we obtain the power spectrum per logarithmic band for the simplest form (quadratic) of the GUP to be

$$\nu|f(-\nu)|^2 = \frac{2\pi}{\kappa}\frac{1}{(e^{\frac{2\pi\nu}{\kappa}}-1)}\left[1+\frac{\beta\hbar^2\omega^4}{c^2\kappa^2(\frac{\nu^2}{\kappa^2}+1)}\right]. \quad (77)$$

It is interesting to note from the above result that the quadratic form of the GUP gives no constraint in contrast to the linear form of the GUP which gives a constraint given by eq. (76). Reassuringly we recover the standard result in the limit α, $\beta \to 0$ [25]

$$\nu|f(-\nu)|^2 = \frac{2\pi}{\kappa}\frac{1}{(e^{\frac{2\pi\nu}{\kappa}}-1)}. \quad (78)$$

This indeed shows that the power spectrum is independent of the energy of the particle.

Conclusions. – In this paper we have proposed a new form of generalized uncertainty principle which contains both the linear as well as the quadratic terms in the momentum, from which we derive the corresponding modified dispersion relation. We compare this with the dispersion relation in rainbow gravity and observe that both have a very similar structure. The new form of the generalized uncertainty principle reduces to the known forms existing in the literature in appropriate limits. The modified velocity of photons is then obtained from the modified dispersion relation and shows that it is energy dependent, and hence allows for a superluminal propagation. We then derive the $(1+1)$-dimensional Klein-Gordon equation taking into account the effects of the modified dispersion relation. Solving this equation iteratively, we obtain the power spectrum arising due to the Unruh effect. The result shows that the power spectrum depends on the energy of the particle owing its origin to the presence of the generalized uncertainty principle. This implies that the energy of the particle provides a back reaction effect due to the generalized uncertainty principle. This is similar to the back reaction effects observed in rainbow gravity [26]. Keeping terms up to leading order in the linear generalized uncertainty principle parameter in the power spectrum result, we observe that a constraint gets imposed on this parameter. However, the quadratic form of the generalized uncertainty principle gives no constraint in contrast to the linear form of the generalized uncertainty principle. Our results capture the effects of both the simplest form and the linear form of the generalized uncertainty principle.

* * *

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