Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Fluctuation theorems and large-deviation functions in systems not featuring a steady state

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Received 27 September 2019
Accepted for publication 8 January 2020
Published 26 February 2020

Online at stacks.iop.org/JSTAT/2020/024009
https://doi.org/10.1088/1742-5468/ab7126

Abstract. Motivated by the time behavior of the functional arising in the variational approach to the KPZ equation, we have adapted a path-integral scheme to deal with *unstable* systems. In a simple mesoscopic model and under two scenarios, we define a suitable mean value of (the exponential of) the entropy production between *arbitrary* initial and final states. This definition leads naturally to an integral fluctuation theorem (FT)—and on the way, to detailed and Crooks' FT. We also find the *general* form of a large-deviation function, as well as its particular form for a particle submitted to a constant force.

Keywords: fluctuation theorems, large deviations in non-equilibrium systems, stochastic thermodynamics

Contents

1. Introduction

It is well known that standard thermodynamics relies on two hypotheses: *thermodynamic equilibrium* and the *thermodynamic limit*. When both hold, a few collective variables (or their Legendre transforms for non-isolated systems) suffice for astonishingly fruitful reasoning. As the first hypothesis is relaxed, the key concept of *entropy production* (EP) emerges. Characterized by minimum EP, *nonequilibrium steady states* enjoy the privileged status that equilibrium states hold in standard thermodynamics and in fact, include the latter as the zero EP case. Within this context, standard thermodynamics' 'second law' has been rephrased into a handful of *fluctuation theorems*.

The *mesoscopic approach* initiated by Langevin, enabled a safe route to relax the second hypothesis: one still considers collective variables, but fluctuating ones. Here, nonequilibrium steady states have divergenceless probability current (which becomes zero for equilibrium states, so fulfilling the detailed-balance condition). This fruitful approach, exploited intensively for more than three decades, has given rise to the so-called 'stochastic thermodynamics' [1–5]. In the 'trajectory version' of this increasingly employed framework, the thermodynamic quantities pick up the fluctuations of the collective variables along their phase-space trajectories.

Besides processes beginning and ending at equilibrium, works on stochastic thermodynamics and fluctuation theorems focus (either explicitly or by assuming divergenceless probability current) on nonequilibrium *steady states* [6–8]. What seems to be missing is the study of fluctuation theorems in systems *not* featuring a steady state, and initial (final) states drawn from *arbitrary* probability distribution functions. Our interest arose from our drive to undertake a stochastic thermodynamics of the KPZ system, starting from its variational approach [9, 10]. As shown in [11], the functional from which the KPZ equation stems decreases linearly with time, once the EW–KPZ crossover is overcome. The latter is interpreted as an activation or escape process toward an unstable state, for which a particle in a constant gravitational field has been considered as a metaphor [11].

Here we show that with a slight adaptation—we define a mean value of the exponential of the EP between *arbitrary* initial and final states—the usual Onsager–Machlup path-integral representation [12] (as developed e.g. in [6–8, 13]) yields meaningful results for unstable mesoscopic systems such as the toy model of [11], at least in the absence of an external protocol. We consider two scenarios: in section 2 the driving noise is white and in section 3, it is Ornstein–Uhlenbeck (OU)[4]. In each we state the formalism and derive detailed and integral fluctuation theorems. In section 4 we write out the general form of a large-deviation function for total entropy production, and its particular form in the case of a particle in a constant gravitational field. Conclusions are drawn in section 5.

2. White driving noise

In the following, we denote functionals as $\mathfrak{f}[x]$. We start from

$$\dot{x} = -V'(x) + \xi(t) \tag{1}$$

and throughout this section, $\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(t') \rangle = 2\,\varepsilon\,\delta(t-t')$. Our immediate focus is the conditional probability density functionals (pdf'l) $\hat{\mathfrak{p}}$ of following a *given* trajectory between the *fixed* ends (x_a, t_a) and (x_b, t_b).

