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Generalized uncertainty principle for Dirac fermion in a torsion field

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Abstract

We derive the uncertainty principle for a Dirac fermion in a torsion field obeying the Hehl–Datta (HD) equation. We find out how the non-linear term in the HD equation modifies the uncertainty principle and how it compares with the generalized uncertainty principle (GUP). We first discuss that there should be a correction factor to the Heisenberg uncertainty principle (HUP) when torsional effects are taken into consideration. We then derive the uncertainty relation from a solitary wave solution of the HD equation in 1 + 1 dimensions. We then introduce the unified length scale L_{CS} (which unifies Compton wavelength and Schwarzschild radius) into the HD equation and see how the probability density of the solution transforms for particles of different masses.

Keywords: generalized uncertainty principle, torsion field, Hehl–Datta equation, unified length scale L_{CS}

(Some figures may appear in colour only in the online journal)

1. Introduction

The background space–time of Einstein's general theory of relativity (GR) is formulated on a Riemannian manifold (V_4) which is torsion-less. If in this space–time continuum, spin angular momentum is introduced and distributed continuously, torsion is produced. The space–time is now a U_4 manifold on which the Einstein–Cartan–Sciama–Kibble (ECSK), or simply Einstein–Cartan (EC) theory is formulated. The affine connection is no longer required to be symmetric and torsion is the anti-symmetric part of the connection. Here, both mass and spin play a dynamical role. While mass 'adds up' on classical length scales due to its monopole nature, spin, being of dipole character, usually averages out in the absence of external forces. Therefore, matter in the macroscopic scale can be dynamically characterized entirely by the energy momentum tensor. In the micro-regime however, a spin density tensor plays an analogous role for spin which exhibits as torsion in the field. In this paper, we consider a minimal coupling of Dirac field on U_4 manifold which is called the Einstein–Cartan–Dirac (ECD) theory [1]. On U_4, the Dirac equation becomes non-linear due to the presence of torsion in the field; and is known as the Hehl–Datta equation [2]. In the ECD theory, the matter field is the spinorial Dirac field ψ, and the four-component spinor is written as:

$$\psi = \begin{bmatrix} P^A \\ \overline{Q}_{B'} \end{bmatrix} \tag{1}$$

where P^A and $\overline{Q}_{B'}$ are two dimensional complex vectors in $\mathbb{C}^2$ space. We redefine the spinors as: $P^0 = F_1$, $P^1 = F_2$, $\overline{Q}^{1'} = G_1$ and $\overline{Q}^{0'} = -G_2$. This is in accordance with [3, 4].

The HD equation is given by:

$$\mathrm{i}\gamma^\mu \partial_\mu \psi = \frac{3}{8} L_{\mathrm{Pl}}^2 \overline{\psi}\gamma^5\gamma_a\psi\gamma^5\gamma^a\psi + \frac{1}{\lambda_{\mathrm{C}}}\psi \tag{2}$$

where L_{Pl} is the Planck length and λ_{C} is the Compton wavelength.

1.1. Generalized uncertainty principle

In the microscopic domain, the Heisenberg uncertainty principle (HUP), is a key feature which states that the uncertainty in position and momentum of a particle must satisfy

$$(\Delta z)(\Delta p) \geqslant \frac{\hbar}{2}$$

where $\hbar$ is the Planck's constant, Δz and Δp are position and momentum dispersion operators respectively.

Two main length scales in relativistic physics are the Compton length $\lambda_{\mathrm{C}} = \frac{\hbar}{Mc}$, corresponding to the uncertainty principle and the Schwarzschild radius $R_{\mathrm{S}} = \frac{2GM}{c^2}$ corresponding to the existence of black holes. These two lines when plotted as a function of M intersect at Planck scales m_{Pl} and L_{Pl}. The uncertainty principle is modified as the energy increases towards the Planck value which leads to the conception of the generalized uncertainty principle [5–13] and is of the form:

$$\Delta z \geqslant \frac{\hbar}{\Delta p} + \alpha L_{\mathrm{Pl}}^2 \left(\frac{\Delta p}{\hbar}\right) \tag{3}$$

where L_{Pl} is the Planck length which is of the order 10^{-35}m, α is a dimensionless constant which depends on the particular model and the factor of 2 in the first term has been dropped.

Formulating the uncertainty relation from Schrödinger equation by calculating the position and momentum dispersion operators from the wave packet solution gives us HUP. Similar is the case for Dirac equation which is given by:

$$\mathrm{i}\gamma^\mu \partial_\mu \psi = \frac{mc}{\hbar}\psi = \frac{1}{\lambda_{\mathrm{C}}}\psi \tag{4}$$

The extra non-linear term in the HD equation as given in (2) comes due to the anti-symmetric nature of the affine connection which is in turn manifested as torsion in the field of the Dirac fermions. We compute the uncertainty relation of Dirac fermions from the HD equation to see how the non-linear term modifies the HUP and how it compares with the generalized uncertainty principle which is a manifestation of the unified expression for the Compton wavelength and the Schwarzschild radius [5–13].

