# Analyticity of the free energy for quantum Airy structures

## Błażej Ruba

Institute of Physics, Jagiellonian University, Prof. Łojasiewicza 11, 30-348 Kraków, Poland

E-mail: blazej.ruba@doctoral.uj.edu.pl

## Abstract

It is shown that the free energy associated to a finite-dimensional Airy structure is an analytic function at each finite order of the $\hbar$-expansion. Its terms are interpreted as objects living on the zero locus of the classical hamiltonians. The geometry of this variety is studied. The structure of singularities of the free energy is described. To this end topological recursion equations are expressed in a form particularly suitable for semiclassical analysis. It involves a differential operator which is a deformation of the exterior derivative. Its local properties are derived. The developed formalism is applied in several examples. Global properties of the obtained partition functions are investigated. In the case of a divergent $\hbar$-series, a simple resummation is performed.

Keywords: topological recursion, Airy structures, semiclassical approximation, quantization, partition function, WKB

## 1. Introduction

Airy structures were introduced in [1] as a reformulation and generalization of the topological recursion [2] for spectral curves. They encode data necessary to formulate certain recursive equations encountered in matrix models [3–5], conformal field theory [6], enumerative geometry [7, 8], and other applications (see [9, 10] for pedagogical reviews) as differential equations. This approach makes manifest symmetry properties of the solutions and strong symplectic flavour of the problem. Further study of Airy structures was undertaken in [11–13].

Recall that an Airy structure is a Lie algebra of differential operators

$$L_i = \hbar \partial_i - \frac{1}{2} A_{ijk} x^j x^k - \hbar B_{ij}^k x^j \partial_k - \frac{\hbar^2}{2} C_i^{jk} \partial_j \partial_k - \hbar D_i, \tag{1.1a}$$

$$[L_i, L_j] = \hbar f_{ij}^k L_k, \tag{1.1b}$$

where indices $i, j, k \in \{1, \dots, n\}$. Repeated indices are summed over in all equations. In general infinite index sets are allowed, but this case will not be considered here. Abbreviation $x = (x^1, \dots, x^n)$, $y = (y_1, \dots, y_n)$ shall often be used. Symbols $A, B, C, D$ and $f$ are tensors obeying a number of relations. Firstly, $A$ is completely symmetric, while $C$ is symmetric in its upper indices. Structure constants are given by the skew-symmetric part of $B$, $f_{ij}^k = B_{ij}^k - B_{ji}^k$. Coefficients $D_i$ are of the form $D_i = \frac{1}{2}B_{ij}^j + \delta_i$, where $\delta_i$ satisfies $f_{ij}^k \delta_k = 0$. Additionally there is a number of complicated quadratic equations, of no use here[1]. All calculations are performed over the field $k = \mathbb{R}$ or $\mathbb{C}$.

To any Airy structure one may associate the set of classical hamiltonians

$$L_i^{\mathrm{cl}}(x, y) = y_i - \frac{1}{2}A_{ijk}x^j x^k - B_{ij}^k x^j y_k - \frac{1}{2}C_i^{jk} y_j y_k. \tag{1.2}$$

They satisfy relations $\{L_i^{\mathrm{cl}}, L_j^{\mathrm{cl}}\} = f_{ij}^k L_k^{\mathrm{cl}}$ with respect to the bracket determined by[2] the symplectic form $\omega = \mathrm{d}x^i \wedge \mathrm{d}y_i$. Denote the Hamiltonian vector field corresponding to $L_i^{\mathrm{cl}}$ by $\xi_i$. By construction, $[\xi_i, \xi_j] = f_{ij}^k \xi_k$.

Let $G$ be a simply-connected Lie group with Lie algebra $\mathfrak{g} = \mathrm{lin}\{L_i\}_{i=1}^n$. Since $L_i^{\mathrm{cl}}$ are at most quadratic, $\xi_i$ may be exponentiated to an affine action of $G$ on $k^{2n}$, which preserves the symplectic form.

It was shown in [1] that to every quantum Airy structure one may associate a free energy—the unique formal series $F$ in $\hbar$ and $x$ such that

$$L_i \cdot \exp\left(\hbar^{-1}F(\hbar, x)\right) = 0, \tag{1.3a}$$

$$F(\hbar, 0) = \partial_i F(0, 0) = \partial_i \partial_j F(0, 0) = 0. \tag{1.3b}$$

Due to the presence of $\hbar^{-1}$ in the exponent, it is not immediately obvious if the partition function $Z = \mathrm{e}^{\hbar^{-1}F}$ may be made sense of as a formal series. However, due to conditions (1.3b), one may obtain a series involving only positive powers of indeterminates by changing variables to $x' = \hbar^{-\frac{1}{2}}x$. Therefore operations defined for formal power series, such as exponentiation or inversion, can be applied without encountering meaningless expressions. In [1] a slightly different approach was proposed: the map $L_i \mapsto Z^{-1}L_i Z$ was regarded as an automorphism of the completed Weyl algebra, while equation (1.3a) was reinterpreted as $(Z^{-1}L_i Z) \cdot 1 = 0$. One may show that $(Z^{-1}L_i Z) \cdot 1 = Z^{-1}(L_i Z)$, so the two interpretations are equivalent.

By construction, the free energy may be expanded in powers of $\hbar$,

$$F(\hbar, x) = \sum_{g=0}^{\infty} \hbar^g F_g(x), \tag{1.4}$$

with $F_g \in k[[x]]$. The main result of the present work is that there exists a neighbourhood $U$ of $0 \in k^n$ such that the series $F_g(x)$ converges for each $x \in U$ and $g \in \mathbb{N}$. The series (1.4) itself diverges in general, which will be demonstrated by an explicit example. Employed techniques are standard in the approach to WKB-type approximations through symplectic geometry [14, chapters 1–3]. From this point of view $\mathrm{e}^{\hbar^{-1}F}$ behaves like a wave function, rather than a partition function.

In the process of proving the claim, the partition function is rewritten as

---

[1] Their detailed form consistent with this work can be found in [13].
[2] See the appendix A for the relevant definitions.

$$
Z = \mathrm{e}^{\hbar^{-1}F_0(x)}\rho_0(x)\underbrace{\left(1 + \sum_{m=1}^{\infty}\hbar^m\psi_m(x)\right)}_{\psi(\hbar,x)}.
\tag{1.5}
$$

The objects $\rho_0$ and $\psi$ are related to the more familiar $F_g$ by

$$
F_1(x) = \log\rho_0(x),
\tag{1.6a}
$$

$$
F_g(x) = -\sum_{k=1}^{g-1}\sum_{m_1,\ldots,m_k=1}^{g-1}\frac{(-1)^k}{k}\delta_{m_1+\ldots+m_k}^{g-1}\prod_{j=1}^{k}\psi_{m_j}(x) \quad \text{for } g \geqslant 2,
\tag{1.6b}
$$

$$
\psi_m(x) = \sum_{j_2,\ldots,j_{m+1}=0}^{m}\delta_m^{j_2+2j_3+\ldots+mj_{m+1}}\prod_{g=2}^{m+1}\frac{F_g(x)^{j_g}}{j_g!} \quad \text{for } m \geqslant 1.
\tag{1.6c}
$$

Terms of the expansion (1.5) are interpreted as local expressions for geometric objects on a Lagrangian submanifold $\Sigma_0 \subseteq k^{2n}$. In a neighbourhood of zero $\Sigma_0$ coincides with the graph $y = \mathrm{d}F_0(x)$. Globally, $\Sigma_0$ admits the Lie group $G$ as a universal cover. It is shown that $F_0$ extends to a holomorphic function on $G$. The canonical bundle of $G$ admits a natural square root[3] $K^{\frac{1}{2}}$, and expression $\rho_0(x)\sqrt{\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n}$ extends to a holomorphic section $\eta$ of $K^{\frac{1}{2}}$. This is in accord with the semiclassical interpretation [14, chapter 4] of quantum states as half-forms on Lagrangian submanifolds. Series $\psi_m(x)$ extend to (possibly multi-valued) holomorphic functions on the complement of the ramification locus of the projection $\pi : \Sigma_0 \ni (x,y) \mapsto x \in k^n$. They satisfy equations

$$
\mathrm{d}\psi_m = \Delta\psi_{m-1},
\tag{1.7}
$$

where $\Delta$ is a second order differential operator of cohomological degree 1, meromorphic on $\Sigma_0$ and satisfying equations

$$
\Delta^2 = 0,
\tag{1.8a}
$$

$$
\mathrm{d}\Delta + \Delta\mathrm{d} = 0.
\tag{1.8b}
$$

As seen from the presented proof, these are the only properties of $\Delta$ needed to demonstrate the existence of $\psi$.

