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Weyl symmetry in stochastic quantum gravity

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Abstract

We propose that the gauge principle of d-dimensional Euclidean quantum gravity is Weyl invariance in its stochastic $(d+1)$-dimensional bulk. Observables are defined as depending only on conformal classes of d-dimensional metrics. We work with the second order stochastic quantization of Einstein equations in a $(d+1)$-dimensional bulk. There, the evolution is governed by the stochastic time, which foliates the bulk into Euclidean d-dimensional leaves. The internal metric of each leaf can be parametrized by its unimodular part and conformal factor. Additional bulk metric components are the ADM stochastic lapse and a stochastic shift. The Langevin equation determines the acceleration of the leaf as the sum of a quantum noise, a drift force proportional to Einstein equations and a viscous first order force. Using Weyl covariant decomposition, this Langevin equation splits into irreducible stochastic equations, one for the unimodular part of the metric and one for its conformal factor. For the first order Langevin equation, the unphysical fields are the conformal factor, which is a classical spectator, and the stochastic lapse and shift. These fields can be gauge-fixed in a BRST invariant way in function of the initial data of the process. One gets observables that are covariant with respect to internal reparametrization in each leaf, and invariant under arbitrary reparametrization of the stochastic time. The interpretation of physical observable at finite stochastic time is encoded in a transitory $(d+1)$-dimensional phase where the Lorentz time cannot be defined. The latter emerges in the infinite stochastic time limit by an abrupt phase transition from quantum to classical gravity.

Keywords: Weyl symmetry, stochastic quantum gravity, Langevin equation

1. Introduction

It has been proposed in [1] that quantum gravity might obey the laws of stochastic quantization [2], governed by a second order rather than first order stochastic equation. One theoretical motivation is the suppression of the problem of Lorentz time, which one systematically encounters in the standard QFT formulations. A physical motivation for using stochastic quantisation in [1] was to predict that Lorentz time can only emerge as a signature of the exit of inflation by an abrupt phase transition from quantum to classical gravity, which gives also a heuristic description to primordial cosmology as well as to the ultra-short distance scattering of point-like particles.

In this paper we refine the definition of the Langevin equation for gravity of [1] making it genuinely geometric in the stochastic bulk. We arrive at the important conclusion on the definition of observables: the statement that observables in 2-dimensional gravity depend only on the conformal classes of metrics [3] can be extended to $d > 2$. We thus claim that the gauge symmetry principle of observables in d-dimensional Euclidean gravity is the invariance under Weyl symmetry in a $(d+1)$-dimensional bulk $\mathcal{M}$, modulo the internal reparametrization of its Euclidean d-dimensional leaves Σ.

Our proposal for the definition of quantum gravity is in fact inspired by the observation that what really matters in classical gravity is the propagation of conformal classes of spatial metrics. To the best of our knowledge, this property was firstly advocated in the physical literature in [4], where some of the Einstein equations of motions were cornered out as dictating only physically irrelevant propagation of constraints. See also the more recent works [5], where the 'relativity of local size' is implemented to find solutions to the Einstein equations and equations in York's conformal technique to solve the initial value problem. This implies that the initial physical data for solving Einstein equations of motion only concern conformal classes of spatial metrics, and our definition of observables of quantum gravity implies this classical property[4]. It will indeed appear that the conformal factor is left invariant in the last steps of the stochastic time evolution where it is not submitted to relevant quantum effects.

In practice, we propose that in quantum gravity observables must be reparametrization covariant functionals of the unimodular part of the metric, that is, the physics of quantum gravity is carried by conformal classes of metrics. This is motivated by the work of York [4], which demonstrated the role of conformal classes of spatial metrics in the description of the evolution in classical gravity and the observables. We show in our quantization procedure that functionals of the unimodular part of the metric possess emblematic properties at late stochastic time. We emphasize that defining unimodular classical gravity has been one of the early ideas of Einstein, in view of the existence a non-vanishing cosmological constant. Papers have investigated the notion of unimodular gravity, as e.g. [7] and enclosed references. There are basically two standard formulation of unimodular gravity: one imposes $\det(g\mu v) = 1$ as a gauge choice while the other imposes it as a constraint. Our point of view is that in a formal perturbative treatment of quantum gravity using the Einstein action as a classical action plus BRST invariant gauge-fixing term, one can do a BRST invariant gauge-fixing of the metrics $g\mu v$ with gauge functions $\sqrt{\det(g\mu v)} - 1$ and $\partial\hat{g}\mu v$, where $\hat{g}\mu v$ is the unimodular part of the metrics, which gives a gauge-fixed BRST local action that defines a perturbation expansion for the correlators of $\hat{g}\mu v$, while $\sqrt{\det(g\mu v)} = 1$. In this way any given classical solution of Einstein equations can be rewritten as a unimodular metric by an appropriate choice of coordinates system. We can then extend it at the perturbative quantum level using a unimodular propagating metric, while the conformal factor is spectator. This is compatible with the conclusion that we draw in this paper by studying the stochastic quantization of gravity.

[4] In fact, mathematicians found already in 1925 the relevance of Weyl symmetry for solving Einstein equations [6].

We postulate the existence of the stochastic bulk, whose leaves host a Euclidean d-dimensional theory. The quantum stochastic bulk correlators asymptote to the ones of classical Euclidean gravity at infinite stochastic time. This is of course a completely different framework than the classical approach of [4], where it was showed that it is the d-dimensional Lorentz spacetime itself that can be foliated by spatial $(d-1)$-dimensional leaves, and only conformal classes of spatial metrics matter when solving the classical Einstein evolution.

It must be clear also that the theory is not conformal gravity at late stochastic time, since the stochastic evolution is based on Einstein equations of motion, which are not Weyl covariant. Having equations of motion that are not scale invariant is actually not in contradiction with the postulate that quantum observables of quantum gravity are Weyl invariant functionals of the metric. In fact this definition of observables does not imply that the dynamical evolution in the stochastic bulk is Weyl invariant.

Because we propose that the gauge principle for quantum gravity observables is that of Weyl symmetry, a BRST invariant gauge-fixing of the unphysical fields will be needed. By using an ADM type parametrization for the stochastic bulk [7], the unphysical geometrical fields will be identified as the conformal factor of the d-dimensional metric (in the first order Langevin theory) and the lapse and shift functions of the $(d+1)$-dimensional stochastic bulk. The gauge-fixing procedure will be implemented in the $(d+1)$-dimensional quantum field theory defined by stochastic quantization, namely within the context of equivariant topological quantum field theory.

Contrarily to standard d-dimensional quantization methods, stochastic quantization possesses the ingredients to clarify under which circumstances can one get an 'emerging' Lorentz time, by checking (a hard task) the possibility of an analytic continuation of the stochastic time correlators of the unimodular metric on each Euclidean d-dimensional leaf, computed at a finite stochastic time.

Let us stress that having an acceleration term in the stochastic time evolution equation of the metric, heuristically motivated in [1], is a new input which in cosmological models could explain the exit from inflation by a sharp transition and the emerging of the classical Lorentz time. Such an acceleration tensor along the normal of each leaf has not been often used in the geometry of foliated spaces and its investigation is quite inspiring for visualizing the dynamics of leaves.

This work makes precise the idea that there is no limit of infinite stochastic time in the quantum phase of gravity except for $\hbar = 0$, modulo some formal perturbations for taking into account the possibility of emitting and absorbing perturbative traceless gravitons (namely, excitations of the unimodular part of the classical backround $\hat{g}_{\mu\nu}$). Such perturbative modifications are in fact compatible with the framework of stochastic quantization. In fact, instead of having a hypothetical d-dimensional equilibrium distribution for correlators of the metric in the $\tau = \infty$ limit and $\hbar \neq 0$, there are oscillations in the stochastic time because of the acceleration term. These oscillations express the dynamics of quantum gravity. They may sharply stop by a brutal transition toward classical gravity where, effectively, the limit at infinite stochastic time can be reached, and the whole theory can be directly computed using standard Euclidean QFT methods in d dimensions. The effects of quantum gravity can thus only manifest themselves at finite stochastic time, within the framework of a $(d+1)$-dimensional quantum field theory, giving a specific ultra-short distance physics [1].

A non-trivial part of the program for building covariant Langevin equations of gravity relies on the equivariant topological supersymmetry hidden in all Langevin equations [2, 8]. Indeed, in the case of stochastic quantization of a system with local symmetries, one needs additional gauge degrees of freedom in the stochastic bulk [9, 10][5].

[5] In appendix , we sketch for the sake of completeness the improvements needed to define stochastic quantization of a theory with gauge invariance and impose a gauge restoring force along its gauge orbits.

In this paper the equivariance is with respect to $\mathrm{Diff}_\Sigma \ltimes \mathrm{Weyl}_\mathcal{M}$, namely the semi-direct product of the *d*-dimensional diffeomorphism symmetry in each leaf and the Weyl symmetry in the whole bulk. In fact $\mathcal{M}$ is asssumed to be $\Sigma \times \mathbb{R}$ and Diff_Σ acts on Σ and hence on $\mathcal{M}$ and on the space of metrics on $\mathcal{M}$. This equivalence allows a proper definition of observables, in the same spirit of the definition of topological observables in a topological quantum field theory. We remark that although the product structure $\Sigma \times \mathbb{R}$ of $\mathcal{M}$ seem to prevent topology changes of the leaf Σ in the τ-evolution, we will postulate that the metric on $\mathcal{M}$ satisfies a highly non-linear second order Langevin equation (3.1). Solutions to such equations typically develop singularities in the course of time evolution, even though the initial data are completely smooth. Some of these singularities can be interpreted as topology changes of the leaf Σ in the τ-evolution. However, we will be mostly concerned with the smooth solutions without topology change in this paper.

To achieve our program, some difficulties have to be overcome by decomposing irreducibly all quantities under the representation of $\mathrm{Diff}_\Sigma \ltimes \mathrm{Weyl}_\mathcal{M}$. This is in fact a necessary task to get the correct covariant expression of the stochastic time acceleration of the metric. For this, we have used the ADM decomposition, substituting basically the Lorentz time of the current ADM formalism [7] with the stochastic time, with a different dynamics led by *d*-dimensional Einstein equations of motions plus additional forces to ensure well-defined drift forces along the gauge orbit directions. To unravel the Weyl covariance and implement the unimodular decomposition we found the recent work of [11] extremely useful, in the spirit of [6]. More recent works have dealt with Weyl symmetry, in a similar fashion as our here, in various disparate domains, such as scale-invariant gravity [12], holography [13] and hydrodynamics [14].

The unimodular decomposition implemented in this paper will be dictated by a change of field variables which is pivotal to our results expressing observables as Weyl invariant functionals. We call this method the 'golden rule', which allows us to decompose neatly all geometrical quantities in function of Weyl invariant Σ-tensors plus terms depending on the conformal factor. In fact, our 'golden rule' bears some technical and conceptual similarities with the 'dressing field method' used in [15].