1.2. Notations and conventions

The following conventions are in use for the remainder of this paper:

- Space–time endowed with torsion is specified by U_4 and V_4 is a non-torsional space–time.
- In the standard theory, the Planck length is given by:

$$l_1 = L_{\mathrm{Pl}} = \sqrt{\frac{G\hbar}{c^3}} \tag{5}$$

  and half Compton wavelength is:

$$l_2 = \frac{\lambda_{\mathrm{C}}}{2} = \frac{\hbar}{2Mc} \tag{6}$$

-

$$\begin{aligned} a(l_1) &= 3\sqrt{2}\pi l_1^2 \\ b(l_2) &= \frac{1}{2\sqrt{2}l_2} \end{aligned} \tag{7}$$

- **A unified length scale L_{CS} in quantum gravity**
  Recent works [5, 14, 15] have provided motivation for unifying the Compton wavelength $\left(\lambda_{\mathrm{C}} = \frac{\hbar}{Mc}\right)$ and Schwarzschild radius $\left(R_{\mathrm{S}} = \frac{2GM}{c^2}\right)$ of a point particle with mass M into one single length scale, the Compton–Schwarzschild length (L_{CS}). Such a treatment suggests us to introduce torsion, and relate the Dirac field to the torsion field. This modified theory is given by: $l_1 = l_2 = L_{\mathrm{CS}}$. So our HD equation becomes:

$$\mathrm{i}\gamma^\mu \partial_\mu \psi = \frac{3}{8} L_{\mathrm{CS}}^2 \overline{\psi}\gamma^5 \gamma_a \psi \gamma^5 \gamma^a \psi + \frac{1}{2L_{\mathrm{CS}}}\psi \tag{8}$$

2. A non-static solution in 1 + 1 dimensions of the HD equation

The HD equation on U_4 in Cartesian coordinate system (ct, x, y, z) given in [4] is as follows:

$$(\partial_0 + \partial_3)F_1 + (\partial_1 + \mathrm{i}\partial_2)F_2 = \mathrm{i}\sqrt{2}[b(l_2) + a(l_1)\xi]G_1 \tag{9}$$

$$(\partial_0 - \partial_3)F_2 + (\partial_1 - \mathrm{i}\partial_2)F_1 = \mathrm{i}\sqrt{2}[b(l_2) + a(l_1)\xi]G_2 \tag{10}$$

$$(\partial_0 + \partial_3)G_2 - (\partial_1 - \mathrm{i}\partial_2)G_1 = \mathrm{i}\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_2 \tag{11}$$

$$(\partial_0 - \partial_3)G_1 - (\partial_1 + \mathrm{i}\partial_2)G_2 = \mathrm{i}\sqrt{2}[b(l_2) + a(l_1)\xi^*]F_1 \tag{12}$$

where $\xi = F_1\overline{G}_1 + F_2\overline{G}_2$ and $\xi^* = \overline{F}_1 G_1 + \overline{F}_2 G_2$. These equations are compared and contrasted with the torsionless Dirac equations in [3], and then we see that the impact of torsion is to include the term $a\xi$ on the right-hand side of (9) and (10), and $a\xi^*$ in (11) and (12).

Now, let us assume the ansatz of the form $F_1 = G_2$ and $F_2 = G_1$ and further assume that the Dirac states are a function of only t and z. The four equations in Cartesian coordinates (9)–(12), reduce to the following two independent equations,

$$\begin{aligned} &\partial_t \psi_1 + \partial_z \psi_2 - \mathrm{i}\sqrt{2}b\psi_1 + \frac{\mathrm{i}a}{\sqrt{2}}(|\psi_1|^2 - |\psi_2|^2)\psi_1 = 0 \\ &\partial_t \psi_2 + \partial_z \psi_1 - \mathrm{i}\sqrt{2}b\psi_2 + \frac{\mathrm{i}a}{\sqrt{2}}(|\psi_1|^2 - |\psi_2|^2)\psi_2 = 0 \end{aligned} \tag{13}$$

where $\psi_1 = F_1 + F_2$ and $\psi_2 = F_1 - F_2$. We use the following solitary wave ansatz:

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} A(z) \\ \mathrm{i}B(z) \end{bmatrix} \exp\left(-\mathrm{i}\Lambda t\right) \tag{14}$$

where $A(z)$ and $B(z)$ are real functions. Substituting in (13), we obtain that [4]:

$$A(z) = \frac{-\mathrm{i}2^{3/4}(\sqrt{2}b - \Lambda)}{\sqrt{a}} \frac{\sqrt{\sqrt{2}b + \Lambda}\ \cosh\left(z\sqrt{2b^2 - \Lambda^2}\right)}{\left[\Lambda\ \cosh\left(2z\sqrt{2b^2 - \Lambda^2}\right) - \sqrt{2}b\right]} \tag{15}$$

$$B(z) = \frac{-\mathrm{i}2^{3/4}(\sqrt{2}b + \Lambda)}{\sqrt{a}} \frac{\sqrt{\sqrt{2}b - \Lambda}\ \sinh\left(z\sqrt{2b^2 - \Lambda^2}\right)}{\left[\Lambda\ \cosh\left(2z\sqrt{2b^2 - \Lambda^2}\right) - \sqrt{2}b\right]} \tag{16}$$

The probability density is given by the zeroth component of the four-vector fermion current $\tilde{J}^0 = \overline{\psi}\gamma^0\psi = \psi^\dagger\psi = (|A|^2 + |B|^2)$.

We define the following dimensionless variables:

$$\begin{aligned} q &= \sqrt{2}bz \\ w &= -\frac{\Lambda}{\sqrt{2}b} \\ A(q) &= \frac{\sqrt{a}}{2\sqrt{b}}A(z) \\ B(q) &= \frac{\sqrt{a}}{2\sqrt{b}}B(z) \end{aligned} \tag{17}$$

Scaled thus, $A(q)$ and $B(q)$ take the form:

$$A(q) = \frac{\mathrm{i}(1+w)\sqrt{1-w}\ \cosh(q\sqrt{1-w^2})}{1 + w\ \cosh(2q\sqrt{1-w^2})} \tag{18}$$

$$B(q) = \frac{\mathrm{i}(1-w)\sqrt{1+w}\ \sinh(q\sqrt{1-w^2})}{1 + w\ \cosh(2q\sqrt{1-w^2})} \tag{19}$$