As a consequence of associativity, any differential operator of the form $[d,\kappa]$ anticommutes with $d$. It is natural to ask if $\Delta$ may be written in such a form. It is proven in section 3 that the answer is affirmative, at least locally. Furthermore it is shown that the operator $d_\Delta = d - \hbar\Delta$ obeys the Poincaré lemma.

In section 4 developed formalism is applied in several examples. They illustrate that the system (1.7) provides an efficient tool for solving for $Z$, provided that the closed form of $F_0$ and $\rho_0$ may be found first. This may be achieved either by solving a system of first order, but nonlinear partial differential equations or by solving a system of polynomial equations and computing integrals. Global properties of $F_0$, $\eta$ and $\psi$ are investigated in each example. Functions $\psi_m$ are found to be meromorphic on $\Sigma_0$. It is an interesting question if this is always true. In one case, the partition function turns out to be a divergent series in $\hbar$. Its resummation is discussed. In each example globally meromorphic operator $\kappa$ is found.

---

[3] See the appendix B.

## 2. Analytic structure of the free energy

It is sufficient to consider the case $k = \mathbb{C}$. Analyticity of $F$ in the real case then follows, because every differential operator of the form (1.1$a$) with real coefficients may as well be considered for complex $x$. Since the associated $F_g$ are analytic, so are their restrictions to real $x$.

### 2.1. Characteristic variety

Consider the affine variety $\Sigma = \{(x, y) \in \mathbb{C}^{2n} | L_i^{\mathrm{cl}}(x, y) = 0, \ i = 1, \ldots, n\}$ and its Zariski open subset $\Sigma_s = \{p \in \Sigma | \ dL_1^{\mathrm{cl}} \wedge \ldots \wedge dL_n^{\mathrm{cl}}|_p \neq 0\}$. $\Sigma_s$ is nonsingular, so its irreducible components are disjoint. Since the irreducible components are connected [15, chapter 7], they coincide with the connected components. Each component of $\Sigma_s$ is a complex manifold of dimension $n$.

Bracket relations satisfied by the hamiltonians $L_i^{\mathrm{cl}}$ imply that $\Sigma_s$ is Lagrangian and invariant under the $G$-action. Thus for every $p \in \Sigma_s$ the set $\{\xi_i|_p\}_{i=1}^n$ is a basis of the tangent space $T_p\Sigma_s$. It follows that orbits of the $G$-action on $\Sigma_s$ are open in $\Sigma_s$. In particular they coincide with the components of $\Sigma_s$.

Let $\Sigma_0$ be the $G$-orbit of 0. Mapping $Q : G \ni g \mapsto g(0) \in \Sigma_0$ is a universal cover. Its fiber $\Gamma = Q^{-1}(0)$ is a discrete subgroup of $G$, which may be identified [16, chapter 1] with $\pi_1(\Sigma_0, 0)$, the fundamental group of $\Sigma_0$ based at 0.

By the implicit function theorem, $x^i$ may be used as local coordinates on some neighbourhood of zero in $\Sigma_0$. In other words, for sufficiently small $x$ equations $L_i^{\mathrm{cl}}(x, y) = 0$ may be solved to express $y_i$ as analytic functions[4] of $x$. The same can be done in a neighbourhood of any $p \in \Sigma_0$ at which $\mathcal{D} = \det\left(\frac{\partial L_i}{\partial y_j}\right)_{i,j=1}^n$ is nonzero. Set $\mathrm{Ram} = \{p \in \Sigma_0 | \mathcal{D}(p) = 0\}$ is called the ramification locus. It coincides with the set of all $p \in \Sigma_0$ such that the differential $d\pi|_p$ is not onto. $\mathrm{Ram}$ is the zero locus of a nonzero polynomial on $\Sigma_0$, so it is either empty or a codimension one subvariety of $\Sigma_0$. Moreover $\Sigma_0 \setminus \mathrm{Ram}$ is connected.

For future reference, note that the vector fields $\xi_i$ restricted to $\Sigma_0$ and expressed in local coordinates take the form

$$\xi_i = \{L_i, x^j\}\frac{\partial}{\partial x^j} = \frac{\partial L_i}{\partial y_j}\frac{\partial}{\partial x^j} = \left(\delta_i^k - B_{ij}^k x^j - C_i^{jk} y_j\right)\frac{\partial}{\partial x^k}. \tag{2.1}$$

This relation may be inverted using the Cramer's rule, yielding

$$\frac{\partial}{\partial x^i} = \mathcal{D}^{-1}M_i^j\xi_j, \tag{2.2}$$

where $M_i^j$ are the restrictions to $\Sigma_s$ of certain polynomials in $x, y$. Since $\xi_i$ are globally holomorphic, this formula shows that each $\mathcal{D}\frac{\partial}{\partial x^i}$ extends to a holomorphic vector field on $\Sigma_0$. In particular $\frac{\partial}{\partial x^i}$ is a meromorphic vector field on $\Sigma_0$.

It is useful to introduce the divergence of $\xi_i$:

$$\mathcal{L}(\xi_i)(dx^1 \wedge \ldots \wedge dx^n) = \mathrm{div}(\xi_i) \cdot dx^1 \wedge \ldots \wedge dx^n, \tag{2.3}$$

where $\mathcal{L}(\xi_i)$ is the Lie derivative with respect to $\xi_i$. Iterating this equation one gets

---

[4] Typically for a given $x$ there are several $y$ such that $L_i^{\mathrm{cl}}(x, y) = 0$. However there is a unique such $y$ depending continuously on $x$ and subject to the initial condition $y(0) = 0$.

$$[\mathcal{L}(\xi_i), \mathcal{L}(\xi_j)](\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n) = (\xi_i(\mathrm{div}(\xi_j)) - \xi_j(\mathrm{div}(\xi_i)))\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n.$$
(2.4)

On the other hand $[\mathcal{L}(\xi_i), \mathcal{L}(\xi_j)] = f_{ij}^k \mathcal{L}(\xi_k)$, so comparison of (2.3) with (2.4) gives

$$\xi_i(\mathrm{div}(\xi_j)) - \xi_j(\mathrm{div}(\xi_i)) = f_{ij}^k \cdot \mathrm{div}(\xi_k).$$
(2.5)

Expression for $\mathrm{div}(\xi_i)$ in local coordinates, derived from (2.1), takes the form:

$$\mathrm{div}(\xi_i) = -B_{ij}^j - C_i^{jk} \frac{\partial y_j}{\partial x^k}.$$
(2.6)

A convenient framing of the holomorphic cotangent bundle of $\Sigma_0$ is provided by the holomorphic 1-forms defined by $\sigma^i(\xi_j) = \delta_j^i$. They satisfy

$$\mathcal{L}(\xi_i)\sigma^j = -f_{ik}^j \cdot \sigma^k, \qquad \mathrm{d}\sigma^i = -\frac{1}{2}f_{jk}^i \sigma^j \wedge \sigma^k,$$
(2.7)

as may be evaluated, say, using the Leibniz rule for Lie derivatives and the Cartan's magical formula[5]. Using this result (2.5) may be rephrased as $d\left(\mathrm{div}(\xi_i)\mathrm{d}\sigma^i\right) = 0$.

