To the blessed memory of my teachers,
Ludwig Dmitrievich Faddeev
and Askold Ivanovich Vinogradov

Etudes of the resolvent

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Abstract. Based on the notion of the resolvent and on the Hilbert identities, this paper presents a number of classical results in the theory of differential operators and some of their applications to the theory of automorphic functions and number theory from a unified point of view. For instance, for the Sturm–Liouville operator there is a derivation of the Gelfand–Levitan trace formula, and for the one-dimensional Schrödinger operator a derivation of Faddeev's formula for the characteristic determinant and the Zakharov–Faddeev trace identities. Recent results on the spectral theory of a certain functional-difference operator arising in conformal field theory are then presented. The last section of the survey is devoted to the Laplace operator on a fundamental domain of a Fuchsian group of the first kind on the Lobachevsky plane. An algebraic scheme is given for proving analytic continuation of the integral kernel of the resolvent of the Laplace operator and the Eisenstein–Maass series. In conclusion there is a discussion of the relationship between the values of the Eisenstein–Maass series at Heegner points and the Dedekind zeta-functions of imaginary quadratic fields, and it is explained why pseudo-cusp forms for the case of the modular group do not provide any information about the zeros of the Riemann zeta-function.

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Keywords: resolvent of an operator, characteristic determinant of an operator, Hilbert identities, Sturm–Liouville operator, Gelfand–Levitan trace formula, Schrödinger operator, functional-difference operator, Laplace operator on the Lobachevsky plane, eigenfunction expansions, Jost solutions, Zakharov–Faddeev trace identities, Eisenstein–Maass series, Dedekind zeta-functions of imaginary quadratic fields, Riemann zeta-function.

Contents

1. Introduction

This survey is an extended and revised version of my talk at the meeting of
the Moscow Mathematical Society on April 1, 2014. The reader is presented with
a collage of classical results in the theory of differential operators, written from
a single viewpoint, together with some applications to automorphic functions and
number theory. The relationship between these topics reflects the unity of mathe-
matics, and their choice reflects the tastes and interests of the author, influenced by
the traditions of the Leningrad–St. Petersburg mathematical school. The survey is
intended for a broad readership, from specialists in operator theory and functional
analysis to algebraic geometers and theoretical physicists. To convey to the mod-
ern reader the elegance and beauty of the results, the achievements of the Soviet
mathematical school, we chose a neoclassical style of presentation.

Let us describe the contents of this survey in more detail. In §2 we recall
the notion of the resolvent, which plays a main role in the theory of self-adjoint
operators acting in a Hilbert space. In §2.2 we present the Hilbert identities for
the resolvent of a self-adjoint operator A, and in §2.3 we give the definition of the
regularized determinant det A. Section 3 is devoted to the classical Sturm–Liouville
theory. Thus, in §3.1 we briefly recall well-known facts, and in §3.2 we derive the
celebrated Gelfand–Levitan trace formula. Section 4 is devoted to the presentation
of the main results for the one-dimensional Schrödinger operator H. In particular,
in §4.1 we introduce the Jost solutions and recall the formula for the resolvent of
the operator H, in §4.2 we derive Faddeev's formula for the regularized determinant
$\det(H - \lambda I)$, and in §4.3 we present the derivation of the Zakharov–Faddeev trace
identities.

Section 5 is based on [43] and is devoted to the spectral analysis of a certain
functional-difference operator, a special pseudodifferential operator H of infinite
order which arises in conformal field theory and in the representation theory of
quantum groups. Thus, in §5.1 we introduce the self-adjoint Weyl operators U and
V in the definition of the operator $H = U + U^{-1} + V$, and in §5.2 we consider the

'unperturbed' operator $H_0 = U + U^{-1}$. In §5.3 we define a solution to the scattering problem and the Jost functions, and we give an explicit formula for the resolvent of the self-adjoint operator H acting in the Hilbert space $L^2(\mathbb{R})$, together with the eigenfunction expansion theorem. Finally, in §5.4 we describe functional-difference operators for mirror curves.

The last section, §6, is devoted to the spectral theory of the Laplace operator A on a fundamental domain of a Fuchsian group of the first kind Γ on the Lobachevsky plane $\mathbb{H}$. Of particular interest here is the case of a non-compact fundamental domain, when A has a continuous spectrum (considered in Faddeev's classical paper [14]). Thus, in §6.1 we follow [14] and give an algebraic scheme for proving a fundamental result on meromorphic continuation with respect to the variable s of the integral kernel of the resolvent $(A - s(1-s)I)^{-1}$ of the operator A and that of the Eisenstein–Maass series $E(z, s)$, to the domain $0 < \operatorname{Re} s \leqslant 1$. Moreover, in the half-plane $\operatorname{Re} s \geqslant 1/2$, the resolvent kernel can have only simple poles and only poles on the line $\operatorname{Re} s = 1/2$ and on the interval $[1/2, 1]$, while the Eisenstein–Maass series can have only poles on the interval $[1/2, 1]$, which immediately gives the eigenfunction expansion theorem for the operator A. As explained in [45], it follows from the celebrated Selberg trace formula that the regularized determinant of the operator A can be expressed in terms of the Selberg zeta-function.

The arithmetic case $\Gamma = \operatorname{PSL}(2, \mathbb{Z})$ — the modular group — is considered in §§6.2 and 6.3. Thus, in §6.2 we discuss as a curiosity the 'sensation' at the end of the 1970s about the connection of the eigenvalues of the Laplace operator with the zeros of the Riemann zeta-function and the L-series by means of so-called pseudo-cusp forms. Using the first Hilbert identity, we explain why pseudo-cusp forms do not provide any information about the location of these zeros.

Finally, §6.3 contains a discussion of the relationship between the values of the Eisenstein–Maass series at Heegner points and the Dedekind zeta-functions of imaginary quadratic fields. Using the uniform distribution of the Heegner points in a fundamental domain of the modular group — the Linnik asymptotics — we naively 'prove' the Riemann hypothesis. Of course, such an application of Linnik's asymptotics is unacceptable, as is confirmed by an analogue of the classical Vinogradov–Gauss formula in the critical strip, obtained in the paper [48] by A. I. Vinogradov and the author. Nevertheless, attempts to relate the Laplace operator, pseudo-cusp forms, and the Heegner points to zeros of Dedekind zeta-functions of imaginary quadratic fields continue to this day. Evidence of this can be found in the papers of Zagier [49] and Colin de Verdière [7], [8] in the early 1980s, as well as in recent studies by Bombieri and Garrett (see the conference talks [3], [18]).

2. Main definitions

In response to questions in quantum mechanics, von Neumann developed a theory of unbounded self-adjoint operators acting in a Hilbert space. According to the Dirac–von Neumann axioms (see [41], for instance), it is self-adjoint operators that correspond to quantum observables, and the simplest of them — the position and the momentum of a particle — are described by unbounded operators.

For the convenience of the reader, we follow the classical monograph [2] and briefly recall the standard notation and basic facts from the theory of self-adjoint operators.

