IOP Publishing Physica Scripta

Phys. Scr. 95 (2020) 035103 (7pp) https://doi.org/10.1088/1402-4896/ab5324

Lie algebraic approach to quantum driven optomechanics

A Paredes-Juárez[1], I Ramos-Prieto[1], M Berrondo[2] and J Récamier[1]

[1] Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Apartado Postal 48-3, 62251 Cuernavaca, Morelos, México

[2] Department of Physics and Astronomy, Brigham Young University, Provo, UT 84602, United States of America

E-mail: iranrp123@gmail.com

Received 9 July 2019, revised 3 September 2019
Accepted for publication 31 October 2019
Published 28 January 2020

Abstract

We present an approximate Lie algebraic method to deal with a forced optomechanical Hamiltonian. We show that the approximations made in order to linearize the interaction Hamiltonian are fully justified by means of a comparison between a purely numerical calculation of the number of photons, phonons and linear entropy using the full Hamiltonian and the results obtained by means of our approximate time evolution operator.

Keywords: quantum optics, optomechanics, Lie algebra

(Some figures may appear in colour only in the online journal)

1. Introduction

The control of quantum states of macroscopic objects and force detection within and very close to the quantum regime, has been achieved in optomechanical systems [1, 2]. The coupling between optical and mechanical microscopic degrees of freedom is the fundamental principle to enable the description of radiation pressure on a mirror, where field and mechanical modes are coupled in a Fabry–Pérot cavity [3]. In this sense, quantum optomechanics essentially leads to two perspectives: quantum control over mechanical motion or, conversely, mechanical control over quantum states [4–6].

The radiation pressure that light exerts on a material object was observed experimentally more than a century ago [7, 8]. Subsequently, in recent decades, an interest in the motion of mechanical oscillators (mass center), coupled to oscillation modes in a cavity has resurfaced [9–11]. Some recent applications of this type of resonators include: the LIGO project that uses gravitational wave interferometers whose optical path is modified due to radiation pressure [12], the cooling of mechanical resonators in the study of the transition between quantum and classical behavior [13] and the amplification and measurement of nanometric scale forces [14, 15], are representative of the very varied applications of this type of systems. One of the most important applications within this framework, was given by Ashkin [16] when he showed that small dielectric balls can be accelerated and trapped using radiation pressure forces via focused laser beams. This idea led to the realization of optical tweezers. Another important development was the laser cooling of ions and neutral atoms, which culminated in the experimental generation of Schrödinger cat states and Bose–Einstein condensates [17–20]. On the other hand, to round off the non-classicality of states at a macroscopic scale, some schemes based on homodyne tomography have been proposed for the reconstruction of quasi-probability distributions within the associated phase space [21–24].

Taking advantage of the fact that the coherent states are the quantum states whose statistical behavior most resemble the classical one, the quantum theory of optomechanical cavities has enabled the generation of these states or even superpositions of them, known as Schrödinger cat states [4, 25–29]. So, in order to obtain the time-evolution operator for the driven optomechanical system, from a theoretical point of view, is a difficult task. Moreover, when the system is interacting with its environment, it is not possible to solve the master equation of the pumped optomechanical system. However, from an operational approach it is possible to obtain analytical approximations for the evolution operator [30]. In this regard, in this work we propose an approach

[3] Author to whom any correspondence should be addressed.

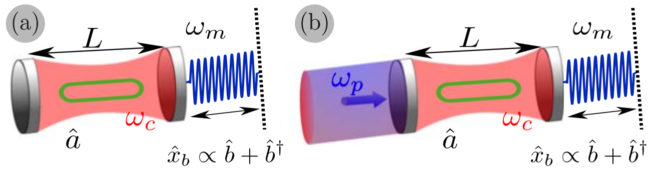

Figure 1. Schematic representation of an optomechanical system of a mirror with variable position $\hat{x}_b$ coupled to a cavity of fixed length L and single-mode frequency ω_c, and mirror frequency ω_m. (a) No driving system, and (b) and driving system of frequency ω_p.

based on Lie algebraic techniques, since the constituent operators of the unperturbed model Hamiltonian turns out to generate a closed Lie algebra. This fact enables us to express the corresponding evolution operator of the whole system as a product of exponentials according to the well-known Wei–Norman theorem [31, 32].

The paper is organized as follows: in section 2 we make use of algebraic techniques to obtain a Hamiltonian in the interaction representation that is amenable to approximations such that its time evolution operator can be written as a product of exponentials; we apply it to an initial pure state written as a product of coherent states for the field and for the mechanical oscillator. In section 3 we evaluate the average number of photons, phonons and linear entropy as a function of time and as a function of ω_p/ω_c. In section 4 we give analytic expressions for the evaluation of the Husimi Q function for the field and for the mechanical oscillator and present numerical results for the Husimi function of the field at some selected instants of time. Finally, in section 5 we give our conclusions.