Compared with a vast literature on quantum gravity, the novelty in our work is to postulate that time is not a fundamental parameter to order phenomena. Rather it may (or may not) emerge, depending on the configuration of the stochastic leaves at a given value of the stochastic time. Moreover, we give a physical meaning to the so-called stochastic time. We claim that the latter is a physical microscopic time that is an alternative to the Lorentz time, when the latter does not exist.

Prior to establishing a rigorous connection between the Dirac observables, say in the Wheeler–de Witt quantization, and the observables that we introduce, we point out a substantial conceptual difference between our approach and the traditional methodology. In the former, one first defines a Euclidean quantum field theory using a more general stochastic quantization process and ask afterward the physical interpretation of the correlators of the metrics in each constant stochastic time leaf. In the latter, one relies on the validity at the quantum level of the classical notions such as Dirac observables, as in [16]. Often, the presentation does not resolve key problems such as constraints posed by Wheeler–de Witt or the indefiniteness of path integrals in gravity.

In cases where the Wheeler–de Witt superspace technique does lead to well defined computations in quantum gravity, we can reproduce these results by the stochastic approach because they are obtained from a well defined 4-dimensional path integral. Some perturbative aspects of quantum gravity enters the discussions of Dirac observables, as in [17], but they can also be accommodated within the context of stochastic quantization, which is generally well defined at infinite stochastic time, order by order in perturbation theory, using relevant cutoffs. In cases where statistical physics is described directly in term of a Boltzmann distribution, Langevin-equation techniques are not needed, and the situation in Euclidean quantum field theory is completely analogous. Notice, however, that for Dirac observables, the gauge symmetry comes only from diffeomorphisms, but the superspace analysis could perhaps be enriched by the use of Weyl invariance. Indeed, this invariance plays a key role in our definition of the observables, as well as in the work of York.

This article is structured as follows. Section 2 introduces the technical structure to describe the stochastic bulk: its ADM decomposition and the speed and acceleration of the leaves. Section 3 is devoted to the Langevin equation and its covariance on the stochastic leaves. We then analyze various contributions from the Langevin equation, both physically and mathematically. To enlighten the Weyl properties of the Langevin equation, section 4 deals with the decompositions into the traceless and trace parts of various quantities. The 'golden rule' described above is made explicit in section 5 to perform the unimodular decomposition of the traceless and trace parts of the Langevin equations. Eventually, section 6 discusses the observables in the quantum stochastic bulk. There we show the presence of a residual x-independent Weyl symmetry which is used, together with the BRST symmetry, to localize the conformal factor. We then specialize to some relevant sub-cases of the Langevin equation. This is possible because we use a second order Langevin equation which involves a physical dimensionful parameter ΔT, whose value defines the relative strength between the acceleration and viscous effects in the second order Langevin equation.

After concluding remarks in section 7, appendix sketches a proof of the invariance of the gauge-invariant-observables evolution under the addition of a gauge fixing restoring term in the first order Langevin equation.

2. ADM decomposition of the stochastic bulk

To put in a geometrical framework the heuristic second order Langevin equation depicted in [1], the language of foliation inside the stochastic bulk is most useful.

If we call x^μ the coordinates of classical Euclidean Einstein d-dimensional gravity with metric $g_{\mu\nu}(x)$ and action $S = \int \mathrm{d}^{\mathrm{d}}x\sqrt{g}R(g_{\mu\nu})(x)$ (in suitable units), the basic idea of stochastic quantization is the extension

$$
\begin{aligned}
&x^\mu \to x^A \equiv (x^\mu, \tau), \\
&g^{\mu\nu}(x) \to (g^{\mu\nu}(x,\tau),\ g^{\mu\tau}(x),\ g^{\tau\tau}(x,\tau)).
\end{aligned}
\tag{2.1}
$$

The aim is to define a quantum field theory in the $\{x, \tau\}$ space, with a flow of correlators of the x- and τ-dependent fields, toward a certain limit when $\tau \to \infty$[6].

To describe this geometrical system, it is appropriate to use the ADM parametrization [7] for the pseudo-Euclidean squared length

$$
\mathrm{d}s^2 = -N^2\mathrm{d}\tau^2 + (\mathrm{d}x^\mu + N^\mu\mathrm{d}\tau)g_{\mu\nu}(\mathrm{d}x^\nu + N^\nu\mathrm{d}\tau)
\tag{2.2}
$$

of any given infinitesimal line element in the stochastic bulk[7]. In this expression $N^\mu = N^2 g^{\mu\tau}$ is the stochastic shift vector and N is the lapse, such that $g_{\tau\tau} = -N^2 + N_\mu N^\mu$. The determinant of the $(d+1)$-dimensional metric is $-N^2 g$, where $g = \det(g_{\mu\nu}) > 0$.

[6] For renormalizable theories such as Yang–Mill, τ-dependent correlators flow smoothly when $\tau \to \infty$ toward the correlators of the standard path integral in d-dimensions, but in gravity the situation is different because there is no equilibrium distribution for $\hbar \neq 0$ and there is no smooth limit to classical gravity.

[7] The signature of the total $(d+1)$-dimensional metric is $(-, +, \cdots, +)$.

There is no demand for full reparametrization invariance in the total space $\mathcal{M} = \{(x^\mu, \tau)\}$. Rather, the pseudo-Euclidean $(d+1)$-dimensional bulk $\mathcal{M}$ is foliated by the stochastic time τ, with equal-stochastic-time Euclidean d-dimensional leaves $\Sigma \equiv \Sigma(x^\mu, \tau)$ having internal Euclidean metric $g_{\mu\nu}(x, \tau)$. Upper indexes $\mu, \nu \ldots$ are lowered by the d-dimensional tensor $g_{\mu\nu}(x, \tau)$ in a way that preserves the reparametrization symmetry of Σ. The foliation using the coordinate τ, gives an absolute meaning to τ, modulo some possible one-dimensional reparametrization $\tau \to \tau = \tau'(\tau)$.

For this reason, we can postulate that $N^\mu = N^2 g^{\mu\tau} = N^\mu(x)$ is τ-independent. This condition is preserved by diffeomorphisms on Σ of the form $x^\mu \to x'^\mu(x)$, whose infinitesimal transformations are represented by the Lie derivative $\pounds^\Sigma_\xi$ along a τ-independent vector field $\xi^\mu(x)$ (the operation $\pounds^\Sigma_\xi$ only involves $g_{\mu\nu}$). Note however that $N_\mu = g_{\mu\tau}$ is x- and τ-dependent, as well as $N(x, \tau)$.

This decomposition is for a different purpose from that of [7], where the Minkowski time was used to define the foliation. Here the foliation parameter is the stochastic time. The physical meaning differs completely.

We will shortly consider Weyl covariance in our presentation: it dilates locally the metric fields but not the coordinates x^μ and τ. Equation (2.2) shows that N^μ is Weyl invariant from the beginning, since $\mathrm{d}x^\mu + N^\mu \mathrm{d}\tau$ must be a Weyl invariant one-form. This condition is the tip of an iceberg, made of all Weyl transformations of fields and curvatures in $\mathcal{M}$. The technology is as in, e.g. [11], where it is actually used in the different context of standard Hamiltonian classical gravity.

Symmetrization is defined as $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$ and ∇_μ is the Levi-Civita connection of Σ with Christoffel coefficients $\Gamma^\gamma_{\alpha\beta}(g_{\mu\nu}(x, \tau))$ (involving no τ derivatives). The Riemann tensor of Σ, $R^\alpha{}_{\beta\gamma\delta}(g_{\mu\nu})$, is derived from the Christoffel symbols by the standard formula, except that $g_{\mu\nu}$ is function of x^μ and τ. We will now define the speed and acceleration of Σ along its normal vector.

2.1. Speed and acceleration of a leaf

Each leaf Σ in the foliation is defined at a fixed value of stochastic time τ, has internal metric $g_{\mu\nu}(x, \tau)$ and normal vector in the stochastic bulk $\boldsymbol{N}$ with corresponding one-form $\boldsymbol{n}$:

$$\boldsymbol{N} \equiv N^A \partial_A = \frac{1}{N}\partial_\tau - \frac{N^\mu}{N}\partial_\mu, \qquad \boldsymbol{n} \equiv N_A \mathrm{d}x^A = -N\mathrm{d}\tau, \tag{2.3}$$

where we have normalized it as $N_A N^A = -1$.

Following e.g. [18], we can define the projector onto Σ as

$$P^A{}_B \equiv \delta^A{}_B + N^A N_B, \quad P^A{}_\mu g_{A\nu} = g_{\mu\nu}, \quad P^A{}_B N_A = 0. \tag{2.4}$$

This projector allows to extend objects defined on the hypersurface to the full stochastic bulk.

The extrinsic curvature (second fundamental form) of a leaf in Σ represents the variation of the internal metric along the hypersurface-orthogonal direction $\boldsymbol{N}$. Hence

$$K_{\mu\nu} \equiv \frac{1}{2}\pounds_N\, g_{\mu\nu} = \frac{1}{2}\big(N^A\partial_A g_{\mu\nu} + \partial_\mu N^A g_{A\nu} + \partial_\nu N^A g_{\mu A}\big), \tag{2.5}$$

which gives the explicit result

$$K_{\mu\nu} = \frac{1}{2N}\big(\partial_\tau g_{\mu\nu} - N^\alpha\partial_\alpha g_{\mu\nu} - \partial_\mu N^\alpha g_{\alpha\nu} - \partial_\nu N^\alpha g_{\mu\alpha}\big) = \frac{1}{2N}(\partial_\tau g_{\mu\nu} - \nabla_\mu g_{\nu\tau} - \nabla_\nu g_{\mu\tau}). \tag{2.6}$$

Its trace is

$$K \equiv g^{\mu\nu} K_{\mu\nu} = \frac{1}{N}(\partial_\tau \ln \sqrt{g} - \nabla_\mu N^\mu). \tag{2.7}$$

The extrinsic curvature can be extended in $\mathcal{M}$ using the projector $P^A{}_B$,

$$K_{AB} \equiv P^\mu{}_A P^\nu{}_B K_{\mu\nu}. \tag{2.8}$$

One can verify

$$K_{\tau\tau} = N^\mu N^\nu K_{\mu\nu}, \qquad K_{\tau\mu} = N^\alpha K_{\alpha\mu} = N^\alpha K_{\mu\alpha} = K_{\mu\tau}. \tag{2.9}$$

It is convenient to define the N-independent stochastic speed $D_\tau g_{\mu\nu}$ of a leaf along its normal as

$$D_\tau g_{\mu\nu} \equiv 2NK_{\mu\nu} = \partial_\tau g_{\mu\nu} - \nabla_\mu g_{\nu\tau} - \nabla_\nu g_{\mu\tau} \tag{2.10}$$

and introduce the rate of evolution of this speed, which we call the acceleration $\boldsymbol{\gamma}_{\mu\nu}$ of the leaf along its normal $\boldsymbol{N}$

$$\boldsymbol{\gamma}_{\mu\nu} \equiv N\pounds_N(D_\tau g_{\mu\nu}) = (\partial_\tau - N^\alpha\partial_\alpha)D_\tau g_{\mu\nu} - 2D_\tau g_{\alpha(\mu}\partial_{\nu)}N^\alpha. \tag{2.11}$$

The acceleration $\boldsymbol{\gamma}_{\mu\nu}$ is the specific part of the $(d+1)$-dimensional Riemann tensor $R^A{}_{BCD}$ that is covariant in the leaf at constant τ and contains the term $\partial^2_\tau g_{\mu\nu}$ but no derivative of the lapse function N.