2.1. Self-adjoint operators. An operator A with dense domain $D(A)$ in a Hilbert space $\mathscr{H}$ is said to be symmetric if

$$(Af, g) = (f, Ag)$$

for all elements $f, g \in D(A)$, where $(\,\cdot\,, \cdot\,)$ is the inner product in $\mathscr{H}$. The adjoint operator A^* to the densely defined operator A is defined as follows: $g \in D(A^*)$ if there is a $g^* \in \mathscr{H}$ such that

$$(Af, g) = (f, g^*)$$

for all $f \in D(A)$, and then $g^* = A^*g$. The operator A is said to be self-adjoint if $A = A^*$. Clearly, every self-adjoint operator is symmetric. An operator A is said to be closed if its graph $\Gamma(A)$ — the set of pairs $\{f, Af\}$ for all $f \in D(A)$ — is a closed subset of $\mathscr{H} \oplus \mathscr{H}$; A admits a closure if the closure of $\Gamma(A)$ in $\mathscr{H} \oplus \mathscr{H}$ is the graph of an operator, that is, there is an operator $\overline{A}$ such that $\overline{\Gamma(A)} = \Gamma(\overline{A})$. A closed operator defined on the whole of $\mathscr{H}$ is bounded.

A symmetric operator A is said to be essentially self-adjoint if its closure $\overline{A}$ is a self-adjoint operator. A typical example is the operator $A = id/dx$ acting in the Hilbert space $L^2(\mathbb{R})$ and defined on the linear space $C_0^\infty(\mathbb{R})$ of smooth functions with compact support. Its closure $\overline{A}$ is a self-adjoint operator with the domain $D(\overline{A}) = W_2^1(\mathbb{R})$, the Sobolev space of absolutely continuous square-integrable functions with square-integrable derivative.

2.2. The resolvent and the spectral theorem. Let A be a closed operator. The values $\lambda \in \mathbb{C}$ for which the operator[1]

$$R_\lambda(A) = (A - \lambda I)^{-1}$$

(the resolvent[2] of A) exists and is defined everywhere on $\mathscr{H}$ are called regular values. The set $\rho(A) \subseteq \mathbb{C}$ of regular values is open and is called the resolvent set. The spectrum of an operator A is the complement to the regular set: $\sigma(A) = \mathbb{C} \setminus \rho(A)$. For a self-adjoint operator, $\sigma(A) \subseteq \mathbb{R}$.

The resolvent of an operator A satisfies the relation

$$R_\lambda(A) - R_\mu(A) = (\lambda - \mu) R_\lambda(A) R_\mu(A), \quad \lambda, \mu \in \rho(A), \tag{2.1}$$

which generalizes the elementary algebraic formula

$$\frac{1}{a - \lambda} - \frac{1}{a - \mu} = \frac{\lambda - \mu}{(a - \lambda)(a - \mu)}$$

and is called the first Hilbert identity. It follows from (2.1) that $R_\lambda(A)$ is a holomorphic function on $\rho(A)$ with values in the Banach algebra $\mathscr{L}(\mathscr{H})$ of bounded

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[1] Here I is the identity operator on $\mathscr{H}$.
[2] The notation $R(\lambda, A)$ is also used.

operators on $\mathscr{H}$. Let A and B be closed operators with a common domain. Then their resolvents satisfy the so-called Hilbert second identity

$$R_\lambda(A) - R_\lambda(B) = R_\lambda(A)(B - A)R_\lambda(B), \qquad \lambda \in \rho(A) \cap \rho(B), \qquad (2.2)$$

which generalizes the algebraic formula

$$\frac{1}{a - \lambda} - \frac{1}{b - \lambda} = \frac{b - a}{(a - \lambda)(b - \lambda)}.$$

The spectral theorem of von Neumann is a fundamental fact in the theory of self-adjoint operators acting in a Hilbert space. In particular, for each self-adjoint operator A there is unique projection-valued countably additive measure E, defined on the σ-algebra $\mathscr{B}$ of Borel subsets of the real line, such that $\mathsf{E}(\varnothing) = 0$, $\mathsf{E}(\mathbb{R}) = I$,

$$D(A) = \left\{ f \in \mathscr{H} : \int_{-\infty}^{\infty} \lambda^2 \, d(\mathsf{E}_\lambda f, f) < \infty \right\},$$

and for $f \in D(A)$

$$Af = \int_{-\infty}^{\infty} \lambda \, d\mathsf{E}_\lambda f,$$

where $\mathsf{E}_\lambda = \mathsf{E}((-\infty, \lambda))$ and the integral is taken as a limit of Riemann–Stieltjes sums in the strong topology on $\mathscr{H}$. The relationship between the projection-valued measure E and the resolvent R_λ of A is given by the formula

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_a^b (R_{\lambda + i\varepsilon} - R_{\lambda - i\varepsilon}) \, d\lambda = \mathsf{E}((a, b)) + \frac{1}{2}(\mathsf{E}(\{a\}) + \mathsf{E}(\{b\})), \qquad (2.3)$$

sometimes called Stone's formula. This formula is an operator version of the classical Sokhotski–Plemelj formula

$$\frac{1}{\lambda - i0} - \frac{1}{\lambda + i0} = 2\pi i \delta(\lambda)$$

in the theory of distributions, and is a basis of the eigenfunction expansion theorem for differential operators.

2.3. The determinant of an operator. Here we briefly recall the notion of the characteristic determinant of a self-adjoint operator (see [22]). In the simplest case, when K is a compact self-adjoint operator with trace (a trace-class or nuclear operator), the Fredholm determinant is given by the simple formula

$$\det(I - \lambda K) = \prod_i (1 - \lambda_i \lambda), \qquad (2.4)$$

where the λ_i are the eigenvalues of the operator K, and this is an entire function.[3] If the operator K is invertible, then from (2.4) we easily obtain

$$\frac{d}{d\lambda} \log \det(I - \lambda K) = -\operatorname{Tr} R_\lambda(A), \quad \text{where} \quad A = K^{-1}.$$

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[3]Since $\operatorname{Tr}|K| = \sum_i |\lambda_i| < \infty.$

This formula can be generalized to a wider class of operators. Namely, if the resolvent $R_\lambda(A)$ is of trace class, then the characteristic determinant $\det(A - \lambda I)$ of A is determined (up to a multiplicative constant) from the relation

$$\frac{d}{d\lambda} \log \det(A - \lambda I) = -\operatorname{Tr} R_\lambda(A). \tag{2.5}$$

As we shall see below, this formula makes sense if we understand Tr to be a properly regularized trace of the resolvent. If, in addition, $R_\lambda(A)$ is an integral operator acting in $L^2(X)$ with integral kernel $R_\lambda(x, y)$ that is continuous on $X \times X$, where the subset $X \subset \mathbb{R}$ is bounded, then by a well-known theorem[4]

$$\operatorname{Tr} R_\lambda(A) = \int_X R_\lambda(x, x)\, dx. \tag{2.6}$$

In the case $X = \mathbb{R}$ the following formula holds:[5]

$$\operatorname{Tr} R_\lambda(A) = \lim_{n \to \infty} \int_{-n}^{n} R_\lambda(x, x)\, dx. \tag{2.7}$$

A more general way of introducing a regularized determinant is based on the notion of the zeta-function of an elliptic operator (see the survey [39], as well as [41] and references there). For simplicity, we assume that A is an elliptic operator with a purely discrete spectrum consisting of non-negative eigenvalues λ_n of finite multiplicity accumulating at infinity. The zeta-function of A is defined by

$$\zeta_A(s) = \sum_{\lambda_n > 0} \frac{1}{\lambda_n^s},$$

where it is assumed that the series converges absolutely for $\operatorname{Re} s > a$ for some $a > 0$. Under fairly general assumptions (for example, for the Sturm–Liouville operator considered below), $\zeta_A(s)$ admits an analytic (meromorphic) continuation to a domain containing the half-plane $\operatorname{Re} s \geqslant 0$ and is regular for $s = 0$. Then the regularized determinant $\det A$ of A is defined by

$$\det A = \exp\{-\zeta_A'(0)\},$$

where the prime indicates the derivative with respect to s. The characteristic determinant $\det(A - \lambda I)$ is defined in a similar way, and in many examples this definition is consistent with the formula (2.5).