2. Lie algebraic approach for a driven optomechanical system

Let us consider a system described by a Hamiltonian consisting of a mechanical harmonic oscillator of frequency ω_m, a field oscillator with frequency ω_c, an optomechanical coupling between the field and the mechanical oscillator given by the third term in equation (1) and a driving of the field mode with frequency ω_p (see figure 1). The simplest form for the pumped optomechanical system is given by the Hamiltonian [30, 33–35]:

$$\frac{\hat{H}_p}{\hbar} = \omega_c\hat{n} + \omega_m\hat{N} - G\hat{n}(\hat{b} + \hat{b}^\dagger) + \Omega\cos(\omega_p t)(\hat{a}^\dagger + \hat{a}), \tag{1}$$

where, Ω is related to the input laser power P and G the coupling constant is given by

$$G = \frac{\omega_c}{L}\left(\frac{\hbar}{2m\omega_m}\right)^{1/2}. \tag{2}$$

When there is no pumping term, Hamiltonian (1) reduces to the time-independent optomechanical Hamiltonian [4, 25, 33],

$$\hat{H}_{\rm opt} = \hbar\omega_c\hat{n} + \hbar\omega_m\hat{N} - \hbar G\hat{n}(\hat{b} + \hat{b}^\dagger). \tag{3}$$

The set of operators appearing in $\hat{H}_{\rm opt}$ has the following commutation relations: ($\hat{n} = \hat{a}^\dagger\hat{a}$, $\hat{N} = \hat{b}^\dagger\hat{b}$):

	$\hat{n}$	$\hat{N}$	$\hat{n}\hat{b}$	$\hat{n}\hat{b}^\dagger$	$\hat{n}^2$
$\hat{n}$	0	0	0	0	0
$\hat{N}$	0	0	$-\hat{n}\hat{b}$	$\hat{n}\hat{b}^\dagger$	0
$\hat{n}\hat{b}$	0	$\hat{n}\hat{b}$	0	$\hat{n}^2$	0
$\hat{n}\hat{b}^\dagger$	0	$-\hat{n}\hat{b}^\dagger$	$-\hat{n}^2$	0	0
$\hat{n}^2$	0	0	0	0	0

In this table, we had to incorporate the operator $\hat{n}^2$ that arises from the commutator between $\hat{n}\hat{b}$ and $\hat{n}\hat{b}^\dagger$. The set of operators appearing in this table is closed under commutation. Thus, the time evolution operator corresponding to $\hat{H}_{\rm opt}$ can be written exactly as a product of exponentials [4, 31],

$$\hat{U}_{\rm opt}(t) = {\rm e}^{\alpha_1\hat{n}}{\rm e}^{\alpha_2\hat{N}}{\rm e}^{(\alpha_3+|\alpha_4|^2/2)\hat{n}^2}\hat{D}_{\hat{b}}(\alpha_4\hat{n}). \tag{4}$$

Here, $\hat{D}_{\hat{A}}(\alpha) = {\rm e}^{\alpha\hat{A}^\dagger - \alpha^*\hat{A}}$ is the Glauber displacement operator. The time-dependent functions α_i are obtained after substitution of equation (4) into Schrödinger's equation, and are given by:

$$\begin{aligned}\alpha_1 &= -{\rm i}\omega_c t,\\ \alpha_2 &= -{\rm i}\omega_m t,\\ \alpha_3 &= -\left(\frac{G}{\omega_m}\right)^2[-{\rm i}\omega_m t + (1 - {\rm e}^{-{\rm i}\omega_m t})],\\ \alpha_4 &= -\frac{G}{\omega_m}(1 - {\rm e}^{{\rm i}\omega_m t}).\end{aligned} \tag{5}$$

However, with the pumping term present, this is not enough. In order to deal with the pumping term, we will use the interaction picture for the complete Hamiltonian:

$$\hat{H}_p = \hat{H}_{\rm opt} + \hbar\Omega\cos(\omega_p t)(\hat{a}^\dagger + \hat{a}), \tag{6}$$

with $\hat{H}_{\rm opt}$ given by equation (3). The full time evolution operator is then factorized as $\hat{U} = \hat{U}_{\rm opt}\hat{U}_{\rm I}$, where $\hat{U}_{\rm opt}$ is given in equation (4) and the time evolution operator in the interaction picture $\hat{U}_{\rm I}$ satisfies the equation

$${\rm i}\hbar\partial_t\hat{U}_{\rm I} = \hbar\Omega\cos(\omega_p t)[\hat{U}_{\rm opt}^\dagger(\hat{a} + \hat{a}^\dagger)\hat{U}_{\rm opt}]\hat{U}_{\rm I} \tag{7}$$

with the initial condition $\hat{U}_{\rm I}(t_0, t_0) = 1$. Applying the transformation given above we obtain

$$\begin{aligned}&\hat{U}_{\rm opt}^\dagger\hat{a}\hat{U}_{\rm opt}\\ &= {\rm e}^{{\rm i}E(t)(2\hat{n}+1)}{\rm e}^{{\rm i}F(t)[\hat{b}^\dagger{\rm e}^{{\rm i}\frac{L}{2}\omega_m} + \hat{b}{\rm e}^{-{\rm i}\frac{L}{2}\omega_m}]}\hat{a}{\rm e}^{-{\rm i}\omega_c t}\\ &\hat{U}_{\rm opt}^\dagger\hat{a}^\dagger\hat{U}_{\rm opt}\\ &= \hat{a}^\dagger{\rm e}^{{\rm i}\omega_c t}{\rm e}^{-{\rm i}F(t)[\hat{b}^\dagger{\rm e}^{{\rm i}\frac{L}{2}\omega_m} + \hat{b}{\rm e}^{-{\rm i}\frac{L}{2}\omega_m}]}{\rm e}^{-{\rm i}E(t)(2\hat{n}+1)},\end{aligned} \tag{8}$$

where

$$F(t) = 2\left(\frac{G}{\omega_m}\right)\sin\left(\frac{\omega_m}{2}t\right)$$
$$E(t) = \left(\frac{G}{\omega_m}\right)^2(\omega_m t - \sin(\omega_m t)). \tag{9}$$

Notice the presence of the operators in the exponentials; this fact prevents the use of Lie algebraic methods. However, the factor G/ω_m is much smaller than one [30] so we will approximate the interaction Hamiltonian by:

$$\hat{H}_1 = \hbar\Omega\cos(\omega_p t)(\hat{a}e^{-i\omega_c t} + \hat{a}^\dagger e^{i\omega_c t}). \tag{10}$$