Both the speed $D_\tau g_{\mu\nu}$ and acceleration $\boldsymbol{\gamma}_{\mu\nu}$ are covariant tensors in the leaf with respect to diffeomorphisms with τ-independent parameters $\xi^\mu(x)$. Moreover N and N^μ are respectively a scalar and a vector for such diffeomorphisms, denoted from now on as Diff_Σ.

Let us stress again that both $D_\tau g_{\mu\nu}$ and $\boldsymbol{\gamma}_{\mu\nu}$ are constructed to be independent on N. This plays an important role for understanding the stochastic evolution of the leaves.

3. The leaf-covariant Langevin equation

The necessity of an acceleration term in the stochastic evolution of the metric $g_{\mu\nu}(x,\tau)$ was suggested in [1] on the basis of physical arguments, so that the Langevin equation equates to a combination of a drift force proportional to Einstein equations of motion, a viscous force containing $\partial_\tau g_{\mu\nu}$ and a noise $\eta_{\mu\nu}$ multiplied by the square root of the Planck constant $\sqrt{\hbar}$. In fact [1], discusses the consequences for primordial cosmology and the short distance behavior of particles of such a second order Langevin equation for the stochastic quantization of gravity. We will make this equation Diff_Σ-covariant.

Beyond specific arguments for Gravity, one may generically justify the need of a stochastic acceleration term as follows: in the stochastic evolution of a massive particle with drift force $U'(x)$, the Langevin theory considers Newton laws of mechanics for a large number of particles with this conserved drift force plus some uncertainties and loss of information on the details of the evolution. The original Langevin equation was actually a second order one

$$m\ddot{x} = -U'(x) - \alpha\dot{x} + \beta\eta.$$

It is often a hard task to prove that one can neglect the 'inertial' term $m\ddot{x}$, getting the simplified Langevin equation

$$\alpha\dot{x} = -U'(x) + \beta\eta,$$

where β can be eventually related to the statistical temperature of the system. One must prove case by case that the acceleration term can be neglected when approaching an equilibrium.

The existence of an equilibrium itself must be demonstrated, depending on the chosen potential U. The possibility of a phase transition is a delicate question. Appendix sketches how to generalize the Langevin equation when there is gauge symmetry, by defining at the same time both the stochastic evolution of gauge degrees of freedom and their stochastic gauge-fixing.

For the stochastic quantization of gravity it was suggested in [1] that the acceleration term of the Langevin equation is essential and cannot be neglected since one cannot softly approach the limit $\tau \to \infty$ because otherwise there would be a Euclidean equilibrium distribution in quantum gravity, which is not the case.

We must address the geometric aspects of the second-order Langevin equation of gravity by giving a leaf-covariant formulation of it, with the understanding that the leaf is the analog of a particle with internal structure $g_{\mu\nu}$ and trajectory parametrized by τ. The Diff_{Σ}-covariance of the stochastic process will be obtained by giving a central role to the leaf-tensors $D_\tau g_{\mu\nu}$ and $\boldsymbol{\gamma}_{\mu\nu}$.

For this purpose, we postulate that the second-order Langevin equation is

$$\Delta T \boldsymbol{\gamma}_{\mu\nu} = -\mathcal{N} G_{\mu\nu\rho\sigma} \frac{\delta S}{\delta g_{\rho\sigma}} - D_\tau g_{\mu\nu} + 2N g_{\mu\nu} + \sqrt{\hbar}\, \eta_{\mu\nu}, \tag{3.1}$$

where ΔT is the dimensionful constant introduced in [1].

The Σ-tensor $G_{\mu\nu\rho\sigma}$ is a function of $g_{\mu\nu}$ that will be defined in equation (4.1), while $\mathcal{N} G_{\mu\nu\rho\sigma}(g_{\alpha\beta})$ is a kernel that factors Einstein equations of motion. A discussion of its decomposition in trace and traceless parts will shortly follow.

In order to get a scalar in each leaf with the right conformal weight, the factor $\mathcal{N}$ must be equal to

$$\mathcal{N} \equiv \alpha N_\mu N^\mu + \beta N^2, \tag{3.2}$$

where α and β are numbers. To have an equilibrium distribution at $\tau = \infty$ independent of the choice of the kernel $\mathcal{N} G_{\mu\nu\rho\sigma}$, the noise distribution is related to the kernel in the following way[8]

$$\langle\!\langle\, \mathcal{F}[\eta_{\mu\nu}] \,\rangle\!\rangle^\tau \equiv \int [\mathrm{d}\eta_{\mu\nu}]\, \mathcal{F}[\eta_{\mu\nu}]\, \exp\Big[-\frac{1}{2} \int \mathrm{d}^d x \mathrm{d}\tau \sqrt{g}\, \eta_{\alpha\beta} \mathcal{N}^{-1}\, G^{-1\alpha\beta\rho\sigma} \eta_{\rho\sigma} \Big]. \tag{3.3}$$

We will use later the freedom in the choice of the coefficients α and β and choose

$$\alpha = 1, \qquad \beta = 0 \tag{3.4}$$

that will facilitate examining the properties of the stochastic process for observables which we will define by demanding Weyl invariance.

The principle of stochastic quantization is that correlation functions of the noise are defined as an input [2], given by (3.3). Correlation functions of the fields are then computable at all possible values of the stochastic time τ because $g_{\mu\nu} = g_{\mu\nu}(\eta_{\alpha\beta})$ is a composite function of τ if $\eta_{\alpha\beta}$ solves the Langevin equation (3.1) with suitable initial conditions.

Indeed, with suitable initial conditions at an arbitrarily chosen initial value of the stochastic time, the differential equation (3.1) determines $g_{\mu\nu}(x,\tau)$ as a function $g_{\mu\nu}[\eta_{\alpha\beta}(x,\tau)]$ of the noise $\eta_{\mu\nu}(x,\tau)$. Notice that for $\hbar = 0$, the Langevin equation is nothing but a flow equation toward the solutions of classical equations of motion, in which case the correlations functions are just exactly centered on the solution of the flow equation. Since all correlation

[8] The theorem about the kernel independence property for observables, and thus of the arbitrariness of the parameters needed in its expression, is a general property [2], and can be proven e.g. by transforming the Langevin equation in a Fokker–Planck equation. See appendix.

functions of the noise $\eta_{\mu\nu}(x,\tau)$ are computable by equation (3.3) after inserting the solution of the Langevin equation in any given functional $\mathcal{O}[g_{\mu\nu}]$, if one substitutes the correlation functions $\mathcal{O}[g_{\mu\nu}](\eta)(x,\tau)$ in place of $\mathcal{F}$ in equation (3.3), one gets

$$\langle\!\langle\, \mathcal{O}[g_{\mu\nu}]\,\rangle\!\rangle^{\tau} = \int [\mathrm{d}\eta_{\mu\nu}]\,\mathcal{O}[g_{\mu\nu}(\eta)]\,\exp\Big[-\frac{1}{2}\int \mathrm{d}^{\mathrm{d}}x\mathrm{d}\tau\sqrt{g}\,\eta_{\alpha\beta}\,\mathcal{N}^{-1}(G^{-1})^{\alpha\beta\rho\sigma}\eta_{\rho\sigma}\Big]. \tag{3.5}$$

Each term in the Langevin equation (3.1) has a natural interpretation as follows. We will be especially concerned with their covariance.

The speed $D_\tau g_{\mu\nu}$ of $(\Sigma, g_{\mu\nu})$ is the completion of the viscous force $\partial_\tau g_{\mu\nu}$ by the term $2\nabla_{(\mu}N_{\nu)}$. The Lie derivative in $\mathcal{M}$, as it is written in (2.5), is a full stochastic bulk operation. However, one can define an intrinsic Lie derivative in Σ, $\mathcal{L}^{\Sigma}_{\boldsymbol{\xi}(x)}g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$, with a parameter equal to the Σ-vector $\xi^\alpha(x)$. The term $2\nabla_{(\mu}g_{\nu)\tau}$ contained in $D_\tau g_{\mu\nu}$ enforces a drift force with parameter $N^\mu = N^2 g^{\mu\tau}$ along the orbits of Diff_Σ in the space of metrics $g_{\mu\nu}$. It reproduces an infinitesimal action of Diff_Σ within Σ on the space of metrics $g_{\mu\nu}$ with a parameter N^μ, which expresses the non-triviality of the foliation. Thus, in the leaf's speed (2.10) of the quantum field stochastic evolution, the term $2\nabla_{(\mu}N_{\nu)}$ provides a gauge restoring force along the gauge orbits of diffeomorphisms for observables that are non-reparametrization invariant (as depending on unphysical longitudinal degrees of freedom, which manifest themselves anyway in virtual processes).

The term $2Ng_{\mu\nu}$ reproduces a Weyl rescaling of the metric with a parameter equal to the lapse N. This gauge restoring force is needed for extracting Weyl dependent observables, and the choice of N will not affect the evolution of Weyl-invariant ones.

The lapse N and shift vector N^μ are thus fields that play the role of parameters for gauge-fixing restoring forces for unphysical degrees of freedom in the Langevin stochastic evolution along orbits of the gauge symmetry $\mathrm{Diff}_\Sigma \ltimes \mathrm{Weyl}_{\mathcal{M}}$.

In fact, N^μ and N are the gravitational analogue of the additional gauge field component A_τ in the stochastic quantization of a gauge field A_μ, which gives a gauge fixing restoring force along the orbits of Yang–Mills transformations and defines the Yang–Mills Parisi–Wu equation in the complete space of gauge field configurations [9, 10, 19]. The choices of N^μ and N do not influence the $\tau = \infty$ limit of the correlation functions of physical observables[9]. So, following the general strategy of [9, 10], we will perform functional integration over $N(x,\tau)$ and $N^\mu(x)$ in the supersymmetric formulation of the Langevin equation, followed by some BRST gauge-fixing on N^μ[10].

