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Minimal determination of a pure qutrit state and four-measurement protocol for pure qudit state

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Abstract

Finding the least measurement settings to determine an arbitrary pure state has been long known as the Pauli problem. Peres conjectured that two unbiased bases may be sufficient to determine a pure state up to some finite ambiguities. Here we find that the Peres conjecture is correct in the case $d = 3$ where a pure state is determined up to at most six candidates states and is incorrect in the case of $d = 4$ for which a counterexample is constructed. We observe that the target state can be picked out from candidates states by an adaptive two-outcome measurement. We thus provided a minimal qutrit tomography protocol constituted of three measurements in contrast to the four measurements required in previous non-adaptive method. We also reduce a five-measurement protocol for a pure qudit state into three full-dimensional measurements and one two-outcome measurement protocol, which exhibits robustness to the usual white noise thus also applies to nearly pure state.

Keywords: information complete measurement setting, four-measurement protocol for pure qudit state, pure state tomography

(Some figures may appear in colour only in the online journal)

Introduction

In quantum mechanics, a quantum state contains maximum information about the system, capable of predicting the outcome statistics of all possible measurements. A converse problem is firstly considered by Pauli, which is referred to as the Pauli problem [1, 2], on what is the minimal number of measurements (each of which is performed sufficient many of runs so that probabilities are obtained without statistical errors), that are needed to specify the pure state uniquely among other pure states [3–11]. The answer to this problem is important as estimating an pure quantum state is one central task in quantum information processing.

In quantum theory, the most general measurement is represented by a positive-operator-valued measure (POVM) $\{\xi_i \geqslant 0\}$, satisfying $\sum_i \xi_i = I$. The probability corresponding to the outcome i is given by Born's rule $P_i = \mathrm{Tr}(\rho\xi_i)$. POVMs are called informationally complete [3–6] if the statistics from them are sufficient to determine any pure state (among other pure states). It has been shown that three measurement bases are not enough in dimensions $d \geqslant 3$. In [5], Jaming constructed POVMs with four properly chosen orthonormal bases, which are prefixed, while the minimal number of measurements needed for an adaptive protocol is still unknown. Adaptive protocol permits optimizing the latter measurement bases with the information obtained from previous measurements. This technique can significantly reduce the resource and shows advantages over the pre-fixed measurements schemes [12, 13]. Goyeneche *et al*, introduced a five measurements adaptive protocol which enables one to estimate a physical state with straightforward calculations. Can we also develop some adaptive protocols with less than or equals to four measurements, i.e. at least as good as the pre-fixed measurement protocols?

By investigating Peres' consideration about pure state determination (for finite dimensional system) [14] we indeed find such an adaptive protocol. Peres considered a particular version of Pauli's problem and conjectured that measurements onto two mutually unbiased bases should be sufficient to determine a pure state to finite candidate states. A justification is that two unbiased measurements yield the same number of independent parameters with those needed to parameterize an unknown physical state. This consideration captures the following question: to what extend the mutually unbiased measurements can determine a finite dimensional pure state. In the case of continuous variables, there is a two-parameter family of wave functions with the same momentum and position distributions [15].

In this paper, we shall consider discrete systems and show that the Peres' conjecture is correct in $d = 3$, in which case we prove that two unbiased measurements can settle the state down to at most six candidate states, while Peres' conjecture is incorrect in the case of $d = 4$ since there is a counter example. Moreover we observe that given a set of finite number of candidate states an adaptive dictomatic measurement can pick out the target state. As applications, we first present a minimal tomography protocol for pure qutrit state with three measurements. Secondly, we can reduce a previous five-measurement adaptive protocol for pure qudit states into a 4-meausrement protocol involving three full-dimensional and a dictomatic measurements. We then show this protocol is robust to white noise and can apply to nearly pure states.

Peres' conjecture in the case of $d = 2, 3, 4$

Peres has conjectured that, as two mutual unbiased measurements yield the same number of independent parameters as those needed for characterizing a pure state, their statistics should be sufficient to determine an arbitrary pure state up to finite ambiguities [14]. If this conjecture is correct, it provides a new viewpoint to solve Pauli's problem. In the following, we shall examine Peres' conjecture in the cases of $d = 2, 3, 4$ in details.

