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Study of Carnot engine in deformed formalism

M Damghani[1,3], H Hassanabadi[1], W S Chung[2] and S Sargolzaeipor[1]

[1] Faculty of Physics, Shahrood University of Technology, Shahrood, Iran, PO Box: 3619995161-316, Shahrood, Iran

[2] Department of Physics and Research Institute of Natural Science, College of Natural Science, Gyeongsang National University, Jinju 660-701, Republic of Korea

E-mail: Mohammad1373da@gmail.com

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Abstract

In this paper, we discuss the q-deformed algebra and study the Schrödinger equation in Carnot cyclic and we obtain the energy eigenvalues and the wave function in Carnot cyclic by an analytical method and the thermodynamic properties such as force parameters and the quantum heat engine including the adiabatic and isothermal quantum processes of the system by using of the energy are calculated. Also, we obtain efficiency in the Carnot engine. Finally, all results have satisfied what we had expected before.

Keywords: the Schrödinger equation, the deformed formalism, the Carnot cyclic, the efficiency

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum algebra has been receiving much attention in physic and mathematics fields. In 1976, Arik and Coon analyzed Hilbert space via deformed algebra [1]. The deformed algebra has been investigated by some authors [2–4]. Hassanabadi *et al* investigated energy eigenvalue of the system for commutative and noncommutative spaces by using of deformed algebra [5]. The nonrelativistic Schrödinger equation is of great importance in nuclear and particle physics [6, 7]. Pedram investigated maximally localized states in presence of perturbative representation of the deformed algebra then the author obtained invariant density of states by using of extend the generalized uncertainty principle [8]. Chung *et al* in [9] obtained the thermodynamics aspects of the q-deformed Tamm–Doncoff oscillator algebra by using of particular Fock spaces under the finite and infinite dimensions. The energy spectrum of the Coulomb potential under minimal length commutation relations are reported in [10]. Stetsko and Takachuk obtained the energy spectrum for the hydrogen atom via deformed Heisenberg algebra [11]. In [12], the authors have obtained the harmonic oscillator spectrum and eigenvectors by using an extension of the techniques of conventional supersymmetric quantum mechanics. On the other hand, the Carnot engine and their efficiency appeared more than forty years ago and introduced for the first time by Carnot [13] and lately investigated by other authors [14–18]. in [19], the authors considered three levels masers as thermodynamic process and they showed the limiting efficiency of a three level maser is that of a Carnot heat engine. Carnot engine is like a piston with movable walls, and a particle is that this system has different energies during isothermal and adiabatic processes. Bender *et al* Used of a single quantum mechanical particle confined to a potential well to describe the quantum heat engine then, they obtained the efficiency of this engine [20]. In another paper, WU *et al* studied the reversible Otto cycle of Schrödinger equation in a potential well also they obtained the relationship between the dimensionless work out put and the efficiency in two states heat engine [21]. Acıkkalp and Caner in 2015 used an irreversible quantum engine to describe thermodynamic parameters including work input, cooling load, exergy destruction and exergy efficiency [22]. Finally, they showed which the sustainability index has a dependency on the constant-temperature heat sink. Abe investigated work out put in a quantum Carnot cycle in two states and calculated the efficiency in power maximum output then, he obtained the efficiency in the problem and showed the efficiency independently of any the

[3] Author to whom any correspondence should be addressed.

parameters in the model [23]. Ahmadi *et al* studied an irreversible Stirling refrigerator cycle then they calculated the coefficient of performance and the ecological coefficient of performance via thermodynamic processes [24]. Our purpose is to solve the deformed Schrödinger equation under a single particle confined in a one-dimensional box. Then, we investigate the thermodynamic engine such as the efficiency by using the energy. Also, a heat engine takes in heat at a high temperature and exhausts heat at a low temperature. In the process of heat flow, some of the input heat is converted to work. The efficiency is the fraction of the heat input at high temperature converted to work. Thermodynamics is the study of the relationships between heat and work. The first law constrains the operation of a heat engine. The first law is the application of conservation of energy to the system.