The way to obtain a supersymmetric representation of correlators $\langle\!\langle\, \mathcal{O}[g_{\mu\nu}]\,\rangle\!\rangle^{\tau}$, where the noise have been integrated out, is standard, as originally stated in [2] and [8]. Formally, it relies on determinant identities and the argument can be made systematic in the context of topological quantum field theory, by imposing in (3.5) the Langevin equation (3.1) relating $\eta_{\mu\nu}$ and $g_{\mu\nu}$ in a (stochastic) equivariant BRST invariant way. This transforms (3.5) into a supersymmetric path integral involving an equivariant Q supersymmetry acting on the field $g_{\mu\nu}$ and its topological ghosts $\Psi_{\mu\nu}$ and $\overline{\Psi}_{\mu\nu}$. One gets a $(d+1)$-dimensional TQFT path integral whose fields variables are $g_{\mu\nu}$, $\Psi_{\mu\nu}$ and $\overline{\Psi}_{\mu\nu}$ with a Q-exact action that localizes the path integral to the solution of the Langevin equation (3.1) [9, 10]. The link between the Langevin equation and its supersymmetric representation explains the choice that physical observables

[9] Appendix sketches a general proof that the first order stochastic evolution of gauge invariant observables is not affected by additional gauge-fixing restoring terms in the Langevin equation. Any given choice of vector field N^μ gives the same evolution for an observable that is reparametrization invariant.

[10] The Weyl gauge restoring force term $2Ng_{\mu\nu}$ in the Langevin equation (3.1) was not discussed in [1]. We now understand that such a Weyl symmetry restoring force is necessary in view of correctly defining observables.

should be the restricted set of functionals $\langle\!\langle\, \mathcal{O}[\hat{g}_{\mu\nu}]\, \rangle\!\rangle^{\tau}$, where $\hat{g}_{\mu\nu}$ is the unimodular part of $g_{\mu\nu}$ to be introduced shortly.

4. Irreducible decomposition of the Langevin equation and its kernel

We will irreducibly decompose each term of the Langevin equation (3.1) in its traceless and trace parts, term by term. We need also to discuss the positivity of the noise distribution, which is fundamental to guarantee convergence.

The presence of the kernel $\mathcal{N}G_{\mu\nu\rho\sigma}$ in front of Einstein equations of motion in (3.1) is necessary. Indeed, one must lower the indexes of the equations of motion appearing in the Langevin equation to get a covariant equation. Since $\mathcal{N}G_{\mu\nu\rho\sigma}$ is a function of $g_{\mu\nu}$, N^{μ} and N, the noise of (3.1) is currently named a multiplicative noise in the language of statistical mechanics [20].

The expression of the kernel reflects the gauge symmetry of the theory and the tensor $G_{\mu\nu\rho\sigma}$ is fixed by requiring Diff_{Σ}-covariance. It is symmetric in $\mu\nu$ and $\rho\sigma$, so its general form is

$$G_{\mu\nu\rho\sigma}(x,\tau) = \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}) - \lambda g_{\mu\nu}g_{\rho\sigma}, \tag{4.1}$$

where λ is a dimensionless constant. This tensor is nothing but a Wheeler–DeWitt metric [21] over the space of metrics $g_{\mu\nu}$ of any given leaf in a linear space with dimension $\frac{d(d+1)}{2}$. In our case, we are free to choose the value of λ, since we have the kernel-independence theorem for the equilibrium distribution, valid for all kernels that give a well-defined evolution. This is in contrast to what happens in the different context when one uses a Wheeler–DeWitt metric and the ADM formalism to tentatively quantize gravity in the temporal gauge. In this other situation, one gets a fixed value of the parameter λ by a decomposition in $d-1$ dimensions of d-dimensional Einstein gravity, which implies the damaging occurrence of a pseudo-Euclidean metric over the space of $(d-1)$-dimensional spatial metrics. In our case, the question is different, and for defining the stochastic quantization of Euclidean gravity, we can use any value of λ that ensures the positivity of the noise distribution.

The choice of the parameter λ actually controls the Euclidean or pseudo-Euclidean signature of the kernel (4.1) through the sign of $1-\lambda d$.

Indeed the $\frac{1}{2}d(d+1)$ eigenvalues of $G_{\mu\nu\rho\sigma}$ are

$$(1-\lambda d, 1, 1, \ldots, 1). \tag{4.2}$$

The positivity of $G_{\mu\nu\rho\sigma}$ is thus warranted if one chooses the parameter λ to satisfy

$$\lambda < \frac{1}{d}. \tag{4.3}$$

Notice that also $\mathcal{N}$ is guaranteed to be positive with our choice (3.4).

For a more transparent discussion of the positivity of the noise weight, one can decompose $G_{\mu\nu\rho\sigma}$ in traceless and trace parts. Since $g^{\mu\nu}g^{\rho\sigma}G_{\mu\nu\rho\sigma} = d(1-\mathrm{d}\lambda)$ one has

$$G_{\mu\nu\rho\sigma} = G^{T}_{\mu\nu\rho\sigma} + \frac{1-\mathrm{d}\lambda}{d}g_{\mu\nu}g_{\rho\sigma}, \tag{4.4}$$

with the traceless part

$$G^{T}_{\mu\nu\rho\sigma} = \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}) - \frac{1}{d}g_{\mu\nu}g_{\rho\sigma}. \tag{4.5}$$

Denoting $\eta \equiv g^{\mu\nu}\eta_{\mu\nu}$, the noise $\eta_{\mu\nu}$ splits in traceless and trace parts:

$$\eta_{\mu\nu} = \eta^T_{\mu\nu} + \frac{1}{d} g_{\mu\nu}\eta. \tag{4.6}$$

Defining the inverse G^{-1} by $(G^{-1})^{\mu\nu\rho\sigma} G_{\rho\sigma\alpha\beta} = \delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)}$, one has

$$(G^{-1})^{\mu\nu\rho\sigma} = \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) + \frac{\lambda}{1-\lambda d} g^{\mu\nu}g^{\rho\sigma}. \tag{4.7}$$

Thus the definition of noise distribution (3.3) is

$$\langle\!\langle\, \mathcal{F}[\eta_{\alpha\beta}]\, \rangle\!\rangle^{\tau} = \int [\mathrm{d}\eta][\mathrm{d}\eta^T_{\alpha\beta}]\, \mathcal{F}[\eta, \eta^T_{\alpha\beta}]\, \exp\Big[-\int \mathrm{d}^d x \mathrm{d}\tau\, \mathcal{N}^{-1}\sqrt{g}\Big(\eta^T_{\mu\nu}\eta^{T\mu\nu} + \frac{1}{d(1-\lambda d)}\eta^2\Big)\Big]. \tag{4.8}$$

This verifies that the Gaussian distribution has a positive definite weight under the already spelled condition (4.3).

The gravity classical drift-force in the Langevin equation (3.1) is the multiplication of the Einstein tensor

$$E^{\rho\sigma} \equiv \frac{\delta S}{\delta g_{\rho\sigma}} = R^{\rho\sigma} - \frac{1}{2} R g^{\rho\sigma} \tag{4.9}$$

by the positive kernel $\mathcal{N} G_{\mu\nu\rho\sigma}$. Its decomposition into the traceless and trace parts is

$$\mathcal{N} G_{\mu\nu\rho\sigma} E^{\rho\sigma} = \mathcal{N} E^T_{\mu\nu} + \frac{(1-\mathrm{d}\lambda)(2-d)}{2d} g_{\mu\nu}\mathcal{N} R, \tag{4.10}$$

where $E^T_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{d} R g_{\mu\nu}$ is indeed traceless.

Analogously, the stochastic speed decomposes into the traceless and trace parts as

$$D_\tau g_{\mu\nu} \equiv D^T_\tau g_{\mu\nu} + \frac{2}{d} g_{\mu\nu} D_\tau \ln\sqrt{g}, \tag{4.11}$$

where we defined

$$D_\tau \ln\sqrt{g} \equiv \partial_\tau \ln\sqrt{g} - \nabla_\mu N^\mu, \tag{4.12}$$

$$D^T_\tau g_{\mu\nu} \equiv \partial_\tau g_{\mu\nu} - 2\nabla_{(\mu}N_{\nu)} - \frac{2}{d} g_{\mu\nu} D_\tau \ln\sqrt{g}. \tag{4.13}$$

The stochastic acceleration $\boldsymbol{\gamma}_{\mu\nu}$ decomposes as

$$\boldsymbol{\gamma}_{\mu\nu} \equiv \boldsymbol{\gamma}^T_{\mu\nu} + \frac{1}{d}\boldsymbol{\gamma} g_{\mu\nu}, \tag{4.14}$$

where the trace and traceless parts are

$$\boldsymbol{\gamma} \equiv g^{\mu\nu}\boldsymbol{\gamma}_{\mu\nu} = g^{\mu\nu}(\partial_\tau - N^\alpha\partial_\alpha) D_\tau g_{\mu\nu} - 2g^{\mu\nu} D_\tau g_{\alpha\mu}\partial_\nu N^\alpha, \tag{4.15}$$

$$\boldsymbol{\gamma}^T_{\mu\nu} \equiv (\partial_\tau - N^\alpha\partial_\alpha) D_\tau g_{\mu\nu} - 2D_\tau g_{\alpha(\nu}\partial_{\mu)} N^\alpha - \frac{1}{d} g_{\mu\nu}\boldsymbol{\gamma}. \tag{4.16}$$

By projecting onto the trace and traceless parts, we have finally decomposed the Langevin equation (3.1) in irreducible representations with respect to the internal diffeomorphism symmetry of the leaves:

$$\Delta T\boldsymbol{\gamma} + 2D_\tau \ln\sqrt{g} = -\frac{\mathcal{N}}{2}(1-\mathrm{d}\lambda)(2-d)R + 2\mathrm{d}N + \sqrt{\hbar}\,\eta, \tag{4.17}$$

$$\Delta T\,\gamma^{T}_{\mu\nu} + D^{T}_{\tau} g_{\mu\nu} = -\mathcal{N} E^{T}_{\mu\nu} + \sqrt{\hbar}\,\eta^{T}_{\mu\nu}. \tag{4.18}$$

These irreducible decompositions play a central role in our analysis of the Langevin equation. We can already observe that there is no dependence on derivatives of the lapse function N.

5. Weyl transformation and unimodular decomposition

We just achieved the decomposition of the Langevin equation in irreducible traceless and trace parts.

We now go a step further, by expressing these equations in function of the unimodular part of the metric, the conformal factor and all other rescaled Weyl-invariant fields.

The change of field variables that extracts explicitly their Weyl weight will illuminate some properties of the Langevin equation of gravity. In particular, it will allow us to decompose neatly the algebraic dependence on N, with additional linear terms to the preexisting one in the gauge restoring force $2Ng_{\mu\nu}$. Using scale invariant fields turns out to be useful to define gravity observables and reveal their Ward identities.

A Weyl transformation of the metric is defined by

$$\mathrm{d}s^2 \mapsto \Omega^2 \mathrm{d}s^2 \qquad \text{or} \qquad g_{AB} \mapsto \Omega^2 g_{AB}, \tag{5.1}$$

where the Weyl factor $\Omega(x,\tau)$ is a function of all the coordinates. Weyl transformations do not act on the coordinates but form a gauge symmetry. There is no practical need to introduce a Weyl gauge field for our purpose, although it would make sense mathematically to do so.