Peres' conjecture is easily seen to be true in the qubit case, i.e. $d = 2$. As shown in figure 1, a pure qubit state can be represented as one point on the surface of the Bloch sphere, and two unbiased measurements σ_x, σ_z determine an unknown pure state up to at most two possible pure states: for an unknown pure state $|\Phi\rangle = \sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi}|1\rangle$, measurement of σ_z determines the angle θ, i.e. state must lie on the circle $c_{0,\varphi}$ while the measurement of observable σ_x restricts the state to the circle $c_{+,\varphi'}$. Since two circles intersect at most at two points, the unknown state is determined up to two candidates $\{|\Phi\rangle, |\Phi'\rangle\}$.

Now, we prove that Peres' conjecture is correct in dimension $d = 3$. For a qutrit we consider two unbiased Von Neuman measurements $\{|0\rangle, |1\rangle, |2\rangle\}$ with and $\{|\theta_0\rangle, |\theta_1\rangle, |\theta_2\rangle\}$ where

$$|\theta_m\rangle = \frac{1}{\sqrt{3}} \sum_{n\in\{0,1,2\}} \mathrm{e}^{\mathrm{i}\frac{2mn\pi}{3}}|n\rangle. \tag{1}$$

An arbitrary pure state can always be cast into the following form

$$|\Phi\rangle = \sum_{n\in\{0,1,2\}} \alpha_n \mathrm{e}^{\mathrm{i}\varphi_n}|n\rangle \tag{2}$$

with $\alpha_n \geqslant 0$, $\sum_n \alpha_n^2 = 1$, and $\varphi_1 = 0$. Via the first measurement one can obtain the coefficients $\alpha_n^2 = |\langle n|\Phi\rangle|^2$ with $n \in \{0, 1, 2\}$. Via the second measurement one obtains probabilities $P_m = |\langle\theta_m|\Phi\rangle|^2$ through which we aim to determine those phases φ_n. For this purpose we introduce a different set of data

$$\begin{aligned} Q_j &= \sum_{m\in\{0,1,2\}} \mathrm{e}^{\mathrm{i}\frac{-2mj\pi}{3}} P_m \\ &= \frac{1}{3} \sum_{m\in\{0,1,2\}} \mathrm{e}^{\mathrm{i}\frac{-2mj\pi}{3}} \left| \sum_{n\in\{0,1,2\}} \alpha_n \mathrm{e}^{\mathrm{i}\varphi_n} \mathrm{e}^{\mathrm{i}\frac{-2mn\pi}{3}} \right|^2 \\ &= \frac{1}{3} \sum_{m,n,k\in\{0,1,2\}} \alpha_n\alpha_k \mathrm{e}^{\mathrm{i}(\varphi_n-\varphi_k)} \mathrm{e}^{\mathrm{i}\frac{2m(k-n-j)\pi}{3}} \\ &= \sum_{n\in\{0,1,2\}} \alpha_n\alpha_{n+j}\mathrm{e}^{\mathrm{i}(\varphi_n-\varphi_{n+j})} \end{aligned} \tag{3}$$

satisfying $Q_0 = 1$ and $Q_j^* = Q_{2-j}$ for $j \neq 0$. In what follows we shall show that this set of data suffices to determine those phases in the case of $d = 3$, which proves Peres' conjecture in this case.