The conservation of energy is given by [21, 22]

$$W = |Q_H| - |Q_C| \to |Q_C| = |Q_H| - W. \tag{1}$$

The efficiency of the cycle can be written as [25, 26].

$$\eta = \frac{W}{Q_H} = 1 - \frac{T_C}{T_H}. \tag{2}$$

The efficiency only depends on the ratio of the absolute temperatures. Finally, by letting the deformed parameter, go to zero we recovered the results for the ordinary thermodynamic engine. This paper has been organized as follows: in section 2 we obtained the energy eigenvalues and the wave function of the q-deformed Schrodinger equation in presence of a single particle confined in one-dimensional box.in section 3 we studied the Adiabatic, free and isothermal quantum processes by using of total energy. In section 4. We discuss the quantum Carnot cycle in two states of isothermal and adiabatic processes. Then, we obtained energy, force and work done in the Carnot cycle and in finally we obtained efficiency in this cycle. Finally, our conclusion appears in section 5.

2. The Schrödinger equation in deformed formalism

As one method for curing some problems in quantum gravity, the generalization of the uncertainty relation has come, which is called a generalized uncertainty principle (GUP) [6, 7]. There has been much development for the GUP formulation and GUP-corrected quantum systems. The generalized uncertainty principle (GUP) is given by the modified commutation between coordinate and its momentum as

$$[X,\ P] = \mathrm{i}\left(1 - \frac{|P|}{k}\right)^2 = \mathrm{i}\,(1 - q\,|P|)^2, \tag{3}$$

where $\left(q = \frac{1}{k}\right) \to 0$ and in agree with ordinary commutation relation in quantum mechanics [27, 28].

We consider the coordinate and momentum operators for the algebra (3) as [7]

$$X = x, \quad P = \frac{p}{1 + q\,|p|} = \frac{\frac{1}{i}\,\partial_x}{1 + q\,\left|\frac{1}{i}\,\partial_x\right|}, \tag{4}$$

where $q \geqslant 0$. The expectation value of an operator $\hat{\mathrm{O}}$ for the wave function $\Psi(x)$ is

$$\langle\hat{\mathrm{O}}\rangle = \langle\Psi(x)|\hat{\mathrm{O}}|\ \Psi(x)\rangle = \int_{-\infty}^{\infty}\Psi^*(x)\hat{\mathrm{O}}\,\Psi(x)\,\mathrm{d}\,x. \tag{5}$$

The Schrödinger equation in the coordinate representation of algebra (3) appears as

$$\begin{aligned}&\left[\frac{P^2}{2m} + V(x)\right]\Psi(x) = E\ \Psi(x),\\ &\frac{1}{2m}\left(\frac{\frac{1}{i}\,\partial_x}{1 + \left|q\frac{1}{i}\partial_x\right|}\right)^2\Psi(x) + V(x)\Psi(x)\\ &= E\ \Psi(x).\end{aligned} \tag{6}$$

We want to consider a single particle with mass m confined in one-dimensional box as bellow

$$V(x) = \begin{cases}0 & 0 < x < L\\ \infty & \text{elsewhere}\end{cases}. \tag{7}$$

By substitution of equation (7) in equation (6) then, the solution of the equation (6) as

$$\Psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}\,x. \tag{8}$$

Then, the. (6) can be written as

$$-\frac{1}{2\,m}\sum_{n=0}^{\infty}(-1)^n(n+1)\,\left|\frac{q}{i}\partial\right|^n\Psi_n(x) = E_n\Psi_n(x). \tag{9}$$

Using the relations

$$\begin{aligned}&\left|\frac{q}{i}\partial\right|^n \mathrm{e}^{\mathrm{i}ax} = (a\ q)^n\mathrm{e}^{\mathrm{i}ax},\\ &\left|\frac{q}{i}\partial\right|^n \mathrm{e}^{-\mathrm{i}ax} = (a\ q)^n\mathrm{e}^{-\mathrm{i}ax}.\end{aligned} \tag{(10a)}$$

We have

$$\left|\frac{q}{i}\partial\right|^n\ \mathrm{Sin}\ a\ x = (aq)^n\ \mathrm{Sin}\ a\ x. \tag{(10b)}$$

