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Detecting non-Gaussianity via nonclassicality

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Abstract

Two prominent classification schemes for Bosonic field states are (i) classical versus nonclassical, and (ii) Gaussian versus non-Gaussian. These two paradigms are closely related yet fundamentally different. In particular, classical states may be Gaussian or non-Gaussian, and Gaussian states may be classical or nonclassical. Both non-Gaussianity and nonclassicality are important resources for quantum information processing. It is desirable to study the interplay between non-Gaussianity and nonclassicality, and seek effective methods of detecting them. In this work, we introduce a quantifier for non-Gaussianity by exploiting and optimizing an information-theoretic quantifier for nonclassicality, and exhibit its basic properties. A criterion for simultaneously detecting non-Gaussianity and nonclassicality follows, and its applications are illustrated through several examples. This unveils some intrinsic relations between non-Gaussianity and nonclassicality.

Keywords: quantum optics, non-Gaussianity, nonclassicality, Wigner–Yanase skew information

(Some figures may appear in colour only in the online journal)

1. Introduction

Bosonic field states are basic ingredients for quantum information processing with continuous variables, for which there are two prominent classification schemes of the quantum states $\mathcal{S}$. The first is Gaussian versus non-Gaussian, which reads as

$$\mathcal{S} = \mathcal{G} \bigcup \mathcal{G}^c, \tag{1}$$

where $\mathcal{G}$ is the set of Gaussian states (characterized by Gaussian characteristic functions, or equivalently Gaussian Wigner functions), while the complement $\mathcal{G}^c$ is the set of non-Gaussian states [1–6]. The second is classical versus nonclassical, which reads

$$\mathcal{S} = \mathcal{C} \bigcup \mathcal{C}^c, \tag{2}$$

where $\mathcal{C}$ is the set of classical states (characterized as probabilistic mixtures of optical coherent states, or equivalently, states whose Glauber–Sudarhsan P functions are genuine probability distributions), while the complement $\mathcal{C}^c$ is the set of nonclassical states [7–9].

Combining the above two classification schemes, (1) and (2), for quantum states, we have the following further classification

$$\mathcal{S} = \mathcal{GC} \bigcup \mathcal{GC}^c \bigcup \mathcal{G}^c\mathcal{C} \bigcup \mathcal{G}^c\mathcal{C}^c,$$

where $\mathcal{GC} = \mathcal{G} \cap \mathcal{C}$, etc. In particular, $\mathcal{G}^c\mathcal{C}^c$ constitutes the set of quantum states which are simultaneously non-Gaussian and nonclassical. All the above sets are nonempty. For example, thermal states and coherent states belong to $\mathcal{GC}$, squeezed coherent states belong to $\mathcal{GC}^c$, generic mixtures of coherent states belong to $\mathcal{G}^c\mathcal{C}$, and Fock states (excluding vacuum) belong to $\mathcal{G}^c\mathcal{C}^c$. Here we are interested in $\mathcal{G}^c\mathcal{C}^c$, i.e. states which are both non-Gaussian and nonclassical.

Due to the increasing importance of non-Gaussian and nonclassical states, it is desirable to have effective methods of

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detecting and quantifying non-Gaussianity and nonclassicality simultaneously. However, in the literature, these two features are often addressed separately.

In the Gaussian versus non-Gaussian paradigm, Gaussian states and associated Gaussian operations are recognized as playing increasingly important roles in both theoretical and experimental investigations of quantum information in the past decade [2–6], and non-Gaussian states are identified as necessary resources for a variety of quantum protocols which display advantage over the classical counterparts [10–15]. Concerning the issue of detecting and quantifying non-Gaussianity, various measures for non-Gaussianity have been introduced in the literature. For example, distance-like measures based on Hilbert–Schmidt distance, relative entropy, and Bures distance are introduced in [16–20]. A measure for non-Gaussianity based on Husimi distributions is proposed in [21]. Resource-theoretic frameworks are proposed in [22–24]. Kurtosis is employed to quantifying non-Gaussianity in [25].

In the classical versus nonclassical paradigm, nonclassical states are studied extensively and intensively in the past half century. A variety of quantifiers for nonclassicality have been developed ever since Mandel characterized nonclassicality as deviation from Poissonion distributions for photon numbers [26]. Now there are distance-based measures [27–30], nonclassical depth [31, 32], quadrature distributions [33, 34], negativity [35, 36], entanglement potential [37], information-theoretic quantifiers [38, 39], etc. All these various approaches capture different aspects of nonclassicality, and may be useful in particular tasks.