The corresponding time evolution operator $\hat{U}_1$ can be written as:

$$\hat{U}_1 = e^{\beta\hat{a}^\dagger}e^{\gamma\hat{a}}e^{\delta}, \tag{11}$$

where the time dependent functions β, γ, and δ satisfy the following set of coupled ordinary differential equations:

$$\dot{\beta} = -i\Omega\cos(\omega_p t)e^{i\omega_c t}$$
$$\dot{\gamma} = -i\Omega\cos(\omega_p t)e^{-i\omega_c t}$$
$$\dot{\delta} = \beta\dot{\gamma} \tag{12}$$

with the initial conditions $\beta(0) = \gamma(0) = \delta(0) = 0$. Integrating we obtain the analytical expressions:

$$\beta(t) = -\frac{\Omega}{2}\left(\frac{e^{i(\omega_p+\omega_c)t}}{\omega_p+\omega_c} - \frac{1}{\omega_p+\omega_c}\right) + \frac{\Omega}{2}\left(\frac{e^{-i(\omega_p-\omega_c)t}}{\omega_p-\omega_c} - \frac{1}{\omega_p-\omega_c}\right),$$
$$\gamma(t) = -\frac{\Omega}{2}\left(\frac{e^{i(\omega_p-\omega_c)t}}{\omega_p-\omega_c} - \frac{1}{\omega_p-\omega_c}\right) + \frac{\Omega}{2}\left(\frac{e^{-i(\omega_p+\omega_c)t}}{\omega_p+\omega_c} - \frac{1}{\omega_p+\omega_c}\right),$$
$$\delta(t) = i\frac{\Omega^2}{4}\left(\frac{1}{\omega_p+\omega_c} - \frac{1}{\omega_p-\omega_c}\right)t + \frac{\Omega^2}{4}\left[\frac{e^{2i\omega_p t}-1}{2\omega_p(\omega_p+\omega_c)} - \frac{e^{i(\omega_p-\omega_c)t}-1}{\omega_p^2-\omega_c^2} + \frac{e^{-i(\omega_p+\omega_c)t}-1}{(\omega_p+\omega_c)^2} + \frac{e^{i(\omega_p-\omega_c)t}-1}{(\omega_p-\omega_c)^2} + \frac{e^{-2i\omega_p t}-1}{2\omega_p(\omega_p-\omega_c)} - \frac{e^{-i(\omega_p+\omega_c)t}-1}{\omega_p^2-\omega_c^2}\right], \tag{13}$$

and in what remains we omit writing the time dependence of the previous functions. Additionally we can notice that $\gamma = -\beta^*$, so that

$$\hat{U}_1 = e^{\delta}e^{\frac{1}{2}|\beta|^2}e^{\beta\hat{a}^\dagger-\beta^*\hat{a}} = e^{\delta+\frac{1}{2}|\beta|^2}\hat{D}_{\hat{a}}(\beta). \tag{14}$$

In the case when $\omega_p = \omega_c$ the integration yields:

$$\beta = -\frac{\Omega}{4\omega_p}(e^{2i\omega_p t} - 1) - i\frac{\Omega t}{2},$$
$$\delta = \frac{\Omega^2}{16\omega_p^2}[(1 - 2i\omega_p t)\cos 2\omega_p t + i(1 + 2i\omega_p t)\sin 2\omega_p t + 1 - 2\omega_p^2 t^2]. \tag{15}$$

Consequently, using (4) and (14), the full time evolution operator is:

$$\hat{U}(t) = e^{\delta+\frac{1}{2}|\beta|^2}e^{\alpha_1\hat{n}}e^{\alpha_2\hat{N}}e^{\left(\alpha_3+\frac{|\alpha_4|^2}{2}\right)\hat{n}^2} \times \hat{D}_{\hat{b}}(\alpha_4\hat{n})\hat{D}_{\hat{a}}(\beta). \tag{16}$$

Consider now an initial state given by the product of coherent states

$$|\Psi(0)\rangle = |\alpha\rangle_c \otimes |\Gamma\rangle_m, \tag{17}$$

where $|\alpha\rangle_c$ is a field coherent state and $|\Gamma\rangle_m$ is a mechanical coherent state. Therefore, using the relationship [36]

$$D_{\hat{a}}(\beta)|\alpha\rangle_c = \hat{D}_{\hat{a}}(\beta)\hat{D}_{\hat{a}}(\alpha)|0\rangle_c = e^{\frac{1}{2}(\beta\alpha^*-\beta^*\alpha)}|\alpha+\beta\rangle_c, \tag{18}$$

and applying the time evolution operator to the initial state we get:

$$|\Psi(t)\rangle = \hat{U}_{\rm opt}(e^{\delta+\frac{1}{2}|\beta|^2}e^{\frac{1}{2}(\beta\alpha^*-\beta^*\alpha)}|\alpha+\beta\rangle_c)\otimes|\Gamma\rangle_m$$
$$= e^{\delta+\frac{1}{2}|\beta|^2+\frac{1}{2}(\beta\alpha^*-\beta^*\alpha)-\frac{1}{2}|\alpha+\beta|^2} \times \sum_p \frac{(\alpha+\beta)^p}{\sqrt{p!}}e^{\left[\alpha_1+\frac{1}{2}(\alpha_4\Gamma^*-\alpha_4^*\Gamma)\right]p} \times e^{\left(\alpha_3+\frac{1}{2}|\alpha_4|^2\right)p^2}|p\rangle_c \otimes |\Gamma_p(t)\rangle_m, \tag{19}$$

with the explicit form for the functions α_i, given by

$$\alpha_3 + \frac{1}{2}|\alpha_4|^2 = i\left(\frac{G}{\omega_m}\right)^2(\omega_m t - \sin\omega_m t),$$
$$\alpha_1 + \frac{1}{2}(\alpha_4\Gamma^* - \alpha_4^*\Gamma) = -i[\omega_c t - \Im(\alpha_4\Gamma^*)], \tag{20}$$

and

$$\Gamma_p(t) = \Gamma e^{-i\omega_m t} + p\frac{G}{\omega_m}(1 - e^{-i\omega_m t}). \tag{21}$$

