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Organisational hierarchy constructions with easy Kuramoto synchronisation

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Abstract

We provide a graph theoretic construction enabling tree-based hierarchies to be modified into structures with enhanced connectivity and synchronisability. Specifically, the construct transforms trees into members of a family of graphs known as *expanders*, which we call 'expander-augmented-hierarchies' or trees. We show that this produces graphs with significantly enhanced synchronisation properties in the context of the Kuramoto model of phase oscillators coupled on networks. When considered as organisational structures these networks enjoy both the managability of simple hierarchies with near regular degree distribution, and low critical coupling by the addition of relatively few extra edges. For the expander augmented tree, we examine the synchronisation properties, computed through the time-averaged Kuramoto order parameter over an ensemble of natural frequencies. We compare this with a range of other networks including hierarchies augmented by random matching of the leaf nodes. For these we compute the ratio Q of smallest to largest Laplacian eigenvalues, the smallness of which has been argued to be an indicator of good synchronisability. While not *the best* of these in Q, the expander-augmented-hierarchy exhibits synchronisability barely distinguishable from others with lower Q within the variance over an ensemble of natural frequencies. However, the expander augmented tree has the advantage that its properties are *automatically designed for* as opposed to the outcome of a random search for low Q-value graphs that in itself scales very poorly.

Keywords: Kuramoto model, synchronisation, complex systems

(Some figures may appear in colour only in the online journal)

1. Introduction

The Kuramoto model [30] was originally motivated by the phenomenon of collective synchronisation whereby a system of coupled oscillating nodes will sometimes lock on to a common frequency despite differences in the natural frequencies of the individual nodes. This model has since been applied to a wide variety of application fields including, biological, chemical, engineering, and social systems (see the survey articles [1, 2, 47], and recently [15] and [45]). While Kuramoto studied the infinite complete network it is natural to consider finite networks of any given topology, and we do this here. In particular, we are motivated by the potential for using the Kuramoto model as a representation for the decision processes in organisations [27] where hierarchies often figure as a formal structure, as will be explained further below. We propose minimalistic additional edges to a hierarchy that improve its capacity for Kuramoto synchronisation.

In this setting of the model we take a cue from numerous models of human decision-making that present the process as fundamentally cyclic: the perception-action cycle of Neisser [35], the situation awareness model of Endsley [16] and the observe-orient-decide-act model of Boyd [39]. Thus, in our formulation the node of a network represents an agent in an organisation; the network is the organisational structure; phase of a Kuramoto oscillator represents the agent's state in a perception-action cycle; intrinsic frequencies are proportional to the natural decision-making speed of the agent; and the coupling strength is the intensity of interaction between connected members of the organisation. By organisations we refer to the structure of a range of systems for distributed human work, from military, business, and administrative entities.

The hierarchy is one of the most ancient and ubiquitous forms of organisational structure [19]. Apart from the psychological dimension of concentrating power in a single individual at the top of the hierarchy, its success also resides in its scalability to large sizes while maintaining an even load on all members through the division of labour [32], and providing a clear mechanism for accountability of every individual in the organisation [42, 43]. However, organisational theorists have long known the limits of the hierarchy in the face of unstable or dynamical environments: for large hierarchies, decision chains become long, and the time for information from the bottom to travel up and decisions to travel down is too large to allow for responsiveness [32]. In this respect, the effort of members in interacting with one another to come to consensus in responding to changes in the environment may be mapped to the property of Kuramoto oscillators achieving as close as possible to phase synchronisation. This specific mapping of the mathematical model to organisational properties was originally proposed in [24], with further elements such as the stochasticity of individual human decision processes introduced in [27]. An alternative mapping of the Kuramoto model to organisations, not used in the present work, has been proposed by [10] but where properties such as phases and frequencies are not identified. Either way, it is well-known that Kuramoto oscillators on tree graphs require relatively high coupling to achieve synchronisation [11]. The alternatives to hierarchies for organisations in dynamical environments generally involve high connectivity between members (for example, all-to-all connected teams) which therefore do not scale well for human decision-making. In this paper we propose modifications to trees that enhance their synchronisation properties.

At this point we note the literature addressing the synchronisation of dynamical systems on tree networks. In [11] a particular closed-form expression is given for the critical coupling of trees, and in turn for several classes of trees tight bounds are given for the expected critical coupling assuming node frequency distributions. In [18] the solvability of the critical coupling for trees is exploited to provide tight approximations to several engineering design problems including power re-dispatch in power grids, sparse network design, robust network design for distributed wireless analog clocks, and the detection of edges leading to the the Braess' paradox in power grids. Tree networks have also been studied as models for sensory brain activity [29], including the specific use of the Kuramoto model [23]. Since we shall be making small additions to tree networks with significant improvements to the synchronization capacity of the network we expect our results may be of wider interest to a variety of application settings.

The key to our paper is exploiting the properties of so-called expander families [22]. These are graphs that have good connectivity properties in that every partition of the nodes into two subsets has a number of boundary edges between them that scales with the size of the smallest subset. We use this to propose modified tree graphs that both scale and exhibit high synchronisability. To test the synchronisation properties we compute the time-averaged Kuramoto order parameter for this network, as well as a range of alternate structures. In particular, it has long been argued in the literature that an appropriate graph theoretic measure that correlates well with high synchronisability for a range of coupled dynamical systems is the ratio of largest and smallest non-zero Laplacian eigenvalues [4], building on the celebrated master stability function of Pecora and Carroll [40]. Arguably, this ratio should be as small as possible. Indeed, optimising graphs for the related Laplacian spectral gap [13] generates well synchronised arrangements [12] but with denser graphs compared to the hierarchy. However we note that the relevant dynamical systems considered here are different from the Kuramoto model which only features the Laplacian explicitly after linearisation. Nevertheless, to test our expander augmented hierarchy we compute the Laplacian eigenvalue ratio for a range of modified hierarchies, augmented by a random matching of leaf nodes but subject to the same degree distribution as our expander augmented tree. Indeed, we find that some graphs with smaller values of this ratio compete slightly better on average than our expander augmented hierarchy, while others fare marginally worse. But when variance over an ensemble of natural frequencies is taken into account these are barely distinguishable. However, the search space for low eigenvalue ratio graphs out of a uniform random sample turns out to be vast, and therefore scales poorly as the number of levels of the hierarchy increases. In the same context as [12], the work of Estrada *et al* [17] developed a method of generating regular graphs that have excellent expansion properties but which increase in degree as they are scaled up. Our expander augmented hierarchy, contrastingly, has fixed degree as the structure is scaled. From an organisational-application perspective this is superior in maintaining the load on individual decision-makers as the organisation increases in size.