In the case of $d = 3$, we have two independent conditions encoded in the following complex data

$$\begin{aligned} Q_1 &= \alpha_1\alpha_0\mathrm{e}^{-\mathrm{i}\varphi_1} + \alpha_2\alpha_1\mathrm{e}^{\mathrm{i}\varphi_1-\mathrm{i}\varphi_2} + \alpha_0\alpha_2\mathrm{e}^{\mathrm{i}\varphi_2} \\ &:= x\mathrm{e}^{-\mathrm{i}(\delta+\epsilon)} + y\mathrm{e}^{\mathrm{i}2\delta} + z\mathrm{e}^{\mathrm{i}(\epsilon-\delta)} \end{aligned} \tag{4}$$

where we have denoted $x = \alpha_1\alpha_0$, $y = \alpha_2\alpha_1$, and $z = \alpha_0\alpha_2$ together with $2\delta = \varphi_1 - \varphi_2$ and $2\epsilon = \varphi_1 + \varphi_2$. We note that $0 \leqslant \varphi_{1,2} < 2\pi$ so that $-\pi < \delta < \pi$ leading to $\sin\delta \neq 0$. If two of $\alpha' s$ are zero then the first measurement determines the state already. If there is one coefficient, e.g. $\alpha_2 = 0$, then we have $y = z = 0$ and the phase φ_1 can be read off directly from the phase of Q_1. In what follows we assume $x, y, z > 0$ and we rewrite condition equation (4) as

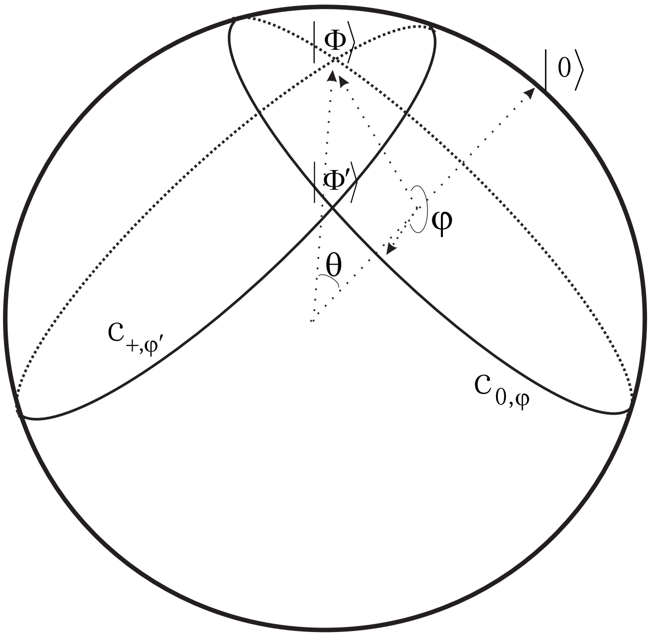

Figure 1. Peres' conjecture in the case of $d = 2$.

$$xe^{-i\epsilon} + ze^{i\epsilon} = Q_1 e^{i\delta} - ye^{i3\delta} \tag{5}$$

which, by denoting $r = \mathrm{Re}\ Q_1$ and $t = \mathrm{Im}\ Q_1$, i.e. $Q_1 = r + it$, gives rise to the following two equations

$$(z + x)\cos\epsilon = r\cos\delta - t\sin\delta - y\cos 3\delta, \tag{6}$$

$$(z - x)\sin\epsilon = r\sin\delta + t\cos\delta - y\sin 3\delta. \tag{7}$$

By eliminating variable ϵ we obtain

$$\begin{aligned}\Delta_+\Delta_- = {} & \Delta_-(r\cos\delta - t\sin\delta - y\cos 3\delta)^2 \\ & + \Delta_+(r\sin\delta + t\cos\delta - y\sin 3\delta)^2,\end{aligned} \tag{8}$$

with $\Delta_\pm = (x \pm z)^2$. Since $\sin\delta \neq 0$ the variable $u = \cot\delta$ is well defined and it holds

$$\frac{\cos 3\delta}{\sin\delta} = \frac{u(u^2 - 3)}{1 + u^2}, \quad \frac{\sin 3\delta}{\sin\delta} = \frac{3u^2 - 1}{1 + u^2} \tag{9}$$