The wedge product $\epsilon = \sigma^1 \wedge \ldots \wedge \sigma^n$ is a nowhere-vanishing section of the canonical bundle $K$ of $\Sigma_0$. By (2.7), it satisfies

$$\mathcal{L}(\xi_i)\epsilon = -f_{ij}^j \cdot \epsilon.$$
(2.8)

Since $K$ is a line bundle, sections $\epsilon$ and $\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ have to be proportional. Relative factor may be computed by contracting with $\xi_1, \ldots, \xi_n$ and using (2.1),

$$\epsilon = \mathcal{D}^{-1} \cdot \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n.$$
(2.9)

Taking the Lie derivative of this equation with respect to $\xi_i$ gives

$$\mathcal{L}(\xi_i)\epsilon = \left(-\mathcal{D}^{-1}\xi_i(\mathcal{D}) + \mathrm{div}(\xi_i)\right)\epsilon.$$
(2.10)

Comparison with (2.8) leads to the result

$$\mathcal{D}^{-1}\xi_i(\mathcal{D}) = \mathrm{div}(\xi_i) + f_{ij}^j.$$
(2.11)

This formula shows that each $\mathcal{D} \cdot \mathrm{div}(\xi_i)$ extends to a holomorphic function on $\Sigma_0$. Moreover it provides an explicit expression for the derivatives of $\mathcal{D}$.

Since the canonical bundle of $\Sigma_0$ is equipped with a distinguished framing, there exists a natural choice of a square root of $K$, with the symbol $\sqrt{\epsilon}$ playing the role of a global trivialization.

### 2.2. Hamilton–Jacobi equation

Let $\theta = y_i \mathrm{d}x^i$. Since $\mathrm{d}\theta = -\omega$ and $\Sigma_0$ is Lagrangian, the restriction of $\theta$ to $\Sigma_0$ is a closed form. For any $p \in \Sigma_0$ let

$$F_0(p) = \int_0^p \theta,$$
(2.12)

with the integral taken over any smooth path in $\Sigma_0$ from 0 to $p$. By the Stokes' theorem, its value is unchanged by continuous deformations of the path. Nevertheless, there is still an ambiguity in the definition of $F_0$ due to the presence of non-contractible loops in $\Sigma_0$. In other

---

[5] $\mathcal{L}(\xi) = \mathrm{d}\iota(\xi) + \iota(\xi)d$, where $\iota(\xi)$ is the interior product with $\xi$.

words, $F_0$ is multi-valued. It is best regarded as a holomorphic function on $G$, the universal cover of $\Sigma_0$.

Notice that certain function was denoted by the same symbol as the formal series appearing in (1.4). This will be justified by showing that the two objects agree when the global version is expressed in local coordinates.

Now restrict attention to a polydisc

$$U = \{(x, y) \in \Sigma_0 | \, |x^i| < r, \, i = 1, \ldots, n\} \tag{2.13}$$

with some $r > 0$ sufficiently small so that $\pi|_U$ is an isomorphism onto its image. By construction, $x^i$ furnish a coordinate system on $U$. $F_0$ is single-valued on $U$, so it may be expressed as a convergent power series.

Definition of $F_0$ implies that $F_0(0) = 0$. Since the differential form $\theta|_{\Sigma_0}$ vanishes at zero, also the differential $\mathrm{d}F_0$ vanishes at zero. Additionally one has

$$\frac{\partial F_0}{\partial x^i} \mathrm{d}x^i = \mathrm{d}F_0 = \theta|_{\Sigma_0} = y_i \mathrm{d}x^i \big|_{\Sigma_0}, \tag{2.14}$$

which shows that $U \cap \Sigma_0$ coincides with the graph

$$y_i = \frac{\partial F_0(x)}{\partial x^i}. \tag{2.15}$$

Since $L_i^{\mathrm{cl}}\big|_{\Sigma_0} = 0$, this entails that $F_0$ satisfies the Hamilton–Jacobi equation

$$L_i^{\mathrm{cl}}(x, \mathrm{d}F_0) = \partial_i F_0 - \frac{1}{2} A_{ijk} x^j x^k - B_{ij}^k x^j \partial_k F_0 - \frac{1}{2} C_i^{jk} \partial_j F_0 \partial_k F_0 = 0. \tag{2.16}$$

Evaluating the derivative of this equation at zero one gets that the second derivatives of $F_0$ vanish at zero. Hence $F_0$ satisfies the same differential equation and the same initial condition as the unique formal series $F_0(x)$ in (1.4). Thus the two objects are equal. In particular $F_0(x)$ in (1.4) is a convergent series on $U$.

## 2.3. Equations for the partition function

Equations for higher order terms in the partition function will now be derived. First observe that the Hamilton–Jacobi equation (2.16) implies that

$$\mathrm{e}^{-\hbar^{-1} F_0(x)} L_i \mathrm{e}^{\hbar^{-1} F_0(x)} = \hbar \xi_i - \frac{\hbar}{2} C_i^{jk} (\partial_j \partial_k F_0) - \frac{\hbar^2}{2} C_i^{jk} \partial_j \partial_k - \hbar D_i, \tag{2.17}$$

where (2.1) and (2.15) were used to collect several terms into a single expression. Further simplification may be achieved using (2.6), once again combined with the fundamental formula (2.15). The final result takes the form

$$\mathrm{e}^{-\hbar^{-1} F_0(x)} L_i \mathrm{e}^{\hbar^{-1} F_0(x)} = \hbar \left( \xi_i + \frac{1}{2} \mathrm{div}(\xi_i) - \delta_i \right) - \frac{\hbar^2}{2} C_i^{jk} \partial_j \partial_k. \tag{2.18}$$

This means that partition function written in the form

$$Z(\hbar, x) = \mathrm{e}^{\hbar^{-1} F_0(x)} \sum_{m=0}^{\infty} \hbar^m \rho_m(x) \tag{2.19}$$

is annihilated by each $L_i$ if and only if the functions $\{\rho_m\}_{m=0}^{\infty}$ satisfy

$$\left(\xi_i + \frac{1}{2}\mathrm{div}(\xi_i) - \delta_i\right)\rho_0 = 0, \tag{2.20a}$$

$$\left(\xi_i + \frac{1}{2}\mathrm{div}(\xi_i) - \delta_i\right)\rho_m = \frac{1}{2}C_i^{jk}\partial_j\partial_k\rho_{m-1}, \quad \text{for } m \geqslant 1. \tag{2.20b}$$

Equation (2.20a) may be used to eliminate $\mathrm{div}(\xi_i)$ from (2.20b), so that

$$\xi_i(\psi_m) = \Delta_i\psi_{m-1}, \quad \text{for } m \geqslant 1, \tag{2.21}$$

where $\psi_m = \frac{\rho_m}{\rho_0}$. Operators $\Delta_i$ are defined by

$$\Delta_i = \frac{1}{2}C_i^{jk}\rho_0^{-1}\partial_j\partial_k\rho_0. \tag{2.22}$$

Equation (2.21) has the form of an inhomogeneous transport equation, with the source term for $\psi_m$ depending only on $\psi_{m-1}$.

### 2.4. Half-form $\rho_0$

It will now be shown that (2.20a) may be solved for $\rho_0 \in \mathcal{O}(U)$. Define

$$\lambda_0 = \left(\delta_i - \frac{1}{2}\mathrm{div}(\xi_i)\right)\sigma^i. \tag{2.23}$$

Equations (2.5) and (2.7) imply that $\lambda_0$ is closed, so one may put

$$\rho_0(x) = \exp\left(\int_0^x \lambda_0\right). \tag{2.24}$$

Now note that $e^{F_1}$ is the unique solution of (2.20a) in the space of formal series in $x$ with constant term 1. Therefore $\rho_0 = e^{F_1}$. In particular $F_1(x)$ converges on $U$.