The probability density is given by:

$$\tilde{J}^0 = \psi^\dagger\psi = \left[\frac{(1+w)^2(1-w)\cosh^2(q\sqrt{1-w^2}) + (1-w)^2(1+w)\sinh^2(q\sqrt{1-w^2})}{[1 + w\ \cosh(2q\sqrt{1-w^2})]^2}\right] \tag{20}$$

Six unique cases (corresponding to the range of values of w) which give different solutions have been studied in [4], of which the case $w \in (0, 1)$, contains no singularities anywhere thus giving us a physically viable solution. Two sub-cases were considered: (a) with $w \in \left(0, \frac{1}{2}\right)$ and (b) with $w \in [\frac{1}{2}, 1)$. (a) has a local minimum at the origin and two global maxima symmetric around the origin at non-zero q. This is given by the blue wave-function in figure 1. On the

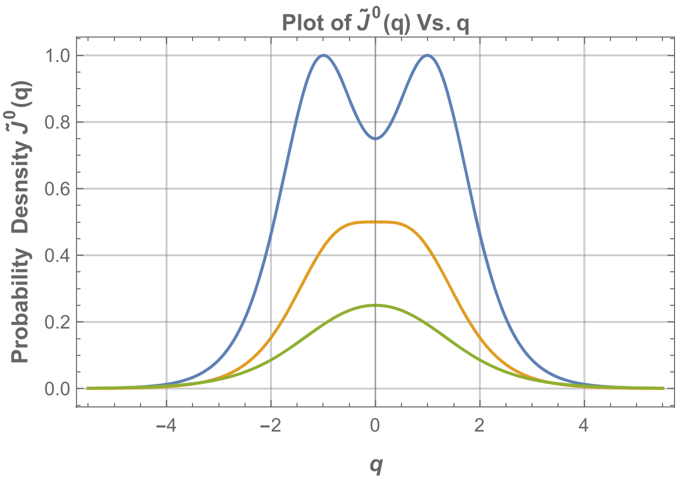

Figure 1. Case (a) *blue*: $w = 0.25$, case (b) *green*: $w = 0.75$, *orange*: $w = 0.5$. Case (a) has local minima at origin and two maximas at two symmetrically opposite sides of the origin at non-zero q and case (b) has global maxima at the origin.

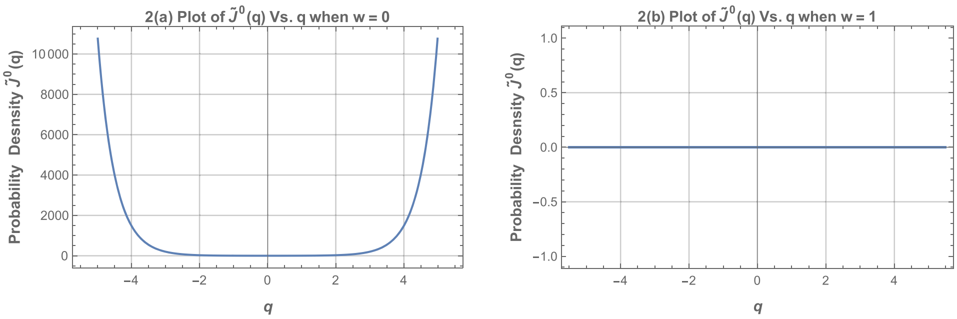

Figure 2. 2(a) is a graph of the probability density when $w = 0$. This produces an unphysical solution. 2(b) is a trivial solution when $w = 1$.

other hand, (b) has a global maxima at the origin and monotonically decays to zero at infinity. This is shown by the orange and green wave-functions in figure 1.

The case $w = 0$ produces an unphysical solution and $w = 1$ gives us a trivial solution. This is shown in figure 2

The substitution of different values of w in the probability density given in (20), corresponds to different Dirac–Hehl–Datta fermions. The non-linearity in the HD equation stems from anti-symmetric nature of the affine connection but the total current is still conserved as in the torsion free case [16]. The normalization factor for all the three cases, considered in figure 1 are different and the probability density integrates to unity in all the cases. Thus the non-static solution (18) and (19) to the HD equation is valid.

3. Calculating the uncertainty relation from the given solitary wave solution

3.1. Standard length scale

Here,

$$w = w_{\mathrm{s}} = -\frac{\Lambda}{\sqrt{2}b} = -2\Lambda l_2 \tag{21}$$

To compute the uncertainty relation, we find the expectation values of q, q^2, p and p^2, where $\hat{p} = -\mathrm{i}\hbar\frac{\partial}{\partial q}$

$$\langle q\rangle = \int_{-\infty}^{\infty} \psi^{\dagger} q\psi \mathrm{d}q = \int_{-\infty}^{\infty} q\psi^{\dagger}\psi \mathrm{d}q = 0 \tag{22}$$

The Gaussian is symmetric with the z axis. Therefore, $\langle q\rangle = 0$

$$\begin{aligned}\langle q^2\rangle &= \int_{-\infty}^{\infty} q^2\psi^{\dagger}\psi \mathrm{d}q \\ &= \int_{-\infty}^{\infty} q^2 \left[(1-w^2)\frac{w+\cosh(2q\sqrt{1-w^2})}{[1+w\ \cosh(2q\sqrt{1-w^2})]^2}\right]\mathrm{d}q\end{aligned} \tag{23}$$

The solution for this integral gives us a conditional expression which assumes all the values of w, real and complex. But since we know that the HD equation produces physical solutions only for $w \in (0, 1)$, w, satisfies all the conditions and thus our answer is of the form:

$$\begin{aligned}\langle q^2\rangle &= \frac{Li_2\left(\frac{-1}{\nu}\right) + Li_2\left(\frac{1}{\nu}\right) + Li_2\left(\frac{-1}{\mu}\right) + Li_2\left(\frac{1}{\mu}\right)}{w\sqrt{1-w^2}} \\ &= g(w)\end{aligned} \tag{24}$$

where, $\nu = \sqrt{\frac{-1+\sqrt{1-w^2}}{w}}$ and $\mu = \sqrt{\frac{-1-\sqrt{1-w^2}}{w}}$ $Li_n(x)$ is a poly-logarithm function also known as Jonquière's function which is of the form:

$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^n}{k^n}$$

The dispersion of position operator is:

$$(\Delta q)^2 = \langle q^2\rangle - \langle q\rangle^2 = g(w) \tag{25}$$

Given in figure 3 is the graph of Δq vs w.