Given the importance of non-Gaussianity and nonclassicality in achieving quantum advantage, the following question arises naturally: How to detect non-Gaussianity and nonclassicality? In this work, by exploiting an information-theoretic quantifier for nonclassicality, we introduce a quantifier for non-Gaussianity which can be used to detect non-Gaussianity and nonclassicality simultaneously for some important states, and further investigate its applications. The remainder of the paper is structured as follows. In section 2, we present a brief review of nonclassicality of Bosonic field states and a quantifier for nonclassicality recently introduced in [39]. In section 3, we introduce a quantifier for non-Gaussianity and investigate its fundamental properties and implications. We evaluate the quantity for some popular states, and discuss its usage in detecting both non-Gaussianity and nonclassicality. We compare our quantifier for non-Gaussianity with others in the literature and indicate its features and convenience in section 4. Finally, a summary and discussion is presented in section 5.

2. Nonclassicality

In quantum optics whose underlying structure is Bosonic fields, classicality is usually phrased in terms of the Glauber–Sudarshan P functions [7, 8]: A state whose P function is a genuine probability distribution is termed classical, otherwise it is nonclassical. More precisely, consider a single-mode Bosonic field described by the canonical commutation relation

$$[a, a^\dagger] = \mathbf{1}$$

for the annihilation operator a and creation operator $a^\dagger$. Coherent states $|\alpha\rangle$, defined as the eigenstates of the annihilation operator, $a|\alpha\rangle = \alpha|\alpha\rangle$, $\alpha \in \mathbb{C}$, are usually regarded as the most classical pure states. Any probabilistic mixture of coherent states is a classical state, while all other states are nonclassical. Basic examples of classical states are coherent states $|\alpha\rangle$ and thermal states

$$\tau_\lambda = (1-\lambda)\sum_{n=0}^{\infty} \lambda^n |n\rangle\langle n|, \qquad 0 \leqslant \lambda < 1 \tag{3}$$

with average photon number $\bar{n} = \mathrm{tr}\tau_\lambda a^\dagger a = \lambda/(1-\lambda)$, while eminent examples of nonclassical states include Fock states $|n\rangle$ (excluding the vacuum state $|0\rangle$) and squeezed coherent states $S_\zeta|\alpha\rangle$, $|\zeta| > 0$, among others. Here $S_\zeta = \mathrm{e}^{\zeta a^{\dagger 2}/2 - \zeta^* a^2/2}$ are the squeezing operators with squeezing parameters $\zeta = r\mathrm{e}^{\mathrm{i}\phi} \in \mathbb{C}$. A pure state is classical if and only if it is a coherent state. All other pure states are nonclassical.

Based on the above classical-nonclassical dichotomy, various quantifiers for nonclassicality are introduced [26–39]. In particular, an information-theoretic quantifier for nonclassicality of single-mode Bosonic field is defined as [39]

$$N(\rho) = \frac{1}{2}\mathrm{tr}[\sqrt{\rho}, a][\sqrt{\rho}, a]^\dagger, \tag{4}$$

where tr denotes operator trace and $[X, Y] = XY - YX$ denotes operator commutator. More explicitly,

$$N(\rho) = \frac{1}{2} + \mathrm{tr}\rho a^\dagger a - \mathrm{tr}\sqrt{\rho} a^\dagger \sqrt{\rho} a.$$

The motivation of the above quantifier for nonclassicality comes from the celebrated Wigner–Yanase skew information

$$I(\rho, H) = -\frac{1}{2}\mathrm{tr}[\sqrt{\rho}, H]^2,$$

introduced as earlier as 1963 by Wigner and Yanase in a seminal study of information content of quantum states [40]. Here H is an observable (Hermitian operator). Although the skew information $I(\rho, H)$ is only defined for quantum states with respect to Hermitian operator H, if we rewrite it as

$$I(\rho, H) = -\frac{1}{2}\mathrm{tr}[\sqrt{\rho}, H]^2 = \frac{1}{2}\mathrm{tr}[\sqrt{\rho}, H][\sqrt{\rho}, H]^\dagger,$$

and replace H by any operator (not necessarily Hermitian) X, then we have

$$I(\rho, X) = \frac{1}{2}\mathrm{tr}[\sqrt{\rho}, X][\sqrt{\rho}, X]^\dagger,$$

which is still a well defined nonnegative quantity. In particular, if we take X to be the annihilation operator a, then we are led to equation (4). Consequently, the nonclassicality quantity $N(\rho)$ is actually a natural extension of the Wigner–Yanase skew information. Due to the remarkable properties of the Wigner–Yanase skew information and its wide applications in quantum information [41–50], it is reasonable to expect that $N(\rho)$ will be useful in capturing nonclassicality of quantum states [39]. Remarkably, $N(\rho)$ has the following three equivalent expressions [39]