From the geometric perspective, $\rho_0$ is a local expression for a certain half-form on $\Sigma_0$. Indeed, define $\eta \in H^0(K^{\frac{1}{2}}, U)$ by the condition

$$\eta \otimes \eta = \rho_0(x)^2 \, dx^1 \wedge \ldots \wedge dx^n, \tag{2.25}$$

with the sign of $\eta$ fixed by $\eta|_0 = \sqrt{\epsilon}|_0$. Then (2.20a) may be reformulated as

$$\mathcal{L}(\xi_i)\eta = \delta_i \cdot \eta. \tag{2.26}$$

This has the advantage of being formulated in terms of objects defined globally on $\Sigma_0$. This leads to the question whether a global solution of (2.26) exists. It will now be demonstrated that the answer is affirmative after pulling back to the universal cover. However $\eta$ may turn out to be multi-valued on $\Sigma_0$. Consider the ansatz $\eta = \widetilde{\rho}_0 \cdot \sqrt{\epsilon}$. Plugging this into (2.26) one obtains

$$\xi_i(\widetilde{\rho}_0) = \nu_i \cdot \widetilde{\rho}_0, \tag{2.27}$$

where $\nu_i = \delta_i - \frac{1}{2}f_{ij}^j$. Iterating this equation gives $f_{jk}^i\nu_i = 0$. Therefore $\nu$ coincides with the derivative at zero of a unique holomorphic homomorphism $\widetilde{\rho}_0$ from $G$ to the multiplicative group $\mathbb{C}^{\times}$ of nonzero complex numbers. By construction, $\widetilde{\rho}_0$ satisfies (2.27). In general $\widetilde{\rho}_0$ is multi-valued on $\Sigma_0$. In this case a global solution of (2.26) does not exist on $\Sigma_0$. Precise condition for $\widetilde{\rho}_0$ to descend to a function on $\Sigma_0$ is that the group $\Gamma$ should be contained in the

kernel of $\widetilde{\rho}_0$. If the Lie algebra spanned by the $L_i$ is such that each element may be represented as a linear combination of commutators, then $\nu_i = 0$ is the only solution of equation $f^i_{jk}\nu_i = 0$. In this situation $\eta$ holomorphic on whole $\Sigma_0$ is guaranteed to exist.

Notice that it follows from (2.9) and the above that $\rho_0$ is, up to multiplication by a function holomorphic on $G$, equal to $\mathcal{D}^{-\frac{1}{2}}$.

### 2.5. Quantum corrections

In this section equation (2.21) shall be discussed. The first step is to derive basic properties of the operators $\Delta_i$. Put $\widetilde{L}_i = \rho_0^{-1}e^{-\frac{F_0}{\hbar}}L_ie^{\frac{F_0}{\hbar}}\rho_0$. By (2.18) and (2.20a) one has

$$\widetilde{L}_i = \hbar\xi_i - \hbar^2\Delta_i. \tag{2.28}$$

The operators $\widetilde{L}_i$ are related to $L_i$ by a similarity transformation, so they satisfy the same commutation relations: $[\widetilde{L}_i, \widetilde{L}_j] = \hbar f^k_{ij}\widetilde{L}_k$. Sorting terms in this equation according to the accompanying power of $\hbar$ one gets that it is equivalent to[6]

$$[\Delta_i, \Delta_j] = 0, \tag{2.29a}$$

$$[\xi_i, \Delta_j] - [\xi_j, \Delta_i] = f^k_{ij}\Delta_k. \tag{2.29b}$$

Formula (2.2) combined with holomorphicity of $\mathcal{D}\rho_0^{-1}\xi_i(\rho_0)$ on $\Sigma_0$ implies that each $\mathcal{D}^3\Delta_i$ extends to a holomorphic differential operator on $\Sigma_0$. In particular each $\Delta_i$ preserves the space of meromorphic functions with the polar set contained in Ram.

Now let $\omega$ be a meromorphic $p$-form defined on an open subset of $\Sigma_0$. Then

$$\omega = \omega_{j_1\ldots j_p}\sigma^{j_1} \wedge \ldots \wedge \sigma^{j_p} \tag{2.30}$$

for some meromorphic functions $\omega_{j_1\ldots j_p}$. Define

$$\Delta\omega = (\Delta_i\omega_{j_1\ldots j_p})\sigma^i \wedge \sigma^{j_1} \wedge \ldots \wedge \sigma^{j_p}. \tag{2.31}$$

By construction, $\Delta$ is a meromorphic differential operator which shifts the degree by one. Using (2.7) one may rewrite properties (2.29) in the neat form

$$\Delta^2 = 0, \tag{2.32a}$$

$$d\Delta + \Delta d = 0. \tag{2.32b}$$

In particular $d_\Delta = d - \hbar\Delta$ satisfies $d_\Delta^2 = 0$. One may regard $d_\Delta$ as a deformation of the exterior derivative. In this language equations satisfied by $\psi$ take the form

$$d_\Delta\psi = 0, \tag{2.33}$$

or equivalently

$$d\psi_m = \Delta\psi_{m-1}. \tag{2.34}$$

To solve (2.34), suppose that $\psi_{m-1} \in \mathcal{O}(U)$. Define $\lambda_m = \Delta\psi_{m-1}$. Then

$$d\lambda_m = d\Delta\psi_{m-1} = -\Delta d\psi_{m-1}. \tag{2.35}$$

Assuming that $\psi_{m-1}$ satisfies $d\psi_{m-1} = \Delta\psi_{m-2}$ for some $\psi_{m-2}$, one has

---

[6] The first equation may be derived also directly from the fact that partial derivatives commute. The second relation is not as obvious.

$$\mathrm{d}\lambda_m = -\Delta^2 \psi_{m-2} = 0. \tag{2.36}$$

It follows that one may put

$$\psi_m(x) = \int_0^x \lambda_m. \tag{2.37}$$

By induction, each $\psi_m$ and hence each $F_g$ is holomorphic on $U$.

Suppose that for some $m \geqslant 1$ the function $\psi_{m-1}$ extends to a (possibly multi-valued) holomorphic function on $\Sigma_0 \setminus \mathrm{Ram}$. Then the pullback of $\lambda_m$ to a universal cover of $\Sigma_0 \setminus \mathrm{Ram}$ is a holomorphic 1-form, so $\psi_m$ extends to a (possibly multi-valued) holomorphic function on $\Sigma_0 \setminus \mathrm{Ram}$. Since the assumption is trivially satisfied for $m = 1$, all $\mathrm{d}\psi_m$ extend to holomorphic 1-forms on $\Sigma_0 \setminus \mathrm{Ram}$, by induction.

It is natural to ask what is the behaviour of $\psi_m$ near the ramification locus. The best statement one could hope for is that there exists $\nu_m \in \mathbb{N}$ such that $\mathcal{D}^{\nu_m} \psi_m$ extends to a holomorphic function on $G$. Suppose that this holds up to order $m - 1$. Then $\mathcal{D}^{\nu_{m-1}+4} \lambda_m \in \Omega^1(G)$. Unfortunately, this does not imply that $\psi_m = \int \lambda_m$ is meromorphic. Already in the calculus of a single complex variable the integral of a meromorphic 1-form $\lambda$ defined on a simply-connected Riemann surface may contain logarithmic terms. This happens if and only if the integral of $\lambda$ over a small loop around one of the poles is nonzero, or equivalently, if $\lambda$ has nonzero residues. Similarly in the present problem, it could happen that $\lambda_m$ has nonzero periods over cycles of $G \setminus Q^{-1}(\mathrm{Ram})$, leading to multi-valuedness of $\psi_m$ on $G \setminus Q^{-1}(\mathrm{Ram})$.

Incidentally, the reasoning presented above suggests that if $\psi_m$ are meromorphic, then the order of the pole of $\psi_m$ at $\mathrm{Ram}$ grows linearly with $m$. This behaviour is indeed observed in the examples studied in section 4.

## 3. Local properties of $\Delta$

### 3.1. Trivialization

It will now be demonstrated that in a neighbourhood of any point of $\Sigma_0 \setminus \mathrm{Ram}$ the operator $\Delta$ admits a representation as a commutator with $d$,

$$\Delta = [d, \kappa]. \tag{3.1}$$

Suppose that $\kappa$ is an operator defined on functions and satisfying

$$\Delta_i = [\xi_i, \kappa]. \tag{3.2}$$

Then equation (3.1) holds, provided that one puts

$$\kappa \left( \omega_{j_1 \ldots j_p} \sigma^{j_1} \wedge \ldots \wedge \sigma^{j_p} \right) = \left( \kappa \omega_{j_1 \ldots j_p} \right) \sigma^{j_1} \wedge \ldots \wedge \sigma^{j_p}. \tag{3.3}$$

Indeed, since both $\Delta$ and $\kappa$ commute with multiplication by $\sigma^j$, it is sufficient to prove that (3.1) holds when acting on functions. This is easy:

$$[d, \kappa] f = [\xi_i, f] \sigma^i = (\Delta_i f) \sigma^i = \Delta f. \tag{3.4}$$

It remains to construct $\kappa$ acting on functions.