Now moving on to the momentum operator given by $\hat{p} = -\mathrm{i}\hbar\frac{\partial}{\partial q}$,

$$\langle p\rangle = -\mathrm{i}\hbar\int_{-\infty}^{\infty} \psi^{\dagger}\frac{\partial}{\partial q}\psi \mathrm{d}q \tag{26}$$

$$\frac{\partial}{\partial q}\psi = \begin{bmatrix}\frac{\partial}{\partial q}A(q) \\ \frac{\partial}{\partial q}\mathrm{i}B(q)\end{bmatrix} \tag{27}$$

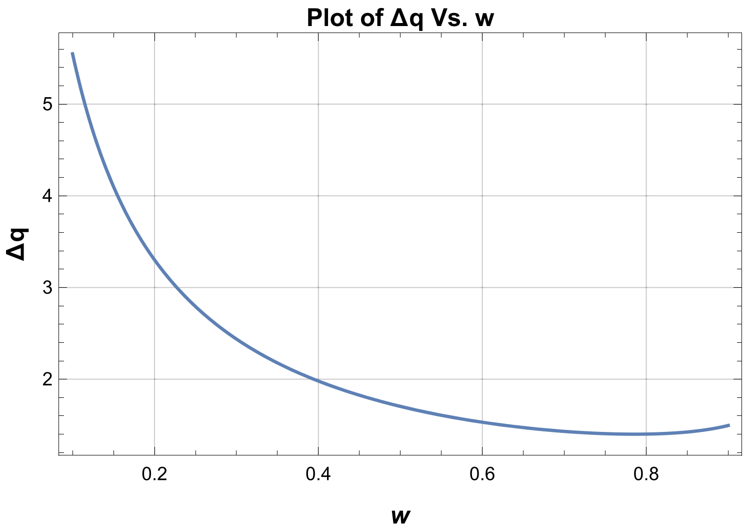

Figure 3. Δq vs w.

$$\frac{\partial}{\partial q}A(q) = \frac{\partial}{\partial q}\left[\frac{\mathrm{i}\sqrt{1-w}(1+w)\cosh(q\sqrt{1-w^2})}{1+w\,\cosh(2q\sqrt{1-w^2})}\right]$$
$$= -\mathrm{i}(1+w)\sqrt{1-w}\sqrt{1-w^2}\left[\frac{(-1+2w+w\,\cosh(2q\sqrt{1-w^2}))\sinh(q\sqrt{1-w^2})}{[1+w\,\cosh(2q\sqrt{1-w^2})]^2}\right] \tag{28}$$

$$\frac{\partial}{\partial q}\mathrm{i}B(q) = \frac{\partial}{\partial q}\left[\frac{-\sqrt{1+w}(1-w)\sinh(q\sqrt{1-w^2})}{1+w\,\cosh(2q\sqrt{1-w^2})}\right]$$
$$= (1-w)\sqrt{1+w}\sqrt{1-w^2}\frac{(-1-2w+w\,\cosh(2q\sqrt{1-w^2}))\cosh(q\sqrt{1-w^2})}{[1+w\,\cosh(2q\sqrt{1-w^2})]^2} \tag{29}$$

$$\psi^\dagger\frac{\partial}{\partial q}\psi = -A(q)\frac{\partial}{\partial q}A(q) + \mathrm{i}B(q)\frac{\partial}{\partial q}\mathrm{i}B(q)$$
$$= (-1+w)(1+w)\sqrt{1-w^2}\frac{(-1+2w^2+w\,\cosh(2q\sqrt{1-w^2}))\sinh(2q\sqrt{1-w^2})}{[1+w\,\cosh(2q\sqrt{1-w^2})]^3} \tag{30}$$

$$\langle p\rangle = -\mathrm{i}\hbar\int_{-\infty}^{\infty}(-1+w)(1+w)\sqrt{1-w^2}\frac{(-1+2w^2+w\,\cosh(2q\sqrt{1-w^2}))\sinh(2q\sqrt{1-w^2})}{[1+w\,\cosh(2q\sqrt{1-w^2})]^3}\mathrm{d}q$$
$$= 0 \tag{31}$$

$\langle p \rangle$ is zero because it is an integral of odd function from $-\infty$ to ∞.

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi^\dagger \frac{\partial^2}{\partial q^2} \psi \mathrm{d}q \tag{32}$$

$$\frac{\partial^2}{\partial q^2}\psi = \begin{bmatrix} \frac{\partial^2}{\partial q^2} A(q) \\ \frac{\partial^2}{\partial q^2} \mathrm{i}B(q) \end{bmatrix} \tag{33}$$

$$\begin{aligned} \frac{\partial^2}{\partial q^2} A(q) = & \frac{\mathrm{i}(1-w)^{3/2}(1+w)^2 \cosh\left(q\sqrt{1-w^2}\right)}{2\left[1+w\cosh\left(2q\sqrt{1-w^2}\right)\right]^3} \Big[2+8w-15w^2 \\ & + 4w(-3+2w)\cosh\left(2q\sqrt{1-w^2}\right) + w^2 \cosh\left(4q\sqrt{1-w^2}\right)\Big] \end{aligned} \tag{34}$$