$$N(\rho) = \frac{1}{2}\mathrm{tr}[\sqrt{\rho}, a][\sqrt{\rho}, a]^\dagger$$
$$= \frac{1}{2}(I(\rho, Q) + I(\rho, P))$$
$$= \frac{1}{2\pi}\int_0^{2\pi} I(\rho, H_\theta)\mathrm{d}\theta,$$

where $Q = (a + a^\dagger)/\sqrt{2}$ and $P = (a - a^\dagger)/(\sqrt{2}\mathrm{i})$ are the canonical conjugate quadratures, while $H_\theta = (\mathrm{e}^{-\mathrm{i}\theta}a + \mathrm{e}^{\mathrm{i}\theta}a^\dagger)/\sqrt{2}$ are the homodyne rotated quadratures. The above coincidence indicates some unique and versatile feature of $N(\cdot)$. We will employ this quantity to study non-Gaussianity.

3. Non-Gaussianity

Recall that any Gaussian state of single-mode Bosonic fields can be expressed as a displaced squeezed thermal state [1]

$$g = D_\alpha S_\zeta \tau_\lambda S_\zeta^\dagger D_\alpha^\dagger, \tag{5}$$

where $D_\alpha = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}$ are the Weyl displacement operators and $S_\zeta = \mathrm{e}^{\zeta a^{\dagger 2}/2 - \zeta^* a^2/2}$ are the squeezing operators, $\alpha, \zeta \in \mathbb{C}$, and τ_λ are thermal states as defined by equation (3). Here we address the issue of quantifying non-Gaussianity by exploiting nonclassicality, and thus establish some links between these two different notions.

To motivate our investigation, first note that any coherent state $|\alpha\rangle$ is Gaussian with minimal nonclassicality [39]

$$N(|\alpha\rangle) = 1/2$$

among pure states. Moreover, coherent states are the only pure states that are classical. In the meantime, although any squeezed coherent state

$$|\alpha_z\rangle = S_z|\alpha\rangle, \qquad z \in \mathbb{C}$$

is Gaussian, yet its nonclassicality is [39]

$$N(|\alpha_z\rangle) = \frac{1}{2}\cosh(2|z|) > \frac{1}{2} \quad \text{for } z \neq 0,$$

which shows that all squeezed coherent states are nonclassical [39]. Thus we cannot use nonclassicality directly to quantify non-Gaussianity since squeezed coherent states are also Gaussian. However, if we take into account the squeezing (which does not alter Gaussianity and non-Gaussianity), and consider minimization over squeezing, then we obtain

$$\min_\zeta N(S_\zeta|\alpha_z\rangle) = \frac{1}{2},$$

which indeed indicates the minimal value of non-classicality among pure states [39]. Accordingly, in terms of $N(\cdot)$, we define a quantifier for non-Gaussianity of quantum state ρ as

$$N_{\mathrm{g}}(\rho) = \min_\zeta N(S_\zeta \rho S_\zeta^\dagger) - \frac{1}{2}. \tag{6}$$

Recall that $S_\zeta = \mathrm{e}^{\zeta a^{\dagger 2}/2 - \zeta^* a^2/2}$ are the squeezing operators with squeezing parameters $\zeta = r\mathrm{e}^{\mathrm{i}\phi}$. Equivalently,

$$N_{\mathrm{g}}(\rho) = \min_\zeta(\mathrm{tr}\rho a_\zeta^\dagger a_\zeta - \mathrm{tr}\sqrt{\rho}a_\zeta^\dagger\sqrt{\rho}a_\zeta),$$

where $a_\zeta = S_\zeta^\dagger a S_\zeta = a\cosh r - a^\dagger \mathrm{e}^{\mathrm{i}\phi}\sinh r$, $\zeta = r\mathrm{e}^{\mathrm{i}\phi}$.

We make a further comment on the motivation of the definition of $N_{\mathrm{g}}(\rho)$. Recall that all Gaussian states can be expressed as displaced squeezed thermal states. While displacements should not have any effect on nonclassicality, the squeezing is crucial, and thus in order to detect non-Gaussianity via nonclassicality, we strip off this kind of squeezing nonclassicality (which is simultaneously of Gaussian nature) from the states by taking minimum over all squeezing, which leads to equation (6).