Finally, by way of introduction, we comment on the relationship of this work to other approaches seeking to enhance synchronisation in the Kuramoto model. Numerous authors seek to explicitly optimise networks for synchronisation. Brede [6] considers placement of network links according to closeness or otherwise of native frequencies; in our case, the designer of an organisational network may not have the luxury of where to place slower, respectively faster oscillators. Hence we seek *insensitivity* to the detail of the frequency placement, as seen when we average over an ensemble of natural frequencies. In [49], optimization of the network is undertaken in the presence of noise however this approach requires significant computational effort beyond $N = 15$; for our work, the ability to scale a method is critical. Significant work has shown that tree structures are ideal for synchronisation in *oriented graphs* [37], with the depth of the tree influencing the time to synchronisation [51]. However, in our application of the model for decision-making where bidirectional information flows are required in the decision-cycle, undirected graphs are the most appropriate. Finally, recent work has developed approaches to test for whether a complex empirical network is hierarchical or not [33, 48, 52]. In our case, we begin with an idealised hierarchy, a tree, and transform it to enhance its synchronisation properties; whether it may be finally deemed 'hierarchical' is immaterial.

This logic of formalising notions of synchronisation, proving expansion properties of modified trees, and then exploring the impact in a concrete dynamical model underlies the structure of this paper. The Kuramoto model is defined and the related notions of synchronisation and critical coupling that we use here are described. Lower bounds for the critical coupling are derived for both general and regular networks. The inspiration for this work is in the relationship with these bounds and the graph expansion number that we identify here. We then prove a series of theorems that are the key to the construction of the modified trees that we propose may be the basis for organisational structures. In this context we describe a graph construction that provides efficient manageability when considered as an organizational structure. We then compute the time-averaged order parameter for synchronisation and compare to other graphs, including adding the randomly chosen matchings of leaf nodes of trees. The paper concludes with an outlook for the future.

2. Defining the Kuramoto model

Each node has an associated phase angle θ_i, as well as its own natural frequency ω_i. The basic governing equation is the differential equation:

$$\dot{\theta}_i = \omega_i + k\sum_{j=1}^{n} A_{ij}\sin(\theta_j - \theta_i), i = 1, .., n. \tag{1}$$

Here A is the 0, 1 adjacency matrix of the network, with a 1 in position $\{i, j\}$ indicating an edge in the network between nodes i and j. The interaction between the nodes is also governed by a coupling constant k. Note that each θ_i is implicitly understood to denote a function of time t.

2.1. Synchronisation

It has been observed that for arbitrary initial phases, networks synchronise in that some of the node phases converge to the same, or nearly the same phase angle, while the phase frequencies $\dot{\theta}_i$ converge to a common value. Meanwhile, the remaining nodes behave non-uniformly or 'drift'. Moreover, at a sufficiently large *critical* coupling constant k, it becomes possible for all nodes to synchronise in that they share the same phase frequencies. By summing over equation (1), the sine terms cancel and the average frequency of the nodes is a constant $\bar{\omega}$ so that

$$\frac{1}{n}\sum_{j=1}^{n}\dot{\theta}_j = \frac{1}{n}\sum_{j=1}^{n}\omega_j = \bar{\omega}. \tag{2}$$

Define a *frequency fixed point* in which we say the network is *frequency synchronised* as a situation in which all the node frequencies are equal and are fixed over time. By equation (2), this is characterized by

$$\dot{\theta}_i = \bar{\omega}, i = 1, .., n. \tag{3}$$

It follows that at a frequency fixed point all phase angle differences $\theta_i - \theta_j$ remain constant. We note that a frequency fixed point may be accompanied by the nodes having the same or nearly the same phase angles. This is however not necessarily the case, and the phase angles of a frequency fixed point may be significantly different. Examples of this kind are the ring networks [38] and tree networks [11]. Combining equations (1) and (3), the phase differences at a frequency fixed point satisfy

$$\bar{\omega} = \omega_i + k \sum_{j=1}^{n} A_{ij} \sin(\theta_j - \theta_i), i = 1, .., n. \tag{4}$$

We define the critical coupling k_c to be the least value of k for which a frequency fixed point occurs. This may be termed the *dynamical systems* notion of criticality; we shall later contrast this with other possible definitions of criticality. Note that while this is one of the main elements indicative of synchronisation that can be considered in the Kuramoto model, it may also be considered in combination with other indicators. In particular *linear stability* (see section 3.1) where all points in phase space sufficiently close to the fixed point converge to it over time, and *phase cohesiveness* where phase angle differences across edges are bounded to some constant less than $\pi/2$ [14] and [15].

2.2. Order parameter and critical points

In studies of the Kuramoto model it is common to measure the degree of phase synchronisation achieved in the dynamics using the order parameter

$$r(t) = \frac{1}{N} \left| \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t)} \right|. \tag{5}$$

This will fluctuate around $1/\sqrt{N}$ at low coupling values, where individual phases are randomly distributed around the unit circle, and converge to unity at high coupling. One may then compute the time-average of $r(t)$ after dispensing with some initial transient for $t \leqslant T_{\mathrm{th}}$ up to a final time T to gain a static measure for the system synchronisation properties:

$$\langle r \rangle_T = \frac{1}{T - T_{\mathrm{th}}} \int_{T_{\mathrm{th}}}^{T} r(t) \mathrm{d}t. \tag{6}$$

Further averaging of this over an ensemble of randomly drawn natural frequencies ω_i is possible, which we apply here.

This then leads to a *statistical* notion of a critical coupling, namely that value of k such that the scaling limit as $N \to \infty$ of $\langle r \rangle_T$ is non-zero [21]. Another marker of criticality, also drawing upon statistical physics and inference ideas, is the Fisher information, studied for the Kuramoto model in [26]. In this paper we will predominately work with the dynamical systems notion of criticality, however in the latter part of the paper we will draw upon the quantity $\langle r \rangle_T$.