Finally, from the equation (9) the energy is given by

$$E_n = \frac{1}{2m}\left(\frac{n\ \pi}{L + q\ n\ \pi}\right)^2. \tag{11}$$

The expectation value of the Hamiltonian as bellow

$$E(L) = \langle\Psi|H|\ \Psi\rangle. \tag{12}$$

The total energy is

$$E_q(L) = \sum_{n=1}^{\infty}|\alpha_n|^2 E_n, \tag{13}$$

where the coefficients α_n satisfy the normalization condition

$$\sum_{n=1}^{\infty}|\alpha_n|^2 = 1. \tag{14}$$

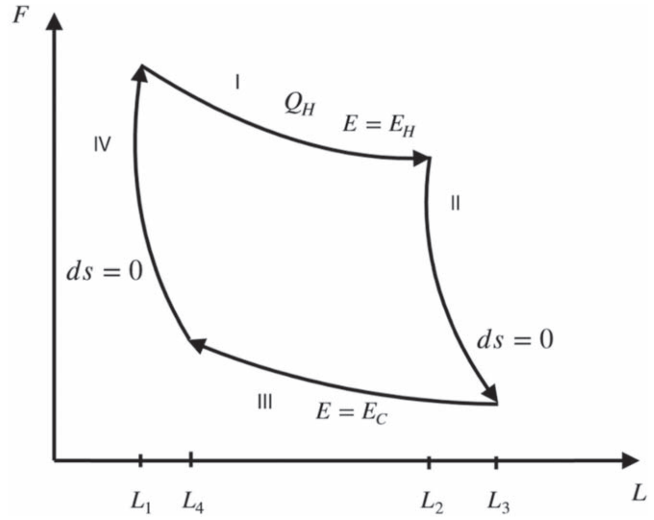

Figure 1. A Carnot cycle consisting of four processes, two adiabatic processes and two isothermal processes.

From equation (12), the force F exerted on the wall of the well is given by

$$F = -\frac{\mathrm{d}E(L)}{\mathrm{d}L}. \tag{15}$$

We can easily find other thermodynamic processes such as Adiabatic, free expansion and Isothermal processes.

3. Thermodynamic processes of the system

3.1. Adiabatic process

An adiabatic process is one in which no heat is gained or lost by the system. The first law of thermodynamics with $Q = 0$ shows that all the change in internal energy is in the form of work done. Now we introduce an adiabatic expansion as bellow

$$\mathrm{d}\,u + \mathrm{d}\,W = 0. \tag{16}$$

The adiabatic condition is $PV^{\gamma} = \mathrm{Const}$, during an adiabatic process the exponent γ is the ration of the specific heats as bellow

$$\gamma = \frac{C_p}{C_v}. \tag{17}$$

The efficiency of the ideal gas can be written as

$$\eta = 1 - \frac{T_C}{T_H} = 1 - \frac{T_2}{T_1} \Rightarrow \eta = 1 - \frac{V_1^{\gamma-1}}{V_2^{\gamma-1}}, \tag{18}$$

where $\gamma = 3$. Therefore, the adiabatic condition becomes

$$PV^3 = \mathrm{Const}. \tag{19}$$

Finally, the equation (15) in an adiabatic process is given by

$$F_q = \sum_{n=1}^{\infty} |\alpha_n|^2 \, \frac{n^2\pi^2}{m(L + q\, n\, \pi)^3}. \tag{20}$$

3.2. Free expansion and Isothermal processes

3.2.1. First process. A free expansion is an irreversible process in thermodynamics in which a volume of gas is kept in one side of a thermally isolated container, with the other side of the container being evacuated. The partition between the two parts of the container is then opened, and the gas fills the whole container. The free expansion process and the final pressure has decreased relative to the initial pressure is written by

$$\begin{aligned} &T = \mathrm{Const} \\ &P_i V_i^{\gamma} = P_f V_f^{\gamma}. \end{aligned} \tag{21}$$

3.2.2. Two process. An isothermal process is a change of a system, in which the temperature remains constant.

In a constant temperature process involving an ideal gas, pressure can be expressed in terms of the volume

$$P\ V = \ N\ K\ T. \tag{22}$$

The isothermal process is $P\ V = \ \mathrm{Const}$.