Recall that on a complex Lie group there are two distinguished framings of the holomorphic tangent bundle: right-invariant $\xi_i$ and left-invariant $\chi_i$, which generate left and right translations, respectively. Since left translations commute with right translations, one has $[\xi_i, \chi_j] = 0$.

Pick an element $p \in \Sigma_0 \setminus \mathrm{Ram}$. Let $U \subseteq \Sigma_0 \setminus \mathrm{Ram}$ be a polydisc centered at $p$. Then $U$ is isomorphic as a $\mathfrak{g}$-manifold to a neighbourhood of the neutral element in $G$, so there exists[7] a holomorphic trivialization $\{\chi_i\}_{i=1}^n$ of $TU$ such that

$$[\xi_i, \chi_j] = 0, \qquad [\chi_i, \chi_j] = f_{ij}^k \chi_k. \tag{3.5}$$

Operators $\Delta_i$ may be expanded as

$$\Delta_i = \frac{1}{2} \alpha_i^{rs} \chi_r \chi_s + \beta_i^r \chi_r + \gamma_i \tag{3.6}$$

with uniquely determined $\alpha_i^{rs}, \beta_i^r, \gamma_i \in \mathcal{O}(U)$ satisfying $\alpha_i^{rs} = \alpha_i^{sr}$. Cocycle condition (2.29b) is equivalent to the statement that the differential forms

$$\alpha^{rs} = \alpha_i^{rs} \sigma^i, \quad \beta^r = \beta_i^r \sigma^i, \quad \gamma = \gamma_i \sigma^i \tag{3.7}$$

are closed. Thus there exist $a^{rs}, b^r, c \in \mathcal{O}(U)$, unique up to constants, such that

$$\alpha^{rs} = \mathrm{d}a^{rs}, \quad \beta^r = \mathrm{d}b^r, \quad \gamma = \mathrm{d}c. \tag{3.8}$$

Now define a differential operator

$$\kappa = \frac{1}{2} a^{rs} \chi_r \chi_s + b^r \chi_r + c. \tag{3.9}$$

Using (3.5) one gets $[\xi_i, \kappa] = \Delta_i$, which completes the proof of existence.

Consider the special case $p = 0$. Then $c - c(0) = \psi_1$. Indeed,

$$\xi_i(\psi_1) = [\xi_i, \kappa](1) = \xi_i(\kappa(1)) = \xi_i(c), \tag{3.10}$$

so $\psi_1 - c$ is a constant.

### 3.2. Poincaré lemma

Let $U$ be a polydisc in $\Sigma_0 \setminus \mathrm{Ram}$ and let $\tau = \sum_{m=0}^\infty \hbar^m \tau_m \in \Omega^p(U)[[\hbar]]$ with some $p \geqslant 1$ be $d_\Delta$-closed, i.e. such that $d_\Delta \tau = 0$. This means that one has

$$\mathrm{d}\tau_0 = 0, \tag{3.11a}$$

$$\mathrm{d}\tau_m = \Delta \tau_{m-1} \quad \text{for } m \geqslant 1. \tag{3.11b}$$

It will now be proven that in this situation there exists $\zeta \in \Omega^{p-1}(U)[[\hbar]]$ such that

$$\tau = \mathrm{d}_\Delta \zeta. \tag{3.12}$$

Equation (3.12) is equivalent to the system

$$\tau_0 = \mathrm{d}\zeta_0, \tag{3.13a}$$

$$\tau_m = \mathrm{d}\zeta_m - \Delta \zeta_{m-1} \quad \text{for } m \geqslant 1. \tag{3.13b}$$

Existence of $\zeta_0 \in \Omega^{p-1}(U)$ such that $\tau_0 = \mathrm{d}\zeta_0$ follows from the standard Poincaré lemma for holomorphic forms. Therefore one has

$$\mathrm{d}\tau_1 = \Delta \tau_0 = \Delta \mathrm{d}\zeta_0 = -\mathrm{d}\Delta \zeta_0, \tag{3.14}$$

so there exists $\zeta_1 \in \Omega^{p-1}(U)$ such that $\tau_1 + \Delta \zeta_0 = \mathrm{d}\zeta_1$. In this situation

---

[7] If $\Gamma$ is a normal subgroup of $G$, then $\Sigma_0$ is a complex Lie group and $\chi_i$ are defined on whole $\Sigma_0$. If $\Gamma$ is not a normal subgroup, $\Sigma_0$ is only locally isomorphic to a Lie group.

$$d\tau_2 = \Delta\tau_1 = \Delta d\zeta_1 - \Delta^2\zeta_0 = -d\Delta\zeta_1, \tag{3.15}$$

so there exists $\zeta_2 \in \Omega^{p-1}(U)$ such that $\tau_2 = d\zeta_2 - \Delta\zeta_1$. Continuing like this inductively one obtains existence of a solution $\zeta = \sum_{m=0}^{\infty} \hbar^m \zeta_m$ to equation (3.12).

## 4. Examples

### 4.1. One-dimensional example

Consider the Airy structure given by a single differential operator

$$L = \hbar\partial_x - \frac{1}{2}x^2 - \frac{\hbar^2}{2}\partial_x^2. \tag{4.1}$$

Replacing derivatives by $y$ and solving a quadratic equation, one finds that $\Sigma_0$ is locally given by $y(x) = 1 - \sqrt{1-x^2}$. Integrating the form $\theta = ydx$ gives

$$F_0(x) = x - \frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\arcsin x. \tag{4.2}$$

When $F_0(x)$ is expressed in terms of the global coordinate $z = \arcsin(x) \in \frac{\mathbb{C}}{2\pi\mathbb{Z}}$ on $\Sigma_0$, it becomes clear that $F_0$ is not single valued on $\Sigma_0$.

The Hamiltonian vector field on $\Sigma_0$ takes the form $\xi = \partial_z$. Since $\delta = 0$, one has

$$\eta = \sqrt{dz} = \sqrt{\frac{dx}{\cos(z)}}, \tag{4.3}$$

so $\rho_0(z) = \frac{1}{\sqrt{\cos(z)}}$. Alternatively, this result could have been derived as follows:

$$\rho_0 = \mathcal{D}^{-\frac{1}{2}} = \left(\frac{\partial L^{cl}}{\partial y}\right)^{-\frac{1}{2}} = (1-y)^{-\frac{1}{2}} = \frac{1}{\sqrt{\cos z}}, \tag{4.4}$$

where the first equality follows from $\nu = 0$. In the last step $y$ has been expressed in terms of the $z$ coordinate. It follows that

$$\Delta = \frac{1}{2}\rho_0^{-1}\partial_z^2\rho_0 = \frac{1}{2}\left(\frac{1}{\cos z}\frac{\partial}{\partial z} + \frac{\sin z}{2\cos^2 z}\right)^2. \tag{4.5}$$

Recursive relation $\psi_m'(z) = \Delta\psi_{m-1}(z)$ may be used to compute arbitrarily many $\psi_m(z)$ without expanding them in power series. For $m = 1$ one gets

$$\psi_1(z) = \frac{\sin(z)(5 + \cos^2(z))}{24\cos^3(z)}. \tag{4.6}$$

If $\psi_1$ is expressed in terms of the $x$ variable, a square root appears in the denominator. This is merely an artifact of the coordinate system. It has been checked by explicit calculation that functions $\{\psi_m\}_{m=1}^{40}$ are meromorphic on $\Sigma_0$, with $\mathcal{D}^{3m}\psi_m$—restriction to $\Sigma_0$ of a polynomial in $x, y$.