and equation (7) becomes a polynomial equation of 6 degree about u

$$\begin{aligned}\Delta_+\Delta_-(1 + u^2)^3 &= \frac{\Delta_+\Delta_-}{\sin^6\delta} \\ &= \Delta_-\left((ru - t)(1 + u^2) - yu(u^2 - 3)\right)^2 \\ &\quad + \Delta_+\left((r + tu)(1 + u^2) - y(3u^2 - 1)\right)^2 \\ &= \sum_{k=0}^{6} c_k u^k,\end{aligned} \tag{10}$$

in which

$$c_0 = \Delta_- t^2 + \Delta_+ (r+y)^2, \tag{11}$$

$$c_1 = -2t\Delta_-(r+3y) + 2t\Delta_+(r+y), \tag{12}$$

$$c_5 = -2t\Delta_-(r-y) + 2t\Delta_+(r-3y), \tag{13}$$

$$c_6 = \Delta_-(r-y)^2 + \Delta_+ t^2. \tag{14}$$

In what follows, we shall show that the above polynomial equation (10) is nontrivial (thus it has finite solutions), i.e. not an identity, in the case of $x, y, z > 0$. If this is not the case one should have $c_1 = c_5 = 0$ since the left-hand-side has only even powers of u, from which it follows $c_1 - c_5 = 32xyzt = 0$ enforcing $t = 0$, which leads to $c_0 = \Delta_+(r+y)^2$, $c_6 = \Delta_-(r-y)^2$, and

$$c_2 = \Delta_-(r+3y)^2 + 2\Delta_+(r+y)(r-3y) \tag{15}$$

$$c_4 = \Delta_+(r-3y)^2 + 2\Delta_-(r-y)(r+3y). \tag{16}$$

To vanish identically we should have $c_0 = c_6$ and $c_2 = c_4$. However it turns out that

$$\begin{aligned} 0 &= c_4 - c_2 \\ &= \Delta_-((r-y)^2 - 16y^2) - \Delta_+((r+y)^2 - 16y^2) \\ &= 64xy^2z > 0, \end{aligned} \tag{17}$$

a contradiction. To sum, as long as $x, y, z > 0$ the polynomial equation (10) is nontrivial which means at most six solutions for u. In the case of $x \neq z$, ϵ can be uniquely determined by equations (6) and (7). In the case of $x = z$, i.e. $\Delta_- = 0$, the polynomial equation (10) becomes a nontrivial polynomial of degree at most three and thus there are at most three solutions for u and from equation (6) two solutions of ϵ are possible so that we have in total at most six solutions in all cases.

It should be noted that the constraints on the non-vanishing of all the amplitudes can be lifted. If there are two α's are zero the first measurement suffice. Suppose that there are one vanishing coefficient, e.g. $\alpha_2 = 0$ such that $y = z = 0$, then we have only to determine $\varphi_1 = \delta + \epsilon$ which is exactly the phase of Q_1. Therefore in this case two unbiased measurements are sufficient to determine a general pure qutrit state up to finite candidate states.

In the case of $d = 4$, we consider the following family of pure states (containing infinite elements) and which disprove the conjecture since all of them yield the same distributions for unbiased measurements, and the measurements fail to determine state to finite ambiguities.

$$|\Phi_\phi\rangle \propto |0\rangle + \mathrm{e}^{\mathrm{i}\phi}|1\rangle + \mathrm{e}^{\mathrm{i}\phi}|2\rangle - |3\rangle \tag{18}$$

with ϕ being arbitrary and two measurements $\{|n\rangle\}$ and $\{|u_n\rangle\}$ where

$$|u_0\rangle = \frac{1}{2}(|0\rangle + |1\rangle + |2\rangle + |3\rangle), \tag{19}$$

$$|u_1\rangle = \frac{1}{2}(|0\rangle + |1\rangle - |2\rangle - |3\rangle), \tag{20}$$

$$|u_2\rangle = \frac{1}{2}(|0\rangle - |1\rangle - |2\rangle + |3\rangle), \tag{21}$$

$$|u_3\rangle = \frac{1}{2}(|0\rangle - |1\rangle + |2\rangle - |3\rangle). \tag{22}$$

It is easy to check that these two measurements are unbiased and both probabilities $|\langle n|\Phi_\phi\rangle|^2$ and $|\langle u_n|\Phi_\phi\rangle|^2$ are independent of ϕ for all $n = 0, 1, 2, 3$, and the two measurement can not determine the state up to finite candidate states.