The conditional pdf'l of tracing the trajectory $[x^F]:= x(t)$ from (x_a, t_a) to (x_b, t_b) is

$$\hat{\mathfrak{p}}^F[x^F] := \hat{\mathfrak{p}}^F([x^F]|x_a, t_a, x_b, t_b) = [P^F(x_b|x_a)]^{-1} \exp(-\mathfrak{S}^+[x^F])$$

namely, the (normalized) integrand in the Onsager–Machlup path-integral representation of the solution to the Fokker–Planck equation (FPE) [12]. The functional

$$\mathfrak{S}^+[x^F] = \int_{t_a}^{t_b} \mathrm{d}t\, L^+(x, \dot{x})$$

is known as 'the stochastic action' and

$$P^F(x_b|x_a) = \int_{x_a}^{x_b} \mathcal{D}[x^F]\, \hat{\mathfrak{p}}^F[x^F],$$

as 'the forward propagator' [12, 14] ($\mathcal{D}[x^F]$ is the integration measure over the forward trajectory between the *fixed* values x_a at t_a, and x_b at t_b). The Onsager–Machlup function or 'stochastic Lagrangian'

$$L^+(x, \dot{x}) = \frac{1}{4\varepsilon}[\dot{x} + V'(x)]^2 \tag{2}$$

is evaluated at $x(t)$, $\dot{x}(t)$.

[4] This choice is motivated by the memory effects arising in the variational approach to the KPZ problem [11].

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The conditional pdf'l of tracing the *same* trajectory back is

$$\hat{\mathfrak{p}}^B[x^B] := \hat{\mathfrak{p}}^B([x^B]|x_a,t_a,x_b,t_b) = [P^B(x_a|x_b)]^{-1}\exp(-\mathfrak{S}^-[x^B]),$$

with $[x^B] := x(t_a+t_b-t)$, and the backward stochastic action and propagator are

$$\mathfrak{S}^-[x^B] = \int_{t_a}^{t_b} \mathrm{d}s\, L^-(x(t),\dot{x}(t)), \quad P^B(x_a|x_b) = \int_{x_b}^{x_a} \mathcal{D}[x^B]\,\hat{\mathfrak{p}}^B[x^B].$$

$L^-(x,\dot{x})$ has the functional form (2) but it is evaluated at $x(s)$, $\dot{x}(s)$, $s=t_a+t_b-t$. It is immediate to verify that $L^-(x(t),\dot{x}(t)) = L^+(x(t),-\dot{x}(t))$.

2.1. Detailed fluctuation theorem and Crooks' theorem

It is also immediate to prove the so-called *detailed fluctuation theorem*

$$\frac{\hat{\mathfrak{p}}^F[x^F]}{\hat{\mathfrak{p}}^B[x^B]} = \exp\left[-\frac{1}{\varepsilon}\int_{t_a}^{t_b} \mathrm{d}t\,\dot{V}\right] = \exp\left(-\frac{\Delta V}{\varepsilon}\right) \tag{3}$$

(here $\Delta V := V(x_b) - V(x_a)$), analogous to $T\Delta S = \Delta U$.

Following [8], we introduce the EP functional

$$\mathfrak{R}^F[x^F] := \ln\left(\frac{\mathfrak{p}^F[x^F]}{\mathfrak{p}^B[x^B]}\right), \tag{4}$$

corresponding to the exchange entropy. For a *single* trajectory $\hat{\mathfrak{R}}^F = \ln(\hat{\mathfrak{p}}^F/\hat{\mathfrak{p}}^B)$ is purely reversible and

$$\hat{\mathfrak{R}}^F[x^F] = -\hat{\mathfrak{R}}^B[x^B]. \tag{5}$$

Now if x_a is drawn at t_a from a *fixed* (arbitrary, not necessarily stationary) distribution $p_a(x_a,t_a)$, we must consider the *joint-conditional* pdf

$$\mathfrak{p}^F[x^F] := \mathfrak{p}^F([x^F];x_a,t_a|x_b,t_b) = p_a(x_a,t_a)\,\hat{\mathfrak{p}}^F([x^F]|x_a,t_a,x_b,t_b), \tag{6}$$

and now $\mathfrak{R}^F$ depends also on x_a. Similarly, if x_b is drawn from a *fixed* distribution $p_b(x_b,t_b)$, we must consider

$$\mathfrak{p}^B[x^B] := \mathfrak{p}^B([x^B];x_b,t_b|x_a,t_a) = p_b(x_b,t_b)\,\hat{\mathfrak{p}}^B([x^B]|x_a,t_a,x_b,t_b), \tag{7}$$

and $\mathfrak{R}^B$ depends also on x_b. For notational compactness, hereafter we omit the conditional dependence of the $\hat{\mathfrak{p}}$'s and the joint-conditional dependence of the $\mathfrak{p}$'s.