Consequently the wave function or the state vector of the complete system at time t is:

$$|\Psi(t)\rangle = e^{\delta+\frac{1}{2}|\beta|^2-\frac{1}{2}|\alpha+\beta|^2}e^{i\Im(\beta\alpha^*)} \sum_p \frac{(\alpha+\beta)^p}{\sqrt{p!}}e^{i(G/\omega_m)^2(\omega_m t-\sin\omega_m t)p^2} \times e^{-i[\omega_c t-\Im(\alpha_4\Gamma^*)]p}|p\rangle_c \otimes |\Gamma_p(t)\rangle_m. \tag{22}$$

We see from equation (21) that the entanglement between the field and the mechanical oscillator is maximal for $\omega_m t = \pi$ and the system returns to its non entangled state at $\omega_m t = 2\pi$.

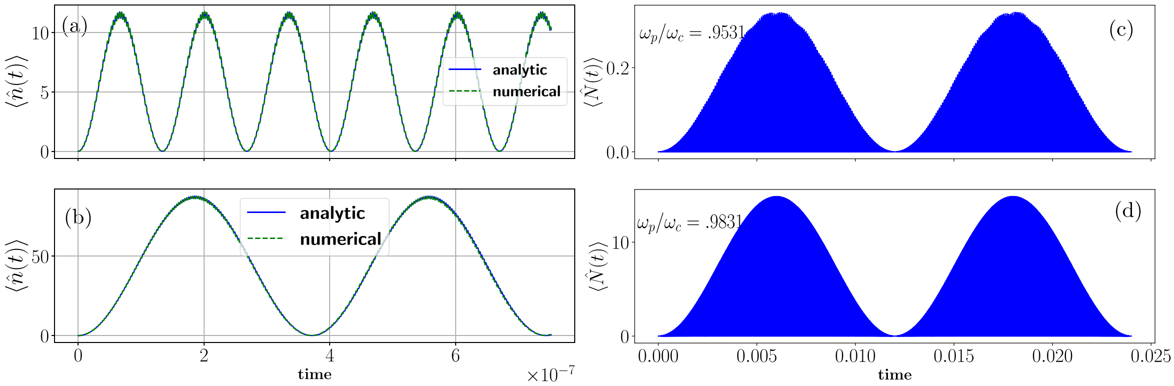

Figure 2. Average value of the photon number as a function of time (left column) with initial coherent state $|\alpha = 0\rangle$ and average value of the phonon number (right column) with initial mechanical coherent state $|\Gamma = 0\rangle$. Hamiltonian parameters: top $\omega_p = 0.9531\omega_c$, bottom $\omega_p = 0.983\,1\omega_c$. Time is given in seconds.

3. Average number of photons, phonons, and linear entropy

The Hamiltonian for a pumped optomechanical system given by equation (1) contains creation and annihilation operators for the field and a coupling term between the number of photons present in the field and the creation–annihilation operators for the mechanical oscillator. Due to the pumping term, the number of photons will be evolving in time and, since the coupling between the field and the mechanical oscillator depends upon the number of photons present, it is of interest to evaluate the time evolution of the average number of photons and phonons. The average number of photons at time t is

$$\begin{aligned}\langle\hat{n}(t)\rangle &= \langle\Psi(t)|\hat{n}|\Psi(t)\rangle\\ &= \mathrm{e}^{2\Re(\delta)+|\beta|^2}|\alpha+\beta|^2 = |\alpha+\beta|^2,\end{aligned} \tag{23}$$

where we have used the unitarity condition $2\Re(\delta) + |\beta|^2 = 0$.

The average number of phonons at time t depends on the number of photons at time t and is given by:

$$\begin{aligned}\langle\hat{N}(t)\rangle &= \langle\Psi(t)|\hat{N}|\Psi(t)\rangle\\ &= \mathrm{e}^{-|\alpha+\beta|^2}\sum_p \frac{|\alpha+\beta|^{2p}}{p!}|\Gamma_p(t)|^2.\end{aligned} \tag{24}$$

For the numerical calculations, we have used a set of Hamiltonian parameters obtained from [37] that is: $\omega_c = 10^9\ \mathrm{s}^{-1}$, $\omega_m = (\pi/6)\ \omega_c\ 10^{-6}$, $L = 10^{-4}$ m, $m = 10^{-13}$ kg so that $G/\omega_m = 2.36\times10^{-2}$. For the forcing term we set $\Omega = (\pi/2)\omega_c\ 10^{-1}$.