Indeed, for any four-dimensional pure state there exist two unbiased measurements not capable of determining the state up to finite ambiguities. This is because any pure state can be obtained by performing a unitary operation U on, for example, $|\Phi_0\rangle$ which is defined in equation (18). Note that the unbiased measurements are still unbiased after the unitary operation U, and they yield the same distributions on the state after U operation.

Adaptive 2+1 protocol for pure qutrit state

Now we proceed with Peres' conjecture holding in $d = 3$ and a pure state $|\Phi\rangle$ is determined up to some finite ambiguities $\{|\Phi_i\rangle\}$ by them. To finally determine the state, we can choose a two-outcome measurement $\{|\phi\rangle\langle\phi|, 1 - |\phi\rangle\langle\phi|\}$, where $|\phi\rangle$ is chosen such that $\{|\langle\Phi_i|\phi\rangle|^2\}$ are all different from each other. As $|\langle\Phi|\phi\rangle|^2$ must lie in the set $\{|\langle\Phi_i|\phi\rangle|^2\}$, the candidate which is compatible with the yielding binary distribution is the target state, and the ambiguities are thus removed. Obviously, this step need information from previous measurement thus it is adaptive. Note, such measurement settings are always available. See that, if state $|\phi\rangle$ does not meet the requirement, then there are i, j so that $|\langle\phi|\Phi_i\rangle| = |\langle\phi|\Phi_j\rangle|$, and $|\phi\rangle$ then can be parameterized by strictly-less independent parameters than that needed for a general state, and each binary combination of candidate states define a lower-dimensional subspace with measure zero, the union of them is also lower-dimensional. This implies that $|\phi\rangle$ can even be picked at random, and it is nearly impossible to take by chance a state violating the requirement.

The above proof and argument lead to a minimal pure qutrit tomography protocol directly. In contrast to the currently non-adaptive method using minimal four measurements with 12 outcomes, our adaptive measurement protocol composites of three measurements with 8 outcomes. By the first unbiased measurement, we can obtain the amplitude. The second unbiased measurement can determine the phases up to a finite set. While there is no general analytic formula available to calculate these phases, we have to find them numerically. For this propose, we can construct a family states with the amplitudes keeping constant (determined by the first measurement) and uniformly assigning random values to phases $\{\varphi_1, \varphi_2\}$. These states predict the probabilities $\{P_m(\{\varphi_n\})\}$ for the second measurement on each assignment. We introduce a cost function

$$c(\{\varphi_n\}) \equiv \sum_m |P_m(\{\varphi_n\}) - P_m|,$$

and it must equal to zero when $\{\varphi_n\}$ corresponds to a candidate state. For the finiteness of the post-processing, we find sufficiently small values of the cost function c which can be regarded as zero within allowed errors. In the following, we apply the method to determine a pure qutrit state.

As an example let us determine the following pure state in the three-dimensional Hilbert space

$$|\Phi\rangle = \frac{|0\rangle + e^{i\frac{\pi}{2}}|1\rangle + e^{i\frac{\pi}{3}}|2\rangle}{\sqrt{3}}.$$

To do so we shall at first perform the measurement on the computational basis $\{|0\rangle, |1\rangle, |2\rangle\}$, yielding probabilities $|\alpha_n|^2 = \frac{1}{3}$ from which the target state is determined up to two unknown phases: $|\Phi_{\varphi_1,\varphi_2}\rangle = (|0\rangle + \mathrm{e}^{\mathrm{i}\varphi_1}|1\rangle + \mathrm{e}^{\mathrm{i}\varphi_2}|2\rangle)/\sqrt{3}$. And then we perform the second measurement on the unbiased basis $\{|\theta\rangle\}$ that yields probabilities $\{P_m\}$. We assign 1000 values evenly from $(0, 2\pi)$ to (φ_1, φ_2), the assignments give 1000×1000 states $\{|\Phi_{\varphi_1,\varphi_2}\rangle\}$. The cost function $c(\varphi_1, \varphi_2)$ is shown in figure 2. It shows six phase vectors,