Although it is clear from equation (6) that

$$N_{\mathrm{g}}(\rho) \leqslant N(\rho) - \frac{1}{2},$$

there is no simple monotonic comparison between the ordering given by $N_{\mathrm{g}}(\cdot)$ and that by $N(\cdot)$ in the sense that $N(\rho_1) < N(\rho_2)$ implies neither $N_{\mathrm{g}}(\rho_1) < N_{\mathrm{g}}(\rho_2)$ nor $N_{\mathrm{g}}(\rho_1) > N_{\mathrm{g}}(\rho_2)$ in general. For example, consider $\rho_1 = |0\rangle\langle 0|$, $\rho_2 = S_\zeta|0\rangle\langle 0|S_\zeta^\dagger$, then $N(\rho_1) = 1/2 < N(\rho_2) = \frac{1}{2}\cosh(2|\zeta|)$, but $N_{\mathrm{g}}(\rho_1) = 0 = N_{\mathrm{g}}(\rho_2)$. On the other hand, for the Fock state $|1\rangle\langle 1|$, although $N(|1\rangle\langle 1|) = 3/2 < N(\rho_2) = \frac{1}{2}\cosh(2|\zeta|)$ for sufficiently large $|\zeta|$, we have $N_{\mathrm{g}}(|1\rangle\langle 1|) = 1 > 0 = N_{\mathrm{g}}(\rho_2)$.

$N_{\mathrm{g}}(\rho)$ has the following desirable properties which render it useful in detecting non-Gaussianity.

(1) For any Gaussian state, as defined by equation (5), we have

$$N_{\mathrm{g}}(g) = -\frac{\sqrt{\lambda}}{1 + \sqrt{\lambda}} \leqslant 0. \tag{7}$$

The equality is saturated if and only if $\bar{n} = \mathrm{tr}\tau_\lambda a^\dagger a = 0$, that is, $\tau_\lambda = |0\rangle\langle 0|$ with $\lambda = 0$, for which ρ is a Gaussian pure state (equivalently, squeezed coherent state). Thus, Gaussian mixed states possess negative non-Gaussianity in terms of $N_{\mathrm{g}}(\cdot)$.

(2) $N_{\mathrm{g}}(\rho) \leqslant 0$ for any Gaussian state or any classical state. Consequently, if $N_{\mathrm{g}}(\rho) > 0$, then the state ρ must be simultaneously non-Gaussian and nonclassical. This supplies a sufficient (although not necessary) criterion for detecting both non-Gaussianity and nonclassicality.

(3) $N_{\mathrm{g}}(\rho)$ is invariant under unitary transformations implementing symplectic transformations in phase space of Bosonic fields. In particular, $N_{\mathrm{g}}(U\rho U^\dagger) = N_{\mathrm{g}}(\rho)$ for $U = D_\alpha$, S_ζ or $\mathrm{e}^{\mathrm{i}\theta a^\dagger a}$, $\alpha, \zeta \in \mathbb{C}$, $\theta \in \mathbb{R}$.

(4) $N_{\mathrm{g}}(\rho)$ is neither convex nor concave with respect to ρ.

Now we outline proofs of the above statements.

Item (1) follows from direct evaluation.

Item (2) follows from item (1) and the nonclassicality criterion in [39], since $N_{\mathrm{g}}(\rho) > 0$ implies that $N(\rho) > 1/2$.

For item (3), if $U = \mathrm{e}^{-\mathrm{i}H}$ is a unitary transformation corresponding to a symplectic transformation in phase space with Hermitian H at most bilinear in the field operators, then U can be expressed as composition of squeezing operators and displacement operators. The invariance of $N(\cdot)$ under displacements and further optimization in equation (6) leads to item (3).

Item (4) will be demonstrated in the following examples.

We now evaluate $N_{\rm g}(\rho)$ for some important quantum states in order to illustrate its basic features and intuitive meaning.

(1) *Fock states*

For the Fock states $|n\rangle\langle n|$, noting the relation

$$S_\zeta^\dagger a S_\zeta = a\cosh r - a^\dagger \mathrm{e}^{\mathrm{i}\phi}\sinh r,$$
$$S_\zeta^\dagger a^\dagger S_\zeta = a^\dagger\cosh r - a\mathrm{e}^{-\mathrm{i}\phi}\sinh r,$$

for $\zeta = r\mathrm{e}^{\mathrm{i}\phi}$, we have

$$N(S_\zeta|n\rangle\langle n|S_\zeta^\dagger) = \left(\frac{1}{2}+n\right)\cosh(2r).$$

It follows that

$$N_{\rm g}(|n\rangle\langle n|) = \min_\zeta N(S_\zeta|n\rangle\langle n|S_\zeta^\dagger) = n,$$

which is a neat expression indicating non-Gaussianity, as well as nonclassicality, of the Fock states for $n \neq 0$.