We see in figure 2 left column the results of our approximate method (full line) and those obtained from a converged numerical calculation (dots) for the photon generation in the case when the initial state of the field is the vacuum state. It can be seen an excellent agreement between both of them. In the top figure (a) we show the case when $\omega_p = 0.9531\omega_c$; at the initial time there are no photons present, as time evolves the average number of photons increases periodically with a period of the order of 10^{-7} s. and an amplitude around 10. In the bottom figure, left side (b), we show a case nearer resonance conditions, $\omega_p = 0.9831\omega_c$. As in the previous case there is an oscillatory behavior, however, the amplitude of the oscillations is much larger attaining values around 60 and notice also the different period for the oscillations, here it is around 3 times that in the previous case. For larger differences between the forcing frequency and the field's frequency (for instance, when $\omega_p = 0.8\omega_c$) the photon generation is much smaller attaining values around 1.25 only. The ratio between frequencies was chosen near resonance (but far enough) in order to guarantee the convergence of time-dependent calculations, since when we are very close to resonance (equation (15)) the average number of photons increases rapidly and this translates into the need of a very large Hilbert space that becomes computationally unsustainable.

In the right column of figure 2 we show the time-evolution of the average number of phonons with initial state the vacuum state. In the top figure (c) we present the results obtained when $\omega_p = 0.9531\omega_c$. In that case, the average number of photons is a periodic function with amplitude around 10 and the entanglement between the field and the mechanical oscillator produces the generation of phonons. In this case we see an oscillatory behavior with a period $T_m = 2\pi/\omega_m \simeq 0.012$ s and very fast oscillations with frequency of the order of $10^7\ \mathrm{s}^{-1}$. These oscillations are so fast that they cannot be distinguished in the figure. We can see from the figure that the average number of phonons attains a maximum value around 0.25. At the bottom (d) we show the case with $\omega_p = 0.9831\omega_c$, there, the photon generation is much larger and the phonon generation increases correspondingly attaining a maximum around 12. Notice that the period for the average number of phonons is the same in both cases since it depends only upon the frequency of the mechanical oscillator.

Let us now compute the linear entropy for a forced system whose state at time t is given by equation (22). In general, we can calculate the linear entropy of any one of the two subsystems by $S^{(x)} = 1 - \mathrm{Tr}_x[\rho_x^2]$, where x is the label of the mirror or cavity. In this sense, the density matrix of the

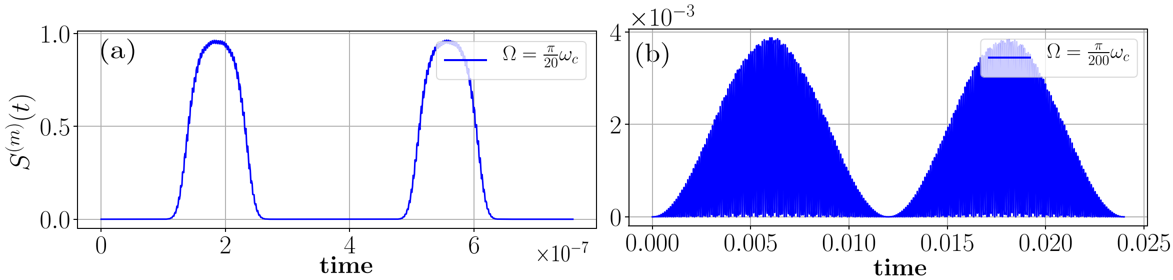

Figure 3. Time evolution of the Linear entropy for the mirror. Initial state $|\Psi(0)\rangle = |\alpha\rangle_c \otimes |\Gamma\rangle_m = |0\rangle_c \otimes |0\rangle_m$. Amplitude of the forcing term (a) $\Omega = \frac{\pi}{2}\omega_c \times 10^{-1}$, (b) $\Omega = \frac{\pi}{2}\omega_c \times 10^{-2}$ Notice the time scale (in seconds).

mirror is given by the partial trace:

$$\begin{aligned}\rho_m(t) &= \mathrm{Tr}_c[\rho] \\ &= \mathrm{e}^{-|\alpha+\beta|^2} \sum_p \frac{|\alpha+\beta|^{2p}}{p!} |\Gamma_p(t)\rangle\langle\Gamma_p(t)|,\end{aligned} \tag{25}$$

where $\mathrm{Tr}_c[\rho]$ is the trace over the field's degrees of freedom of the full density matrix ρ and $\Gamma_p(t)$ is defined in equation (21). From this expression we get

$$\begin{aligned}\rho_m^2(t) &= \mathrm{e}^{-2|\alpha+\beta|^2} \sum_{p,p'} \frac{|\alpha+\beta|^{2p}}{p!} \frac{|\alpha+\beta|^{2p'}}{p'!} \\ &\times \mathrm{e}^{-\frac{|\Gamma_p(t)|^2}{2}} \mathrm{e}^{-\frac{|\Gamma_{p'}(t)|^2}{2}} \mathrm{e}^{\Gamma_p^*(t)\Gamma_{p'}(t)} |\Gamma_p(t)\rangle\langle\Gamma_{p'}(t)|,\end{aligned} \tag{26}$$

and taking the trace we obtain:

$$\begin{aligned}\mathrm{Tr}_m[\rho_m^2(t)] &= \mathrm{e}^{-2|\alpha+\beta|^2} \\ &\sum_{p,p'} \frac{(|\alpha+\beta|^2)^{p+p'}}{p!p'!} \mathrm{e}^{-|\Gamma_p(t)-\Gamma_{p'}(t)|^2},\end{aligned} \tag{27}$$

so that, the linear entropy for the mirror is given by:

$$S^{(m)}(t) = 1 - \mathrm{Tr}_m[\rho_m^2(t)]. \tag{28}$$

In figure 3 we plot the linear entropy for the mirror for initial states $|\alpha\rangle_c \otimes |\Gamma\rangle_m = |0\rangle_c \otimes |0\rangle_m$ near resonance $\omega_p = 0.9831\omega_c$ for two cases where the difference relies in the magnitude of the forcing amplitude Ω. In (a) we show the case when $\Omega = \frac{\pi}{20}\omega_c$ and in (b) the case when $\Omega = \frac{\pi}{200}\omega_c$.