$$\begin{gathered}\{\frac{\pi}{2}, \frac{\pi}{6}\}, \{\frac{5\pi}{3}, \frac{\pi}{6}\}, \{\frac{\pi}{2}, \frac{\pi}{3}\}, \\ \{\frac{11\pi}{6}, \frac{\pi}{3}\}, \{\frac{5\pi}{3}, \frac{3\pi}{2}\}, \{\frac{11\pi}{6}, \frac{3\pi}{2}\}\end{gathered} \tag{23}$$

as the numerical solution to $c(\varphi_1, \varphi_2) = 0$.

Lastly, we perform a third measurement which is a dichotomic POVM $\{|\phi\rangle\langle\phi|, 1 - |\phi\rangle\langle\phi|\}$ with, for a instance, $|\phi\rangle = \frac{\sqrt{2}}{2}|0\rangle + \frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|1\rangle + \frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}|2\rangle$. The resulting probabilities read $\{0.81, 0.39, 0.74, 0.54, 0.04, 0.12\}$, respectively, corresponding to six candidate states. The statistics of the third measurement should be compatible with $|\langle\Phi|\phi\rangle|^2 = 0.74$ from which the state can be uniquely determined. In total, this protocol only requires measuring two mutually unbiased and one dichotomic observables while at least four measurements are required in a pre-fixed measurement schemes [7].

We now apply the analytic approach to this case. As $x = z$, the equations (6) and (10) become

$$2\cos\epsilon = \frac{\sqrt{3}+1}{2}\cos\delta - \frac{\sqrt{3}-1}{2}\sin\delta - \cos 3\delta, \tag{24}$$

$$(3 + 2\sqrt{3})u^3 + u^2 - (1 - 2\sqrt{3})u + 1 = 0, \tag{25}$$

with three solutions $u = \cot\delta \in \{-1, \sqrt{3}, 2 + \sqrt{3}\}$ such that $\delta \in \{-\pi/4, \pi/6, \pi/12\}$ to the cubic equation (25). For each δ equation (24) gives rise to two possible $\epsilon \in \{\pi/12, \pi/3, 5\pi/12\}$ and $\pi - \epsilon$ from which the numerical solution equation (23) can be reproduced by noting $\varphi_{1,2} = \epsilon \pm \delta$.

Adaptive 3+1 protocol for pure qudit state

For given d whenever Peres' conjecture holds true, we always have an adaptive $2 + 1$ protocol similar to qutrit case. For higher dimensional cases, however, Peres' conjecture remains a conjecture, here we shall propose instead an adaptive $3 + 1$ protocol. It has been shown that any pure state can be determined by statistics from the following five measurements [3]:

$$B_1 = \{|2j - 1\rangle, |2j\rangle\}, \tag{26}$$

$$B_2 = \{\frac{1}{\sqrt{2}}(|2j - 1\rangle \pm |2j\rangle)\}, \tag{27}$$

$$B_3 = \{\frac{1}{\sqrt{2}}(|2j\rangle \pm |2j + 1\rangle)\}, \tag{28}$$

$$B_4 = \{\frac{1}{\sqrt{2}}(|2j - 1\rangle \pm \mathrm{i}|2j\rangle)\}, \tag{29}$$