(2) *Mixtures of vacuum and Fock states*

For the mixtures

$$\rho_t = (1-t)|0\rangle\langle 0| + t|n\rangle\langle n|, \qquad 0 < t \leqslant 1 \tag{8}$$

we have

$$N_{\rm g}(\rho_t) = \begin{cases} t - \sqrt{t(1-t)}, & n = 1 \\ nt & n \geqslant 2 \end{cases}. \tag{9}$$

Consequently, for $n = 1$, $N_{\rm g}(\rho_t) > 0$ when $t > 1/2$, and for $n \geqslant 2$, we always have $N_{\rm g}(\rho_t) > 0$, which implies non-Gaussianity and nonclassicality in these cases.

By the way, we show that $N_{\rm g}(\cdot)$ is not concave in general. From the above results we have

$$N_{\rm g}((1-t)|0\rangle\langle 0| + t|1\rangle\langle 1|) = t - \sqrt{t(1-t)},$$

and

$$(1-t)N_{\rm g}(|0\rangle\langle 0|) + tN_{\rm g}(|1\rangle\langle 1|) = t,$$

it follows that

$$N_{\rm g}((1-t)|0\rangle\langle 0| + t|1\rangle\langle 1|) < (1-t)N_{\rm g}(|0\rangle\langle 0|) + tN_{\rm g}(|1\rangle\langle 1|),$$

and consequently, $N_{\rm g}(\cdot)$ cannot be concave. Another simple counterexample to concavity arises from considering the thermal state defined by equation (3). By equation (7) and $N_{\rm g}(|n\rangle\langle n|) = n$, we have

$$N_{\rm g}(\tau_\lambda) = -\frac{\sqrt{\lambda}}{1+\sqrt{\lambda}} < 0 < \sum_{n=0}^{\infty}(1-\lambda)\lambda^n N_{\rm g}(|n\rangle\langle n|),$$

which shows that $N_{\rm g}(\cdot)$ cannot be concave. We will show that $N_{\rm g}(\cdot)$ cannot be convex in the next example.

The mixture ρ_t given by equation (8) provides a nice and simple example to illustrate the effect of mixing with vacuum on non-Gaussianity. In this context, we have $N_{\rm g}(\rho_t) = nt = tN_{\rm g}(|n\rangle\langle n|)$, $n \geqslant 2$, which indicates decreasing of non-Gaussianity since $0 < t \leqslant 1$.

(3) *ON states*

The ON states

$$|0n_+\rangle = \sqrt{1-t}\,|0\rangle + \sqrt{t}\,|n\rangle, \qquad 0 < t < 1,\ n = 1, 2, \cdots$$

and

$$|0n_-\rangle = \sqrt{1-t}\,|0\rangle - \sqrt{t}\,|n\rangle, \qquad 0 < t < 1,\ n = 1, 2, \cdots$$

are superpositions of the vacuum and the Fock states, and can serve as resource units for universal quantum computation [51]. By direct evaluation, we have

$$N_{\rm g}(|0n_\pm\rangle) = \begin{cases} \frac{1}{2}(\sqrt{1+8t^3}-1), & n = 1 \\ \frac{1}{2}(\sqrt{1+24t^2}-1), & n = 2 \\ nt, & n = 3, 4, \cdots \end{cases} \tag{10}$$

Consequently, $N_{\rm g}(|0n_\pm\rangle) > 0$, which implies non-Gaussianity and nonclassicality of the ON states. Moreover, we see that as n increases, non-Gaussianity increases.

In this context, we show that $N_{\rm g}(\cdot)$ cannot be convex. Take $\rho_1 = |02_+\rangle\langle 02_+|$, $\rho_2 = |02_-\rangle\langle 02_-|$ with

$$|02_+\rangle = \sqrt{1-t}\,|0\rangle + \sqrt{t}\,|2\rangle,$$
$$|02_-\rangle = \sqrt{1-t}\,|0\rangle - \sqrt{t}\,|2\rangle,$$

then

$$\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 = (1-t)|0\rangle\langle 0| + t|2\rangle\langle 2|,$$

and by equation (9),

$$N_{\rm g}\left(\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2\right) = N_{\rm g}((1-t)|0\rangle\langle 0| + t|2\rangle\langle 2|) = 2t.$$

On the other hand, by equation (10),

$$N_{\rm g}(\rho_1) = N_{\rm g}(\rho_2) = \frac{1}{2}(\sqrt{1+24t^2}-1).$$

Since

$$2t > \frac{1}{2}(\sqrt{1+24t^2}-1), \qquad 0 < t < 1,$$

we conclude that

$$N_{\rm g}\left(\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2\right) > \frac{1}{2}N_{\rm g}(\rho_1) + \frac{1}{2}N_{\rm g}(\rho_2),$$

which shows that $N_{\rm g}(\cdot)$ cannot be convex.