We note that for a sufficiently large coupling constant frequency synchronisation at a frequency fixed point must occur for any connected network. However the various notions of phase synchronisation are in general matters of degree, and true phase synchronisation where all phase angles are equal is only possible for homogeneous systems where all the natural frequencies ω_i are equal.

3. Critical coupling for frequency fixed points

We now develop lower bounds for both the critical coupling for a frequency fixed point and its expected value where the natural frequencies ω_i are assumed to follow a distribution. This will inform the explicit construction of networks that are practical as organisational structures while also having low critical coupling.

3.1. Lower bound on the critical coupling

In the following for any subset of nodes X let $\partial(X)$ be the number of edges with one end node in X and the other in X^c the complement of X.

Theorem 1.

$$k_c \geqslant \max_{|X|\leqslant\frac{n}{2}} \frac{1}{\partial(X)} |\sum_{i\in X}(\bar{\omega} - \omega_i)|. \tag{7}$$

Proof. We can rearrange equation (4) as

$$\bar{\omega} - \omega_i = k\sum_{j=1}^{n} A_{ij}\sin(\theta_j - \theta_i), i = 1, .., n. \tag{8}$$

If we sum both sides of equation (8) over the elements of any node subset X we obtain

$$\sum_{i\in X}(\bar{\omega} - \omega_i) = k\sum_{i\in X}\sum_{j=1}^{n} A_{ij}\sin(\theta_j - \theta_i), \tag{9}$$

$$= k\sum_{(i,j)\in(X,X^c)} A_{ij}\sin(\theta_j - \theta_i), \tag{10}$$

where each pair of terms on the right hand side of equation (9) $A_{ij}\sin(\theta_j - \theta_i)$ and $A_{ji}\sin(\theta_i - \theta_j)$ with $i, j \in X$ cancel each-other out. Since the sine terms have magnitude at most 1 we have

$$|\sum_{i\in X}(\bar{\omega} - \omega_i)| \leqslant k\partial(X). \tag{11}$$

As this is true for any node subset X we must have

$$k \geqslant \max_{X} \frac{1}{\partial(X)} |\sum_{i\in X}(\bar{\omega} - \omega_i)|. \tag{12}$$

Since this must be true for any coupling constant k at a frequency fixed point, it is also true for k_c. Lastly we note that

$$|\sum_{i\in X}(\bar{\omega} - \omega_i)| = |\sum_{i\in X^c}(\bar{\omega} - \omega_i)| \text{ and } \partial(X) = \partial(X^c), \tag{13}$$

and so it is sufficient to consider only those X with at most $n/2$ nodes in inequality (12). This completes the proof of the theorem. □

We note that when the underlying network is a tree the expression on the right hand side of equation (7) is maximised when the subnetworks induced on X and X^c are both subtrees and $\partial(X) = 1$. Inequality equation (7) then becomes an equality and the correct expression for the critical coupling of any tree by a result of [11].

3.2. Expected critical coupling

By using theorem 1 we can obtain lower bounds for the expected critical coupling where the natural frequencies ω_i are assumed to follow a distribution. In the following we assume that the natural frequencies follow a normal distribution with mean 0 and variance σ^2. The general case with mean μ can be translated into the former case through a rotating frame of reference corresponding to a change of variable $\omega_i' = \omega_i - \bar{\omega}$, which shows the critical coupling is independent of the mean.

Theorem 2. *For normally distributed natural frequencies the expected critical coupling $E(k_c)$ has the lower bound*

$$E(k_c) \geqslant \max_{|X| \leqslant \frac{n}{2}} \frac{\sqrt{|X|(n-|X|)}}{\partial(X)} \frac{\sqrt{2}}{\sqrt{\pi n}} \sigma. \tag{14}$$

Proof. By expanding the mean frequency $\bar{\omega}$ in inequality (7) we obtain

$$k_c \geqslant \max_{|X| \leqslant \frac{n}{2}} \frac{1}{\partial(X)} \left| \left(\frac{|X|}{n} - 1 \right) \sum_{i \in X} \omega_i + \frac{|X|}{n} \sum_{i \in X^c} \omega_i \right|. \tag{15}$$

For any node subset X let $R(X)$ be the random variable defined by

$$R(X) = \frac{1}{\partial(X)} \left[\left(\frac{|X|}{n} - 1 \right) \sum_{i \in X} \omega_i + \frac{|X|}{n} \sum_{i \in X^c} \omega_i \right]. \tag{16}$$

As a linear function of normally distributed independent random variables $R(X)$ is itself normally distributed with a mean of 0 and a variance of

$$Var(R(X)) = \frac{1}{\partial(X)^2} \left[\left(\frac{|X|}{n} - 1 \right)^2 |X| \sigma^2 + \frac{|X|^2}{n^2} |X^c| \sigma^2 \right], \tag{17}$$

$$= \frac{\sigma^2}{n^2 \partial(X)^2} \left[(|X| - n)^2 |X| + |X|^2 (n - |X|) \right], \tag{18}$$

$$= \frac{|X|(n-|X|)\sigma^2}{n^2 \partial(X)^2} [n - |X| + |X|], \tag{19}$$

$$= \frac{|X|(n-|X|)}{\partial(X)^2} \frac{\sigma^2}{n}. \tag{20}$$

Thus the expected absolute value of $R(X)$ is given by [41]

$$E(|R(X)|) = \left[\frac{\sqrt{|X|(n-|X|)}}{\partial(X)} \right] \frac{\sqrt{2}}{\sqrt{\pi n}} \sigma. \tag{21}$$

The result follows from combining inequality (15) and equality (21) which completes the proof. □

Remarks. The bound of theorem 2 is sharpest when some cut-set X dominates the maximum expression of inequality (14).

3.3. Regular networks

When the network is regular we can obtain lower bounds on the critical coupling that may improve theorem 2. We note that the case where the network is complete, and so regular, has been studied previously [7], and their results align with the following ($d = n - 1$).