4. Carnot cycle in quantum mechanics

In this section, we want to study a cyclic heat engine in quantum mechanics by using the adiabatic and the isothermal processes which discussed in section 3. we want to consider two of the Eigenstates of the potential well then, This point can be derived that this ground-state wave function in a well of width L_1 produces the force on the wall as

$$F_q = \frac{\pi^2}{m(L_1\ + q\ \pi)^3}. \tag{23}$$

The expectation value of the Hamiltonian

$$E_{H-q} = \frac{1}{2m}\left(\frac{\pi}{L_1 + q\ \pi}\right)^2. \tag{24}$$

The equation (24) is the generalized energy relation of the Carnot cycle, whose energy value is for the adiabatic and the isothermal processes. Now, we consider cycle set the limits to the efficiency of a heat engine operating between two temperatures which the cycle consists of fourth reversible steps such as Isothermal expansion at T_h, Adiabatic expansion from T_h to T_c, Isothermal compression at T_c, Adiabatic compression from T_c to T_h. We shows the four-step cyclic process in figure 1. At first we discuss first step where in this process the piston to expand isothermally. Then, the state of the system is a linear combination of the lowest two energy eigenstates

$$\Psi_{n=1,2}(x) = \sqrt{\frac{2}{L}}\,\alpha_1(L)\sin\frac{\pi}{L}x + \sqrt{\frac{2}{L}}\,\alpha_2(L)\sin\frac{2\pi}{L}x. \quad (25)$$

From equation (14), we can achieve the constraint

$$|\alpha_1|^2 + |\alpha_2|^2 = 1. \quad (26)$$

The Hamiltonian is written by

$$E_1(L)_q = \frac{1}{2m}|\alpha_1|^2\left[\left(\frac{\pi}{L+q\,\pi}\right)^2 - \left(\frac{2\pi}{L+2\,q\,\pi}\right)^2\right] + \frac{1}{2m}\left(\frac{2\pi}{L+2\,q\,\pi}\right)^2. \quad (27)$$

With equality of equations (24) and (27), when ($q \to 0$) we have

$$\frac{\pi^2}{2\,m\,L_1^2} = \frac{\pi^2}{2\,m\,L^2}(4 - 3\,|\alpha_1|^2) \to L^2 = L_1^2(4 - 3\,|\alpha_1|^2). \quad (28)$$

Thus, we consider $L = L_2$ for the system in the second energy eigenstate, so the maximum value of L in this isothermal expansion, when $|\alpha_1| = 0$ is achieved $L_2 = 2L_1$.

During this isothermal expansion process the force by using of equation (27) is written by

$$F_{1-q}(L) = -\frac{\mathrm{d}E_1(L)_q}{\mathrm{d}L} = |\alpha_1|^2\left(\frac{\pi^2}{m\,(L+q\,\pi)^3} - \frac{4\,\pi^2}{m\,(L+2\,q\,\pi)^3}\right) + \frac{4\,\pi^2}{m(L+2\,q\,\pi)^3}. \quad (29)$$

When ($q \to 0$) and $|\alpha_1| = 1$, the equation (29) changed as

$$F_{1-(q\to 0)}(L) = -\frac{\mathrm{d}E_1(L)_{q\to 0}}{\mathrm{d}L} = \frac{\pi^2}{m\,L_1^2 L}, \quad (30)$$

where $L\,F_{1-q}$ is a constant.

For the second step we the piston to expand adiabatically from L_2 goes L_3, Expectation value of the Hamiltonian and the force in the process are as bellow

$$E_{c-q}(L) = \frac{1}{2m}\left[\frac{2\pi}{L+2\,q\,\pi}\right]^2, \quad (31)$$

$$F_{2-q}(L) = \frac{4\pi^2}{m(L+2\,q\,\pi)^3} \xrightarrow{q\to 0} L^3F_2(L) = \frac{4\pi^2}{m}. \quad (32)$$

For the Third step we the piston to compress isothermally from L_3 goes L_4, we can obtained the The expectation value of the Hamiltonian and the force in the process as following

$$E_{3-q}(L) = \frac{1}{2m}\left[\frac{2\,\pi}{L_3+2\,q\,\pi}\right]^2, \quad (33)$$

$$F_{3-\,q}(L) = \frac{4\pi^2}{m\,(L_3+2\,q\pi)^3} \xrightarrow{q\to 0} L_3^2F_3(L) = \frac{4\pi^2}{m\,L}. \quad (34)$$

Finally, for the fourth step we compress adiabatically L_4 until we return to the starting point L_1