3. Sturm–Liouville problem

3.1. The resolvent and the eigenfunction expansion. Following the classical monograph [33], we consider the simplest problem of finding all non-trivial solutions of the Sturm–Liouville equation

$$-y'' + v(x)y = \lambda y, \qquad 0 \leqslant x \leqslant \pi, \tag{3.1}$$

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[4]See the monograph [22], § III.10, as well as [31], § 30.5, Theorem 12.

[5]It is sufficient to approximate $R_\lambda(A)$ by the operators $P_n R_\lambda(A) P_n$, where the P_n are the orthogonal projections onto $L^2(-n, n) \subset L^2(\mathbb{R})$, and to use Theorem 6.3 in Chap. III in [22].

with zero boundary conditions

$$y(0) = y(\pi) = 0,$$

where $v(x)$ is a continuous real-valued function on the interval $[0, \pi]$. The differential operator

$$\mathcal{L} = -\frac{d^2}{dx^2} + v(x)$$

is symmetric[6] on the subspace $C_0^\infty(0, \pi)$ of smooth functions with compact support. Its Friedrichs extension is a self-adjoint Sturm–Liouville operator L in $L^2(0, \pi)$ with the domain

$$D(L) = \{y \in W_2^2(0, \pi) \colon y(0) = y(\pi) = 0\},$$

where $W_2^2(0, \pi)$ is the Sobolev space of square-integrable functions on $(0, \pi)$ with square-integrable generalized derivatives up to second order.

Let $y_1(x, \lambda)$ and $y_2(x, \lambda)$ be solutions of equation (3.1) with the boundary conditions

$$y_1(0, \lambda) = 0, \quad y_1'(0, \lambda) = 1 \quad \text{and} \quad y_2(\pi, \lambda) = 0, \quad y_2'(\pi, \lambda) = 1,$$

and let $d(\lambda) = y_1(\pi, \lambda)$. The function $d(\lambda)$ is entire of order $1/2$ with simple zeros λ_n corresponding to the simple eigenvalues of the operator L and tending to infinity. When $v(x) \in C^1(0, \pi)$, the following asymptotics hold:

$$\lambda_n = n^2 + c + O\left(\frac{1}{n}\right), \quad \text{where} \quad c = \frac{1}{\pi} \int_0^\pi v(x)\, dx. \tag{3.2}$$

The operator L has a purely discrete spectrum and the corresponding eigenfunction expansion theorem follows from (2.3). Specifically, the resolvent $R_\lambda = (L - \lambda I)^{-1}$ of L is an operator-valued meromorphic function with simple poles at $\lambda = \lambda_n$ and with residues that are projection operators onto the one-dimensional subspaces corresponding to the eigenfunctions. For $\lambda \neq \lambda_n$ the resolvent R_λ is a bounded integral operator on $L^2(0, \pi)$ with integral kernel

$$R_\lambda(x, \xi) = \frac{1}{W(y_1, y_2)(\lambda)}(y_1(x, \lambda)y_2(\xi, \lambda)\theta(\xi - x) + y_1(\xi, \lambda)y_2(x, \lambda)\theta(x - \xi)), \tag{3.3}$$

where $W(f, g) = f'g - fg'$ is the Wronskian of functions f and g, so that the Wronskian of two solutions of (3.1) does not depend on x and $W(y_1, y_2)(x, \lambda) = -d(\lambda)$, and $\theta(x)$ is the Heaviside function: $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x \geqslant 0$.

Indeed, the kernel $R_\lambda(x, \xi)$ satisfies the equation

$$\left(-\frac{\partial^2}{\partial x^2} + v(x) - \lambda\right)R_\lambda(x, \xi) = \delta(x - \xi), \qquad 0 < x, \xi < \pi \tag{3.4}$$

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[6]The symmetric operator $\mathcal{L}$ acting in $L^2(0, \pi)$ has defect indices $(2, 2)$, and its self-adjoint extensions are described by the Sturm–Liouville boundary conditions.

(where $\delta(x)$ is the Dirac delta function), which follows from (3.1) and the elementary formula

$$\theta'(x) = \delta(x)$$

in the theory of distributions. Using (3.4), we easily show that the range of R_λ is $D(L)$ and $(L - \lambda I)R_\lambda = I$.

3.2. Characteristic determinant and trace identities. It follows from (3.2) that the operator R_λ is of trace class when $\lambda \neq \lambda_n$, so that by using the definition of $\det(L - \lambda I)$ in terms of the operator zeta-function, it is not difficult to prove the formula (2.5) (see [41], §5.5.1, for example). Since the integral kernel $R_\lambda(x, \xi)$ of the trace-class operator R_λ is continuous on $[0, \pi] \times [0, \pi]$, we get from (2.6) that

$$\operatorname{Tr} R_\lambda = \int_0^\pi R_\lambda(x, x)\, dx = -\frac{1}{d(\lambda)} \int_0^\pi y_1(x, \lambda) y_2(x, \lambda)\, dx. \tag{3.5}$$

It is easy to compute the integral in (3.5) using the following classical trick [13]. Specifically, we differentiate (3.1) for $y_1(x, \lambda)$ with respect to λ:

$$-\ddot{y}_1'' + v(x)\dot{y}_1 = \lambda\dot{y}_1 + y_1,$$

where the dot means the λ-derivative. We multiply this equation by $y_2(x, \lambda)$ and subtract the equation for $y_2(x, \lambda)$, multiplied by $\dot{y}_1(x, \lambda)$. As a result, we obtain the identity

$$y_1 y_2 = \dot{y}_1 y_2'' - \dot{y}_1'' y_2 = -W(\dot{y}_1, y_2)'. \tag{3.6}$$

Recalling the definition of the solutions y_1 and y_2, we get from this that

$$\int_0^\pi y_1(x, \lambda) y_2(x, \lambda)\, dx = \dot{y}_1(\pi, \lambda) = \dot{d}(\lambda),$$

and comparing (2.5) with (3.5), we obtain

$$\det(L - \lambda I) = C d(\lambda)$$

with some constant C. By computing the asymptotics as $\lambda \to -\infty$ in this formula it is easy to get that $C = 2$ (see [41], §5.5.1). For example,

$$\det\left(-\frac{d^2}{dx^2} - \lambda I\right) = 2\,\frac{\sin \pi\sqrt{\lambda}}{\sqrt{\lambda}}.$$

Summarizing, we obtain the following Hadamard product representation for the entire function $d(\lambda)$:

$$d(\lambda) = \frac{\det L}{2}(-\lambda)^\delta \prod_{\lambda_n \neq 0}\left(1 - \frac{\lambda}{\lambda_n}\right), \tag{3.7}$$

where $\delta = 1$ if $\lambda = 0$ is an eigenvalue for L, and $\delta = 0$ otherwise.