Theorem 3. *Let the network A_{ij} be regular of degree d. Then for normally distributed natural frequencies the expected critical coupling $E(k_c)$ has the lower bound*

$$E(k_c) \geqslant \frac{n-1}{\mathrm{d}n}\left[\sqrt{2\log n} - \frac{\frac{1}{2}\log(4\pi\log n)}{\sqrt{2\log n}} - O(\frac{1}{\log n})\right]\sigma \tag{22}$$

$$\to \frac{\sqrt{2\log n}}{d}\sigma \text{ as } n \to \infty. \tag{23}$$

Proof. Let X_i denote the ith node of the network so that by equation (16)

$$R(X_i) = \frac{1}{d}\left[\left(\frac{1}{n} - 1\right)\omega_i + \frac{1}{n}\sum_{j \neq i}\omega_j\right] \tag{24}$$

let E_k be the event that $|\omega_k|$ is the largest absolute value among $\{|\omega_i|, i = 1, .., n\}$. Then E_k occurs with probability $Pr(E_k) = 1/n$ and in this case the expected value of $|\omega_k|$ is at least $[\sqrt{2\log n} - (1/2)\log(4\pi\log n)/\sqrt{2\log n}) - O(1/\log n)]\sigma$ [9] (in fact this lower bound is the expression for the maximum value among $\{\omega_i, i = 1, .., n\}$). In this case the $\omega_i, i \neq j$ are symmetrically distributed about zero between $-\omega_k$ and ω_k and so the expected value of the sum $\sum_{i\neq k}^{n}\omega_i$ is zero in equation (24). Thus the conditional expectation $E[\max_i[|R(X_i)|]|E_k]$ has the lower bound

$$E[\max_i[|R(X_i)|]|E_k] \geqslant E[|R(X_k)|]|E_k] \tag{25}$$

$$\geqslant \frac{n-1}{\mathrm{d}n}\left[\sqrt{2\log n} - \frac{\frac{1}{2}\log(4\pi\log n)}{\sqrt{2\log n}} - O(\frac{1}{\log n})\right]\sigma. \tag{26}$$

Since by inequality equation (13) $k_c \geqslant \max_i |R(X_i)|$, and by using equation (26) we have

$$E(k_c) \geqslant E[\max_i |R(X_i)|], \tag{27}$$

$$= \sum_{k=1}^{n} Pr(E_k)E[\max_i[|R(X_i)|]|E_k], \tag{28}$$

$$\geqslant \sum_{k=1}^{n}\frac{1}{n}\frac{n-1}{\mathrm{d}n}\left[\sqrt{2\log n} - \frac{\frac{1}{2}\log(4\pi\log n)}{\sqrt{2\log n}} - O(\frac{1}{\log n})\right]\sigma, \tag{29}$$

$$= \frac{n-1}{\mathrm{d}n}\left[\sqrt{2\log n} - \frac{\frac{1}{2}\log(4\pi\log n)}{\sqrt{2\log n}} - O(\frac{1}{\log n})\right]\sigma, \tag{30}$$

which completes the proof. □

4. Expansion number and critical coupling for frequency synchronisation

The *edge expansion number* $h(G)$ (also called the Cheeger number and isoparametric number) of a network G (see [3, 8, 22]) is defined as

$$h(G) = \min_{|X| \leqslant \frac{n}{2}} \frac{\partial(X)}{|X|}, \tag{31}$$

and measures the size of the interface between any partitioning of the nodes into two subsets. This has well studied relationships with the eigenvalues of the graph Laplacian [31], which we define below. These parameters relate in turn to other topological features such as homogeneous degree, betweenness, large girth, large average shortest loops, and small diameter [12].

We illustrate how the expansion number relates directly to the frequency fixed point critical coupling through the inequalities of theorems 1 and 2. Firstly note that $1/h(G)$ can be expressed as

$$\frac{1}{h(G)} = \max_{|X| \leqslant \frac{n}{2}} \frac{|X|}{\partial(X)}. \tag{32}$$

Let G^* be a Kuramoto network with topology A and natural frequencies ω_i, and G be the corresponding graph defined only by A. Define $u(G^*)$ to be the bound from theorem 1

$$u(G^*) = \max_{|X| \leqslant \frac{n}{2}} \frac{1}{\partial(X)} |\sum_{i \in X} (\bar{\omega} - \omega_i)|. \tag{33}$$

Then $u(G^*)$ is a lower bound for the critical coupling k_c and can be thought of as a node weighted variation of the reciprocal of the edge expansion number $1/h(G)$ where a small cut $\partial(X)$ is sought combined with a sum of natural frequencies over X as far as possible from $\bar{\omega}|X|$. Similarly let $v(G)$ be the expression from theorem 2 defined by

$$v(G) = \max_{|X| \leqslant \frac{n}{2}} \frac{\sqrt{|X|(n - |X|)}}{\partial(X)}. \tag{34}$$

Then $v(G)$ is the network topological coefficient of the lower bound for the expected critical coupling $E(k_c)$. It can be thought of as a variation of the reciprocal of the edge expansion number where a small cut $\partial(X)$ is sought combined with $|X|$ as close to $n/2$ as possible, thereby maximizing $|X|(n - |X|)$.

At an intuitive level large $h(G)$ means that G is well connected across any node partitioning X,X^c. This in turn tends to produce ease of synchronisation through the many edges of each cut acting to 'balance' the natural frequencies in each part. This action is clear when we assume that the nodes are all in the same half circle of phase angles $[0, \pi]$ (note that critical coupling is defined as the presence of a frequency fixed point for any phase angles).

5. A construction: the expander augmented hierarchy

We now describe a network construction that combines efficient manageability when considered as an organizational structure, with low critical coupling when considered as a network of coupled oscillating nodes. In the following for a network with an even number of nodes, a *matching* is a set of edges that include all the nodes of the network and where no two edges have a common node. Thus matchings correspond to a non-overlapping pairing of the node set. As we shall show, by adding such a matching to the bottom layer of a binary tree the resulting network has a high expansion number and a low critical coupling. Define an infinite family $\mathcal{F}$ of networks as an *expander family* if there is some constant c so that $h(G) \geqslant c$ for all $G \in \mathcal{F}$.

Construction 1. Let p be a prime number. Then $G[p] = (V, E)$ has node set $V = \mathbb{Z}_p$ and edges that connect each node x with nodes: $x+1, x-1$, and the multiplicative inverse of x x^{-1} (where all arithmetic is mod p and we define $0^{-1} = 0$).

We note that this construction will produce three loop edges on the nodes $p-1, 0, 1$, and may also produce up to two double edges. However for our purposes these loop and double edges are unimportant for the construction and so loop edges can be omitted and only one copy of a double edge retained. Importantly the collection of networks $G[p], p$ prime is an expander family [50]. We can use construction 1 as the basis of constructing a cubic graph $R[m]$ for any even positive number m.