$$\begin{aligned} \frac{\partial^2}{\partial q^2} \mathrm{i}B(q) = & \frac{(-1+w)^2(1+w)^{3/2} \sinh\left(q\sqrt{1-w^2}\right)}{2\left[1+w\cosh\left(2q\sqrt{1-w^2}\right)\right]^3} \Big[-2+8w+15w^2 \\ & + 4w(3+2w)\cosh\left(2q\sqrt{1-w^2}\right) - w^2 \cosh\left(4q\sqrt{1-w^2}\right)\Big] \end{aligned} \tag{35}$$

$$\begin{aligned} \psi^\dagger \frac{\partial^2}{\partial q^2}\psi = & -A(q)\frac{\partial^2}{\partial q^2}A(q) + \mathrm{i}B(q)\frac{\partial^2}{\partial q^2}\mathrm{i}B(q) = \frac{(1-w^2)^2}{4[1+w\cosh(2q\sqrt{1-w^2})]^4} \\ & \times \Big[(8w-22w^3) + (4-21w^2)\cosh(2q\sqrt{1-w^2}) + (2w(-6+5w^2)) \\ & \times \cosh(4q\sqrt{1-w^2}) + + w^3 \cosh(6q\sqrt{1-w^2})\Big] \end{aligned} \tag{36}$$

Since the function is an even function, the integral finally becomes,

$$\begin{aligned} \langle p^2 \rangle = -\hbar^2 \Bigg[& \int_0^\infty \frac{(1-w^2)^2(8w-22w^3)\mathrm{d}q}{2[1+w\cosh(2q\sqrt{1-w^2})]^4} \\ & + \int_0^\infty \frac{(1-w^2)^2(4-21w^2)\cosh(2q\sqrt{1-w^2})\mathrm{d}q}{2[1+w\cosh(2q\sqrt{1-w^2})]^4} \\ & + \int_0^\infty \frac{(1-w^2)^2 2w(-6+5w^2)\cosh(4q\sqrt{1-w^2})\mathrm{d}q}{2[1+w\cosh(2q\sqrt{1-w^2})]^4} \\ & + \int_0^\infty \frac{(1-w^2)^2 w^2 \cosh(6q\sqrt{1-w^2})\mathrm{d}q}{2[1+w\cosh(2q\sqrt{1-w^2})]^4} \Bigg] \end{aligned} \tag{37}$$

Evaluation of the individual integrals gives us:

$$\int_0^\infty \frac{(1-w^2)^2(8w-22w^3)}{2[1+w\,\cosh(2q\sqrt{1-w^2})]^4}\mathrm{d}q = -w(1-w^2)^{5/2}(-4+11w^2)$$
$$\times\frac{\sqrt{-1+w^2}(11+4w^2)+6(2+3w^2)\arctan\left(\frac{1}{\sqrt{-1+w^2}}\right)-6(2+3w^2)\arctan\left(\frac{1+w}{\sqrt{-1+w^2}}\right)}{12(-1+w^2)^{7/2}}$$
$$= f_1(w) \tag{38}$$

$$\int_0^\infty \frac{(1-w^2)^2(4-21w^2)\cosh(2q\sqrt{1-w^2})}{2[1+w\,\cosh(2q\sqrt{1-w^2})]^4}\mathrm{d}q = (4-21w^2)\sqrt{1-w^2}$$
$$\times\frac{-\sqrt{-1+w^2}(2+13w^2)-6w^2(4+w^2)\arctan\left(\frac{1}{\sqrt{-1+w^2}}\right)+6w^2(4+w^2)\arctan\left(\frac{1+w}{\sqrt{-1+w^2}}\right)}{24w(-1+w^2)^{7/2}}$$
$$= f_2(w) \tag{39}$$

$$\int_0^\infty \frac{(1-w^2)^2 2w(-6+5w^2)\cosh(4q\sqrt{1-w^2})}{2[1+w\,\cosh(2q\sqrt{1-w^2})]^4}\mathrm{d}q = (1-w^2)^{3/2}(-6+5w^2)$$
$$\times\frac{\sqrt{-1+w^2}(-2+9w^2+8w^4)+30w^4\,\arctan\left(\frac{1}{\sqrt{-1+w^2}}\right)-30w^4\,\arctan\left(\frac{1+w}{\sqrt{-1+w^2}}\right)}{12w(-1+w^2)^{7/2}}$$
$$= f_3(w) \tag{40}$$

$$\int_0^\infty \frac{(1-w^2)^2 w^2\,\cosh(6q\sqrt{1-w^2})}{2[1+w\,\cosh(2q\sqrt{1-w^2})]^4}\mathrm{d}q = (1-w^2)^{3/2}$$
$$\times\frac{\sqrt{-1+w^2}(-8+26w^2-33w^4)-30w^6\,\arctan\left(\frac{1}{\sqrt{-1+w^2}}\right)+30w^6\,\arctan\left(\frac{1+w}{\sqrt{-1+w^2}}\right)}{24w(-1+w^2)^{7/2}}$$
$$= f_4(w) \tag{41}$$

$\langle p^2\rangle$ is completely in terms of w. So let us call $\langle p^2\rangle = -\hbar^2 f(w)$, where

$$f(w) = f_1(w) + f_2(w) + f_3(w) + f_4(w) \tag{42}$$

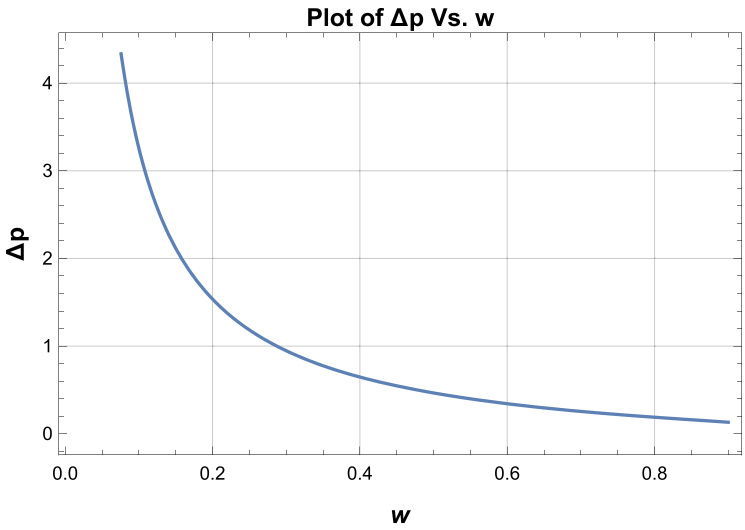

Figure 4. Δp vs w.