The way $\mathrm{d}s^2$ transforms implies the following Weyl transformation laws of all metric field components of the $(d+1)$-dimensional ADM parametrization (2.2):

$$N \mapsto \Omega N, \qquad g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}, \qquad N^\mu \mapsto N^\mu, \qquad \sqrt{g} \mapsto \Omega^d \sqrt{g}. \tag{5.2}$$

The infinitesimal version of Weyl transformations, δ_ω, with ω the infinitesimal abelian parameter, is

$$\delta_\omega N = \omega N, \qquad \delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}, \qquad \delta_\omega N^\mu = 0 \qquad \delta_\omega \frac{1}{d}\log\sqrt{g} = \omega. \tag{5.3}$$

To extract Weyl-independent components in all fields, we use the decomposition of the metric $g_{\mu\nu}$ in its unimodular part $\hat{g}_{\mu\nu}$ and conformal factor $a \equiv \sqrt{g}^{\frac{1}{d}}$[11].

The field $\phi \equiv \log a$ is in fact convenient to express the conformal factor dependence. One has

$$\hat{g}_{\mu\nu} \equiv (\sqrt{g})^{-\frac{2}{d}} g_{\mu\nu}, \qquad a \equiv (\sqrt{g})^{\frac{1}{d}} \equiv \exp\phi \tag{5.4}$$

and hence

$$\delta_\omega \hat{g}_{\mu\nu} = 0, \qquad \delta_\omega \phi = \omega. \tag{5.5}$$

As already stated, the shift vector N^μ is Weyl invariant from the beginning.

The ADM fields of the $(d+1)$-dimensional space—except ϕ—can be transformed into Weyl-invariant fields by appropriate rescaling:

$$\hat{N}^\mu \equiv N^\mu, \quad \hat{N} \equiv a^{-1} N, \quad \hat{g}_{\mu\nu} \equiv a^{-2} g_{\mu\nu}, \quad \hat{g}^{\mu\nu} \equiv a^2 g^{\mu\nu}. \tag{5.6}$$

[11] We checked that the unimodular decomposition of all curvature tensors agree with [11]. Notice however that our definition of $\hat{K}$ differs from the one in [11], due to our 'golden rule'.

All the curvatures and Lie derivatives involved in the Langevin equation of gravity can be re-expressed using these fields. Modulo a rescaling factor, they are made of terms involving only the hat fields and of other terms involving derivatives of ϕ.

In what follows, every time an index μ is raised or lowered on a hatted quantity, using the unimodular part of the metric, one gains a factor of a^2 or a^{-2} respectively (e.g. $\hat{N}_\mu \equiv a^{-2}N_\mu$). To perform the unimodular decomposition of the Langevin trace and traceless equations, one must proceed by successive steps. Hatted quantities are defined by applying the following 'golden rule': for any given tensor W, $\hat{W}$ is defined from W replacing all fields by their hat rescaled ones. In particular, since $\phi = \log a = \frac{1}{d}\log\sqrt{g}$, one has $\hat{\phi} = \frac{1}{d}\log\sqrt{\hat{g}} = 0$ because $\hat{g}_{\mu\nu}$ has unit determinant. In general, the application of this rule guarantees the Weyl-invariance of the hatted quantities, since they are built out of Weyl-invariant objects only.

A useful observation is that the Christoffel symbols decompose as $\Gamma^\mu_{\nu\rho} = \hat{\Gamma}^\mu_{\nu\rho} + \Sigma^\mu_{\nu\rho}$ [6], with

$$\hat{\Gamma}^\mu_{\nu\rho} \equiv \frac{1}{2}\hat{g}^{\mu\alpha}(\partial_\nu\hat{g}_{\alpha\rho} + \partial_\rho\hat{g}_{\alpha\nu} - \partial_\alpha\hat{g}_{\nu\rho}), \tag{5.7}$$

$$\Sigma^\mu_{\nu\rho} \equiv \delta^\mu_\rho\partial_\nu\phi + \delta^\mu_\nu\partial_\rho\phi - \hat{g}^{\mu\alpha}\hat{g}_{\nu\rho}\partial_\alpha\phi. \tag{5.8}$$

Let us start with the unimodular decomposition of the various terms in (4.17) and (4.18). Consider first the Ricci tensor and curvature scalar of $g_{\mu\nu}$. One has

$$R_{\mu\nu} = \hat{R}_{\mu\nu} - (d-2)\hat{\nabla}_\mu\partial_\nu\phi - \hat{g}_{\mu\nu}\hat{\nabla}_\alpha\hat{g}^{\alpha\beta}\partial_\beta\phi + (d-2)\partial_\mu\phi\partial_\nu\phi - (d-2)\hat{g}_{\mu\nu}\partial_\alpha\phi\hat{g}^{\alpha\beta}\partial_\beta\phi, \tag{5.9}$$

$$R = g_{\mu\nu}R^{\mu\nu} = \exp(-2\phi)\Big(\hat{R} - 2(d-1)\hat{g}^{\mu\nu}\big(\hat{\nabla}_\mu\partial_\nu\phi + \frac{d-2}{2}\partial_\mu\phi\partial_\nu\phi\big)\Big). \tag{5.10}$$

From now on, the 'hat Σ-covariant derivative' $\hat{\nabla}_\mu$ is defined as ∇_μ with the Christoffel symbols $\Gamma^\mu_{\nu\rho}$ replaced by $\hat{\Gamma}^\mu_{\nu\rho}$. Likewise $\hat{R}_{\mu\nu}$ and $\hat{R}$ are obtained just like $R_{\mu\nu}$ and R but using the hatted quantities $\hat{\Gamma}^\mu_{\nu\rho}$ and $\hat{g}_{\mu\nu}$. Defining $\hat{E}^T_{\mu\nu} \equiv \hat{R}_{\mu\nu} - \frac{1}{d}\hat{R}\hat{g}_{\mu\nu}$, one has

$$\begin{aligned} E^T_{\mu\nu} &= \hat{R}_{\mu\nu} - \frac{1}{d}\hat{g}_{\mu\nu}\hat{R} - (d-2)(\hat{\nabla}_\mu\partial_\nu\phi - \partial_\mu\phi\partial_\nu\phi) + \frac{d-2}{d}\hat{g}_{\mu\nu}\hat{g}^{\alpha\beta}\big(\hat{\nabla}_\alpha\partial_\beta\phi - \partial_\alpha\phi\partial_\beta\phi\big), \\ &= \hat{E}^T_{\mu\nu} - (d-2)(\hat{\nabla}_\mu\partial_\nu\phi - \partial_\mu\phi\partial_\nu\phi)^T. \end{aligned} \tag{5.11}$$

The scalar factor $\mathcal{N}$ decomposes as

$$\mathcal{N} = \exp(2\phi)\,\hat{\mathcal{N}}. \tag{5.12}$$

The trace of the stochastic speed is

$$D_\tau \ln\sqrt{g} = d(\partial_\tau - \hat{N}^\mu\partial_\mu)\phi - \hat{\nabla}_\mu\hat{N}^\mu = d\tilde{\partial}_\tau\phi - \hat{\nabla}_\mu\hat{N}^\mu, \tag{5.13}$$

where $\hat{\nabla}_\mu\hat{N}^\mu = \partial_\mu\hat{N}^\mu$ because $\hat{g}_{\mu\nu}$ is unimodular and

$$\tilde{\partial}_\tau \equiv \partial_\tau - \hat{N}^\mu\partial_\mu. \tag{5.14}$$

The traceless part of the stochastic speed is

$$D^T_\tau g_{\mu\nu} = \exp(2\phi)\Big(\partial_\tau\hat{g}_{\mu\nu} - 2\hat{\nabla}_{(\mu}\hat{N}_{\nu)} + \frac{2}{d}\hat{g}_{\mu\nu}\hat{\nabla}_\alpha\hat{N}^\alpha\Big) \equiv \exp(2\phi)\hat{D}^T_\tau\hat{g}_{\mu\nu}. \tag{5.15}$$

One has thus the following decomposition:

$$D_\tau g_{\mu\nu} = \exp(2\phi)\Big(\hat{D}_\tau \hat{g}_{\mu\nu} + 2\hat{g}_{\mu\nu}\tilde{\partial}_\tau\phi\Big). \tag{5.16}$$

The first term in the right hand side of this equation is an example of the golden rule, with

$$\hat{D}_\tau \hat{g}_{\mu\nu} \equiv \partial_\tau \hat{g}_{\mu\nu} - 2\hat{\nabla}_{(\mu}\hat{N}_{\nu)}. \tag{5.17}$$

Notice in particular that, while the trace term is not Weyl covariant due to the ϕ dependence, the traceless part is Weyl covariant. Stated differently, a Weyl transformation affects only the trace of the stochastic speed.

Consider now the acceleration $\boldsymbol{\gamma}_{\mu\nu}$ as it is defined in (2.11). Its unimodular decomposition is

$$\boldsymbol{\gamma}_{\mu\nu} = \exp(2\phi)\Big(\hat{\gamma}_{\mu\nu} + 2\hat{g}_{\mu\nu}\tilde{\partial}^2_\tau\phi + 4\hat{g}_{\mu\nu}(\tilde{\partial}_\tau\phi)^2 + 4\hat{D}_\tau\hat{g}_{\mu\nu}\tilde{\partial}_\tau\phi\Big). \tag{5.18}$$

Here again the golden rule is at work:

$$\hat{\gamma}_{\mu\nu} = \tilde{\partial}_\tau \hat{D}_\tau \hat{g}_{\mu\nu} - 2\hat{D}_\tau \hat{g}_{\alpha(\mu}\partial_{\nu)}\hat{N}^\alpha. \tag{5.19}$$

Consequently the trace part is

$$\gamma = \hat{\gamma} + 2d\tilde{\partial}^2_\tau\phi + 4d(\tilde{\partial}_\tau\phi)^2 - 8\tilde{\partial}_\tau\phi\hat{\nabla}_\mu\hat{N}^\mu, \tag{5.20}$$

and the traceless part is

$$\gamma^T_{\mu\nu} = \mathrm{e}^{2\phi}\Big(\hat{\gamma}^T_{\mu\nu} + 4\tilde{\partial}_\tau\phi\hat{D}^T_\tau\hat{g}_{\mu\nu}\Big), \tag{5.21}$$

where $\hat{D}^T_\tau\hat{g}_{\mu\nu}$ is defined in (5.15). Notice that $\hat{\gamma}$ and $\hat{\gamma}^T_{\mu\nu}$, which (in our knowledge) are yet unknown quantities in the literature, are also given by the golden rule.