When $v(x) \in C^2(0, \pi)$, one can thoroughly investigate the asymptotics of the function $d(\lambda)$ as $\lambda \to -\infty$ both with the help of the differential equation (3.1)

and with the help of a Hadamard product and the asymptotics (3.2) of the eigenvalues with the remainder term $O(n^{-2})$. Specifically, put $\lambda = -k^2$, where $k > 0$. The differential equation (3.1) with respect to y_1 is equivalent to the Liouville integral equation

$$y_1(x, \lambda) = \frac{\sinh kx}{k} + \frac{1}{k} \int_0^x \sinh\{k(x-t)\} v(t) y_1(t, \lambda)\, dt.$$

Solving it by the method of successive approximations and integrating by parts, we get after simple calculations that as $k \to \infty$

$$d(\lambda) = \frac{e^{\pi k}}{2k} \left\{ 1 + \frac{\pi c}{2k} + \frac{1}{8k^2}\left(\pi^2 c^2 - 2(v(0) + v(\pi))\right) + O\left(\frac{1}{k^3}\right) \right\}. \qquad (3.8)$$

On the other hand, using the Euler formula for the function $\sinh \pi k$, we rewrite the right-hand side of (3.7) as[7]

$$\Phi(\lambda) = \frac{\det L}{2} \frac{\sinh \pi k}{\pi k} \prod_{n=1}^{\infty} \frac{n^2}{\lambda_n} \prod_{n=1}^{\infty} \frac{k^2 + \lambda_n}{k^2 + n^2} = C_1 \frac{\sinh \pi k}{\pi k} \varphi(k),$$

where

$$C_1 = \frac{\det L}{2} \prod_{n=1}^{\infty} \frac{n^2}{\lambda_n} \quad \text{and} \quad \varphi(k) = \frac{\sinh \pi k}{\pi k} \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n^2 - n^2}{k^2 + n^2} \right).$$

Put $s_\lambda = \sum_{n=1}^{\infty} (\lambda_n - n^2 - c)$. It follows from (3.2) with the remainder term $O(n^{-2})$ that

$$\sum_{n=1}^{\infty} \frac{\lambda_n^2 - n^2}{k^2 + n^2} = c \sum_{n=1}^{\infty} \frac{1}{k^2 + n^2} + \frac{1}{k^2} s_\lambda + O\left(\frac{1}{k^3}\right)$$

$$= \frac{\pi c \coth \pi k}{2k} - \frac{c}{2k^2} + \frac{1}{k^2} s_\lambda + O\left(\frac{1}{k^3}\right).$$

From this it is now simple to show (see [33]) that as $k \to +\infty$

$$\Phi(\lambda) = \frac{C_1 e^{\pi k}}{2\pi k} \left\{ 1 + \frac{\pi c}{2k} + \frac{1}{8k^2}\left(\pi^2 c^2 - 4c + 8s_\lambda\right) + O\left(\frac{1}{k^3}\right) \right\}. \qquad (3.9)$$

Comparing the coefficients in the asymptotic formulae (3.8) and (3.9), we obtain

$$C_1 = \pi \quad \text{and} \quad s_\lambda - \frac{c}{2} = -\frac{v(0) + v(\pi)}{4}.$$

The first of these formulae gives the expression

$$\det A = 2\pi \prod_{n=1}^{\infty} \frac{\lambda_n}{n^2} = 2\pi \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n - n^2}{n^2} \right)$$

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[7]Here we assume that $\delta = 0$ in (3.7), which can always be achieved by shifting $v(x)$.

for the regularized determinant of the Sturm–Liouville operator, while the second formula, written in the form[8]

$$\sum_{n=1}^{\infty} \left(\lambda_n - n^2 - \frac{1}{\pi} \int_0^\pi v(x)\,dx \right) = \frac{1}{2\pi} \int_0^\pi v(x)\,dx - \frac{v(0) + v(\pi)}{4},$$

is the celebrated Gelfand–Levitan trace formula [21] for the regularized trace of the operator L! In the case where $v(x) \in C^\infty(0, \pi)$, the formulae for the regularized traces of all positive-integer powers of the operator L were obtained in Dikii's[9] classical paper [10] (see also the survey [37] and references there).

4. One-dimensional Schrödinger operator

Leaving aside the case of the radial Schrödinger equation (see the survey [12], the monograph [35], and references there), we consider the Schrödinger equation on the whole of the real line

$$- y'' + v(x)y = \lambda y, \qquad -\infty < x < \infty. \tag{4.1}$$

Here the potential—a measurable, real-valued function $v(x)$—is assumed to be rapidly decaying:

$$\int_{-\infty}^{\infty} (1 + |x|)|v(x)|\,dx < \infty. \tag{4.2}$$

Without loss of generality, we assume $v(x)$ to be continuous. Under condition (4.2), the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + v(x)$$

is defined on the functions $\psi \in L^2(\mathbb{R})$ that are twice differentiable on $\mathbb{R}$ and such that $-\psi'' + v(x)\psi \in L^2(\mathbb{R})$, and it is self-adjoint in $L^2(\mathbb{R})$. It is convenient to write

$$H = H_0 + V,$$

where $H_0 = -d^2/dx^2$ is the free Schrödinger operator with $D(H_0) = W_2^2(\mathbb{R})$, and V is the operator of multiplication by $v(x)$ in $L^2(\mathbb{R})$. The operator H has an absolutely continuous spectrum of multiplicity 2 filling the semi-axis $[0, \infty)$, and finitely many simple negative eigenvalues $\lambda_1, \ldots, \lambda_n$. Let us examine this more carefully (see [13], [15], [35], and [41] for details).

4.1. Jost solutions and the resolvent. It is convenient to use the parametrization $\lambda = k^2$, in which the complex λ-plane cut along $[0, \infty)$ corresponds to the upper half-plane of the variable k, the so-called 'physical sheet' of the Riemann surface of the function $k = \sqrt{\lambda}$. Under the condition (4.2) the Jost solutions are

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[8]We take the opportunity to note that this formula corrects a typing error in the corresponding formula in § 2.2 of [42].