Construction 2. Let m be any positive even integer. Let p be the smallest prime greater than or equal to $m+3$. Consider the list of nodes $2, 3, .., p-2$ (dispensing with the nodes $p-1, 0, 1$ which produce loop edges). For $p-3 > m$ beginning with node $p-3$ delete node $p-3$ and its inverse $(p-3)^{-1}(\mathrm{mod}\ p)$. Repeat this for the next largest node on the list, etc until we have a list l of m numbers between 2 and $p-2$, say $v_1, v_2, .., v_m$. Then $R[m]$ is formed by a cycle connecting v_1 to v_2, v_2 to v_3,..,v_{m-1} to v_m, and v_m to v_1, and also connecting each v_i to $v_i^{-1}(\mathrm{mod}\ p)$.

We illustrate these constructions in figure 1 with the Graphs $G[13]$ and $R[14]$.

As $R[m]$ is formed from relatively few modifications to $G[p]$, where p-prime is of expected size at most $m+3+\log(m+3)$ (following from the prime number theorem [20]), it is expected that $R[m]$ will inherit the good expansion properties of $G[p]$. We use construction 2 as part of a further construction. Let H_r for r a positive integer be a cubic network on 2^r nodes formed by adding a matching to a cycle. Let the *expander augmented hierarchy* network F_r be constructed by adding the chord edges from H_r to the bottom row of a binary tree with depth $r-1$ as shown in figure 2, where the dotted lines indicate the cycle of H_r which is however omitted in F_r.

Theorem 4. *If $\{H_r\}$ is an expander family then $\{F_r\}$ is an expander family.*

This is our key result, and it is a little surprising in that only the chord edges (not on the cycle) from each H_r are needed to give F_r its expander properties. This can be explained in simple terms by the hierarchical structure of the binary tree serving a similar function to the cycle of H_r in its expansion properties.

Proof. We fix a particular integer r. For a subset of nodes X we shall use the terms $\partial(X, F_r)$ and $\partial(X, H_r)$ to distinguish between X and X^c where the complement graph is in relation to F_r and H_r respectively. Let $\{H_r\}$ be an expander family with expansion constant α so that $|\partial(S, H_r)|/|S| > \alpha$ for any subset S of the nodes $V(H_r)$ of H_r, $|S| \leqslant |H_r|/2 = 2^{r-1}$. Let $X = X_1 \cup X_2 \cup ..X_r$ where $a_i = |X_i|$ be any subset of $V(F_r)$ with at most $|F_r|/2$ nodes, and where X_i is the subset of nodes in X that are at a distance $i-1$ from the apex of the tree (see figure 2). We shall assume that $|X| \geqslant 6$ for otherwise it is easy to verify that $|\partial(X)|/|X| > 6/5$ for $|X| \leqslant 5$. In the following we shall consider a number of cases and show that in each case $|\partial(X)|/|X| >$min(1/3, α/24).

Observe that for each i there must be at least $2a_i - a_{i+1}$ edges from X_i that are not between edges of X and so must be in $\partial(X, F_r)$. Thus

$$\partial(X, F_r) \geqslant (2a_1 - a_2) + (2a_2 - a_3) + .. + (2a_{r-1} - a_r) = |X| - 2a_r. \qquad (35)$$

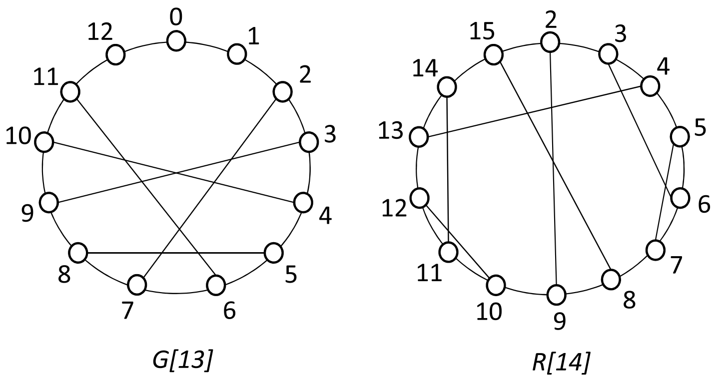

Figure 1. Illustration of the cubic graphs $G[13]$ and $R[14]$. Here nodes have a chord edge across the cycle to their multiplicative inverses mod 13 and mod 17, respectively.

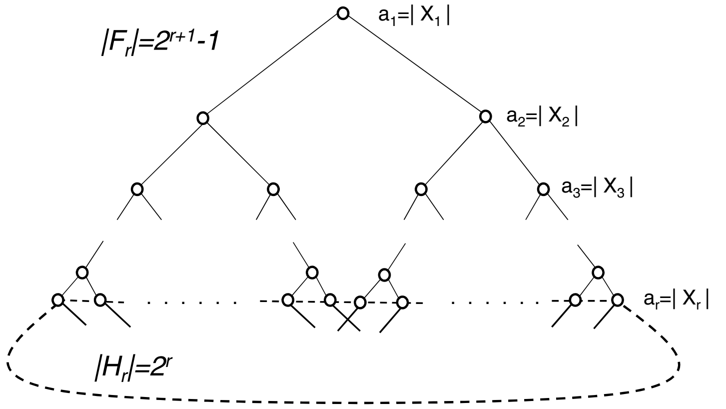

Figure 2. Illustration of the graphs F_r and H_r. F_r is formed by adding the chord edges from H_r to the bottom row of a binary tree.

Thus if $a_r < |X|/4$ then $|\partial(X, F_r)|/|X| > 1/2$. We shall assume therefore that $a_r \geqslant |X|/4$. Similarly

$$\partial(X, F_r) \geqslant (a_2 - 2a_1) + (a_3 - 2a_2) + .. + (a_r - 2a_{r-1}) = 2a_r - |X| - a_1. \tag{36}$$

Now $a_1 \leqslant 1$ and $|X| \geqslant 6$ so that $\partial(X, F_r) \geqslant |X|/3$ and $|\partial(X, F_r)|/|X| > 1/3$. We may therefore assume for the remainder of the proof that $3|X|/4 \geqslant a_r \geqslant |X|/4$. Let X_r consist of k runs $R_1, R_2, .., R_k$ of consecutive nodes around the cycle of H_r. Note that by our expansion assumption about H_r we have $|\partial(X_r, H_r)| \geqslant a_r\alpha$. We consider two cases:

Case 1: $|\partial(X_r, H_r)| \geqslant 3k$. Then $\partial(X_r, F_r)$ contains a number of chord edges not on the cycle of at least

$$|\partial(X_r, H_r)| - 2k \geqslant \frac{|\partial(X_r, H_r)|}{3} \geqslant \frac{\alpha a_r}{3} \geqslant \frac{\alpha|X|}{12}. \tag{37}$$

Thus $|\partial(X, F_r)|/|X| \geqslant |\partial(X, F_r)|/|X| \geqslant \alpha/12$.