As promised earlier, $\Delta$ may be written as $[\xi, \kappa]$ with

$$\kappa = \frac{1}{2}\tan(z)\partial_z^2 + \frac{1}{2}\tan^2(z)\partial_z + \psi_1(z). \tag{4.7}$$

### 4.2. Borel subalgebra of $\mathfrak{sl}_2$

In this section two inequivalent Airy structures for a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{sl}_2$ will be obtained and analyzed. Whole $\mathfrak{sl}_2$ algebra will be lifted to the quantum level, leading to uniqueness of the quantization. Only generators of $\mathfrak{b}$ will have the form required by the definition of an Airy structure, while the $F$ operator is going to have wrong linear term and also a possibly nonzero constant term. Interestingly, both Airy structures are derived from the same set of classical hamiltonians. The point is that the locus $\Sigma_s$ is disconnected, and each connected component gives rise to a distinct Airy structure.

The starting point is a triple of hamiltonians

$$H = -z_3 z_{-3} + z_1 z_{-1}, \tag{4.8a}$$

$$E = z_3 z_{-1} - z_1^2, \tag{4.8b}$$

$$F = -z_1 z_{-3} + z_{-1}^2, \tag{4.8c}$$

where $z_j$ are linear coordinates on a vector space $W$ equipped with symplectic form

$$\omega = -\frac{1}{3} \cdot dz_3 \wedge dz_{-3} + dz_1 \wedge dz_{-1}. \tag{4.9}$$

Hamiltonians $H, E, F$ form an $\mathfrak{sl}_2$ triple, i.e. one has

$$\{H, E\} = 2E, \quad \{H, F\} = -2F, \quad \{E, F\} = H. \tag{4.10}$$

Subalgebra $\mathfrak{b}$ spanned by $H, E$ is the Lie algebra of the closed subgroup $B \subseteq \mathrm{SL}_2$ of matrices of the form $g(t, s) = \begin{pmatrix} t & ts \\ 0 & t^{-1} \end{pmatrix}$ with $t \in \mathbb{C}^\times$, $s \in \mathbb{C}$. It will be useful to know that action of an element $g(t, s) \in B$ on $W$ is given by the formula

$$z_j \mapsto t^j \sum_{k=0}^{\frac{3-j}{2}} \binom{\frac{3-j}{2}}{k} s^k z_{j+2k} \quad \text{for } j \in \{\pm 1, \pm 3\}. \tag{4.11}$$

To construct Airy structures for the subalgebra of $\mathfrak{sl}_2$ spanned by $H$ and $E$, one has to find the zero locus $\Sigma_s$ and divide it into its connected components. First observe that the points

$$p: \quad z_3 = 1, z_1 = z_{-1} = z_{-3} = 0, \tag{4.12a}$$

$$q: \quad z_{-1} = 1, z_3 = z_1 = z_{-3} = 0 \tag{4.12b}$$

belong to $\Sigma_s$. Denote the corresponding orbits by $\Sigma_p$ and $\Sigma_q$. Coordinate $z_3$ vanishes identically on $\Sigma_q$, so $\Sigma_p \cap \Sigma_q = \emptyset$. It remains to show that $\Sigma_s = \Sigma_p \cup \Sigma_q$. Suppose that $r \in \Sigma_s$ has $z_3 \neq 0$. Then the orbit of $r$ contains an element with $z_3 = 1$ and $z_1 = 0$. In this situation equations $E(z) = H(z) = 0$ imply that $z_{-1} = z_{-3} = 0$, so $r \in \Sigma_p$. Now suppose that $r \in \Sigma_s$ is a point with $z_3 = 0$. Equation $E(z) = 0$ gives $z_1 = 0$. If also $z_{-1} = 0$, then $dH \wedge dE|_r = 0$, contradicting $r \in \Sigma_s$. Thus $z_{-1} \neq 0$, so $r \in \Sigma_q$.

Maps $\mathbb{C}^2 \ni (u, s) \mapsto g(e^u, s)r \in \Sigma_r$ are universal covers for $r = p, q$. In both cases the fiber over $r$ is isomorphic to $\mathbb{Z}$ and $\Sigma_r \cong \mathbb{C}^\times \times \mathbb{C}$ as a complex manifold.

*4.2.1. The first orbit.* To quantize the orbit $\Sigma_p$, introduce coordinates

$$y_1 = -z_{-3}, \quad y_2 = z_{-1}, \quad x^1 = \frac{1}{3}(z_3 - 1), \quad x^2 = z_1, \tag{4.13}$$

in which symplectic form takes the standard form (A.1). Replacing variables $y$ by derivatives according to the Weyl prescription gives

$$H = \hbar(1 + 3x^1)\partial_1 + \hbar x^2 \partial_2 + 2\hbar, \tag{4.14a}$$

$$E = \hbar(1 + 3x^1)\partial_2 - (x^2)^2, \tag{4.14b}$$

$$F = \hbar x^2 \partial_1 + \hbar^2 \partial_2^2. \tag{4.14c}$$

The partition function may be found by solving the relevant differential equations directly. It takes the form

$$Z(x^1, x^2) = \frac{\exp\left(\frac{1}{3\hbar} \frac{(x^2)^3}{1+3x^1}\right)}{(1 + 3x^1)^{\frac{2}{3}}}. \tag{4.15}$$

This expression has an essential singularity on the zero locus $1 + 3x^1 = 0$. However, one has $1 + 3x^1 \neq 0$ on $\Sigma_p$.

Mapping $B \ni g \mapsto g \cdot p \in \Sigma_p$ is a threefold cover[8] of $\Sigma_p$. Indeed, the fiber over $p$ consists of elements $g(t, s)$ with $s = 0$ and $t^3 = 1$. Pulling back $Z$ to $B$ yields

$$Z(g(t, s) \cdot p) = t^{-2} \exp\left(\frac{s^3}{3\hbar}\right). \tag{4.16}$$

This expression is globally holomorphic. Absence of poles may be traced back to the fact that in this example $\mathrm{Ram} = \emptyset$. This does not mean that $Z$ extends to a holomorphic function on $\Sigma_p$. In fact $Z$ is multi-valued on $\Sigma_p$. However it is true that $Z(x^1, x^2)\sqrt{dx^1 \wedge dx^2}$ extends to a holomorphic half-form on $\Sigma_p$. Its pullback to $B$ takes the form $\exp\left(\frac{s^3}{3\hbar}\right)\sqrt{\frac{dt}{t} \wedge ds}$. Notice that $\frac{dt}{t} \wedge ds$ is the unique (up to scalars) left-invariant section of the canonical bundle of $B$.

In this example $H$ and $E$ are first order differential operators, so $\Delta = \kappa = 0$.

*4.2.2. The second orbit.*   For the second orbit, introduce coordinates

$$y_1 = z_1, \quad y_2 = z_3, \quad x^1 = 1 - z_{-1}, \quad x^2 = \frac{1}{3} z_{-3}. \tag{4.17}$$

Classical hamiltonians $H, E$ do not contain any terms not proportional to $y_1$ or $y_2$, so 0 is a solution of the Hamilton–Jacobi equation. It satisfies the initial conditions obeyed by $F_0$, so (by uniqueness of $F_0$) one has $F_0 = 0$. Equivalently, $y = 0$ on $\Sigma_q$. This could have been inferred also directly from (4.11). There exist other solutions of the Hamilton–Jacobi equation. They form a one-parameter family

$$F_0(x^1, x^2) = -\frac{1}{9} \frac{(1 - x^1)^3}{x^2} + \mathrm{const.} \tag{4.18}$$

These solutions are singular for $x^2 = 0$. In particular they do not satisfy initial conditions required for $F_0$.

After Weyl quantization, one gets

$$H = \hbar(1 - x^1)\partial_1 - 3\hbar x^2 \partial_2 - 2\hbar, \tag{4.19a}$$

---

[8] But not a universal cover, since $\pi_1(B) = \mathbb{Z}$.

$$E = \hbar(1 - x^1)\partial_2 - \hbar^2\partial_1^2, \tag{4.19b}$$

$$F = (1 - x^1)^2 - 3\hbar x^2\partial_1. \tag{4.19c}$$

Equations $HZ = 0$, $EZ = 0$ do not admit a solution which is a holomorphic function and satisfies correct boundary conditions. Indeed, if there was such a solution, it would coincide with the partition function. On the other hand, it will soon be seen that the partition function is a divergent formal series.