(4) *Cat states*

For the even cat states

$$|\alpha_+\rangle = \frac{1}{(2+2\mathrm{e}^{-2|\alpha|^2})^{1/2}}(|\alpha\rangle + |-\alpha\rangle), \quad \alpha \in \mathbb{C},\ \alpha \neq 0$$

we have

$$N_{\rm g}(|\alpha_+\rangle\langle\alpha_+|) = \sqrt{\left(\frac{1}{2}+|\alpha|^2\tanh|\alpha|^2\right)^2 - |\alpha|^4} - \frac{1}{2} > 0,$$

and for the odd cat states

$$|\alpha_-\rangle = \frac{1}{(2 - 2e^{-2|\alpha|^2})^{1/2}}(|\alpha\rangle - |-\alpha\rangle), \quad \alpha \in \mathbb{C}, \alpha \neq 0$$

we have

$$N_g(|\alpha_-\rangle\langle\alpha_-|) = \sqrt{\left(\frac{1}{2} + |\alpha|^2 \coth|\alpha|^2\right)^2 - |\alpha|^4} - \frac{1}{2} > 0.$$

Consequently, both even and odd cat states are always non-Gaussian and nonclassical, and moreover,

$$N_g(|\alpha_-\rangle\langle\alpha_-|) > N_g(|\alpha_+\rangle\langle\alpha_+|),$$

which indicates that the odd cat states are more non-Gaussian than the corresponding even cat states. This is in accordance with the fact that $|\alpha_-\rangle$ tends to the single-photon number state $|1\rangle$, while $|\alpha_+\rangle$ tends to the vacuum state $|0\rangle$, when $|\alpha|$ tends to zero.

(5) *Photon-added coherent states*

For the photon-added coherent state

$$|\psi_{\text{pac}}\rangle = \frac{1}{\sqrt{1 + |\alpha|^2}} a^\dagger |\alpha\rangle,$$

we have

$$N_g(|\psi_{\text{pac}}\rangle) = \frac{1}{2}\sqrt{1 + \frac{8}{(1 + |\alpha|^2)^3}} - \frac{1}{2} > 0,$$

which implies that such states are always non-Gaussian and nonclassical, and moreover, the degree of non-Gaussianity is a decreasing function of $|\alpha|$.

(6) *Photon-added thermal states*

For the photon-added thermal state

$$\rho_{\text{pth}} = \frac{a^\dagger \tau_\lambda a}{\text{tr} a^\dagger \tau_\lambda a} = \frac{(1 - \lambda)^2}{\lambda} \sum_{n=1}^{\infty} \lambda^n n |n\rangle\langle n|, \quad 0 \leqslant \lambda < 1,$$

we have

$$N_g(\rho_{\text{pth}}) = \frac{1 + \lambda}{1 - \lambda} - \frac{(1 - \lambda)^2}{\lambda^{3/2}} \sum_{n=1}^{\infty} n^{3/2}(n - 1)^{1/2} \lambda^n.$$

Numerical analysis shows that $N_g(\rho_{\text{pth}}) > 0$ for $\lambda < \lambda_c \approx 0.223$, which implies non-Gaussianity as well as nonclassicality in this situation.

(7) *Fock-diagonal states*

For the Fock-diagonal states

$$\rho_F = \sum_{n=0}^{\infty} p_n |n\rangle\langle n|, \tag{11}$$

we have

$$N_g(\rho_F) = \frac{1}{2}\sum_{n=0}^{\infty} (\sqrt{p_n} - \sqrt{p_{n+1}})^2 (n + 1) - \frac{1}{2}. \tag{12}$$

In particular, for the truncated thermal states

$$\tau_{\hat{0}} = \frac{1 - \lambda}{\lambda} \sum_{n=1}^{\infty} \lambda^n |n\rangle\langle n|, \tag{13}$$

which are obtained from the thermal states by removing the vacuum component $|0\rangle\langle 0|$, by putting

$$p_0 = 0, \qquad p_n = (1 - \lambda)\lambda^{n-1}, \qquad n \neq 0$$

into equation (12), we have

$$N_g(\tau_{\hat{0}}) = \frac{1}{1 + \sqrt{\lambda}} - \sqrt{\lambda}.$$

This indicates non-Gaussianity and nonclassicality of $\tau_{\hat{0}}$ when $\lambda < (3 - \sqrt{5})/2$.