Case 2: $|\partial(X_r, H_r)| < 3k$. Then

$$k \geqslant \frac{|\partial(X_r, H_r)|}{3} \geqslant \frac{\alpha a_r}{3} \geqslant \frac{\alpha|X|}{12}. \tag{38}$$

For each run $R_i, i = 1,..,k-1$ consider the unique path in the binary tree between the last node of R_i and the next node on the cycle. This path leads from a node of X_r to a node outside X_r and so must contain at least one edge between a node of X and X^c and so in $\partial(X, F_r)$. The same applies to the node directly before R_{i+1} and the first node of R_{i+1}. Now any edge of the binary tree can appear in at most two paths between consecutive nodes on the cycle. This is illustrated in figure 3 with the two paths in bold between nodes a and b, and c and d, respectively. These two paths share the common edge shown in double bold. It follows that there must be at least $k-1$ edges of the binary tree within $\partial(X, F_r)$ and so $|\partial(X, F_r)| \geqslant \alpha|X|/12 - 1$. Finally we consider two further cases. If $|X| \geqslant 24/\alpha$, $|\partial(X, F_r)|/|X| \geqslant \alpha/24$. If $|X| < 24/\alpha$ then since F_r is 2-connected it follows that $|\partial(X, F_r)| \geqslant 2$ and so $|\partial(X, F_r)|/|X| > \frac{2}{24/\alpha} = \alpha/12$. This completes the proof. □

6. Random leaf node matchings and Laplacian eigenvalues

In [4], it was proposed that for networks of coupled dynamical systems the ratio of largest to least (non-zero) eigenvalues of the graph Laplacian,

$$L_{ij} = D_{ij} - A_{ij} \tag{39}$$

is a useful graph algebraic measure that correlates well with synchronisability; here D_{ij} is the diagonal matrix of degrees. This applies also to the Kuramoto system insofar as its linearised form reduces to the generic model studied there [25]. The eigenvalue ratio in question is

$$Q \equiv \frac{\lambda_n}{\lambda_2} \tag{40}$$

for eigenvalues λ_r to L_{ij}, where we index $r = 1, \ldots, n$ noting that the lowest eigenvalue is always zero, and for a connected graph $\lambda_2 > \lambda_1 = 0$. Note that, more generally, the degeneracy of the zero eigenvalue gives the number of components to which a disconnected graph decomposes. That there should be some correlation between Q and the expansion properties of a graph arises through inequality bounds of the edge expansion number and the lowest non-zero Laplacian eigenvalue [36]. The proposition of [4] is that graphs with smaller Q synchronise better.

To compare our expander augmented hierarchy we consider random matchings of leaf nodes in a binary tree classified according to their value of Q, where we apply the same number of edges between leaf nodes of the tree as there are in the expander form of the previous section. For the remainder of this paper we consider a *seven level hierarchy* built on a binary tree. This means for the pure tree structure we have $V = 127$ nodes and $E = 126$ edges. The modified trees with random matchings at the leaf nodes all have $E = 158$ edges which is the same number of edges for the expander-modified trees derived earlier, here denoted F_6 for a seven level hierarchy.

We perform a massive search to find examples of random matched trees with the lowest possible ratio Q; in the end we searched over 7×10^7 modified trees. The reason for the extensive number is because of the distribution of values of Q for such graphs given in figure 4. Evidently it is easy to find cases with $Q \approx 70$. In fact, labelling the graph cases by the random seed number the best case we find out of the 70 million has $Q = 40.511$ and occurs at position

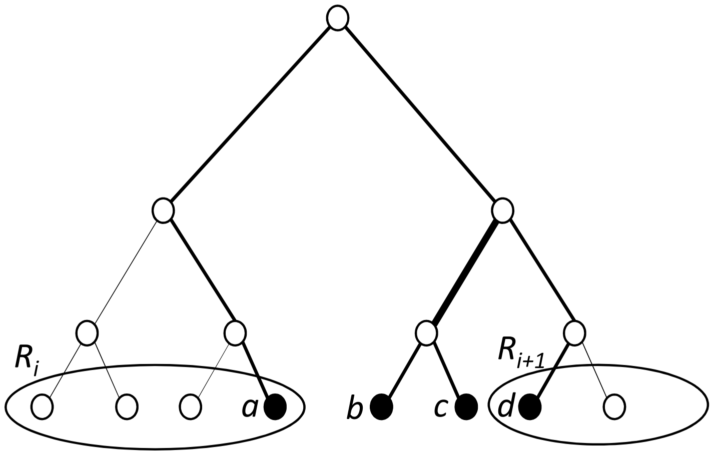

Figure 3. Any edge of the binary tree can appear in at most two paths between consecutive nodes on the cycle. This figure shows paths (in bold) between pairs of consecutive nodes, ab and cd, respectively, with a common edge shown in double bold.

40,526,834 in the search. We have provided *.csv* files for all the graphs considered in the following at the site https://github.com/khoek/graph-paper.