The Hamiltonian vector fields projected onto the $x$ space take the form

$$\xi_1 = (1 - x^1)\frac{\partial}{\partial x^1} - 3x^2\frac{\partial}{\partial x^2}, \tag{4.20a}$$

$$\xi_2 = (1 - x^1)\frac{\partial}{\partial x^2}. \tag{4.20b}$$

Solving equations $(\xi_i + \frac{1}{2}\mathrm{div}(\xi_i))\rho_0 = 0$ gives

$$\rho_0 = \frac{1}{(1 - x^1)^2}. \tag{4.21}$$

Differential operators $\Delta$ are given by

$$\Delta_1 = 0, \qquad \Delta_2 = (1 - x^1)^2\partial_1^2\frac{1}{(1 - x^1)^2}. \tag{4.22}$$

Therefore the $\psi_m$ satisfy a hierarchy of partial differential equations,

$$\left((1 - x^1)\partial_1 - 3x^2\partial_2\right)\psi_m = 0, \tag{4.23a}$$

$$(1 - x^1)\partial_2\psi_m = (1 - x^1)^2\partial_1^2\frac{1}{(1 - x^1)^2}\psi_{m-1}. \tag{4.23b}$$

The first equation states merely that each $\psi_m$ may be expressed as

$$\psi_m(x^1, x^2) = f_m\left(\frac{x^2}{(1 - x^1)^3}\right). \tag{4.24}$$

Plugging this into the second equation one gets

$$f_m(z) = \int_0^z \mathrm{d}\zeta \left(6 + 24\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta} + 9\zeta^2\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\right) f_{m-1}(\zeta). \tag{4.25}$$

This formula implies that if $f_{m-1}(z)$ is a monomial of degree $d$, then $f_m(z)$ is a monomial of degree $d + 1$. Thus, by induction on $m$, $f_m(z) = c_m z^m$ for some coefficients $c_m$. Plugging this result into the equation above one gets a recurrence relation $c_m = 3(3m - 1)c_{m-1}$. Its solution may be found using the functional equation satisfied by the gamma function. The final result is the formal series

$$Z(x^1, x^2) = \frac{1}{(1 - x^1)^2}\sum_{m=0}^{\infty}\frac{\Gamma\left(m + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\left(\frac{9\hbar x^2}{(1 - x^1)^3}\right)^m. \tag{4.26}$$

Proceeding as in section 4.2.1 one may show that each term of this series extends to a holomorphic function on $\Sigma_q$. However the series in $\hbar$ is divergent.

Possibility of interpreting the divergent series $\psi(w) = \sum_{m=0}^{\infty} \frac{\Gamma\left(m+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} w^m$ in (4.26) as an asymptotic expansion of some function $Z^{\mathrm{res}}$ satisfying equations $L_i Z^{\mathrm{res}} = 0$ will now be discussed. First notice that $\psi$ satisfies a differential equation

$$w^2 \psi'(w) + \left(\frac{2}{3}w - 1\right)\psi(w) = -1. \tag{4.27}$$

This condition has a unique solution in $\mathcal{O}(\mathbb{C}^\times)$,

$$\psi^{\mathrm{res}}(w) = \frac{3\mathrm{e}^{-\frac{1}{w}}}{w} {}_1F_1\left[\begin{matrix} \frac{1}{3} \\ \frac{4}{3} \end{matrix}; \frac{1}{w}\right], \tag{4.28}$$

where ${}_1F_1$ is the confluent hypergeometric function. $\psi^{\mathrm{res}}(w)$ has an essential singularity at $w = 0$. By the well-known results [17, p 508] on the asymptotics of hypergeometric functions, $\psi(w)$ is an asymptotic expansion of $\psi^{\mathrm{res}}(w)$ for $w \to 0$, up to a correction of the form $\psi^{\mathrm{cut}}(w) = \Gamma\left(\frac{1}{3}\right)\mathrm{e}^{\frac{i\pi}{3}}\mathrm{e}^{-\frac{1}{w}}\left(\frac{1}{w}\right)^{\frac{2}{3}}$. This term is negligibly small for $w \to 0$, provided that the real part of $w$ is positive. Function $\psi^{\mathrm{cut}}(w)$ is annihilated by the differential operator $w^2\frac{\mathrm{d}}{\mathrm{d}w} + \left(\frac{2}{3}w - 1\right)$, so $\psi^{\mathrm{res}}(w) - \psi^{\mathrm{cut}}(w)$ satisfies (4.27). This function is approximated by the formal series $\psi(w)$ on a larger set than $\psi^{\mathrm{res}}(w)$, but it is multi-valued.

Simple calculation shows that the differential equation (4.27) satisfied by $\psi^{\mathrm{res}}(w)$ is a sufficient condition for the function

$$Z^{\mathrm{res}}(x^1, x^2) = \frac{1}{(1-x^1)^2}\psi^{\mathrm{res}}\left(\frac{x^2}{(1-x^1)^3}\right) \tag{4.29}$$

to be annihilated by the operators $H$ and $E$. Its solutions form a one parameter family. On the other hand, for $EZ^{\mathrm{res}} = HZ^{\mathrm{res}} = 0$ to hold it is not necessary to have (4.27). A necessary and sufficient condition takes the form

$$3w^2\psi''(w) + (8w - 3)\psi'(w) + 2\psi(w) = 0. \tag{4.30}$$

This equation coincides with the derivative of (4.27). Its general solution is a combination $c_1\psi^{\mathrm{res}}(w) + c_2\psi^{\mathrm{cut}}(w)$ with $c_1, c_2 \in \mathbb{C}$. This function has correct asymptotic expansion only for $c_1 = 1$, which is equivalent to the equation (4.27).

One may show that $Z^{\mathrm{res}}$ extends to a holomorphic function on the complement in $\Sigma_0$ of the hypersurface $x^2 = 0$. It is not clear if this is related to the form of singularities of solutions (4.18) of the Hamilton–Jacobi equation.

As in the previous examples, the cocycle $\Delta$ may be trivialized explicitly:

$$\kappa = \rho_0^{-1}\left(f\partial_1^2 - f^2\partial_1\partial_2 + \frac{1}{3}f^3\partial_2^2\right)\rho_0, \tag{4.31}$$

where $f(x^1, x^2) = \frac{x^2}{1-x^1}$.

## 5. Summary and outlook

Analytic properties of the partition functions associated to finite-dimensional Airy structures were investigated. It was shown that $F_g(x)$ are convergent series, which extend to objects defined on the algebraic variety $\Sigma_0$, possibly singular on a subvariety Ram. Question whether they are always meromorphic in the sense explained in the main text remains open. This is true in all examples considered in this paper, possibly due to their simplicity. It is an interesting

problem for the future to obtain description of the behaviour of the free energy near Ram. This issue is intimately connected with questions about geometric properties of Ram. In the language of classical WKB expansions it is related to focal points.

Recursive equations for the partition function are neatly formulated in terms of a differential operator $d_\Delta$, which is a deformation of the exterior derivative on $\Sigma_0$. It is shown that on $\Sigma_0 \setminus \mathrm{Ram}$ is locally of the form $d - \hbar[d, \kappa]$ for some differential operator $\kappa$. Furthermore $d_\Delta$ obeys a version of the Poincaré lemma.

At the moment it is not clear if it is always possible to find a globally defined $\kappa$ and what are its singularities. Evidently, this is connected with the questions about the free energy. Finally, it would be interesting to obtain a description of the cohomology of $d_\Delta$. Several results of this paper may be summarized by the statement that for a polydisc $U$ contained in $\Sigma_0 \setminus \mathrm{Ram}$ the cohomology of $d_\Delta$ acting on $\Omega(U)$ is of the form

$$H_{d_\Delta}^p(U) \cong \begin{cases} \mathbb{C}[[\hbar]] & \text{for } p = 0, \\ 0 & \text{othewise.} \end{cases} \tag{5.1}$$

## Acknowledgments

## Appendix A. Review of complex symplectic geometry

In this appendix basic notions of symplectic geometry are recalled, with definitions adjusted to holomorphic forms on complex manifolds. The main purpose is to fix notations. More detailed exposition may be found in [18, chapters 1–2].