It is worth to remark that p_b *is not* the one resulting from time evolution starting from p_a, and vice versa. For instance, if the support Ω_a of $p_a(x_a,t_a)$ were finite, its image under $[x^F]$ (a function of x_a) would not necessarily be so. Similarly, if the support Ω_b of $p_b(x_b,t_b)$ were finite, its image under $[x^B]$ would not necessarily be so. In other words, and at variance with previous works, they are completely arbitrary. Since the integration of bounded-support distributions can be extended to the real axis at no cost, hereafter we omit the limits of integrals over x_a and x_b, understanding that they go from $-\infty$ to $+\infty$. For a *given* trajectory

$$\hat{\mathfrak{R}}^F[x^F] = \mathfrak{R}^F[x^F] + R_0, \quad \hat{\mathfrak{R}}^B[x^B] = \mathfrak{R}^B[x^B] - R_0, \tag{8}$$

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R_0 being the ratio between the formation entropies of the final and initial states

$$R_0 := -\ln\left[\frac{p_b(x_b,t_b)}{p_a(x_a,t_a)}\right].$$

Our next concern are the distributions [8]

$$\varrho^F(R) := \mathbb{N}^F \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}[x^F]\,\mathfrak{p}^F[x^F]\,\delta(\mathfrak{R}^F[x^F]-R+R_0),$$
$$\varrho^B(R) := \mathbb{N}^B \int_{x(t_b)=x_b}^{x(t_a)=x_a} \mathcal{D}[x^B]\,\mathfrak{p}^B[x^B]\,\delta(\mathfrak{R}^B[x^B]-R+R_0),$$

of the total EP value R in *ensembles* of trajectories

—$\varrho^F(R)$ from (x_a,t_a) to (x_b,t_b), with x_a drawn from $p_a(x_a,t_a)$ and x_b fixed,

—$\varrho^B(R)$ from (x_b,t_b) to (x_a,t_a), with x_b drawn from $p_b(x_b,t_b)$ and x_a fixed,

with

$$\mathbb{N}^F = \left[\int \mathrm{d}x_a\,p_a\,P^F\right]^{-1}, \quad \mathbb{N}^B = \left[\int \mathrm{d}x_b\,p_b\,P^B\right]^{-1}.$$

From equations (6) and (8),

$$\varrho^F(R) = p_a(x_a,t_a)\,\mathbb{N}^F \int_{x_a}^{x_b} \mathcal{D}[x^F]\,\hat{\mathfrak{p}}^F[x^F]\,\delta(\mathfrak{R}^F[x^F]+R_0-R).$$

Using equation (3),

$$\varrho^F(R) = p_a(x_a,t_a)\,\mathbb{N}^F$$
$$\int_{x_a}^{x_b} \mathcal{D}[x^F]\,\hat{\mathfrak{p}}^B[x^B]\exp(\hat{\mathfrak{R}}^F[x^F])\,\delta(\mathfrak{R}^F[x^F]-(R-R_0)).$$

Using the property $f(x)\,\delta(x)=f(0)\,\delta(x)$ and recalling that p_a, p_b and R are *fixed*,

$$\varrho^F(R) = p_a(x_a,t_a)\exp(R-R_0)\,\mathbb{N}^F$$
$$\int_{x_a}^{x_b} \mathcal{D}[x^F]\,\hat{\mathfrak{p}}^B[x^B]\,\delta(\hat{\mathfrak{R}}^F[x^F]-(R-R_0))$$
$$= p_b(x_b,t_b)\exp(R)\,\mathbb{N}^F \int_{x_a}^{x_b} \mathcal{D}[x^F]\,\hat{\mathfrak{p}}^B[x^B]\,\delta(\hat{\mathfrak{R}}^F[x^F]-(R-R_0)).$$