One must also decompose using unimodular variables the noise, for which we require

$$\eta_{\mu\nu} = \mathrm{e}^{2\phi}\hat{\eta}_{\mu\nu}, \quad \eta^T_{\mu\nu} = \mathrm{e}^{2\phi}\hat{\eta}^T_{\mu\nu}, \quad \eta = \hat{\eta}. \tag{5.22}$$

Putting everything together, one gets the following expression for the traceless part of the second order Langevin equation (4.18):

$$\Delta T\Big(\hat{\gamma}^T_{\mu\nu} + 4\tilde{\partial}_\tau\phi\hat{D}^T_\tau\hat{g}_{\mu\nu}\Big) + \hat{D}^T_\tau\hat{g}_{\mu\nu} = -\hat{\mathcal{N}}(\hat{E}^T_{\mu\nu} - (d-2)(\hat{\nabla}_\mu\partial_\nu\phi - \partial_\mu\phi\partial_\nu\phi)^T) + \sqrt{\hbar}\,\hat{\eta}^T_{\mu\nu}. \tag{5.23}$$

As a result of our choice for the kernel, this equation is homogeneous in ϕ with an overall linear dependence on $\hat{\mathcal{N}}$.

The trace part of (4.18) is:

$$\begin{aligned}\Delta T(\hat{\gamma} + 2d\tilde{\partial}^2_\tau\phi + 4d(\tilde{\partial}_\tau\phi)^2 - 8\tilde{\partial}_\tau\phi\hat{\nabla}_\mu\hat{N}^\mu) + 2(d\tilde{\partial}_\tau\phi - \hat{\nabla}_\mu\hat{N}^\mu)\\ = \frac{\hat{\mathcal{N}}}{2}(1-\mathrm{d}\lambda)(d-2)\Big(\hat{R} - 2(d-1)\hat{g}^{\mu\nu}\big(\hat{\nabla}_\mu\partial_\nu\phi + \frac{d-2}{2}\partial_\mu\phi\partial_\nu\phi\big)\Big) + 2d\exp(\phi)\hat{N} + \sqrt{\hbar}\,\hat{\eta}.\end{aligned} \tag{5.24}$$

In $d = 2$ the traceless part does not depend on ϕ, and so are the observables we are looking for. In this case the use of the unimodular part $\hat{g}_{\mu\nu}$ amounts to that of the Beltrami differential. See the comment just below.

For $d > 2$, the traceless parts depends on ϕ, but it is homogeneous, a property that we shall shortly interpret.

For completion, by using the unimodular parametrization of the noise, one gets the noise distribution in the following form:

$$\langle\!\langle\, \mathcal{F}[\hat{\eta}_{\mu\nu}]\,\rangle\!\rangle^{\tau} = \int [\mathrm{d}\hat{\eta}][\mathrm{d}\hat{\eta}^{T}_{\mu\nu}]\, \mathcal{F}[\hat{\eta}, \hat{\eta}^{T}_{\mu\nu}] \exp\Big[-\frac{1}{2}\int \mathrm{d}^{d}x\mathrm{d}\tau \exp(-2\phi)\mathcal{N}^{-1}\big(\hat{\eta}^{T}_{\alpha\beta}\hat{\eta}^{T\alpha\beta} + \frac{1}{1-\lambda d}\hat{\eta}^{2}\big)\Big]. \tag{5.25}$$

The result of [4] that the classical evolution is determined by the conformal classes of metric $g_{\mu\nu}$ suggests that we may consider $\hat{g}_{\mu\nu}$ and ϕ as the independent field variables of gravity, where only $\hat{g}_{\mu\nu}$ is physical. If correct, and we will show it in the first order Langevin theory, this would mean that the stochastic time evolution of the conformal factor is basically irrelevant for physical observables, which we claim being Weyl invariant.

We remark that defining observables as only depending on the unimodular part of $g_{\mu\nu}$ ensures from the beginning that the physical quantum graviton, seen as an excitation of $g_{\mu\nu}$, must be traceless and symmetric. This traceless property is obvious in classical and semi-classical gravity, but it must rely on some symmetry principle in any given attempt of defining quantum gravity. Our definition of quantum observables is compatible with the classical property.

What has been done above for $d > 2$ is quite a striking generalization of the case $d = 2$ [22], where one can use efficiently the Beltrami differential $\mu \equiv \mu^{z}{}_{\bar{z}}$ as the only fundamental field variable of the 2-dimensional metric, with $\hat{g}_{\mu\nu} = \frac{1}{1-\mu\bar{\mu}}\begin{pmatrix}1 & \mu\\ \bar{\mu} & 1\end{pmatrix}$. Despite of the fact that in 2 dimensions $\hat{g}_{\mu\nu}$ does not propagate but intervenes only through its constant moduli, the (complex) Beltrami 1-form $\mathrm{d}z + \mu\mathrm{d}\bar{z}$ is actually the physical field in 2-dimensional gravity because the conformal factor is decoupled, and possibly substituted with an additional Liouville field in non critical dimensions. If needed, the latter is seen as a Wess–Zumino field within the context of a conformal theory, for one has to compensate the conformal anomaly [3, 22]. For $d > 2$, one has of course the very non-trivial propagation of $\hat{g}_{\mu\nu}$ in contrast to the case $d = 2$, but, nonetheless, the definition of observables is analogous.

6. Observables

From now on we will focus on the choice that $\mathcal{N}$ satisfies (3.4), i.e.

$$\mathcal{N} = N^{\alpha}N_{\alpha}. \tag{6.1}$$

This will allow us solve algebraically for $\hat{N}$ in the regime $\Delta T = 0$ and obtain a consistent definition of the expectation value of observables $\mathcal{O}[\hat{g}_{\mu\nu}]$ so that it depends only on the unimodular part of the metric. The analysis is simpler when $\Delta T = 0$, a situation that we now detail.

6.1. $\Delta T = 0$

Consider the Langevin equations (5.23) and (5.24) with $\Delta T = 0$. In the supersymmetric formulation, the path integral of correlators of the unimodular part of the metric is

$$\begin{aligned}\langle\!\langle\, \mathcal{O}[\hat{g}_{\mu\nu}]\,\rangle\!\rangle^{\tau} &= \int [\mathrm{d}\hat{N}][\mathrm{d}\hat{N}^{\mu}][\mathrm{d}\hat{g}_{\mu\nu}][\mathrm{d}\phi]\, \mathcal{O}[\hat{g}_{\mu\nu}]\, \exp\Big[-\frac{1}{2\hbar}\int \mathrm{d}^{\mathrm{d}}x\mathrm{d}\tau \frac{\exp(-2\phi)}{\hat{N}^{\alpha}\hat{N}_{\alpha}}\\ &\big(\|\hat{D}^{T}_{\tau}\hat{g}_{\rho\sigma} + \hat{N}^{\alpha}\hat{N}_{\alpha}(\hat{E}^{T}_{\rho\sigma} - (d-2)(\hat{\nabla}_{\rho}\partial_{\sigma}\phi - \partial_{\rho}\phi\partial_{\sigma}\phi)^{T})\|^{2} + \frac{1}{1-\lambda d}\Big\|2(d\tilde{\partial}_{\tau}\phi - \hat{\nabla}_{\beta}\hat{N}^{\beta}) -\\ &- \frac{\hat{N}^{\alpha}\hat{N}_{\alpha}}{2}(1-\mathrm{d}\lambda)(d-2)\Big(\hat{R} - 2(d-1)\big(\hat{\nabla}^{\beta}\partial_{\beta}\phi + \frac{d-2}{2}\partial^{\beta}\phi\partial_{\beta}\phi\big)\Big) - 2d\exp(\phi)\hat{N}\Big\|^{2} + \text{susy terms}\big)\Big].\end{aligned} \tag{6.2}$$

We have only made explicit the bosonic part of the $(d+1)$-dimensional action. The part *susy terms* contains fermionic terms that form a fermionic path integral that takes care of the

Jacobian of the map between the fields and the noise, which ensures the stochastic supersymmetry. See e.g. [8–10] for details[12]. There is no need here to make these terms explicit.

Because $\mathcal{O}[\hat{g}_{\mu\nu}]$ is Weyl invariant, this path integral is invariant under the Weyl transformation defined in (5.6), but it depends on the parameter τ, as readily seen in (6.2).

This x-independent Weyl symmetry of the path integral representation of observables $\langle\!\langle\, \mathcal{O}[\hat{g}_{\mu\nu}]\,\rangle\!\rangle^{\tau}$ is nothing but a dilatation of the lapse function N by the same factor everywhere in any given leaf. It is a symmetry under any reparametrization of the stochastic time $\tau \to \tau'(\tau)$, which is a sophisticated generalization of the worldline reparametrization invariance of the relativistic particle theory.

The invariance of the path integral (6.2) can be indeed explained by a BRST symmetry operation s using an abelian anticommuting ghost $\omega(\tau)$, which completes the anticommuting vector ghost $\xi^{\mu}(x)$ accounting for diffeomorphisms in each leaf. The operator s acts on all fields as a nilpotent graded differential operation with

$$\begin{aligned} s\hat{g}_{\mu\nu} &= \pounds^{\Sigma}_{\xi}\hat{g}_{\mu\nu} \\ s\phi &= \pounds^{\Sigma}_{\xi}\phi + \omega(\tau) \\ s\hat{N}^{\mu} &= \pounds^{\Sigma}_{\xi}\hat{N}^{\mu} = \xi^{\nu}\partial_{\nu}\hat{N}^{\mu} - \hat{N}^{\nu}\partial_{\nu}\xi^{\mu}. \end{aligned} \tag{6.3}$$

The condition $s^2 = 0$ on $\hat{g}_{\mu\nu}$ and ϕ implies the following transformations of the ghosts

$$\begin{aligned} s\xi^{\mu}(x) &= \xi^{\nu}\partial_{\nu}\xi^{\mu}, \\ s\omega(\tau) &= 0. \end{aligned} \tag{6.4}$$

The s invariance of the action is achieved provided $\hat{N}$ transforms as

$$\begin{aligned} s\hat{N} = s\Big(&\exp(-\phi)(\tilde{\partial}_{\tau}\phi - \frac{1}{d}\hat{\nabla}_{\beta}\hat{N}^{\beta}) - \frac{\exp(-\phi)\hat{N}^{\alpha}\hat{N}_{\alpha}}{4d} \\ &(1-\mathrm{d}\lambda)(d-2)\Big(\hat{R} - 2(d-1)\big(\hat{\nabla}^{\beta}\partial_{\beta}\phi + \frac{d-2}{2}\partial^{\beta}\phi\partial_{\beta}\phi\big)\Big)\Big), \end{aligned} \tag{6.5}$$

Since $s\hat{N}$ is an s-exact expression of $\hat{g}_{\mu\nu}$, $\hat{N}^{\mu}$, ϕ, ξ^{μ} and ω and since $s^2 = 0$ on these fields, one has eventually $s^2\hat{N} = 0$ on all fields. We will shortly eliminate $\hat{N}$ by its equation of motion, so there is no need to write $s\hat{N}$ explicitly.