[9]*Editor's note.* In his translation, the author refers to Дикий as 'Dikii', and this is indeed how his name was transliterated in the translations of his Russian papers. However, later he always wrote 'Dickey', and this is also reflected in some more recent references.

defined, namely, the functions $f_1(x, k)$ and $f_2(x, k)$ satisfying (4.1) and having the asymptotics

$$f_1(x, k) = e^{ikx} + o(1) \quad \text{as } x \to \infty$$

and

$$f_2(x, k) = e^{-ikx} + o(1) \quad \text{as } x \to -\infty.$$

The proof is based on the integral equation of Volterra type

$$f_1(x, k) = e^{ikx} - \int_x^\infty \frac{\sin k(x - t)}{k} v(t) f_1(t, k) \, dt,$$

and the analogous equation for $f_2(x, k)$. The following estimate holds for $k \neq 0$:

$$|e^{-ikx} f_1(x, k) - 1| \leqslant \frac{\sigma(x)}{k} \exp\left\{\frac{1}{|k|} \sigma(x)\right\}, \quad \text{where} \quad \sigma(x) = \int_x^\infty |v(t)| \, dt, \quad (4.3)$$

and moreover $\sigma \in L^1(a, \infty)$ for each $a \in \mathbb{R}$ and $\lim_{x \to \infty} \sigma(x) = 0$. Similarly,

$$|e^{ikx} f_2(x, k) - 1| \leqslant \frac{\tilde{\sigma}(x)}{k} \exp\left\{\frac{1}{|k|} \tilde{\sigma}(x)\right\}, \quad \text{where} \quad \tilde{\sigma}(x) = \int_{-\infty}^x |v(t)| \, dt, \quad (4.4)$$

and moreover $\tilde{\sigma} \in L^1(-\infty, a)$ for each $a \in \mathbb{R}$ and $\lim_{x \to -\infty} \tilde{\sigma}(x) = 0$.

For fixed x the Jost solutions $f_1(x, k)$ and $f_2(x, k)$ admit analytic continuation to the half-plane $\text{Im } k > 0$, and for fixed k they satisfy the estimates (4.3) and (4.4). For real k

$$f_2(x, k) = a(k) f_1(x, -k) + b(k) f_1(x, k), \quad (4.5)$$

where $\overline{a(k)} = a(-k)$, $\overline{b(k)} = b(-k)$, and

$$|a(k)|^2 = 1 + |b(k)|^2.$$

The functions $a(k)$ and $b(k)$ are called the transition coefficients.[10] For the coefficient $a(k)$ we have the formula

$$a(k) = \frac{1}{2ik} W(f_1(x, k), f_2(x, k)), \quad (4.6)$$

implying that $a(k)$ admits analytic continuation to the upper half-plane $\text{Im } k > 0$ and satisfies

$$a(k) = 1 + O\left(\frac{1}{|k|}\right) \quad \text{as } k \to \infty.$$

The function $a(k)$ in the upper half-plane $\text{Im } k > 0$ has finitely many simple zeros $i\varkappa_j$ on the imaginary semi-axis, and the $\lambda_j = -\varkappa_j^2$ are the eigenvalues of the operator H with the eigenfunctions $\psi_j(x) = f_1(x, i\varkappa_j)$, $j = 1, \ldots, n$. Furthermore, it follows from the Poisson–Schwarz formula that $a(k)$ satisfies the so-called dispersion relation

$$a(k) = \exp\left\{\frac{1}{\pi i} \int_{-\infty}^\infty \frac{\log |a(q)|}{q - k} \, dq\right\} \prod_{j=1}^n \frac{k - i\varkappa_j}{k + i\varkappa_j}, \quad \text{Im } k > 0. \quad (4.7)$$

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[10]In quantum mechanics the functions $t(k) = 1/|a(k)|^2$ and $r(k) = |b(k)|^2/|a(k)|^2$ are called the transmission and reflection coefficients, respectively.

The resolvent $R_\lambda = (H - \lambda I)^{-1}$ of the Schrödinger operator H is defined on

$$\rho(H) = \mathbb{C} \setminus \{[0, \infty) \cup \{\lambda_1, \ldots, \lambda_n\}\}$$

and is an integral operator acting in $L^2(\mathbb{R})$ with the integral kernel

$$R_\lambda(x, y) = -\frac{1}{2ika(k)}(f_1(x, k)f_2(y, k)\theta(x - y) + f_1(y, k)f_2(x, k)\theta(y - x)), \quad (4.8)$$

where $k = \sqrt{\lambda}$. In particular, the integral kernel of the resolvent R_λ^0 of the free operator H_0 takes the form

$$R_\lambda^0(x, y) = -\frac{e^{ik|x-y|}}{2ik}, \qquad \operatorname{Im} k > 0. \tag{4.9}$$

As in the case of the Sturm–Liouville operator, the integral kernel $R_\lambda(x, y)$ satisfies the same equation (3.4),

$$\left(-\frac{\partial^2}{\partial x^2} + v(x) - \lambda\right)R_\lambda(x, y) = \delta(x - y), \tag{4.10}$$

where now $-\infty < x, y < \infty$.

The eigenfunction expansion for the operator H follows from the formulae (2.3) and (4.5)–(4.8). In particular, denote by P the orthogonal projection from $\mathscr{H} = L^2(\mathbb{R})$ onto the subspace spanned by $\psi_1, \ldots, \psi_n$, and denote by $\mathfrak{H}_0$ the Hilbert space $L^2([0, \infty), \mathbb{C}^2; |a(k)|^{-2}\, dk)$. The operator $\mathscr{U} : \mathscr{H} \to \mathfrak{H}_0$ defined by the formula

$$(\mathscr{U}f)_l(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)f_l(x, k)\, dx, \qquad l = 1, 2,$$

is a partial isometry of the Hilbert spaces $\mathscr{H}$ and $\mathfrak{H}_0$:

$$\mathscr{U}^*\mathscr{U} = I - P, \quad \mathscr{U}\mathscr{U}^* = I_0,$$

where I_0 is the identity operator on $\mathfrak{H}_0$. The operator $\mathscr{U}H\mathscr{U}^*$ is the operator of multiplication by k^2 in $\mathfrak{H}_0$. The eigenfunction expansion for the free Schrödinger operator is the Fourier transform.

4.2. The characteristic determinant. Since the operator H has an absolutely continuous spectrum, the formula (2.5) no longer makes sense, and there now arises the problem of defining $\det(H - \lambda I)$. In the case of the radial Schrödinger operator, this problem was solved by Buslaev and Faddeev in [5], which subsequently led to the concept of the perturbation determinant [22]. Here we consider the case of the one-dimensional Schrödinger operator and, for simplicity of presentation, instead of (4.2) we impose a stronger condition on the potential $v(x)$.