4. Comparison

In this section, we make a comparative study of our quantifier for non-Gaussianity with several existing ones, and illustrate their respective characteristics. First, we recall four important measures for non-Gaussianity in the literature.

(i) *Hilbert–Schmidt distance*

In terms of the Hilbert–Schmidt distance between a state ρ and its reference Gaussian state ρ_g, which is defined as the unique Gaussian state with the same mean and covariance matrix as ρ, the following measure for non-Gaussianity

$$N_H(\rho) = \frac{1}{2}\frac{\text{tr}(\rho - \rho_g)^2}{\text{tr}\rho^2}$$

was introduced by Genoni *et al* [16–18]. For the Fock-diagonal states defined by equation (11), it is known that

$$N_H(\rho_F) = \frac{1}{2}\left(1 + \frac{\sum_{n=0}^{\infty}(b_n^2 - 2b_n p_n)}{\sum_{n=0}^{\infty} p_n^2}\right)$$

with

$$b_n = \frac{1}{1 + \bar{n}}\left(\frac{\bar{n}}{\bar{n} + 1}\right)^n, \quad \bar{n} = \text{tr}\rho_F a^\dagger a = \sum_{n=0}^{\infty} n p_n.$$

(ii) *Relative entropy*

In terms of the relative entropy $S(\rho|\rho_g)$ between a state ρ and its associated reference Gaussian state ρ_g, the following measure of non-Gaussianity

$$N_R(\rho) = S(\rho|\rho_g) = S(\rho_g) - S(\rho),$$

was studied in [16–18]. Here $S(\rho) = -\text{tr}\rho\ln\rho$ is the von Neumann entropy. It turns out that $N_R(\rho) = \inf_\tau S(\rho|\tau)$ where the inf is over all Gaussian states τ [19]. In particular,

$$N_R(\rho_F) = (\bar{n} + 1)\ln(\bar{n} + 1) - \bar{n}\ln\bar{n} + \sum_{n=0}^{\infty} p_n \ln p_n$$

for the Fock-diagonal states defined by equation (11). Here $\bar{n} = \text{tr}\rho_F a^\dagger a = \sum_{n=0}^{\infty} n p_n$.

(iii) *Wehrl entropy*

By virtue of the Wehrl entropy

$$S_W(\rho) = -\frac{1}{\pi}\int \langle\alpha|\rho|\alpha\rangle \ln\langle\alpha|\rho|\alpha\rangle \, d^2\alpha$$

of the Husimi function $\langle\alpha|\rho|\alpha\rangle$ with $|\alpha\rangle$ being the coherent states, Ivan *et al* proposed the following measure for non-Gaussianity [21]

$$N_W(\rho) = S_W(\rho_g) - S_W(\rho),$$

where ρ_g is the associated reference Gaussian state of ρ. It can

Table 1. Comparing quantifiers for non-Gaussianity.

	$\|n\rangle\langle n\|$	$\frac{1-\lambda}{\lambda}\sum_{n=1}^{\infty}\lambda^n\|n\rangle\langle n\|$
N_{H}	$\frac{n+1}{2n+1}-\frac{1}{n+1}\left(\frac{n}{n+1}\right)^n$	$\frac{5-\lambda}{2(2-\lambda)(3-\lambda)}$
N_{R}	$(n+1)\ln(n+1)-n\ln n$	$\frac{(2-\lambda)\ln(2-\lambda)+\lambda\ln\lambda}{1-\lambda}$
N_{W}	$\ln(n+1)-n-\ln n!+n\psi(n)$	$1-\frac{1}{\pi}\int\bar{p}(\alpha)\ln\bar{p}(\alpha)\mathrm{d}^2\alpha+\frac{2-\lambda}{1-\lambda}\ln(\frac{2-\lambda}{1-\lambda})$
N_{F}	$1-\sqrt{\frac{n^n}{(n+1)^{n+1}}}$	$\frac{1}{2}(1-\sqrt{\frac{\lambda}{2-\lambda}})$
N_{g}	n	$\frac{1}{1+\sqrt{\lambda}}-\sqrt{\lambda}$

be evaluated that

$$N_{\mathrm{W}}(\rho_{\mathrm{F}})=1+\ln(\bar{n}+1)-\frac{1}{\pi}\int\bar{p}(\alpha)\ln\bar{p}(\alpha)\mathrm{d}^2\alpha$$

with $\psi(n)=\sum_{k=1}^{n}k^{-1}-\gamma$, $\gamma\approx 0.577\,216$ is the Euler constant, and

$$\bar{p}(\alpha)=\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\mathrm{e}^{-|\alpha|^2}p_n.$$