Using the x-independent Weyl invariance and the BRST symmetry we just defined, one can localize ϕ in an s-invariant way to the gauge choice

$$\phi(x,\tau) = \phi(x) \equiv \phi_{\{x\}}. \tag{6.6}$$

We thus reach the conclusion that the conformal factor is a spectator in the stochastic evolution of observables, which is just given by some initial condition. More precisely, we found that, for observables $\langle\!\langle\, \mathcal{O}[\hat{g}_{\mu\nu}]\,\rangle\!\rangle^{\tau}$, we can use a conformal factor $\phi_{\{x\}}$ that is independent of τ in a BRST invariant way. Therefore in the τ evolution it remains equal to some arbitrarily chosen initial data $\phi_{\{x\}}$ which is thus a stochastic time independent background for the evolution

[12] The total action, including fermionic terms, is Q exact under the topological stochastic BRST operator $Q\hat{g}_{\mu\nu} = \hat{\Psi}_{\mu\nu} + \pounds^{\Sigma}_{\xi}\hat{g}_{\mu\nu}$, $Q\xi^{\mu} = -\Phi^{\mu} + \xi^{\nu}\partial_{\nu}\xi^{\mu}$, $Q\hat{\Phi}^{\mu} = \xi^{\nu}\partial_{\nu}\Phi^{\mu} - \Phi^{\nu}\partial_{\nu}\xi^{\mu}$, following the same pattern as the stochastic quantization of the Yang–Mills case. The details of the by-now standard method is not worth being displayed here, since we only want to come back to the Langevin equation after the decoupling of the conformal factor by a consistent gauge-fixing. In fact this decoupling preserves Q symmetry, so that we can go back to the Langevin equation after showing how it works in the bosonic sector only.

of observables. In the classical limit, which is the only possible limit when $\tau = \infty$, it becomes the standard conformal factor in general relativity.

Notice furthermore that $\hat{N}$ has a τ evolution that is basically governed by the internal and external conformal scalar curvatures of the leaf $\hat{R}(\hat{g}_{\mu\nu}(x,\tau))$ and $\hat{K}(\hat{g}_{\mu\nu}(x,\tau))$, according to its equation of motion

$$\hat{N} = \exp(-\phi)(\tilde{\partial}_\tau\phi - \frac{1}{d}\hat{\nabla}_\beta\hat{N}^\beta) - \frac{\exp(-\phi)\hat{N}^\alpha\hat{N}_\alpha}{4d}(1-\mathrm{d}\lambda)(d-2)\Big(\hat{R} - 2(d-1)(\hat{\nabla}^\beta\partial_\beta\phi + \frac{d-2}{2}\partial^\beta\phi\partial_\beta\phi)\Big). \tag{6.7}$$

For unphysical correlators, which are ϕ dependent, one can also proceed to the elimination of $\hat{N}$, but one obtains a path integral that is not s-invariant: ϕ is no more a spectator and has a τ-evolution.

After the elimination of the lapse function $\hat{N}$ by its algebraic equation of motion, the path integral (6.2) reads (skipping the ghost term dependence that ensures Weyl BRST invariance)

$$\langle\!\langle\, \mathcal{O}[\hat{g}_{\mu\nu}]\,\rangle\!\rangle^\tau = \int [\mathrm{d}\hat{N}^\mu][\mathrm{d}\hat{g}_{\mu\nu}][\mathrm{d}\phi_{\{x\}}]\,\mathcal{O}[\hat{g}_{\mu\nu}]\,\exp\Big[-\frac{1}{2\hbar}\int \mathrm{d}^\mathrm{d}x\mathrm{d}\tau\frac{\exp(-2\phi_{\{x\}})}{\hat{N}^\alpha\hat{N}_\alpha}$$

$$\Big(\|\hat{D}^T_\tau\hat{g}_{\rho\sigma} + \hat{N}^\alpha\hat{N}_\alpha(\hat{E}^T_{\rho\sigma} - (d-2)(\hat{\nabla}_\rho\partial_\sigma\phi_{\{x\}} - \partial_\rho\phi_{\{x\}}\partial_\sigma\phi_{\{x\}})^T)\|^2 + \text{susy terms}\Big)\Big]. \tag{6.8}$$

This is eventually the genuine path integral definition for physical observables.

Because one functionally integrates over all possible $\hat{N}^\mu(x)$, and because the action is invariant under diffeomorphisms in each leaf, one must do a BRST invariant gauge fixing on $\hat{N}^\mu(x)$ to get a propagation with no zero modes in $\hat{g}_{\mu\nu}$. Indeed, exploiting the reparametrization invariance of the observables and the action, we can use for example the gauge fixing choice $\hat{N}^\mu(x) = \partial_\nu\hat{g}^{\mu\nu}(x,\tau_0)$. This gauge fixing is a good candidate for fixing the internal reparametrization gauge in (6.8), as can be seen for instance by doing an expansion of $\hat{g}_{\mu\nu}$ around a classical background with a small excitation. To regularize potential singularities in space of $\hat{g}_{\mu\nu}(x,\tau_0)$, one may in fact add to the gauge fixing function a nowhere vanishing constant vector n^μ chosen in the initial leaf, that is,

$$\hat{N}^\mu(x) = \partial_\nu\hat{g}^{\mu\nu}(x,\tau_0) + n^\mu. \tag{6.9}$$

Having obtained (6.8) and defined the gauge choice (6.9), we can come back to the Langevin equation and eliminate the *susy terms* by doing the usual change of variables that connects the Langevin equation to its supersymmetric representation. This equation, which only depends on a transverse noise has the form

$$\hat{D}^T_\tau\hat{g}_{\rho\sigma} = -\hat{N}^\alpha\hat{N}_\alpha(\hat{E}^T_{\rho\sigma} - (d-2)(\hat{\nabla}_\rho\partial_\sigma\phi_{\{x\}} - \partial_\rho\phi_{\{x\}}\partial_\sigma\phi_{\{x\}})^T + \sqrt{\hbar}\,\eta^T_{\mu\nu}, \tag{6.10}$$

with a positive $\hat{g}_{\mu\nu}$-Gaussian-norm for the noise $\eta^T_{\mu\nu}$. In (6.10), the τ-Weyl symmetry is manifest and $\phi_{\{x\}}$ is just a spectator. It could be called the Parisi–Wu equation of gravity. The regime with $\Delta T = 0$ should hold near the end of the transition where gravity becomes classical. The Langevin equation (6.10) could have been postulated from the beginning, for it is a consistent equation, if one postulates the τ-Weyl symmetry of observables. Nonetheless, it is a rewarding fact that it has been extracted from the geometrical equation (3.1), involving all ingredient of the foliated $(d+1)$-dimensional space.

A non-trivial and interesting feature that has been developed in this section is that for computing physical observables, one can fix the lapse and shift of the foliation (6.5) and (6.9) in function of $\phi_{\{x\}}$ and $\hat{g}_{\mu\nu}(x,\tau_0)$. The conformal factor $\phi_{\{x\}} = \phi(x,\tau_0)$ is a spectator in the stochastic process, fixed by its initial condition. In contrast, the lapse $N(x,\tau)$ has a τ-evolution, determined by both the extrinsic and intrinsic scalar curvatures of each leaf.

6.2. $\Delta T \neq 0$

This situation has a profound difference with respect to the previous one, and it is the one that was heuristically predicted in [1], because of the acceleration term that provides oscillations of functions of τ, at a scale of ΔT, giving creations and annihilation of quanta in the $(d+1)$-dimensional theory.

Indeed, in the acceleration term of the traceless part of the Langevin equation, there is a term proportional to $\partial_\tau \phi$:

$$\Delta T\Big(\hat{\gamma}^T_{\mu\nu} + 4\tilde{\partial}_\tau\phi\hat{D}^T_\tau\hat{g}_{\mu\nu}\Big) + \hat{D}^T_\tau\hat{g}_{\mu\nu} = -\hat{\mathcal{N}}(\hat{E}^T_{\mu\nu} - (d-2)(\hat{\nabla}_\mu\partial_\nu\phi - \partial_\mu\phi\partial_\nu\phi)^T) + \sqrt{\hbar}\,\hat{\eta}^T_{\mu\nu}. \tag{6.11}$$

The additional term $\Delta T(\hat{\gamma}^T_{\mu\nu} + 4\tilde{\partial}_\tau\phi\hat{D}^T_\tau\hat{g}_{\mu\nu})$ brings new insightful physics. Two main regimes can be distinguished.

1. The regime $\hat{\gamma}^T_{\mu\nu} \gg 4\tilde{\partial}_\tau\phi\hat{D}^T_\tau\hat{g}_{\mu\nu}$. In this case, ϕ is still non dynamical, as it is for the first order equation. Consequently it can be gauge fixed again as in (6.6). This scenario, although technically harder and conceptually different due to the second order term, can be treated using the same analysis as for $\Delta T = 0$. That is, one can solve algebraically the trace part for $\hat{N}$, inject the result in the traceless part and gauge fix ϕ. The result is a traceless second-order Langevin equation for $\hat{g}_{\mu\nu}$. Notice however that there are oscillations here due to the second order term, such that an equilibrium distribution is not reachable. This indicates that this regime is probing deeper in the stochastic bulk, but it is an intermediate step in the stochastic evolution, because the conformal factor is still non-dynamical. Mathematically this is encoded in the fact that, although the leaves are oscillating, the conformal factor is dictated by its initial value. To further analyze this case, we report explicitly its traceless Langevin equation in the simplified situation where N^α and $\phi_{\{x\}}$ are constant:

$$\Delta T(\tilde{\partial}^2_\tau\hat{g}_{\mu\nu})^T + \tilde{\partial}_\tau\hat{g}_{\mu\nu} = -\hat{\mathcal{N}}\Big(\hat{R}_{\mu\nu} - \frac{1}{d}\hat{R}\hat{g}_{\mu\nu}\Big) + \sqrt{\hbar}\,\hat{\eta}^T_{\mu\nu}. \tag{6.12}$$

   This situation is thus a typical second order Langevin theory [1], but for the unimodular part of the metric.