Namely, suppose that $v(x)$ is a bounded function on the real axis and $v(x) = O(|x|^{-3-\varepsilon})$ as $|x| \to \infty$ for some $\varepsilon > 0$. It follows from the first condition that V is a bounded operator on $L^2(\mathbb{R})$, and the second condition means that in the

estimates (4.3), (4.4) we can replace $\sigma(x)$ and $\tilde{\sigma}(x)$ by $O(|x|^{-2-\varepsilon})$. By analogy with (2.5), the regularized determinant of the operator $H - \lambda I$ is given by

$$-\frac{d}{d\lambda} \log \det(H - \lambda I) = \mathrm{Tr}(R_\lambda - R_\lambda^0), \qquad \lambda \in \rho(H), \tag{4.11}$$

where $R_\lambda - R_\lambda^0$ is a trace-class operator. Indeed, from the second Hilbert identity (2.2) we obtain

$$R_\lambda - R_\lambda^0 = R_\lambda V R_\lambda^0. \tag{4.12}$$

Denote by $\sqrt{V}$ the operator of multiplication by the function $\sqrt{v(x)}$, where

$$\sqrt{v(x)} = \sqrt{|v(x)|}\, e^{i\delta},$$

$\delta = 0$ if $v(x) \geqslant 0$, and $\delta = i\pi/2$ if $v(x) < 0$. Since $\sqrt{v(x)} \in L^2(\mathbb{R})$, it follows from (4.8) and (4.9) that the operators $R_\lambda \sqrt{V}$ and $\sqrt{V} R_\lambda^0$ are Hilbert–Schmidt operators, and therefore the operator $R_\lambda - R_\lambda^0$ is of trace class.

Remarkably, the trace on the right-hand side of (4.11) can be calculated explicitly. In the case of the radial Schrödinger equation the corresponding formula was given by Buslaev and Faddeev in [5], and in the case of the whole axis this is the last formula in §1.1 of Faddeev's survey [15] (see also Problem 2.6 in §3.2.2 of Chap. 3 in [41]). In particular, the following relation holds:

$$\mathrm{Tr}(R_\lambda - R_\lambda^0) = -\frac{d}{d\lambda} \log a(\sqrt{\lambda}). \tag{4.13}$$

As far as we know, no complete derivation of this beautiful formula exists in the literature. For the convenience of the reader, we present it here.

Proof of (4.13). Recall that $k = \sqrt{\lambda}$. For $\mathrm{Im}\, k > 0$ the Jost solution $f_1(x, k)$ decays exponentially for large x. For such k a solution $g(x, k)$ of (4.1) linearly independent from $f_1(x, k)$ is found from the relation $W(f_1, g) = 2ik$, and it grows exponentially for large x:

$$e^{ikx} g(x, k) = 1 + O(x^{-1-\varepsilon}) \quad \text{and} \quad e^{ikx} g'(x, k) = -ik + O(x^{-1-\varepsilon}) \quad \text{as } x \to \infty.$$

The functions $f_1(x, k)$ and $g(x, k)$ form a basis in the solution space, and from the condition $W(f_1, g) = 2ik$ and (4.6) we get that

$$f_2(x, k) = a(k)g(x, k) + c(k)f_1(x, k).$$

Therefore, as $x \to \infty$ we have

$$e^{ikx} f_2(x, k) = a(k) + O(x^{-1-\varepsilon}) \quad \text{and} \quad e^{ikx} f_2'(x, k) = -ika(k) + O(x^{-1-\varepsilon}). \tag{4.14}$$

Similarly, as $x \to -\infty$,

$$e^{-ikx} f_1(x, k) = a(k) + O(|x|^{-1-\varepsilon}) \quad \text{and} \quad e^{-ikx} f_1'(x, k) = ika(k) + O(|x|^{-1-\varepsilon}). \tag{4.15}$$

The trace-class operator $R_\lambda - R_\lambda^0$ has an integral kernel $\widetilde{R}_\lambda(x, y)$ that is continuous on $\mathbb{R} \times \mathbb{R}$, and

$$\widetilde{R}_\lambda(x, x) = -\frac{1}{2ika(k)}(f_1(x, k)f_2(x, k) - a(k)).$$

Thus, using formula (2.7), we obtain

$$\mathrm{Tr}(R_\lambda - R_\lambda^0) = \frac{i}{2ka(k)} \lim_{n \to \infty} \int_{-n}^{n} (f_1(x, k)f_2(x, k) - a(k))\, dx. \qquad (4.16)$$

As in the case of a Sturm–Liouville operator, the integral in (4.16) can be evaluated explicitly. Namely, write (3.6) in the form

$$f_1(x, k)f_2(x, k) = -\frac{1}{2k}W(\dot{f}_1(x, k), f_2(x, k))' = \frac{1}{2k}W(f_1(x, k), \dot{f}_2(x, k))', \quad (4.17)$$

where the dot now stands for the k-derivative. From the integral equation for $f_1(x, k)$ we obtain

$$e^{-ikx}\dot{f}_1(x, k) = ix + O(x^{-\varepsilon}) \quad \text{and} \quad e^{-ikx}\dot{f}_1'(x, k) = i - kx + O(x^{-\varepsilon}) \quad \text{as } x \to \infty,$$

and therefore for such x we get by using (4.14) that

$$W(\dot{f}_1(x, k), f_2(x, k)) = (i - 2kx)a(k) + O(x^{-\varepsilon}).$$

Similarly, as $x \to -\infty$,

$$W(f_1(x, k), \dot{f}_2(x, k)) = (i + 2kx)a(k) + O(|x|^{-\varepsilon}).$$

Using these formulae and (4.17), we obtain

$$\int_0^n f_1(x, k)f_2(x, k)\, dx = \left(-\frac{i}{2k} + n\right)a(k) + \frac{1}{2k}W(\dot{f}_1(0, k), f_2(0, k)) + O(n^{-\varepsilon})$$

and

$$\int_{-n}^0 f_1(x, k)f_2(x, k)\, dx = \left(-\frac{i}{2k} - n\right)a(k) + \frac{1}{2k}W(f_1(0, k), \dot{f}_2(0, k)) + O(n^{-\varepsilon}).$$

Adding the last two formulae and using the relation

$$\dot{a}(k) = -\frac{a(k)}{k} + \frac{1}{2ik}W(\dot{f}_1(0, k), f_2(0, k)) + \frac{1}{2ik}W(f_1(0, k), \dot{f}_2(0, k))$$

following from (4.6), we have

$$\lim_{n \to \infty} \int_{-n}^{n} (f_1(x, k)f_2(x, k) - a(k))\, dx = i\dot{a}(k).$$

Substitution of this relation into (4.16) gives us the desired formula (4.13). $\square$

4.3. Trace identities. It follows from (4.13) that the regularized determinant of the Schrödinger operator is given by the formula

$$\det(H - \lambda I) = a(\sqrt{\lambda}),$$

is a holomorphic function on the complex λ-plane cut along the non-negative semi-axis, and has zeros at the eigenvalues of the operator H. Under the assumption that the potential $v(x)$ is a function in the Schwartz class,[11] we easily obtain from (4.13) (as in §3) the trace identities for the one-dimensional Schrödinger operator H.

