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On conditions for an operator to be in the class $\mathscr{S}_p$

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The Schatten classes $\mathscr{S}_p$ of operators on a Hilbert space $\mathscr{H}$ play a significant role in recent studies in quantum information theory (for instance, see [1]–[3] and references therein). In the present paper we give sufficient conditions for an operator to belong to $\mathscr{S}_p$ in the case when the Hilbert space is the space of an irreducible representation of the canonical commutation relations (CCR) (for example, $\mathscr{H} = L^2(\mathbb{R}^s)$ — the case often occurring in applications). The problem is closely related to the asymptotics of the singular values, and in this question we rely on the classical treatise [4] (see also the papers [5] and [6], devoted to the case $\mathscr{H} = L^2(G)$, where G is a bounded domain in $\mathbb{R}^s$).

Let $\mathscr{H}$ be a separable Hilbert space, and $\mathscr{S}_p$ $(p > 0)$ the Schatten class consisting of all compact operators τ on $\mathscr{H}$ whose singular values $s_n(\tau)$ satisfy the condition $\sum_{n=1}^{\infty} |s_n(\tau)|^p < \infty$. In what follows $\mathscr{H}$ will be the space of an irreducible representation of the CCR:

$$a_j a_k^* - a_k^* a_j = \delta_{jk} I, \qquad a_j a_k - a_k a_j = 0.$$

The *annihilation operators* a_j and *creation operators* a_j^* are necessarily unbounded, and, strictly speaking, these relations hold on a common invariant dense domain $\mathscr{D}$ (for instance, see [7] and [8] for details and rigorous treatments). It is well known that the operators

$$N_j = a_j^* a_j = a_j a_j^* - I \tag{1}$$

are essentially selfadjoint and have pure point spectrum with eigenvalues $n_j = 0, 1, 2, \dots$. The operators N_j, $j = 1, \dots, s$, commute, so that the eigenvalues of the *total number operator* $N = \sum_{j=1}^{s} N_j$ have the form $\sum_{j=1}^{s} n_j$.

Theorem. *Let $\tau \in \mathscr{S}_2$ and assume that for some $k = 1, 2, \dots$ the range of τ is contained in the domains* $\operatorname{dom} a_j^k$ *for all $j = 1, \dots, s$ and that $a_j^k \tau \in \mathscr{S}_2$. Then*

$$s_n(\tau) = o(n^{-[k/(2s)+1/2]}), \tag{2}$$

which implies that $\tau \in \mathscr{S}_p$ for all $p > 2s/(k+s)$.

The idea of the proof is to use the decomposition $\tau = (I+N)^{-k/2}\sigma$, where the operator $\sigma = (I+N)^{k/2}\tau$ is well defined and can be shown to be Hilbert–Schmidt under the conditions of the theorem, so that by equation (III.7.12) in [4]

$$s_n(\sigma) = o(n^{-1/2}). \tag{3}$$

To prove that $\sigma \in \mathscr{S}_2$ we use the inequality

$$(I+N)^k \leqslant (s+1)^{k-1}\biggl(I + \sum_{j=1}^{s} N_j^k\biggr) \tag{4}$$

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and the identity

$$N_j^k=\sum_{r=1}^{k}c_r(a_j^*)^r a_j^r \qquad (c_r>0,\ c_k=1) \tag{5}$$

(following from the second equality in (1)), both holding on an appropriate domain containing $\mathscr{D}$.

The operator $(I+N)^{k/2}$ has a compact inverse $(I+N)^{-k/2}$, so that by the properties of the singular values (see [4], Corollary II.2.2)

$$s_{2n}(\tau)\leqslant s_{2n-1}(\tau)\leqslant s_n\bigl((I+N)^{-k/2}\bigr)s_n(\sigma). \tag{6}$$

Arguing along the lines in [6], we obtain

$$s_n\bigl((I+N)^{-k/2}\bigr)=O(n^{-k/(2s)}). \tag{7}$$

Indeed, the eigenvalues of $(I+N)^{-k/2}$ have the form $\bigl(1+\sum_{j=1}^{s}n_j\bigr)^{-k/2}$, $n_j\geqslant 0$. The number $\#(R)$ of integer points $(n_1,\dots,n_s)$ in the simplex $\sum_{j=1}^{s}x_j\leqslant R$, $x_j\geqslant 0$, is estimated by its volume $R^s/s!$. Given n, take R such that $s_n\bigl((I+N)^{-k/2}\bigr)=(1+R)^{-k/2}$; then $n\leqslant\#(R)=O(R^s)=O\bigl(s_n\bigl((I+N)^{-k/2}\bigr)^{-2s/k}\bigr)$, hence (7) follows. Combining (7), (3), and (6), we get (2).

In the Schrödinger representation, $\mathscr{H}=L^2(\mathbb{R}^s)$ and

$$a_j\psi=\frac{1}{\sqrt{2}}\biggl(x_j+\frac{\partial}{\partial x_j}\biggr)\psi=\frac{1}{\sqrt{2}}\,\mathrm{e}^{-x_j^2/2}\frac{\partial}{\partial x_j}\,\mathrm{e}^{x_j^2/2}\psi,$$

hence $a_j^k\psi=2^{-k/2}\mathrm{e}^{-x_j^2/2}\partial^k\mathrm{e}^{x_j^2/2}\psi/\partial x_j^k$ (all the derivatives are generalized). Then the condition of the theorem takes the following form: the kernel $\tau(x,y)$ of the operator τ, as well as all the functions

$$\mathrm{e}^{-x_j^2/2}\frac{\partial^k}{\partial x_j^k}\,\mathrm{e}^{x_j^2/2}\tau(x,y), \qquad j=1,\dots,s,$$

are square integrable with respect to (x,y). Similar formulations are obtained for the Bargmann–Berezin representation in the space of holomorphic functions and for the 'noncommutative Fourier transform' of the operator τ [8].

The action of the operator a is especially simple when τ has the P-representation with respect to the coherent states: $\tau=\int_{\mathbb{C}^s}P(z)|z\rangle\langle z|\,d^sz$ (here $P(z)$ is a complex-valued function; see [8], for instance). Then the condition of the theorem takes the form

$$\int_{\mathbb{C}^s}\int_{\mathbb{C}^s}\overline{P(z)}P(w)\biggl[1+\sum_{j=1}^{s}(z_j^*w_j)^k\biggr]\exp(-|z-w|^2)\,d^sz\,d^sw<\infty.$$

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