7. Order parameter results for modified seven level hierarchies

For these graphs we now present calculations of the time-averaged order parameter further averaged over an ensemble of 30 sets of natural frequencies drawn from a normal distribution with mean 2 and standard deviation 1/2, namely $\mathcal{N}(2, \frac{1}{2})$. We numerically solve the Kuramoto system up to time $T = 2000$ for discrete values of coupling k for F_6 and the other matchings referred to above. We use the NSolve package in Mathematica using its StiffnessSwitching method which dynamically determines the number of points in discrete time at which to solve the evolution. As indicated earlier, we exclude the transient, here the first 200 time-units. We thus compute $\langle r \rangle_T$ as a function of coupling for a range of graphs, with results shown in figure 5. Here we consider not just the expander-modified tree F_6 and lowest Q random matched trees, but other random matchings with a range of values of Q across the distribution in figure 4. Moreover, we also compare with the intuitively easiest sparse graph to synchronise, the star, whose only disadvantage for 126 nodes is the burden it places on the centre node. In figure 5 we show the full dependence of the order parameter across k values from $k = 0$ to high levels of synchronisation. We separately plot curves for the mean over the ensemble of frequency choices (upper plot), and bands around the means indicating the variance across the 30 instances (lower plot). (For 30 instances this is indistinguishable from the standard error.) We indicate also the value of the graph diameter D for each graph, defined as the greatest distance in numbers of hops between any pair of vertices. We note that for all the random matching cases the diameter is the same, $D = 11$ while the expander hierarchy has $D = 12$. In the same plots of figure 5 we indicate through vertical lines the value of coupling at which a frequency fixed point occurs, namely at which $|\dot{\theta}_i - \dot{\theta}_j| = 0\ \forall\ i, j$. We determine this here by computing for each coupling value $\sum_{j\neq 1} |\dot{\theta}_1(T) - \dot{\theta}_j(T)|$, with T the final time-step at which we solve the system, and extract the coupling at which this is of order 10^{-6}. In the upper plot of figure 5 we show the mean over a frequency ensemble and in the lower plot provide a band based on

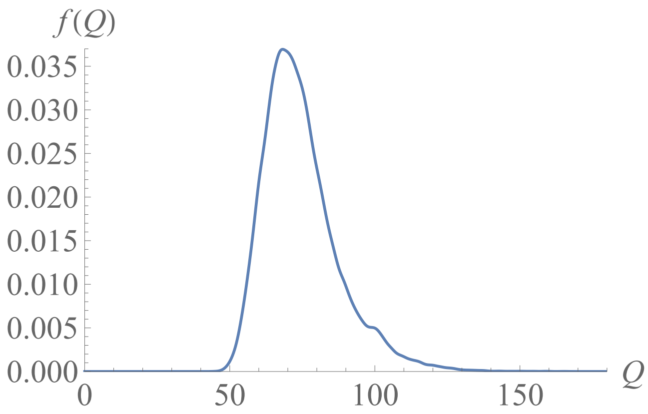

Figure 4. Probability density function f of eigenvalue ratio Q for random leaf matchings of seven level binary trees based on a search over 7×10^7 instances where the degree distribution is fixed to match the expander augmented tree F_6.

the variance. Arrows superimposed on the plot indicate the trend for increasing Q, vertically for the order parameter and horizontally for the critical coupling for a frequency fixed point.

We observe a range of phenomena in these plots. Firstly, our intuition is confirmed, and the star graph does exceptionally well with a rapid, if nearly linear rise in $\langle r \rangle_T$ flattening out to a maximum value at $k \approx 1.7$. Below this, the best performing graphs are those with $Q \in [40, 50]$, below them those with $Q \approx 70$, and finally those with high Q below them, and increasingly spread out. This confirms that indeed Q is a good predictor of synchronisation performance for all the graph types formed by adding a matching to the leaf nodes, while the star is exceptional in that it has very high value of Q and yet has the fewest possible number of edges for any connected graph.

Most importantly, the expander augmented hierarchy F_6, with 'sub-optimal' $Q = 49.65$ and higher diameter D, performs almost as well as the random matchings with lower Q and D. Specifically, in the upper plot of figure 5 the mean result for $\langle r \rangle_T$ for F_6 is barely distinguishable from those with $Q = 40.55$ and $Q = 41.60$; only at intermediate coupling does the curve for the random matching with $Q = 41.60$ lie higher than the other two. When the variance over the natural frequency ensemble is taken into account (lower plot) the three results are almost indistinguishable. In other words, eigenvalue ratio Q and graph diameter D are somewhat 'blunt' measures of performance for synchronisation for graphs with the same degree distribution after averaging over an ensemble of frequencies. We emphasise here that the proposition that Q is a good measure arises from a different form of coupled dynamical system [4] where explicit Lyapunov instabilities may appear and natural frequencies are not part of the formulation. For the Kuramoto model, Q is only an heuristic, and figure 5 shows that degree distributions being equal, it is a good comparator of synchronisability for well-separated Q values. This is emphasised by the vertical arrow superimposed on the plot. In [25], one of us has noted that linear stability analysis of the Kuramoto model does not deliver a sharp instability criterion for phase synchronisation, and that a more appropriate measure of phase synchronisability is the ratio $\omega^{(r)}/\sigma\lambda_r$ for non-zero modes $r \neq 0$. Here, $\omega^{(r)}$ is the projection on the rth Laplacian eigenvector of the natural frequencies. Averaging over an ensemble of frequencies further dulls this criterion.

This leads us to the critical coupling for a frequency fixed point for which we derived bounds whose form suggested the expander approach. In figure 5, we see that the three graphs