The dispersion of momentum operator is thus,

$$\begin{aligned}(\Delta p)^2 &= \langle p^2\rangle - \langle p\rangle^2 = -\hbar^2 f(w)\\ -f(w) &= \frac{(\Delta p)^2}{\hbar^2}\end{aligned} \tag{43}$$

The graph for Δp vs w is given in figure 4.

Though equations (38)–(41) contain the term $\sqrt{-1+w^2}$, after evaluation of all the terms together, the end result is purely real. We would not have been able to plot the above figure for complex values.The uncertainty relation is:

$$(\Delta q)^2(\Delta p)^2 = -f(w)g(w)\hbar^2 \tag{44}$$

Let $-f(w)g(w) = h^2(w)$. The generalized uncertainty principle is obtained as:

$$\begin{aligned}(\Delta q)(\Delta p) &= h(w)\hbar\\ &= (1+h(w)-1)\hbar\\ &= \hbar + \alpha(w)\hbar\\ &= \hbar + \left(\frac{\alpha(w)}{-f(w)}\right)(-f(w))\hbar\\ &= \hbar + \beta(w)(-f(w))\hbar\\ &= \hbar + \beta(w)\frac{(\Delta p)^2}{\hbar^2}\hbar\\ (\Delta q) &= \frac{\hbar}{(\Delta p)} + \beta(w)\frac{\hbar}{(\Delta p)}\frac{(\Delta p)^2}{\hbar^2}\\ (\Delta q) &= \frac{\hbar}{(\Delta p)} + \beta(w)\frac{(\Delta p)}{\hbar}\end{aligned} \tag{45}$$

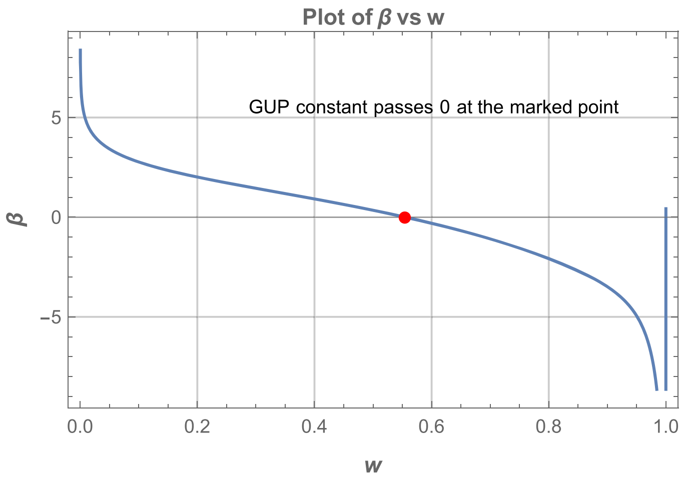

Figure 5. $\beta(w)$ passes through 0 when $w = 0.555\,542$ approximately.

Table 1. Values of functions $f(w), g(w), h(w), \alpha(w), \beta(w)$.

w	$f(w)$	$g(w)$	$h(w)$	$\alpha(w)$	$\beta(w)$
0.1	−3.25722	30.7774	10.0124	9.01243	2.76691
0.2	−1.53665	10.9016	4.09292	3.09292	2.01277
0.3	−0.946858	5.94122	2.37181	1.37181	1.4488
0.4	−0.647309	3.91751	1.59243	0.592432	0.915222
0.5	−0.465485	2.90075	1.16201	0.162005	0.348035
0.6	−0.342801	2.34209	0.896031	−0.103969	−0.303292
0.7	−0.254672	2.04638	0.721911	−0.278089	−1.09195
0.8	−0.188492	1.96371	0.608394	−0.391606	−2.07757
0.9	−0.131402	2.23559	0.541997	−0.458003	−3.48551

Thus, we are able to get uncertainty relation in the form of GUP for Dirac fermions in torsion field. We can therefore assert that torsion has the same effect on uncertainty principle as modified length L_{CS}.

We can also say that HUP is an approximation of GUP. This happens when $h(w) = 1$ or $\beta(w) = 0$. The graph of the modification term $\beta(w)$ with respect to w is given in figure 5.

3.1.1. Checking for different values of w. We are considering that w lies in the range $(0, 1)$. Let us take 0.1 as our step function and find the values of $f(w)$, $g(w)$, $h(w)$, $\alpha(w)$ and $\beta(w)$ in this range.

Table 1 specifies the values of all the functions for the respective w values.

For demonstration purposes, we take $w = 0.1$ and show how we got the GUP equation:

$$\begin{aligned}(\Delta q)(\Delta p) &= h(w)\hbar = 10.0124\hbar \\ &= \hbar + 9.012\,43\hbar \\ (\Delta q) &= \frac{\hbar}{(\Delta p)} + 2.766\,91\frac{(\Delta p)}{\hbar}\end{aligned} \tag{46}$$

Δq vs Δp for the GUP with $w \in (0, 1)$ (figure 6).