Let $M$ be a complex manifold. For any open subset $U \subseteq M$ let $\mathcal{O}(U)$ be the algebra of holomorphic functions on $U$ and $\mathcal{X}(U)$ the Lie algebra of holomorphic vector fields on $U$. Closed, holomorphic 2-form $\omega$ on $M$ is said to be a symplectic form if it is non-degenerate, in the sense that for any open subset[9] $U \subseteq M$ vanishing of the differential form $\omega(\xi, \cdot)$ with $\xi \in \mathcal{X}(U)$ implies that $\xi = 0$. If there exists a symplectic form on $M$, the complex dimension of $M$ is even. Put $\dim_{\mathbb{C}} M = 2n$.

From now on assume that some symplectic form $\omega$ on $M$ is fixed. Let $U \subseteq M$ be an open subset and let $f \in \mathcal{O}(U)$. There exists a unique $\xi_f \in \mathcal{X}(U)$, called the hamiltonian vector field associated to $f$, such that $df = \omega(\xi_f, \cdot)$.

Let $U \subseteq M$ be an open subset and let $f, g \in \mathcal{O}(U)$. Poisson bracket is defined by $\{f, g\} = \xi_f(g)$. With this convention one has $\xi_{\{f,g\}} = [\xi_f, \xi_g]$.

Suppose that $\{x^i, y_i\}_{i=1}^n$ are coordinates on an open subset $U \subseteq M$ such that

$$\omega = \mathrm{d}x^i \wedge \mathrm{d}y_i. \tag{A.1}$$

Then Poisson bracket of $f, g \in \mathcal{O}(U)$ takes the form

---

[9] In contrast to real symplectic geometry, it is not sufficient to assume that this condition is satisfied for $U = M$. The point is that due to non-existence of holomorphic partitions of unity, there might not exist enough holomorphic vector fields defined on whole $M$ to probe local properties of $\omega$. Such vector fields may always be constructed in local coordinate patches.

$$\{f,g\} = \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y_i}. \tag{A.2}$$

Submanifold $N \subseteq M$ is said to be isotropic if restriction of $\omega$ to $N$ vanishes. If in addition $\dim_{\mathbb{C}} N = n$, $N$ is said to be Lagrangian.

Let $U \subseteq M$ be an open subset and let $g_1, \dots, g_n \in \mathcal{O}(U)$ be such that $dg_1 \wedge \dots \wedge dg_n$ has no zeros in $U$. Then by the implicit function theorem, the zero locus

$$\Sigma = \{p \in U | g_1(p) = \dots = g_n(p) = 0\} \tag{A.3}$$

is a submanifold of $M$ of dimension $n$. Suppose that $g_i$ are such that there exist $\{f_{ij}^k\}_{i,j,k=1}^n \subseteq \mathcal{O}(U)$ such that $\{g_i, g_j\} = f_{ij}^k g_k$. Then $\Sigma$ is Lagrangian. Indeed, $\xi_{g_i}(g_j)|_\Sigma = 0$, so each $\xi_{g_i}$ is tangent to $\Sigma$. Since $dg_i$ are linearly independent at every point of $U$, the same is true for $\xi_{g_i}$. Therefore $\xi_{g_i}$ span each tangent space of $\Sigma$. Moreover for any $p \in \Sigma$ one has

$$\omega(\xi_{g_i}, \xi_{g_j})\big|_p = -\{g_i, g_j\}\big|_p = -f_{ij}^k(p)g_k(p) = 0. \tag{A.4}$$

## Appendix B. The canonical bundle and half forms

Let $M$ be a complex manifold of dimension $n$. The $n$th exterior power of the holomorphic cotangent bundle of $M$ is denoted by $K$ and called the canonical bundle. By construction, $K$ is a line bundle. Line bundle $\mathfrak{L}$ equipped with an isomorphism $\mathfrak{L}^{\otimes 2} \cong K$ is said to be a square root of the canonical bundle. Not every complex manifold admits a square root of the canonical bundle. If some square root exists, it is typically not unique.

Suppose that some square root of the canonical bundle, denoted by $K^{\frac{1}{2}}$, is chosen. There exists a unique way to define Lie derivatives of elements of $K^{\frac{1}{2}}$ consistent with the Leibniz rule and identification $K^{\frac{1}{2}} \otimes K^{\frac{1}{2}} = K$. Indeed, let $U \subseteq M$ be open and let $\eta$ be a section of $K^{\frac{1}{2}}$ over $U$, $\eta \in H^0(K^{\frac{1}{2}}, U)$. Suppose first that $U$ is such that there exists a non-vanishing section $s \in H^0(K^{\frac{1}{2}}, U)$. Then there exist unique $f, g \in \mathcal{O}(U)$ such that $\eta = f \cdot s$ and $\mathcal{L}(\xi)(s^{\otimes 2}) = g \cdot s^{\otimes 2}$. It follows that the only consistent definition of the Lie derivative of $\eta$ with respect to $\xi \in \mathcal{X}(U)$ is

$$\mathcal{L}(\xi)\eta = \left(\frac{1}{2}fg + \xi(f)\right) \cdot s. \tag{B.1}$$

A short calculation shows that the right hand side of this equation does not depend on the choice of $s$. This means that for more general open sets $U$, the Lie derivative of $\eta$ may be computed first on a sufficiently fine open cover of $U$ and consistently glued together to yield a section $\mathcal{L}(\xi)\eta \in H^0(K^{\frac{1}{2}}, U)$.

## ORCID iDs

Błażej Ruba ⬤ https://orcid.org/0000-0002-7086-4504

## References

[1] Kontsevich M and Soibelman Y 2018 Airy structures and symplectic geometry of topological recursion *Proceedings of Symposia in Pure Mathematics* vol 100 p 433
[2] Eynard B and Orantin N 2007 Invariants of algebraic curves and topological expansion *Commun. Number Theory Phys.* **1** 347–452
[3] Eynard B 2004 Topological expansion for the 1-hermitian matrix model correlation functions *J. High Energy Phys.* JHEP11(2004)031
[4] Borot G *et al* 2015 Abstract loop equations, topological recursion and new applications *Commun. Number Theory Phys.* **09** 51–187
[5] Ciosmak P *et al* 2018 From CFT to Ramond super-quantum curves *J. High Energy Phys.* JHEP05(2018)133
[6] Kostov I and Orantin N 2010 CFT and topological recursion *J. High Energy Phys.* JHEP11(2010)056
[7] Alexandrov A *et al* 2018 Weighted Hurwitz numbers and topological recursion: an overview *J. Math. Phys.* **59** 081102
[8] Bouchard V *et al* 2008 Hurwitz numbers, matrix models and enumerative geometry *Proc. Symp. Pure Math.* **78** 263–83
[9] Eynard B 2014 A short overview of the 'Topological recursion' *Proc. Seoul ICM 2014* (arXiv:1412.3286)
[10] Borot G 2017 Lecture notes on topological recursion and geometry (arXiv:1705.09986)
[11] Andersen J E *et al* 2017 The ABCD of topological recursion (arXiv:1703.03307)
[12] Borot G *et al* 2018 Higher Airy structures, $\mathcal{W}$ algebras and topological recursion (arXiv:1812.08738)
[13] Bouchard V *et al* 2019 Super quantum Airy structures (arXiv:1907.08913)
[14] Bates S and Weinstein S 1997 *Lectures on the Geometry of Quantization* (Providence, RI: American Mathematical Society)
[15] Shafarevich I 2013 *Basic Algebraic Geometry 2, Schemes and Complex Manifolds* (Berlin: Springer)
[16] Hatcher A 2010 *Algebraic Topology* (Cambridge: Cambridge University Press)
[17] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (Washington, DC: National Bureau of Standards)
[18] da Silva A C 2008 *Lectures on Symplectic Geometry* (Berlin: Springer)