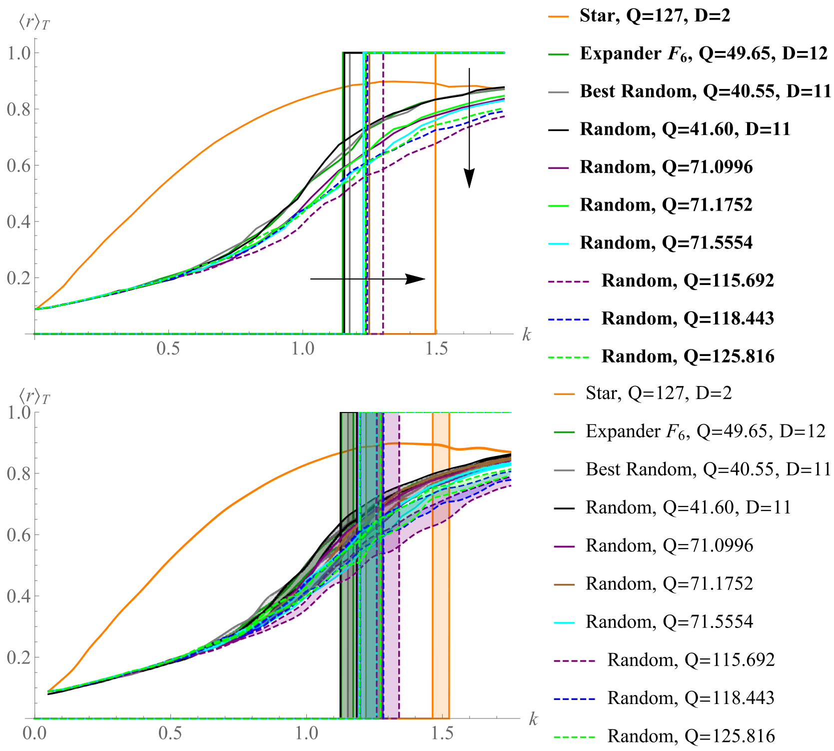

Figure 5. Average order parameter $\langle r \rangle_T$ as a function of coupling k for a range of graphs: top the mean values and bottom bands indicating $\pm$ the variance across 30 frequency instances. The Laplacian eigenvalue ratio Q and graph diameter D for each graph are also indicated. Vertical lines indicate the mean coupling at which a frequency fixed point occurs (upper plot) with bands indicating $\pm$ the variance over the frequency ensemble (lower plot). The vertical arrow indicates the trend of decreasing $\langle r \rangle_T$ with increasing Q (excepting the star), and the horizontal arrow indicates the analogous trend for the frequency fixed point.

with lowest critical coupling are again the expander tree F_6 and the two lowest Q graphs. Looking at the means over natural frequencies, F_6 and the $Q = 41.60$ graph almost coincide. Again, the graph with the lowest Q has slightly higher critical coupling. Taking the variances into account these three cases are again almost indistinguishable. Beyond that, critical couplings increase as Q increases. The star, despite its high order parameter, has quite high critical coupling for a frequency fixed point; we discuss this case further below.

One point of nuance is at which point in the curves of figure 5 is of greatest importance for the quality of synchronisation? One choice is the point of maximum $\langle r \rangle_T$, namely the coupling k for which the value unity is achieved. Another choice is the value of k at which an inflection point in the curve is reached. A third is the frequency fixed point itself, which lies between the first two possible choices. Clearly, at the maximum point in $\langle r \rangle_T$ all of the curves for $Q \in [40, 50]$ have converged in the upper plot of figure 5 and show little variance, as seen in the lower plot of figure 5. At the inflection point, the average values of these same curves show some deviation, at $k \approx 0.9$. However, all begin to exhibit variance in this regime, so that within 'errors' around the natural frequency values they are also consistent. The frequency fixed point lies at the point of maximum variance. This is intuitively clear, as different layouts of frequencies on the same graph may require different couplings to achieve frequency synchronisation. Nevertheless, within the variance the expander augmented hierarchy coincides with the lowest Q value random-matchings of trees, Namely, the expander augmented hierarchy exhibits favourable synchronisation at lowest possible coupling.

For the star graph we can make a number of exact statements. The graph synchronizes due to its excellent expansion properties, having an expansion number of exactly 1. On the other hand it is known that large 3-regular graphs must have an expansion number bounded above by a number arbitrarily close to $\frac{1}{6}$ [34], and these graphs have even more edges than the graphs shown in figure 5, including F_6. We recall that a close relationship between the Laplacian eigenvalue ratio Q and the expansion number holds for regular graphs [22]. In particular, low Q corresponds broadly with high expansion number. However this seems less so for non-regular graphs with the star graph having a poor Q value (127), much higher than that for the best modified hierarchy graphs shown in figure 4, but a good expansion number, at least 6 times as large. All of this emphasises that if degree homogeneity is relaxed any number of graphs with superior synchronisation properties are possible. A star network represents an extreme case of impractical organisational structure with one individual required to maintain relationships with all others.

We conclude that the expander augmented hierarchy, as a *designed* structure is *superior* to even an optimal (in Q) random matching of leaf nodes of the hierarchy with equivalent degree distribution. Fundamentally this is because of the poor scaling properties of the search space for matchings of lowest possible Q with increasing numbers of levels of the hierarchy.

8. Conclusions

We have developed a construction for augmented binary trees that provably constitutes an expander family. Computing the average Kuramoto order parameter as a measure of phase synchronisation for the dynamical evolution of Kuramoto phase oscillators we have shown that this structure has significantly enhanced synchronisation properties. In particular, it performs as well as a random matching of leaf nodes with the same number of edges even when the latter has lower Laplacian eigenvalue ratio Q. For the latter structure, such low values of Q occur in the leading tail of the distribution of random graphs and thus require large search times to find, with those times growing as the number of levels of the binary tree increase. In this respect, the expander-hierarchy is a superior structure from the point of view of scalability, synchronisation and load balance across all nodes. In fact, our sense is that the expansion constant is in practice much better than the proof would suggest. In other words the proof shows that the construction is scalable—but it also seems to have a good expansion constant as the excellent synchronisation properties would suggest. However this requires more sophisticated proof methods than we have achieved thus far.

Though we have chosen the simplest case for a hierarchy in the binary tree, the result can be generalised for complete K-ary trees. These results are useful for any context where trees are naturally occuring structures though we have focused on human organisations as an example. Clearly the application we intend for such a construction is to create targeted edges at the ‘bottom’ of a human organisation, such as an administrative, business or military structure, that enhances its cohesiveness as a decision-making unit. For example, such edges can be established at induction of new members into the organisation. The challenge is to extend the expander construction to the messy nature of real organisations. Even when such organisations are pure hierarchies, they typically are not complete in the sense of every layer being occupied, or with non-uniform degrees across the levels and branches of the structure.

We anticipate ways forward on this challenge. Clearly, the question of validation of such an idealised model as the Kuramoto for human organisations is important. On this point, firstly there are more elements in this context than have been included in the present model, such as stochasticity [27]; future work will test whether the modified expander hierarchy is also robust against noise. However, there is evidence of a strong anti-correlation between the synchronisation on networks under increasing coupling with increasing noise 'strength' for both uniform [28], Gaussian and Lévy noise [44], which we anticipate would apply here. Secondly, there is the added layer of a mathematical representation of the business of the organisation—what it is it makes decisions *about*—in which we are conducting ongoing work. Above all, the important test of whether a design principle, such as we have developed here, is valid for implementation in a real organisational setting is whether it is robust against these variations in the model, and cross-validation [5, 46] against other approaches to modelling similar phenomena. This paper is then a first step in developing new paradigms for organisational design with a deep graph theoretic principle construct at its heart.

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