In (a) the linear entropy is a periodic function where each oscillation has the same amplitude along the evolution. The periodicity of the oscillations depends on that of the average photon number, of the order of 2×10^{-7} section Since the forcing amplitude is large, the average number of photons increases rapidly and there is an important mixing present between the field and the mechanical oscillator so that the linear entropy gets close to one.

In (b) we see a different behavior, an overall oscillatory behavior with a period $T_m = 2\pi/\omega_m$ due to the fact that at those times the mechanical system is disentangled from the field and the density matrix corresponds to that of a pure state. If we look closer, we notice the fast oscillations present in (a) but with variable amplitude due to the fact that since Ω is smaller the average number of photons is also small so that there is less entanglement and the linear entropy is small.

When $\omega_p - \omega_c$ is large (far from resonance condition), there is almost no photon generation, the field and the mechanical oscillator are disentangled and the density matrix is almost that corresponding to a pure state. It should be noted that the numerical simulation was done using the python tool called QuTiP, where the coefficients are functions of time and a cubic spline interpolation is done [38].

4. Husimi-*Q* function

The formulation of quantum mechanics in phase space has played a fundamental role in the clarification of the non-classical behavior of a quantum system [39]. In this sense, the representation of quantum fields in phase space in terms of quasiprobabilities is widely used in quantum optics with particular emphasis given to the Wigner function and the Husimi Q function. The computation of quasiprobabilities is often a tedious task which involves integration over phase-space variables, however, when the density matrix of the system is known, the computation of quasiprobabilities can be done from a series representation [40]. The simplest to evaluate is the Q function which is simply expressed as [41]:

$$\begin{aligned}Q^{[\rho]}(\alpha; t) &= \frac{1}{\pi} \mathrm{Tr}[\rho(t)|\alpha\rangle\langle\alpha|] \\ &= \frac{1}{\pi}\langle\alpha|\rho(t)|\alpha\rangle = \frac{1}{\pi}|\langle\alpha|\psi(t)\rangle|^2,\end{aligned} \tag{29}$$

where $|\alpha\rangle$ is a coherent state and $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ is the density matrix for a pure state at time t. Notice that the Husimi function is always positive, bounded and normalized.

For the system under consideration the Husimi function for the mechanical oscillator will be evaluated as

$$Q^{[\rho_m]}(\mu; t) = \frac{1}{\pi}\langle\mu|\rho_m(t)|\mu\rangle, \tag{30}$$

where $|\mu\rangle$ is a mechanical coherent state and $\rho_m(t)$ is the partial trace given by equation (25). As a result we obtain:

$$\begin{aligned}Q^{[\rho_m]}(\mu; t) &= \frac{1}{\pi}\mathrm{e}^{-|\mu|^2-|\alpha+\beta|^2} \\ &\times \sum_{p=0}^{\infty} \frac{|\alpha+\beta|^{2p}}{p!} \mathrm{e}^{-|\Gamma_p(t)|^2} \mathrm{e}^{2\Re(\mu^*\Gamma_p(t))}.\end{aligned} \tag{31}$$

The Husimi function for the field is given by

$$Q^{[\rho_c]}(\phi; t) = \frac{1}{\pi}\langle\phi|\rho_c(t)|\phi\rangle, \tag{32}$$

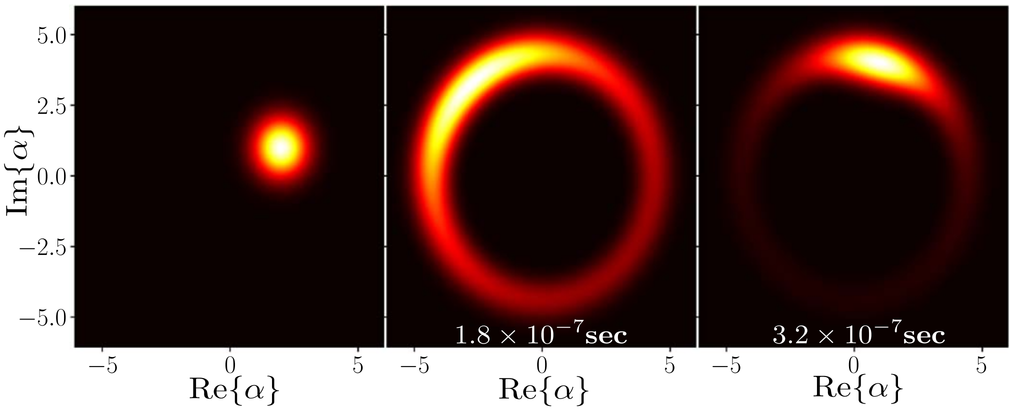

Figure 4. Pictographic representation of the Husimi function $Q^{[\rho_c]}(\phi;\, t)$ for a coherent state. The times that we consider representative were selected with respect to figure 3(a). Where $|\alpha = 3\rangle$ and $|\Gamma = 0\rangle$ as the initial state of the field and mirror, respectively.

where $|\phi\rangle$ is a field coherent state and $\rho_c(t) = \mathrm{Tr}_m[\rho(t)]$. As a result we obtain:

$$Q^{[\rho_c]}(\epsilon;\, t) = \frac{1}{\pi}\mathrm{e}^{-|\alpha+\beta|^2-|\epsilon|^2} \sum_{p,q} F_p(\alpha,\, \Gamma,\, \epsilon,\, \beta;\, t)F_q^*(\alpha,\, \Gamma,\, \epsilon,\, \beta;\, t)\mathrm{e}^{\Gamma_p(t)\Gamma_q^*(t)}, \tag{33}$$

where

$$F_p(\alpha,\, \Gamma,\, \epsilon,\, \beta;\, t) = \frac{[\epsilon\,(\alpha + \beta)]^p}{p!} \times \mathrm{e}^{\frac{p}{2}(\alpha_4\Gamma * -\alpha_4^*\Gamma)}\mathrm{e}^{p\alpha_1}\mathrm{e}^{\left(\alpha_3+\frac{|\alpha_4|^2}{2}\right)p^2}\mathrm{e}^{-\frac{|\Gamma_p(t)|^2}{2}} \tag{34}$$

with α_1, α_3 and α_4 as determined by equation (5).