Then, as similarly we can write the expectation value of the Hamiltonian and the force in the process as

$$E_{4-\,q}(L) = \frac{1}{2m}\left[\frac{\pi}{L+q\,\pi}\right]^2, \quad (35)$$

$$F_{4-q}(L) = \frac{\pi^2}{m\,(L+q\,\pi)^3} \xrightarrow{q\to 0} L^3F_4(L) = \frac{\pi^2}{m}. \quad (36)$$

Now, by using of the four step, we calculate the work done by the engine as

$$W_{q\to 0} = \int_{L_1}^{2L_1}\frac{\pi^2}{m(L+\pi\,q)^3}\,\mathrm{d}\,L + \int_{2L_1}^{L_3}\frac{4\,\pi^2}{m(L+2\,\pi\,q)^3}\,\mathrm{d}L + \int_{L_3}^{\frac{L_3}{2}}\frac{4\,\pi^2}{m(L_3+2\,\pi\,q)^3}\mathrm{d}L + \int_{\frac{L_3}{2}}^{L_1}\frac{\pi^2}{m(L+q\,\pi)^3}\,\mathrm{d}\,L$$
$$= \int_{L_1}^{2L_1}F_{1-(q\to 0)}(L)\,\mathrm{d}\,L + \int_{2L_1}^{L_3}F_{2-(q\to 0)}(L)\,\mathrm{d}\,L + \int_{L_3}^{\frac{L_3}{2}}F_{3-(q\to 0)}(L)\,\mathrm{d}\,L + \int_{\frac{L_3}{2}}^{L_1}F_{4-(q\to 0)}(L)\,\mathrm{d}\,L$$
$$= \frac{\pi^2}{m}\left(\frac{1}{L_1^2} - \frac{4}{L_3^2}\right)Ln2. \quad (37)$$

This quantity of energy is given by

$$Q_{H-(q\to 0)} = \int_{L_1}^{2L_1}\frac{\pi^2}{m(L+\pi\,q)^3}\,\mathrm{d}\,L = \int_{L_1}^{2L_1}F_{1-(q\to 0)}(L)\,\mathrm{d}\,L = \frac{\pi^2}{mL_1^2}Ln2. \quad (38)$$

This quantity of energy $Q_{C-\,q}$ is given back by

$$Q_{c-(q\to 0)} = \int_{L_3}^{\frac{L_3}{2}}\frac{4\,\pi^2}{m\,(L_3+2\,\pi\,q)^3}\mathrm{d}L = \int_{L_3}^{\frac{L_3}{2}}F_{3-(q\to 0)}(L)\mathrm{d}L = \frac{4\,\pi^2}{m\,L_3^2}Ln\left(\frac{1}{2}\right). \quad (39)$$

Then, the efficiency of our two-state quantum heat engine is given by

$$\eta_q = 1 - \frac{Q_{C-(q\to 0)}}{Q_{H-(q\to 0)}}. \quad (40)$$

We can rewrite the equation (40) as

$$\eta_q = 1 - \frac{E_{3-q}}{E_{H-q}} = 1 - \frac{\frac{1}{2m}\left(\frac{2\pi}{L_3 + 2\,q\,\pi}\right)^2}{\frac{1}{2m}\left(\frac{\pi}{L_1 + \,q\,\pi}\right)^2} \xrightarrow{q\to 0} \eta_{(q\to 0)}$$

$$= 1 - 4\left(\frac{L_1}{L_3}\right)^2. \tag{41}$$

With remove deformed parameter is recover ordinary the efficiency of the cycle [10].

5. Conclusion

In this paper, after introducing deformed formalism and the Carnot cycle, Schrödinger equation in this formalism was derived. The first we obtained the energy eigenvalues and the wave function in the deformed Schrödinger equation. Then, we consider the Carnot cycle problem in the formalism and we investigated the adiabatic and isothermal quantum processes of the system. Also, we obtained the thermodynamic properties such as force parameters, the efficiency, and work done in the quantum Carnot cycle. Finally, It was shown that by removing the deformation, the ordinary results in the quantum Carnot cycle were recovered.

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ORCID iDs

M Damghani https://orcid.org/0000-0003-3918-4778
W S Chung https://orcid.org/0000-0002-1358-6384
S Sargolzaeipor https://orcid.org/0000-0001-6951-9730

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