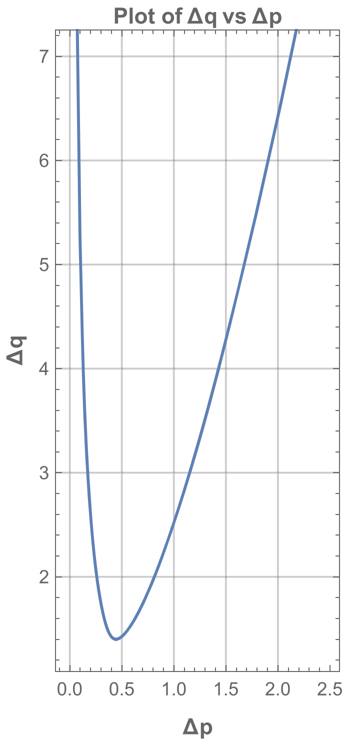

Figure 6. Δq vs Δp for GUP with $w \in (0, 1)$.

The above plot for GUP is similar to the one in [5], thus confirming our result that torsion has the same effect on uncertainty principle as modified length L_{CS}.

3.2. Modified length scale

A relativistic particle of mass m has two length scales associated with it: the half compton line, $\lambda_{\mathrm{C}} = \frac{\hbar}{2Mc}$ and Schwarzschild radius, $R_{\mathrm{S}} = \frac{2GM}{c^2}$. The particle either obeys the relativistic Dirac equation or the field equations of general relativity. This is known experimentally. But theoretically however, both these concepts hold for objects of all masses. The Dirac equation experimentally holds for particles with masses $m \ll m_{\mathrm{Pl}}$ ($\lambda_{\mathrm{C}} \gg L_{\mathrm{Pl}}$), and field equations of GR holds for $m \gg m_{\mathrm{Pl}}$ ($R_{\mathrm{S}} \gg L_{\mathrm{Pl}}$), where m_{Pl} is Planck mass having a value of about 10^{-8} kg.

There is a need for one universal length such that it always stays higher than Planck length, because it is the smallest meaningful length, which limits to λ_{C} in the Planck regime and R_{S} in the classical regime. This Compton–Schwarzschild length, L_{CS} introduced in [5, 9, 14, 17] is given in the following form:

$$\frac{L_{\mathrm{CS}}}{2L_{\mathrm{Pl}}} = \frac{1}{2}\left(\frac{2m}{m_{\mathrm{Pl}}} + \frac{m_{\mathrm{Pl}}}{2\,m}\right) \tag{47}$$

This can also be written as,

$$\begin{aligned} \frac{L_{\mathrm{CS}}}{2L_{\mathrm{Pl}}} &= \frac{m_{\mathrm{Pl}}}{4\,m}\left(1+\frac{4\,m^2}{m_{\mathrm{Pl}}^2}\right) \\ L_{\mathrm{CS}} &= \frac{\lambda_{\mathrm{C}}}{2}\left(1+\frac{R_{\mathrm{S}}^2}{L_{\mathrm{Pl}}^2}\right) \end{aligned} \tag{48}$$

L_{CS} takes the value λ_{C} for $m \ll m_{\mathrm{Pl}}$ and R_{S} for $m \gg m_{\mathrm{Pl}}$.

Now, in our theory, $l_1 = l_2 = L_{\mathrm{CS}}$, the modified HD equation given in (8). This implies, $b(l_2)$, takes the form:

$$b(l_2) = b(L_{\mathrm{CS}}) = \frac{1}{2\sqrt{2}L_{\mathrm{CS}}} \tag{49}$$

Thus, w in the standard theory which was $-\frac{\Lambda}{\sqrt{2}b}$ denoted by w_{s}, now takes the form,

$$w_m = -2\Lambda L_{\mathrm{CS}} \tag{50}$$

$\Rightarrow L_{\mathrm{CS}} = -\frac{w_m}{2\Lambda}$, where l_2 was $-\frac{w_{\mathrm{s}}}{2\Lambda} = \frac{\lambda_{\mathrm{C}}}{2}$

Substituting this in (48), we get,

$$\begin{aligned} L_{\mathrm{CS}} &= \frac{\lambda_{\mathrm{C}}}{2}\left(1+\frac{R_{\mathrm{S}}^2}{L_{\mathrm{Pl}}^2}\right) \\ &= l_2\left(1+\frac{R_{\mathrm{S}}^2}{L_{\mathrm{Pl}}^2}\right) \\ -\frac{w_m}{2\Lambda} &= -\frac{w_{\mathrm{s}}}{2\Lambda}\left(1+\frac{R_{\mathrm{S}}^2}{L_{\mathrm{Pl}}^2}\right) \end{aligned} \tag{51}$$

Thus, we get our modified w to be of the form,

$$w_m = w_{\mathrm{s}}\left(1+\frac{R_{\mathrm{S}}^2}{L_{\mathrm{Pl}}^2}\right) \tag{52}$$

where, $R_{\mathrm{S}} = \frac{2GM}{c^2}$, the gravitational constant $G = 6.674\times10^{-11}\mathrm{Nm}^2\mathrm{kg}^{-2}$, $L_{\mathrm{Pl}} \approxeq 1.6\times 10^{-35}$m.

$$R_{\mathrm{S}} = 1.483\,11\times10^{-27}\times M \tag{53}$$

$$\frac{R_{\mathrm{S}}^2}{L_{\mathrm{Pl}}^2} = 8.592\,26\times10^{15}\times M^2 \tag{54}$$

Let $\frac{R_{\mathrm{S}}^2}{L_{\mathrm{Pl}}^2} = \eta$. Then, $\eta = (8.592\,26\times10^{15})M^2$.

w_m is a function of two variables, M, mass of a particle and w_{s}.

$$w_m = w_{\mathrm{s}}(1+(8.592\,26M^2\times10^{15})) = w_{\mathrm{S}}(1+\eta) \tag{55}$$

As $\eta\to0$, i.e., $M \ll m_{\mathrm{Pl}}$, $w_m \to w_{\mathrm{s}}$, the standard value for which the GUP form is already derived.