In figure 4 we show the Husimi function for the field $Q^{[\rho_c]}(\phi;\, t)$ at times stated in the figure for an initial state of the field $|\alpha = 3\rangle$ with the same set of Hamiltonian parameters as in figure 3(a). At $t = 0$ the state is a coherent state, as time evolves we can see a delocalization of the probability distribution. This is in agreement with the behavior of the linear entropy where we see that it attains values close to one, corresponding to situation with large entanglement, this means that we have a highly non-classical state. At a larger time we see a relocalization of the state and as we can see from figure 3(a), the entanglement and disentanglement behavior is periodic. A similar conduct but with a different intermediate dynamics in phase space will occur with the Husimi function of the mirror. It is important to mention that the reconstruction of quantum mechanical motional states is still an open problem and requires the use of varied techniques [24].

5. Conclusions

In this work we have presented an algebraic method to deal with the Hamiltonian of a driven optomechanical system. Since the time evolution operator for the optomechanical Hamiltonian $\hat{H}_{\mathrm{opt}}$ can be written *exactly* as a product of exponentials, we split the full Hamiltonian as $\hat{H}_p = \hat{H}_{\mathrm{opt}} + \hat{V}$ with $\hat{V}$ the pumping term, and transform to an interaction picture with the operator $\hat{U}_{\mathrm{opt}}$. As a result, the field creation–annihilation operators acquire a complicated structure with exponentials whose exponents are operators. At this point we linearize the transformed creation–annihilation operators and obtain an approximate interaction Hamiltonian whose time evolution operator may be written *exactly* as a product of exponentials. Then, we can write the full time evolution operator in a product form by means of the Wei–Norman theorem. Notice that the approximation is done on the interaction Hamiltonian, once it has been linearized the corresponding time evolution operator is exact. The terms we have neglected to get the approximate interaction Hamiltonian are proportional to G/ω_m and $(G/\omega_m)^2$ which take numerical values smaller than one for the set of parameters considered in this work. We evaluated the average number of photons and phonons as a function of time and of the ratio ω_p/ω_c between the forcing frequency ω_p and the field's frequency ω_c. Far from resonance there is almost no photon generation but near resonance $\omega_p/\omega_c \simeq 1$ there is a rapid increase in the number of photons. The number of phonons depends on the number of photons present in the cavity and when the number of photons is different from zero, the generation of phonons is large even under non resonance conditions. The average number of phonons is a periodic function with a frequency given by that of the mechanical oscillator. To test the validity of our approximations we made a purely numerical calculation of the average number of photons and phonons using the full Hamiltonian $\hat{H}_p$ and we found an excellent agreement between both calculations. We also got analytic expressions for the Husimi Q function of the mechanical oscillator and for the field and we evaluated the one corresponding to the field taking as initial state a coherent state.

Acknowledgments

We acknowledge partial support from DGAPA UNAM project PAPIIT IN111119.

ORCID iDs

I Ramos-Prieto https://orcid.org/0000-0001-8838-3541

References

[1] Aspelmeyer M, Meystre P and Schwab K 2012 Quantum optomechanics *Phys. Today* **65** 29–35

[2] Kippenberg T J and Vahala K J 2008 Cavity optomechanics: back-action at the mesoscale *Science* **321** 1172–6

[3] Aspelmeyer M, Kippenberg T J and Marquardt F 2014 Cavity optomechanics *Rev. Mod. Phys.* **86** 1391–452

[4] Mancini S, Man'ko V I and Tombesi P 1997 Ponderomotive control of quantum macroscopic coherence *Phys. Rev.* A **55** 3042–50

[5] Teufel J D, Donner T, Li D, Harlow J W, Allman M S, Cicak K, Sirois A J, Whittaker J D, Lehnert K W and Simmonds R W 2011 Sideband cooling of micromechanical motion to the quantum ground state *Nature* **475** 359 EP

[6] Wei L F, Liu Y-X, Sun C P and Nori F 2006 Probing tiny motions of nanomechanical resonators: classical or quantum mechanical? *Phys. Rev. Lett.* **97** 237201

[7] Lebedew P 1901 Untersuchungen über die druckkräfte des lichtes *Ann. Phys.* **311** 433–58

[8] Nichols E F and Hull G F 1901 A preliminary communication on the pressure of heat and light radiation *Phys. Rev.* **13** 307–20

[9] Corbitt T, Chen Y, Innerhofer E, Müller-Ebhardt H, Ottaway D, Rehbein H, Sigg D, Whitcomb S, Wipf C and Mavalvala N 2007 An all-optical trap for a gram-scale mirror *Phys. Rev. Lett.* **98** 150802

[10] Metzger C H and Karrai K 2004 Cavity cooling of a microlever *Nature* **432** 1002 EP –

[11] Thompson J D, Zwickl B M, Jayich A M, Marquardt F, Girvin S M and Harris J G E 2008 Strong dispersive coupling of a high-finesse cavity to a micromechanical membrane *Nature* **452** 72