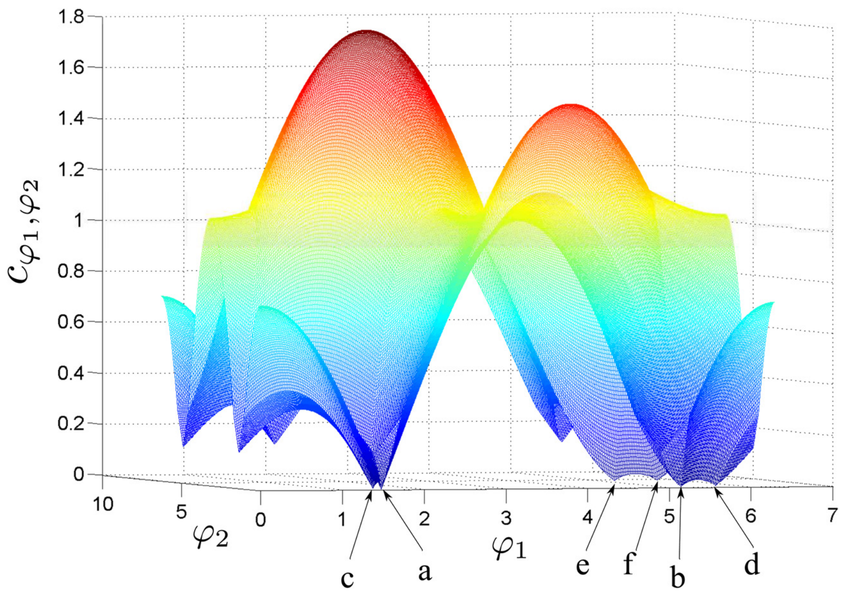

Figure 2. The discriminant equation c_{φ_1,φ_2} is depicted. Any zero value indicates a candidate state. This figure shows six candidate states compatible with the second measurements.

$$B_5 = \{\frac{1}{\sqrt{2}}(|2j\rangle \pm \mathrm{i}|2j+1\rangle)\}. \tag{30}$$

One can assume d is even (generalizing to the case of odd d is trivial), where $j \in \{1, 2, \cdots \frac{d}{2}\}$ and the addition of labels is carried out modulo in d. As statistics from the measurement on B_1 determine the amplitudes, one can construct the other four bases with $\{|2j-1\rangle, |2j\rangle\}$ which have non-zero amplitudes. With the statistics from all the measurements, one can easily deduce an analytical expression for all the parameters [3].

We shall now show below that three measurements B_1, B_2, and B_3 determine the unknown state up to some finite ambiguities, thus we can give a $3+1$ measurements protocol. According to the amplitudes determined by measurement B_1, we relabel the amplitudes to ensure that $\alpha_i \geqslant \alpha_{i+1}$. This arrange would bring advantage in the following calculations. See that,

$$\left\{P_{2j,\pm} = \frac{1}{2}\left|\alpha_{2j} \pm \alpha_{2\mathrm{i}-1}\mathrm{e}^{\mathrm{i}(\varphi_{2j-1}-\varphi_{2j})}\right|^2\right\} \tag{31}$$

obtained from measurement B_2 determine each relative phase up to two possibilities:

$$\varphi_{2j} - \varphi_{2j-1} = \pi \pm (\pi - \arccos\beta_{2j}) \tag{32}$$

where $\beta_j := \frac{P_{j,+}-P_{j,-}}{2\alpha_j\alpha_{j-1}}$. Similarily, the measurement B_3 determines the relative phases as

$$\varphi_{2j+1} - \varphi_{2j} = \pi \pm (\pi - \arccos\beta_{2j+1}). \tag{33}$$

Solving the linear system equations (32) and (33) iteratively, the phases can be determined up to the 2^{d-1} ambiguities denoted by phase vectors $\overrightarrow{\varphi}_\mu$, with each vector corresponding to state $|\Phi_\mu\rangle$.

To remove the ambiguities, we also introduce a two-outcome measurement $\{|\phi\rangle\langle\phi|, 1-|\phi\rangle\langle\phi|\}$ and, require $|\phi\rangle$ to satisfy the constraint: $\mu \neq \nu$, $|\langle\phi|\Phi_\mu\rangle|^2 \neq |\langle\phi|\Phi_\nu\rangle|^2$. By introducing this procedure, we reduce the five bases down to three, with the resulting protocol referred to as $3+1$ protocol here.