Namely, for such $v(x)$ the coefficient $b(k)$ is a Schwartz-class function, and therefore from (4.7) we immediately obtain an asymptotic expansion as $k \to \infty$:

$$\log a(k) = \sum_{l=1}^{\infty} \frac{c_l}{k^l} + O(|k|^{-\infty}), \qquad \operatorname{Im} k > 0, \qquad (4.18)$$

which is an analogue of the expansion (3.9) for the characteristic determinant of the Sturm–Liouville operator. Moreover, $c_{2l} = 0$ and

$$c_{2l+1} = -\frac{1}{\pi i} \int_{-\infty}^{\infty} k^{2l} \log |a(k)| \, dk - \frac{2}{2l+1} \sum_{j=1}^{n} (i\varkappa_j)^{2l+1}. \qquad (4.19)$$

An analogue of the asymptotic expression (3.8) is obtained by means of the following beautiful argument (here we follow the famous paper [50] by Zakharov and Faddeev). From (4.3) it follows that the function $\chi(x, k) = \log f_1(x, k)$ is well defined for large k with $\operatorname{Im} k > 0$, and

$$\chi(x, k) = ikx + o(1), \quad x \to \infty,$$

and

$$\chi(x, k) = \log a(k) + ikx + o(1), \quad x \to -\infty.$$

As follows from (4.1), the function

$$\sigma(x, k) = \frac{d}{dx} \chi(x, k) - ik$$

is a solution of the Riccati equation

$$\sigma' + \sigma^2 - v + 2ik\sigma = 0,$$

decays as $|x| \to \infty$, and satisfies

$$\log a(k) = -\int_{-\infty}^{\infty} \sigma(x, k) \, dx. \qquad (4.20)$$

Now it is not difficult to verify the asymptotic expansion

$$\sigma(x, k) = \sum_{l=1}^{\infty} \frac{\sigma_l(x)}{(2ik)^l} + O(|k|^{-\infty}). \qquad (4.21)$$

---

[11] That is, it is smooth and rapidly decaying with all derivatives as $|x| \to \infty$.

The coefficients $\sigma_l(x)$ are polynomials in the function $v(x)$ and its derivatives at x and are determined by the recurrence relation

$$\sigma_l(x) = -\sigma'_{l-1}(x) - \sum_{j=1}^{l-1} \sigma_{l-j-1}(x)\sigma_j(x), \quad \sigma_1(x) = v(x);$$

moreover, the $\sigma_{2l}(x)$ are total x-derivatives. Comparing the formulae (4.18), (4.19) with (4.20), (4.21), we obtain the Zakharov–Faddeev trace identities

$$\frac{1}{\pi i} \int_{-\infty}^{\infty} k^{2l} \log|a(k)|\, dk + \frac{2}{2l+1} \sum_{j=1}^{n} (i\varkappa_j)^{2l+1} = \left(\frac{1}{2i}\right)^{2l+1} \int_{-\infty}^{\infty} \sigma_{2l+1}(x)\, dx.$$

In [50], the reader can find a remarkable application of these formulae to the proof of the complete integrability of the Korteweg–de Vries equation.

Comparing (4.13) with (4.18), (4.19), we see that $\text{Tr}(R_\lambda - R_\lambda^0)$ can be expanded as $\lambda \to -\infty$ in an asymptotic series in inverse odd powers of $\sqrt{\lambda}$. In the Gelfand–Dikii paper [19] this was proved directly, in both the rapidly decreasing case and the periodic case. Namely, rewriting the second Hilbert identity (4.12) in the form

$$R_\lambda(I - VR_\lambda^0) = R_\lambda^0,$$

we obtain

$$R_\lambda = R_\lambda^0 + \sum_{n=1}^{\infty} R_\lambda^0(VR_\lambda^0)^n,$$

where the infinite series is understood as an asymptotic series as $\lambda \to -\infty$. By using the explicit formula (4.9) for the free resolvent, it is not difficult to obtain the asymptotic expansion

$$R_\lambda(x,x) = \sum_{l=1}^{\infty} \frac{R_l(x)}{\lambda^{l+1/2}} + O(|\sqrt{\lambda}|^{-\infty}).$$

The coefficients $R_l(x)$ are easily found from the third-order differential equation

$$\left(-\frac{d^3}{dx^3} + 4(v(x) - \lambda)\frac{d}{dx} + 2v'(x)\right) R_\lambda(x,x) = 0,$$

for the product of two solutions of the second-order equation (4.1). Details of these beautiful calculations can be found in [19].

5. A certain functional-difference operator

Consider the functional-difference equation

$$\psi(x + ib) + \psi(x - ib) + e^{2\pi bx}\psi(x) = \lambda\psi(x),$$

where

$$b > 0 \quad \text{and} \quad -\infty < x < \infty,$$

and the function $\psi(x)$ admits analytic continuation into the strip

$$\Pi_b = \{z = x + iy \in \mathbb{C} \colon |y| < b\}.$$

A functional-difference operator

$$H = U + U^{-1} + V$$

is associated with this equation, where U and V are the self-adjoint Weyl operators acting in $L^2(\mathbb{R})$. The operator H arises in conformal field theory and in the representation theory of the quantum group $\mathrm{SL}_q(2, \mathbb{R})$. In [43] there is a spectral analysis of this unbounded self-adjoint operator acting in $L^2(\mathbb{R})$. We give a detailed presentation of these results.

5.1. Weyl operators. The quantum mechanical Weyl operators are unitary operators $U(u)$ and $V(v)$ on $L^2(\mathbb{R})$, $u, v \in \mathbb{R}$, defined by the formulae

$$(U(u)\psi)(x) = \psi(x - u) \quad \text{and} \quad (V(v)\psi)(x) = e^{-ivx}\psi(x), \qquad \psi \in L^2(\mathbb{R})$$

(for example, see [41], Chap. 2, where the Planck constant $\hbar$ is set to be 1). The operators $U(u)$ and $V(v)$ satisfy the Weyl commutation relations

$$U(u)V(v) = e^{iuv}V(v)U(u).$$

In the representation theory of the quantum group $\mathrm{SL}_q(2, \mathbb{R})$ one uses complex values of u and v, for which the Weyl operators $U(u)$ and $V(v)$ become unbounded self-adjoint operators acting in $L^2(\mathbb{R})$.

Namely, consider the operators U and V given formally by

$$(U\psi)(x) = \psi(x + ib), \quad (V\psi)(x) = e^{2\pi bx}\psi(x) \tag{5.1}$$

and satisfying the relation

$$UV = q^2 VU, \qquad q = e^{\pi i b^2}, \tag{5.2}$$

on the common domain of U and V. The operators U and V defined by (5.1) are unbounded self-adjoint operators acting in $L^2(\mathbb{R})$. Specifically, U is a self-adjoint operator acting in $L^2(\mathbb{R})$ with the domain

$$D(U) = \{\psi(x) \in L^2(\mathbb{R}) \colon e^{-2\pi bp}\hat{\psi}(p) \in L^2(\mathbb{R})\},$$

where

$$\hat{\psi}(p) = \mathscr{F}(\psi)(p) = \int_{-\infty}^{\infty} \psi(x)e^{-2\pi ipx}\,dx$$

is the Fourier transform[12] in $L^2(\mathbb{R})$. Equivalently, the domain $D(U)$ consists of the functions $\psi(x)$ which admit analytic continuation into the strip

$$\Pi_b^+ = \{z = x + iy \in \mathbb{C} \colon 0 < y < b\}$$

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[12]We are using the normalization of the Fourier transform that is customary in analytic number theory.