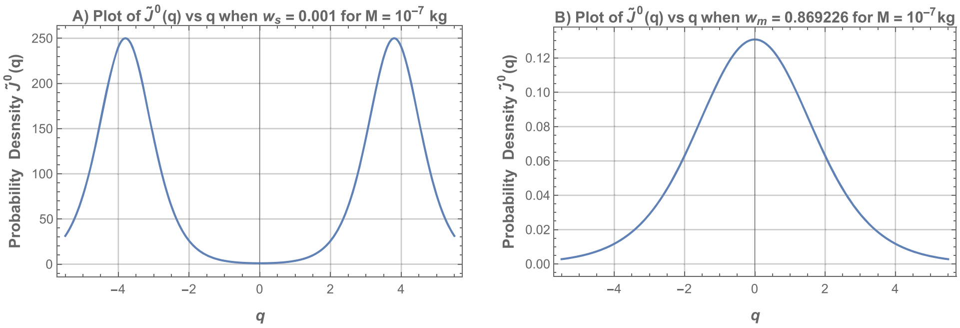

Figure 7. In both these graphs, we have taken the mass of the particle to be a little over Planck mass i.e., 10^{-7}kg. In graph (A), according to the standard length scale, when $w_\mathrm{s} = 0.001$, the probability density has a minima at the origin. Transforming this standard w_s to modified according to (52), $w_m = 0.869\,226$. The probability density of this graph (B) has a maxima at the origin.

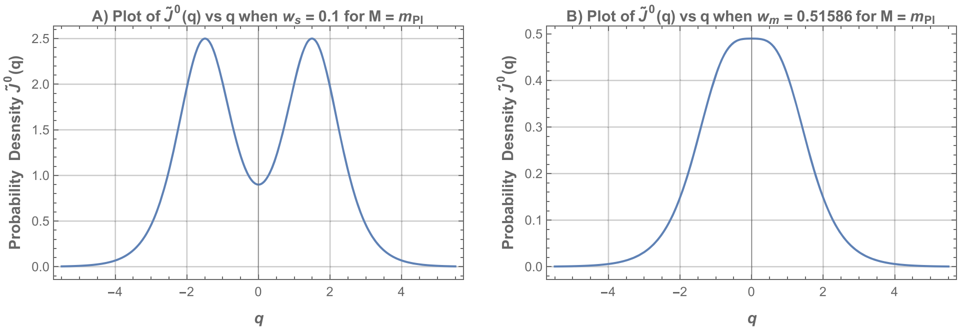

Figure 8. $M = m_{\mathrm{Pl}}$. $w_\mathrm{s} = 0.1 \rightarrow w_m = 0.51$.

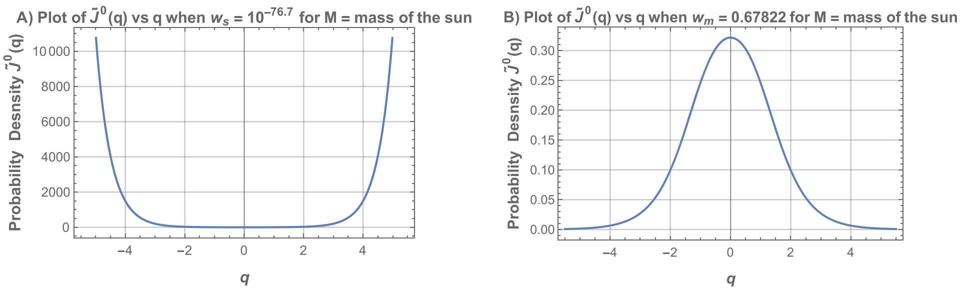

Figure 9. $M =$ mass of the Sun. We can deduce that as $M \rightarrow \infty, w_\mathrm{s} \rightarrow 0$. $w_m = 0.67$ for this particular standard w value.

Now, as $M \gg m_{\mathrm{Pl}}$, $\eta > 1$. Let us take an example. Suppose $M = 10^{-7}$ kg, then $\eta = 85.9226$. If we consider $w_\mathrm{s} = 0.001$, which produces a double-headed wave with local minima at the origin, w_m takes the value $0.869\,226$ which produces the wave to give global maxima at the origin. Figure 7 depicts this transformation (figure 8).

Let us now understand how the probability distribution changes when $M = m_{\mathrm{Pl}} = 2.2 \times 10^{-8}$kg. This is explained in figure 9. We see that at m_{Pl}, the probability distribution when $w_s = 0.1$, is a double-headed wave function which transforms to a wave functions having maxima at the origin using the formula of modified w.

Now, suppose we take the value of M to be quite large, say the mass of the Sun, w_s value must be extremely small of the order 10^{-77} in order to get a physical solution in terms of w_m. Figure 9 explains this.

The above transformations of the probability density can be seen for $M \geqslant m_{\mathrm{Pl}}$. Those values of w_s that produced double-headed wave solutions with minima at the origin, upon introducing the modified length scale L_{CS}, now produces viable solutions with maxima at the origin.

4. Conclusion

We derived the uncertainty principle for a Dirac fermion in a torsion field obeying the HD equation. We found that the non-linear term in the HD equation modifies the Heisenberg uncertainty principle and this modification is of the form of generalized uncertainty principle. Therefore, torsion has the same effect on uncertainty principle as modified length L_{CS}. Upon incorporating the modified length in our solution, we find a transformation of system for particles of mass greater than the Planck mass with double-headed wave solutions with minima at origin to maxima at origin and hence, providing viable solutions.

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