There are unavoidable processing errors, such as white noise, in a real experiment. Here we show that our $3+1$ protocol is robust to white noise and thus can apply to nearly pure states as (with $1 \gg \lambda$)

$$\rho' = (1-\lambda)|\Phi\rangle\langle\Phi| + \frac{\lambda}{d}I. \tag{34}$$

Note that a pure state is mainly determined by the its large amplitudes and the corresponding phases. We can restrict ourselves on the accuracy of the estimation of these essential terms. In the presence of white noise, the statistics from measurements can be obtained by averaging the probability p from measurement on $|\Phi\rangle\langle\Phi|$ with a white noise into $p' \rightarrow (1-\lambda)p + \frac{\lambda}{d}$. So the amplitudes would be estimated as $\alpha_i'^2 = (1-\lambda)\alpha_i^2 + \frac{\lambda}{d}$, and the relative error of amplitude reads

$$\frac{|\alpha_i' - \alpha_i|}{\alpha_i} = \frac{|1-\mathrm{d}\alpha_i^2|}{2\mathrm{d}\alpha_i^2}\lambda. \tag{35}$$

For the phases estimation using equations (32) and (33), the phases are estimated iteratively in an increasing order of amplitudes, from α_1 to α_{n-1}. In a real experiment, the outcome statistics is subject to statistical errors, and these errors would accumulate in the recursive estimations. Note that the phases corresponding to large amplitudes are more important than those corresponding to the small ones, we can reduce the error accumulations by estimating them in early stage of estimations. For this, we rearrange the label after the first measurement to ensure that $\alpha_i \geqslant \alpha_{i+1}$ and then construct the second and the third measurement with this rearranged setting. For the state equation (34) and the phase estimations equations (32) and (33), we have $P'_{j,+} - P'_{j,-} = (1-\lambda)(P_{j,+} - P_{j,-})$ and the relative error of β_j on the right hand side of equations (32) and (33) reads

$$\frac{|\beta_j' - \beta_j|}{\beta_j} = \frac{\lambda}{2d}\left(\frac{1}{\alpha_j^2} + \frac{1}{\alpha_{j-1}^2}\right). \tag{36}$$

When $2\mathrm{d}\alpha_i^2 \gg \lambda$ the relative errors equations (35) and (36) are small and the estimation are accurate. The errors are significant when $2\mathrm{d}\alpha_i^2 \simeq \lambda$, however, we need not worry about them since the amplitudes are negligible, of order $\lambda/2d$, so that the corresponding phases are inessential.

Our method applies to the cases when the estimated state is known to be pure or almost pure. The three measurements B_1, B_2, and B_3 determine $\{\rho_{ii}\}$ and the real parts in $\rho_{i,i+1}, \rho_{i+1,i}$, which, together with the statistics from the measurement $\{|\phi\rangle\langle\phi|, 1-|\phi\rangle\langle\phi|\}$, are sufficient to determine a pure state while insufficient for a mixed one. For a general case without the purity assumption, the statistics from the three measurements allow a weak purity certification, $\mathrm{tr}(\rho^2) \geqslant \sum_{i=1,\cdots,d}\alpha_i^2 + \sum_{j=1,\cdots,\frac{d}{2}}\frac{1}{2}[(p_{2j,+} - p_{2j,-})^2 + (p_{2j+1,+} - p_{2j+1,-})^2]$, (here, the last two terms come from the real parts in $\rho_{i,i+1}, \rho_{i+1,i}$). This certification is not sufficient strong, hence the assumption on the purity is generally needed.

In summary, we have proved that Peres' conjecture is correct in the case of $d=3$ that enables us to propose a minimal adaptive protocol to determine an unknown pure qutrit state. In the mean time we have shown that Peres' conjecture is incorrect in the case of $d=4$. Also, we have shown that our two-outcome measurement to pick out the target state out of a finite number of states used in the qutrit determination protocol can also apply to arbitrary dimensional pure state tomography. As an application we improve the five-measurement protocol into a $3 + 1$ protocol, i.e. three full dimensional measurements and one two-outcome measurement. By analysing the effect of white noise to our $3 + 1$ protocol, we find that our protocol is robust and therefore also applies to nearly pure state.

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