(iv) *Fidelity*

In terms of the fidelity $F(\rho,\rho_{\mathrm{g}})=\mathrm{tr}((\sqrt{\rho}\rho_{\mathrm{g}}\sqrt{\rho})^{1/2})$ between ρ and its associated reference Gaussian state ρ_{g}, the following measure for non-Gaussianity

$$N_{\mathrm{F}}(\rho)=1-F(\rho,\rho_{\mathrm{g}})$$

is proposed by Ghiu *et al* [20]. For the Fock-diagonal states, it is known that

$$N_{\mathrm{F}}(\rho_{\mathrm{F}})=1-\sum_{n=0}^{\infty}\sqrt{\frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n p_n}.$$

Now by considering Fock states and truncated thermal states defined by equation (13), we summarize a comparison of the above measures with our $N_{\mathrm{g}}(\cdot)$ in table 1, in which

$$\psi(n)=\sum_{k=1}^{n}k^{-1}-\gamma,\ \ \bar{p}(\alpha)=\frac{1-\lambda}{\lambda}\mathrm{e}^{-|\alpha|^2}(\mathrm{e}^{\lambda|\alpha|^2}-1).$$

For the Fock states $|n\rangle\langle n|$, all other measures yield rather complicated expressions for non-Gaussianity, and the measure $N_{\mathrm{W}}(\cdot)$ based on the Wehrl entropy even involves harmonic series and the unexpected Euler constant. In sharp contrast, our quantifier yields a neat expression which captures the nature of Fock states succinctly and cannot be simpler. For the truncated thermal states, our quantifier also yield a relatively simple expression. We see certain simplicity and intuitive meaning of $N_{\mathrm{g}}(\cdot)$. Although $N_{\mathrm{g}}(\cdot)$ is not a genuine measure for non-Gaussianity, it captures non-Gaussianity effectively for many nonclassical states, exhibits considerably nice features, and complements other measures for non-Gaussianity, as displayed in table 1.

5. Discussion

Due to the increasing importance of non-Gaussian states in continuous variable quantum information, there is an urgent need of detecting and quantifying non-Gaussianity. There are several approaches to this issue pursued by many authors, and we have added to this endeavor an information-theoretic means of detecting non-Gaussianity. A remarkable feature of our criterion is that it can detect non-Gaussianity and nonclassicality simultaneously for a wide class of states. The method invokes optimization over squeezing of a quantifier for nonclassicality, and thus establishes some intrinsic relations between nonclassicality and non-Gaussianity, both of which are valuable resources in quantum information processing.

It seems difficult to measure the quantifier directly due to the square root involved in the Wigner–Yanase skew information. However, by quantum tomography of the state, the quantifier can be evaluated.

We emphasize that our quantifier only yields a sufficient criterion for detecting non-Gaussianity and nonclassicality, and it is desirable to seek genuine measure for non-Gaussianity and nonclassicality, and investigate further the interplay between them.

Since mixtures of Gaussian states are in general not Gaussian states, the set of Gaussian states is not convex. We may further classify non-Gaussian states according to whether it can be expressed as a probabilistic mixture of Gaussian states or not, that is, non-Gaussian states can be further divided into classical non-Gaussian and quantum non-Gaussian (or genuine non-Gaussian): A state is called classical non-Gaussian if it is non-Gaussian and in the meantime can be expressed as probabilistic mixtures of Gaussian states. If a state cannot be expressed as any probabilistic mixture of Gaussian states, then it is called quantum non-Gaussian. Thus the set of all classical non-Gaussian states together with the Gaussian states constitute a convex set, whose complement is the set of quantum non-Gaussian states. There are relative few studies on detecting and quantifying quantum non-Gaussianity, with some significant advances [22–24, 52–60]. In particular, a criterion for quantum non-Gaussianity based on photon number probabilities is proposed in [52], and Wigner functions are used to quantifying quantum non-Gaussianity in [53]. Witnesses of quantum non-Gaussianity based on s-parameterized quasi-probability functions in phase space are addressed in [56], and demarginalization method is used to detect quantum non-Gaussianity in [58]. It will be desirable to further study relations between nonclassicality and quantum non-Gaussianity from an information-theoretic perspective.

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