Using equation (5),

$$\varrho^F(R) = p_b(x_b,t_b)\exp(R)\,\mathbb{N}^F \int_{x_a}^{x_b} \mathcal{D}[x^F]\,\hat{\mathfrak{p}}^B[x^B]\,\delta(\hat{\mathfrak{R}}^B[x^B]+(R-R_0)).$$

Finally, using equations (7) and (8), and recalling that $\mathcal{D}[x^F]\equiv\mathcal{D}[x^B]$,

$$\varrho^F(R) = \frac{\mathbb{N}^F}{\mathbb{N}^B}\exp(R)\,\varrho^B(-R).$$

Hence for any initial condition and any pair of times t_a and t_b,

$$\frac{\varrho^F(R)}{\varrho^B(-R)} = \frac{\mathbb{N}^F}{\mathbb{N}^B}\exp(R). \tag{9}$$

Let $p^F(R) := \int \mathrm{d}x_b\, \varrho^F(R)$ and $p^B(R) := \int \mathrm{d}x_a\, \varrho^B(R)$ be respectively the pdfs of EP R in the forward and backward evolutions. If we denote $\overline{\bullet} := \int \mathrm{d}x\, \bullet$, then $p^F(R) = \overline{\varrho^F(R)}$ and $p^B(R) = \overline{\varrho^B(R)}$. Using equation (9),

$$\begin{aligned}
p^F(R) &= \mathbb{N}^F \int \mathrm{d}x_b \int \mathrm{d}x_a\, p_a(x_a,t_a) \int_{x_a}^{x_b} \mathcal{D}[x^F]\, \hat{\mathfrak{p}}^F[x^F]\, \delta(\mathfrak{R}^F[x^F] - R + R_0) \\
&= \frac{\mathbb{N}^F}{\mathbb{N}^B}\mathbb{N}^B \int \mathrm{d}x_a \int \mathrm{d}x_b\, p_b(x_b,t_b) \\
&\quad \int_{x_a}^{x_b} \mathcal{D}[x^F]\, \hat{\mathfrak{p}}^B[x^B] \exp(\mathfrak{R}^F[x^F])\, \delta(\mathfrak{R}^F[x^F] - R + R_0) \\
&= \exp(R)\, \mathbb{N}^B \int \mathrm{d}x_a \int \mathrm{d}x_b\, p_b(x_b,t_b) \int_{x_a}^{x_b} \mathcal{D}[x^B]\, \hat{\mathfrak{p}}^B[x^B]\, \delta(\mathfrak{R}^B[x^B] + R - R_0).
\end{aligned}$$

Since $p^B(-R) := \overline{\varrho^B(-R)} = \mathbb{N}^B \int \mathrm{d}x_a\, \varrho^B(-R)$, we obtain an out-of-equilibrium analog of the Crooks' theorem [15],

$$p^F(R) = \exp(R)\, p^B(-R), \tag{10}$$

related in turn with the 'Gallavotti–Cohen' theorem [16, 17]. As t_b can be taken arbitrarily large, we can make a connection with the *large-deviation function* [13, 18] through the relation

$$\zeta(R) = -\lim_{t_b\to\infty} t_b^{-1} \ln p^F(R). \tag{11}$$

We will come back to this point later.