2. The regime $\hat{\gamma}^T_{\mu\nu} \ll 4\tilde{\partial}_\tau\phi\hat{D}^T_\tau\hat{g}_{\mu\nu}$. This means that, deep in the stochastic bulk, the Weyl part of the symmetry (6.3), allowing to set (6.6), is lost. Therefore the stochastic temporal evolution of the unimodular part of the metric is influenced by the evolution of the conformal factor, which is no more a spectator. We must thus compute the evolution of observables doing the path integral over the field $\phi(x,\tau)$ as well. This is a non-trivial task, although it is a well-defined problem. The idea is the following: the stochastic evolution starts with some initial configuration at a fixed stochastic time τ_0 in the bulk. Stochastic second order quantum effects make the temporal evolution of $\hat{g}_{\mu\nu}(\tau_0, x)$ and $\phi(\tau_0, x)$ rapidly oscillate. At a late time, an abrupt transition brings the full Langevin equation to the first order one. There, the evolution of $\hat{g}_{\mu\nu}$ does not depend on the one of ϕ, and we retrieve the physics discussed in the $\Delta T = 0$ case, with a well-defined equilibrium distribution at late stochastic time. The simplified case with N^α and $\phi_{\{x\}}$ constant now reads

$$\tilde{\partial}_\tau \hat{g}_{\mu\nu}(1 + 4\Delta T \tilde{\partial}_\tau \phi) = -\hat{\mathcal{N}}\left(\hat{R}_{\mu\nu} - \frac{1}{d}\hat{R}\hat{g}_{\mu\nu}\right) + \sqrt{\hbar}\,\hat{\eta}^T_{\mu\nu}. \tag{6.13}$$

It is clear that, as long as $\Delta T \neq 0$, there is a coupling between the evolution of $\hat{g}_{\mu\nu}$ and ϕ. In other words, the stochastic evolution is not unimodular in the deep bulk. Here observables are therefore functionals of ϕ and the discussions of the previous section are no longer applicable. This regime constitutes an appealing direction of investigation, with its complete understanding yet to unravel.

In both cases, one must separates the regime where the acceleration term dominates the friction and vice-versa. In the former the theory is dominated by oscillations, where there cannot be an equilibrium, i.e. the theory remains $(d+1)$-dimensional with no possibility of defining a Lorentz time in each leaf. See [1] for some heuristic description of the resulting physics for $\Delta T \neq 0$. The latter is instead the situation treated in the previous section, where it is certain that an equilibrium solution at late time exists. This regime can stop when there is a fluctuation where effectively gravity becomes classical, so one can neglect the noise because it is factorized by $\sqrt{\hbar}$, in which case one quickly reaches the equilibrium at large values of τ, such that the theory can be computed in the bulk, with the possibility of using a Lorentz time, just by solving classical equations of motion. This, in its cosmological application, means that we passed the phase transition marked by the inflation.

7. Conclusion

In the series of work [4] it was clearly mentioned that what matters when solving the classical Einstein equations of motion is the propagation of conformal classes of spatial metrics. In fact the issue of giving a role to Weyl symmetry for the Einstein theory can be traced back to a time as remote as 1925 [6]. In order to be consistent with the well-established property that the equations of motion of classical gravity make no relevant difference between metrics in the conformal class, although the gravity action is not Weyl invariant, we raised as a principle the definition of quantum gravity observables of the metrics as covariant functionals of their unimodular parts.

In fact, beyond technical difficulties, the definition of the observables of a theory is a notion that goes prior to the choice of the method that one chooses to quantize a classical theory. So, we pointed out that the gauge symmetry that determines the observables of gravity is generally Weyl invariance.

To render this explicit, we proposed to use stochastic quantization for defining quantum gravity, a feature that we originally introduced with the motivation that it allows to bypass the question of the impossibility of defining the Lorentz time when quantum gravity is switched on. We have proposed a seemingly consistent way to define the observables of Euclidean quantum gravity.

We have shown that the various properties of stochastic quantization define a process where Weyl symmetry is maintained for physical observables along their propagation in stochastic time. The theory predicts the stochastic lapse and shift being determined as a function of initial conditions. Ward identities imply that the conformal factor is physically irrelevant in quantum gravity, at least at late stochastic time. Specifically, this holds for the first order Langevin equation and the second order in the regime where the $\hat{g}_{\mu\nu}$ fluctuations are greater than $\partial_\tau \phi$. We found also another second order regime, opposite to the one just depicted, where Weyl symmetry is absent due to temporal evolution of ϕ. This scenario is not treatable with the analysis of this paper. It deserves further study.

The physical irrelevance of the conformal factor in first order stochastic quantum gravity is an unexpected and pleasant generalization of the soluble case of two dimensions. We now believe that Weyl invariance should be postulated as the gauge symmetry principle of gravity in general. On the other hand reparametrization invariance is an internal freedom of the theory encoding the fact that one can choose mathematically any (consistent) set of coordinates in each leaf of a foliation of the $(d+1)$-dimensional space $\{x, \tau\}$, determined by the stochastic time evolution.

One cannot exclude the possibility that a conformal anomaly can occur. If it is the case, there will be no conceptual difficulty to establish the conformal invariance of quantum gravity by introducing a Wess–Zumino field, following basically the same pattern as [3], where the Liouville field for non-critical strings in the case $d = 2$ was introduced.

We conclude with a remark concerning the geometrical setup of this work. Having a d-dimensional theory connected to a $(d+1)$-dimensional one is a feature shared by many modern constructions. For instance this happens in holography, where one defines a gravitational theory in $d+1$ dimensions which is related to a d-dimensional matter theory living on its boundary. Although here we use an extra dimension to discuss the stochastic enhancement of the boundary theory, we have a stochastic flow analogous to a holographic RG flow.

Appendix. On the gauge fixing restoring forces

In this appendix we sketch a proof that for the first order Langevin equation additional gauge fixing restoring terms do not modify the evolution of gauge invariant observables, as it was introduced for the Yang–Mills theory in [19].

Consider the generalization of the first order Langevin equation $\alpha\dot{q} = -U'(q) + \beta\eta$ when we have a quantum field theory with a gauge symmetry. Replace q by a $\varphi(x, \tau)$ and $U(q)$ by its action $I[\varphi]$. Let $\delta^{\text{gauge}}_{\epsilon}(x, \tau)$ be the gauge transformation of φ with a local parameter ϵ. The gauge invariance of the action means $\delta^{\text{gauge}}_{\epsilon}(I) = 0$ and a gauge invariant observable is a functional $\mathcal{O}_{G-I}$ with

$$\int \mathrm{d}\tau \mathrm{d}x\, \delta^{\text{gauge}}_{\epsilon}(\varphi(x,\tau)) \frac{\delta \mathcal{O}_{G-I}}{\delta\varphi(x,\tau)} = 0. \tag{A.1}$$

The generalization of the Langevin equation with a kernel K and a gauge restoring force along the orbits of the gauge transformation depending on an arbitrarily chosen functional $v(\varphi)$ is

$$\frac{\partial\varphi}{\partial\tau} = K\Big(\frac{\delta I}{\delta\varphi} + \delta^{\text{gauge}}_{v}(\varphi)\Big) + \eta, \tag{A.2}$$

with the probability distribution for the noise

$$\langle\!\langle\, \mathcal{F}[\eta(x,\tau)]\,\rangle\!\rangle^{\tau} = \int [\mathrm{d}\eta]_{x,\tau}\, \mathcal{F}[\eta(x,\tau)]\, \exp\Big[-\int \mathrm{d}\tau'\mathrm{d}x'\eta(x',\tau')K^{-1}\eta(x',\tau')\Big]. \tag{A.3}$$

The products by K of $\frac{\delta I}{\delta\varphi}$ and $\delta^{\text{gauge}}_{v}(\varphi)$ are respectively drift forces along the physical excitations of φ and unphysical gauge excitations of φ, respectively. $\delta^{\text{gauge}}_{v}(\varphi)$ can be called a stochastic gauge fixing force with field parameter v.

Equation (A.2) implies a Fokker–Planck equation that computes equal-stochastic-time correlators, with

$$\langle\!\langle\, \mathcal{O}[\varphi(x,\tau)]\,\rangle\!\rangle^{\tau,v} = \int [\mathrm{d}\varphi]_{y}\, P^{v}(\varphi_y, \tau)\, \mathcal{O}[\varphi_y] \tag{A.4}$$

and

$$\frac{\partial P^v(\varphi,\tau)}{\partial\tau} = \Big[\int \mathrm{d}y\,\frac{\delta}{\delta\varphi_y}K\Big(\frac{\delta}{\delta\varphi_y}+\frac{\delta I}{\delta\varphi_y}+\delta_v^{\text{gauge}}(\varphi_y)\Big)\Big]P^v(\varphi,\tau). \tag{A.5}$$

If v is a local function of φ, the equilibrium distribution is defined when $\tau\to\infty$ and depends on v in a non-local way. Equation (A.5) shows also that if there is a normalizable stationary distribution $P^v(\varphi,\tau=\infty)$ for the equilibrium distribution that is reached smoothly, it is independent on K.

On the other hand, if one performs a supersymmetric representation of the Langevin equation (A.2), locality can be enforced by functionally integrating over all possible choices of v, introduced as an independent field. One must then proceed to a BRST-invariant gauge fixing of v, a task that can be done in a way that is compatible with stochastic supersymmetry.

Now we can compute $\langle\!\langle \frac{\partial\mathcal{O}[\varphi(x,\tau)]}{\partial\tau}\rangle\!\rangle^\tau$ using the Fokker–Planck equation (A.5). After an integration by parts one gets

$$\Big\langle\!\!\Big\langle\frac{\partial\mathcal{O}[\varphi(x,\tau)]}{\partial\tau}\Big\rangle\!\!\Big\rangle^\tau = \int[\mathrm{d}\varphi]\,\Big[\int\mathrm{d}y\,K\Big(\frac{\delta}{\delta\varphi_y}-\frac{\delta I}{\delta\varphi_y}-\delta_v^{\text{gauge}}(\varphi_y)\Big)\frac{\delta}{\delta\varphi_y}\Big]\mathcal{O}[\varphi(x,\tau)]. \tag{A.6}$$

Since on the right hand side $\int\mathrm{d}y\,\delta_v^{\text{gauge}}(\varphi_y))\frac{\delta}{\delta\varphi_y}$ acts as a gauge transformation with parameter v on functionals of φ, we see that if $\mathcal{O}$ is a gauge invariant functional, the last term cancels and the evolution $\langle\!\langle \frac{\partial\mathcal{O}[\varphi(x,\tau)]}{\partial\tau}\rangle\!\rangle^\tau$ of $\mathcal{O}[\varphi(x,\tau)]$ is independent on v.

On the contrary, the evolution of non gauge invariant observables depends on the choice of v, whose presence is actually necessary in order to define the evolution itself.

To compute both gauge-invariant and non-gauge-invariant correlators, one either defines a clever choice of v or considers v as an independent field and integrates over all possibilities with a BRST-invariant gauge fixing of v.

Both strategies are legitimate, provided the choice of function v or gauge fixing gives a well defined result.

As always, there are good choices of gauge versus bad choices. One expects good classes of gauges governed by some parameters. For instance in the Yang–Mills case, the class of gauges $v\equiv A_5=\alpha\partial_\mu A_\mu$ determines a perfectly well defined stochastic gauge-fixing, with α a free parameter. In this case, one can in fact prove rigorously in perturbation theory that physical correlators are α-independent [10].

In this paper devoted to gravity, we did the gauge choice of equations (6.6) and (6.9).

When we have acceleration, the dependence on v is more subtle. However, when the evolution is dominated by the friction and we are near the equilibrium, the theorem applies.

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