[12] Corbitt T and Mavalvala N 2004 Review: quantum noise in gravitational-wave interferometers *J. Opt.* B **6** S675–83

[13] Gigan S, Böhm H R, Paternostro M, Blaser F, Langer G, Hertzberg J B, Schwab K C, Bäuerle D, Aspelmeyer M and Zeilinger A 2006 Self-cooling of a micromirror by radiation pressure *Nature* **444** 67

[14] Carmon T, Rokhsari H, Yang L, Kippenberg T J and Vahala K J 2005 Temporal behavior of radiation-pressure-induced vibrations of an optical microcavity phonon mode *Phys. Rev. Lett.* **94** 223902

[15] Metzger C, Ludwig M, Neuenhahn C, Ortlieb A, Favero I, Karrai K and Marquardt F 2008 Self-induced oscillations in an optomechanical system driven by bolometric backaction *Phys. Rev. Lett.* **101** 133903

[16] Ashkin A 1970 Acceleration and trapping of particles by radiation pressure *Phys. Rev. Lett.* **24** 156–9

[17] Monroe C, Meekhof D M, King B E and Wineland D 1996 A Schrödinger cat superposition state of an atom *Science* **272** 1131

[18] Nogues G *et al* 1999 Seeing a single photon without destroying it *Nature* **400** 239–49

[19] Davis K B, Mewes M O, Andrews M R, van Druten N J, Durfee D S, Kurn D M and Ketterle W 1995 Bose–Einstein condensation in a gas of sodium atoms *Phys. Rev. Lett.* **75** 3969–73

[20] Anderson M H, Ensher J R, Matthews M R, Wieman C E and Cornell E A 1995 Observation of Bose–Einstein condensation in a dilute atomic vapor *Science* **269** 198–201

[21] Smithey D T, Beck M, Raymer M G and Faridani A 1993 Measurement of the Wigner distribution and the density matrix of a light mode using optical homodyne tomography: application to squeezed states and the vacuum *Phys. Rev. Lett.* **70** 1244

[22] Breitenbach G, Schiller S and Mlynek J 1997 Measurement of the quantum states of squeezed light *Nature* **387** 471–5

[23] Hofheinzet M *et al* 2009 Synthesizing arbitrary quantum states in a superconducting resonator *Nature* **459** 546

[24] Vanner M R, Pikovski I and Kim M S 2015 Towards optomechanical quantum state reconstruction of mechanical motion *Ann. Phys.* **527** 15– 26

[25] Bose S, Jacobs K and Knight P L 1997 Preparation of nonclassical states in cavities with a moving mirror *Phys. Rev.* A **56** 4175–86

[26] Dodonov V V, Malkin I A and Man'ko V I 1974 Even and odd coherent states and excitations of a singular oscillator *Physica* **72** 597–615

[27] Perez-Leija A, Szameit A, Ramos-Prieto I, Moya-Cessa H and Christodoulides D N 2016 Generalized Schrödinger cat states and their classical emulation *Phys. Rev.* A **93** 053815

[28] Zhang J, Peng K and Braunstein S L 2003 Quantum-state transfer from light to macroscopic oscillators *Phys. Rev.* A **68** 013808

[29] Récamier J and Jáuregui R 2003 Construction of even and odd combinations of Morse-like coherent states *J. Opt.* B **5** S365–70

[30] Ventura-Velázquez C, Rodríguez-Lara B M and Moya-Cessa H M 2015 Operator approach to quantum optomechanics *Phys. Scr.* **90** 068010

[31] Wei J and Norman E 1964 On global representations of the solutions of linear differential equations as a product of exponentials *Proc. Am. Math. Soc.* **15** 327–34

[32] de los Santos-Sánchez O and Récamier J 2011 Nonlinear coherent states for nonlinear systems *J. Phys. A: Math. Theor.* **44** 145307

[33] Law C K 1995 Interaction between a moving mirror and radiation pressure: a hamiltonian formulation *Phys. Rev.* A **51** 2537–41

[34] Vitali D, Gigan S, Ferreira A, Böhm H R, Tombesi P, Guerreiro A, Vedral V, Zeilinger A and Aspelmeyer M 2007 Optomechanical entanglement between a movable mirror and a cavity field *Phys. Rev. Lett.* **98** 030405

[35] Ghobadi R, Bahrampour A R and Simon C 2011 Quantum optomechanics in the bistable regime *Phys. Rev.* A **84** 033846

[36] Moya-Cessa H and Soto-Eguibar F 2011 *Introduction to Quantum Optics* (Paramus, NJ: Rinton Press Inc.)

[37] Verhagen E, Deléglise S, Weis S, Schliesser A and Kippenberg T J 2012 Quantum-coherent coupling of a mechanical oscillator to an optical cavity mode *Nature* **482** 63

Kockum A F, Miranowicz A, De Liberato S, Savasta S and Nori F 2019 Ultrastrong coupling between light and matter *Nat. Rev. Phys.* **1** 19

[38] Johansson J R, Nation P D and Nori F 2013 QuTiP 2: a Python framework for the dynamics of open quantum systems *Comput. Phys. Commun.* **184** 1234–40

[39] Zachos C, Fairlie D and Curtright T 2005 *Quantum Mechanics in Phase Space* (Singapore: World Scientific)

[40] Moya-Cessa H and Knight P L 1993 Series representation of quantum-field quasiprobabilities *Phys. Rev.* A **48** 2479–81

[41] Haroche S and Raimond J M 2006 *Exploring the Quantum: Atoms, Cavities, and Photons* (Oxford: Oxford University Press)