2.2. Integral fluctuation theorem

Denoting the usual ensemble average by $\langle\bullet\rangle := \int \mathrm{d}x\, p(x)\, \bullet$, we define the following mean value

$$\begin{aligned}
&\overline{\langle \exp(-\hat{\mathfrak{R}}^F)\rangle} := \\
&\mathbb{N}^F \int \mathrm{d}x_a\, p_a(x_a,t_a) \int \mathrm{d}x_b \int_{x_a}^{x_b} \mathcal{D}[x^F]\, \hat{\mathfrak{p}}^F[x^F] \exp(-\hat{\mathfrak{R}}^F[x^F]) \\
&= \int \mathrm{d}x_a\, p_a(x_a,t_a) \int \mathrm{d}x_b\, \frac{p_b(x_b,t_b)}{p_a(x_a,t_a)} \frac{\mathbb{N}^F}{\mathbb{N}^B}\mathbb{N}^B \int_{x_a}^{x_b} \mathcal{D}[x^B]\, \hat{\mathfrak{p}}^B[x^B] \\
&= \mathbb{N}^B \int \mathrm{d}x_a \int \mathrm{d}x_b\, p_b(x_b,t_b) \int_{x_a}^{x_b} \mathcal{D}[x^B]\, \mathfrak{p}^B[x^B].
\end{aligned}$$

The integrand in the last line is the transition pdf from x_b to x_a, still conditioned to the fixed distribution $p_b(x_b,t_b)$. Calling

$$\tilde{p}^B(x_b,t_b|x_a,t_a) = \mathbb{N}^B \int \mathrm{d}x_a \int \mathrm{d}x_b\, p_b(x_b,t_b) \int_{x_a}^{x_b} \mathcal{D}[x^B]\, \mathfrak{p}^B[x^B],$$

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we have

$$
\begin{aligned}
&\overline{\langle \exp(-\hat{\mathfrak{R}}^F)\rangle} \\
&= \mathbb{N}^F \int \mathrm{d}x_a \, p_a(x_a, t_a) \int \mathrm{d}x_b \int_{x_a}^{x_b} \mathcal{D}[x^F] \, \mathfrak{p}^F[x^F] \, \exp(-\hat{\mathfrak{R}}^F[x^F]) \\
&= \mathbb{N}^B \int \mathrm{d}x_a \int \mathrm{d}x_b \, p_b(x_b, t_b) \int_{x_a}^{x_b} \mathcal{D}[x^B] \, \mathfrak{p}^B[x^B] = \tilde{p}^B(x_b, t_b | x_a, t_a).
\end{aligned}
\tag{12}
$$

Since the last quantity is $\equiv 1$, equation (12) yields the usual form of the *integral theorem*

$$
\overline{\langle \exp(-\hat{\mathfrak{R}}^F)\rangle} \equiv 1.
$$

3. Ornstein–Uhlenbeck driving noise

With a little additional effort, we may extend our analysis to the case in which our toy model is submitted to an OU external noise with self-correlation time τ [19],

$$
\dot{x} = -V'(x) + \eta(t), \quad \langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t') \rangle = \frac{\varepsilon}{\tau} \exp\left(-\frac{|t - t'|}{\tau} \right). \tag{13}
$$

This can in turn be generated dynamically, driven by a white noise $\xi(t)$,

$$
\dot{\eta} = -\frac{1}{\tau}[\eta + \xi(t)], \tag{14}
$$

where as in (1), $\xi(t)$ fulfills $\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = 2\,\varepsilon\,\delta(t - t')$. Qualitatively, for $t \gtrsim \tau$, one expects to recover the picture of section 2.

The FPE associated to the Langevin equations (13) and (14) is

$$
\partial_t p = \partial_x [V'(x)\, p] + \frac{1}{\tau} \partial_\eta (\eta p) + \frac{\varepsilon}{\tau^2} \partial_\eta^2 p,
$$

where $p \equiv p(x, \eta, t | x_0, \eta_0, t_0)$—$x_0$, η_0 being the initial conditions—is a Markov process in (x, η) space and admits a path-integral representation. The difficulty of the diffusion matrix being singular is overcome by performing the functional integral in the *phase space* of the variables x, η and their canonically conjugate momenta. Once functional integration over the latter and over η have been performed, we retrieve the analog of equation (2),

$$
L^+(x, \dot{x}, \ddot{x}(t)) = \frac{1}{4\varepsilon} [\tau \ddot{x} + \dot{x} + V'(x)]^2.
$$

Again since $\ddot{x}(t)$ does not change its sign, $L^-(x(t), \dot{x}(t), \ddot{x}(t)) = L^+(x(t), -\dot{x}(t), \ddot{x}(t))$.

Following the steps of section 2, we define

$$
\hat{\mathfrak{p}}^F[x^F, \eta^F] = \mathbb{N}^B \exp(-\mathfrak{S}^+[x^B, \dot{x}^B, \ddot{x}^B]),
$$

and similarly for $\hat{\mathfrak{p}}^B[x^F, \eta^B]$. The analogs of equations (3) and (5) are now

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$$\frac{\hat{\mathfrak{p}}^F[x^F,\eta^F]}{\hat{\mathfrak{p}}^B[x^B,\eta^B]} = \exp[-(\tau\Delta T + \Delta V)/\varepsilon], \tag{15}$$

$$\mathfrak{R}^F[x^F,\eta^F] = -\mathfrak{R}^B[x^B,\eta^B] = (\tau\Delta T + \Delta V)/\varepsilon \tag{16}$$

with $\Delta T := \frac{1}{2}[\dot{x}^2(t_b) - \dot{x}^2(t_a)]$. Note that for $t_b - t_a < \tau$ and $\tau \gg 1$, the kinetic-like energy difference dominates over the potential one. The extra terms come from memory effects related with the EP involved in the preparation of those states.

The forward and backward propagators are

$$P^F(x_b,\eta_b,t_b|x_a,\eta_a,t_a) = \int_{x_a,\eta_a}^{x_b,\eta_b} \mathcal{D}[x^F,\eta^F]\,\hat{\mathfrak{p}}^F[x^F,\eta^F],$$
$$P^B(x_a,\eta_a,t_a|x_b,\eta_b,t_b) = \int_{x_b,\eta_b}^{x_a,\eta_a} \mathcal{D}[x^B,\eta^B]\,\hat{\mathfrak{p}}^B[x^B,\eta^B],$$

with *fixed* (x_a,η_a), (x_b,η_b). Using a similar notation as before (recalling again that p_b *is not* the one resulting from time evolution starting from p_a) equation (9) becomes

$$\frac{\varrho^F(R)}{\varrho^B(-R)} = \frac{\mathbb{N}^F}{\mathbb{N}^B}\,\frac{p_a(x_a,\eta_a,t_a)}{p_b(x_b,\eta_b,t_b)}\,\exp(R), \tag{17}$$

with fixed R. The procedure leading to equation (17) parallels that of equation (9), but the functional integrations are performed over $[x]$ and $[\eta]$. With the definitions

$$p^F(R) := \int \mathrm{d}x_b\,\mathrm{d}\eta_b\,\varrho^F(R), \quad p^B(R) := \int \mathrm{d}x_a\,\mathrm{d}\eta_a\,\varrho^B(R),$$

equation (10)—the analog of the Crooks' theorem—stays the same. Now the reversible entropy, equation (8), reads

$$\hat{\mathfrak{R}}^F[x^F,\eta^F] = \mathfrak{R}^F[x^F,\eta^F] + \ln\left[\frac{p_b(x_b,\eta_b,t_b)}{p_a(x_a,\eta_a,t_a)}\right],$$

where $p_b(x_b,\eta_b,t_b)$ and $p_a(x_a,\eta_a,t_a)$ are *arbitrary but fixed* as in the previous case.

Again, suitable definitions of $\langle\exp(-\hat{\mathfrak{R}}^F)\rangle$ and $p^B(x_b,\eta_b,t_b|x_a,\eta_a,t_a)$ lead to the *integral theorem*

$$\overline{\langle\exp(-\hat{\mathfrak{R}}^F)\rangle} = \hat{p}^B(x_b,\eta_b,t_b|x_a,\eta_a,t_a) \equiv 1. \tag{18}$$

4. Large-deviation function

From equation (4) we can write

$$\langle\mathfrak{R}^F\rangle = \langle\ln\mathfrak{p}^F[x^F]\rangle - \langle\ln\mathfrak{p}^B[x^B]\rangle,$$

resembling the Gallavotti–Cohen theorem [16, 17]. As shown by equation (10), the second term on the r.h.s. is exponentially small as compared to the first one, so for $t_b \to \infty$ and fixed t_a,

$$\zeta(R) = \lim_{t_b \to \infty} t_b^{-1} \Delta V/\varepsilon = - \lim_{t_b \to \infty} t_b^{-1} \langle \ln \mathfrak{p}^F[x^F] \rangle. \tag{19}$$

In order to proceed further, some information (or at least an assumption) is required on the time behavior of the solution to the FPE. In the scaling regime of a large enough KPZ system (between the Edwards–Wilkinson/KPZ crossover and saturation) the interface evolves at constant speed v_∞ and its roughness width scales as t^β. Hence $\mathfrak{p}(h,t) \to \mathfrak{p}(\tilde{h})$, with $\tilde{h} = (h - v_\infty t)/t^\beta$[5]. Mimicking this case, we have considered $V(x) = -Fx$, the solution to whose FPE is the self-similar, diffusion-like expression

$$P(x,t|0,0) = \frac{\exp[-(x - Ft)^2/4\varepsilon t]}{\sqrt{4\pi\varepsilon t}}. \tag{20}$$

The asymptotic speed is $v_\infty = F$, the variance grows $\propto t$, and the normalization changes from one instant to another!

Since as $t_b \to \infty$, the relative standard deviation of x around Ft decreases as $t_b^{-1/2}$, equation (19) reads

$$\zeta(R) = \frac{F^2}{\varepsilon}. \tag{21}$$

This LDF being constant, it has no singularities that could indicate a phase transition.

In the case of OU-noise driving, the condition $t_b \gg \tau$ is fulfilled and equation (23), analogously to (21), reads again

$$\zeta(R) = \frac{F^2}{\varepsilon}. \tag{22}$$

From equation (16), the analog of equation (19) turns out to be

$$\zeta(R) = \lim_{t_b \to \infty} t_b^{-1} (\tau \Delta T + \Delta V)/\varepsilon = - \lim_{t_b \to \infty} t_b^{-1} \langle \ln \mathfrak{p}^F[x^F] \rangle, \tag{23}$$

for fixed t_a.

5. Conclusions

Motivated by the time behavior of the functional from which the KPZ equation stems [9, 10] and intending to undertake a stochastic thermodynamics of such a system, we have focused on develop the ground for obtaining detailed and integral fluctuation theorems (as well as large-deviation functions) for systems not featuring a steady state (namely, unstable systems), and initial (final) states drawn from *arbitrary* probability distribution functions. As stated in [11], the interface evolution can be interpreted as an activation or escape process toward an unstable state, for which a particle in a constant gravitational field is a suitable metaphor. Here we have shown that by just

[5] Moreover, h fluctuations' correlation length grows as $t^{1/z}$, leading to a dependence of $\tilde{h}$ on the scaling variable $|x|/t^{1/z}$.

slightly adapting the usual Onsager–Machlup path-integral representation [8, 12, 13], it is possible to get meaningful results for unstable mesoscopic systems like the toy model introduced in [11]. Among the next steps, we consider redoing the calculations under a time-varying protocol. A simple possibility would be to consider a force that depends linearly on time: $F = F_o + (F_1 - F_o)(t - t_a)/(t_b - t_a)$.

The feasibility of our main objective, namely a direct thermodynamic analysis of the KPZ equation within the framework of the variational approach [9, 10], depends now only on adapting the present path-integral methodology to a spatially extended system (which, as a matter of fact, is straightforward). Such an analysis is underway, and will be the subject of a forthcoming work.

Acknowledgments

The authors wish to pay tribute to the memory of Professor Christian Van den Broeck, a great physicist and a friend, and thank Miguel A Rodríguez warmly for invaluable advice. RRD acknowledges support by UNMdP, through project EXA935/19–15/E893. HSW acknowledges financial support received from the Agencia Estatal de Investigación (AEI, MCIU, Spain) and Fondo Europeo de Desarrollo Regional (FEDER, UE), under Project PACSS (RTI2018-093732-B-C21/C22) and the Maria de Maeztu Program for units of Excellence in R&D (MDM-2017-0711).

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https://doi.org/10.1088/1742-5468/ab7126