with the property that $\psi(x + iy) \in L^2(\mathbb{R})$ for all $0 \leqslant y < b$ and that the limit

$$\psi(x + ib - i0) = \lim_{\varepsilon \to 0^+} \psi(x + ib - i\varepsilon)$$

exists in the sense of convergence in $L^2(\mathbb{R})$. Furthermore, for $\psi \in D(U)$ we have

$$(U\psi)(x) = \psi(x + ib - i0).$$

The domain $D(U^{-1})$ of the inverse operator U^{-1} is defined similarly, and we have $(U^{-1}\psi)(x) = \psi(x - ib + i0)$. The domain $D(V)$ of the self-adjoint operator V consists of the functions $\psi(x) \in L^2(\mathbb{R})$ for which $e^{2\pi bx}\psi(x) \in L^2(\mathbb{R})$. Thus, we have

$$U^{-1} = \mathscr{F}^{-1}V\mathscr{F},$$

where the inverse Fourier transform is given by the formula

$$\psi(x) = \int_{-\infty}^{\infty} \hat{\psi}(p)e^{2\pi ipx}\, dp.$$

5.2. The operator H_0. The free operator $H_0 = U + U^{-1}$ is an unbounded self-adjoint operator acting in $L^2(\mathbb{R})$ and defined on $D(H_0) = D(U) \cap D(U^{-1})$ by the formula

$$(H_0\psi)(x) = \psi(x + ib - i0) + \psi(x - ib + i0), \qquad \psi \in D(H_0).$$

Obviously, for $b \to 0$ the operator $b^{-2}(H_0 - 2I)$ turns into the operator $-d^2/dx^2$. In terms of the Fourier transform the operator $\widehat{H}_0 = \mathscr{F}H_0\mathscr{F}^{-1}$ is the operator of multiplication by $2\cosh(2\pi bp)$, and thus the domain $D(H_0)$ admits an equivalent description:

$$D(H_0) = \left\{\psi(x) \in L^2(\mathbb{R}) : \int_{-\infty}^{\infty} \cosh^2(2\pi bp)|\hat{\psi}(p)|^2\, dp < \infty\right\},$$

and it is a 'hyperbolic analogue' of the Sobolev space $W_2^2(\mathbb{R})$.

For $\lambda \in \mathbb{C} \setminus [2, \infty)$ the resolvent of the operator $\widehat{H}_0$,

$$\widehat{R}_\lambda^0 = (\widehat{H}_0 - \lambda I)^{-1},$$

is the operator of multiplication by the function $(2\cosh(2\pi bp) - \lambda)^{-1}$, and it is bounded on $L^2(\mathbb{R})$. Because the function $2\cosh(2\pi bp)$ is a two-to-one map of the real axis $-\infty < p < \infty$ onto $[2, \infty)$, the spectrum of $\widehat{H}_0$ is absolutely continuous and fills the semi-infinite interval $[2, \infty)$ with multiplicity 2. Correspondingly, for $\lambda \in \mathbb{C} \setminus [2, \infty)$ the resolvent

$$R_\lambda^0 = (H_0 - \lambda I)^{-1}$$

of H_0 is an integral operator acting in $L^2(\mathbb{R})$ with integral kernel depending on the difference of the arguments,

$$(R_\lambda^0\psi)(x) = \int_{-\infty}^{\infty} R_\lambda^0(x - y)\psi(y)\, dy, \tag{5.3}$$

where

$$R_\lambda^0(x) = \int_{-\infty}^{\infty} \frac{e^{2\pi i p x}}{2\cosh(2\pi b p) - \lambda}\, dp. \tag{5.4}$$

It is convenient to use the parametrization (cf. §4.1)

$$\lambda = 2\cosh(2\pi b k),$$

in which the resolvent set $\mathbb{C} \setminus [2, \infty)$ becomes the 'physical sheet'—the strip $0 < \operatorname{Im} k \leqslant 1/(2b)$—and the continuous spectrum $[2, \infty)$ is doubly covered by the real axis $-\infty < k < \infty$. The integral (5.4) is easily calculated, and we obtain

$$R_\lambda^0(x) = \frac{i}{2b\sinh(2\pi b k)}\left(\frac{e^{-2\pi i k x}}{1 - e^{2\pi x/b}} + \frac{e^{2\pi i k x}}{1 - e^{-2\pi x/b}}\right), \qquad \lambda = 2\cosh(2\pi b k). \tag{5.5}$$

The function $R_\lambda^0(x)$ is regular at $x = 0$, and for $0 < \operatorname{Im} k \leqslant 1/(2b)$ the following estimate holds:

$$|R_\lambda^0(x)| \leqslant C e^{-2\pi \operatorname{Im} k |x|},$$

where $C > 0$ is a constant,[13] so that for $\lambda \notin [2, \infty)$ the formulae (5.3) and (5.5) do indeed define a bounded operator on $L^2(\mathbb{R})$.

It is instructive to rewrite (5.5) in terms of the solutions of the equation

$$\psi(x + ib - i0, k) + \psi(x - ib + i0, k) = 2\cosh(2\pi b k)\psi(x, k) \tag{5.6}$$

for the continuous-spectrum eigenfunctions of the operator H_0, that is, in terms of the solutions $f_\pm(x, k) = e^{\pm 2\pi i k x}$, which are analogues of the Jost solutions in the theory of the one-dimensional Schrödinger operator (see §4.1). Namely,

$$R_\lambda^0(x - y) = \frac{i}{bC(f_-, f_+)(k)}\left(\frac{f_-(x, k)f_+(y, k)}{1 - e^{2\pi(x-y)/b}} + \frac{f_-(y, k)f_+(x, k)}{1 - e^{-2\pi(x-y)/b}}\right), \tag{5.7}$$

where

$$C(f, g)(x, k) = f(x + ib, k)g(x, k) - f(x, k)g(x + ib, k)$$

is the so-called Casorati determinant, which is an analogue of the Wronskian for solutions of the functional-difference equation (5.6). It is a periodic function of x with period ib, and in the case of the Jost solutions

$$C(f_-, f_+)(x, k) = 2\sinh(2\pi b k).$$

There is a remarkable similarity between (5.7) and the formulae (3.3) and (4.8), where instead of the Heaviside function $\theta(x)$ a smoothed analogue of it is involved, namely, the function $\theta_b(x)$ defined by the formula[14]

$$\theta_b(x) = \frac{1}{1 - e^{-2\pi x/b}}\,.$$

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[13]Here and below we use C to denote various constants.

[14]As noted by A. M. Polyakov, the function $\theta_b(x)$, after identification of x with the energy ϵ and identification of $2\pi/b$ with the inverse temperature $1/(kT)$, coincides with the one-particle partition function $\mathcal{Z} = (1 - e^{-\epsilon/(kT)})^{-1}$ in the Bose–Einstein statistics.

In this case, the analogue of the relation $\theta'(x) = \delta(x)$ is the formula

$$\frac{1}{ib}(\theta_b(x - i0) - \theta_b(x + i0)) = \delta(x)$$

for real x, which is equivalent to the Sokhotski–Plemelj formula. The following simple formula also holds:

$$\frac{1}{ib}\theta_b(x \pm i0) = \frac{1}{ib}\,\text{p. v. }\theta_b(x) \mp \frac{1}{2}\delta(x), \tag{5.8}$$

where the distribution $\theta_b(x)$ is understood as the Cauchy principal value. From this we obtain for $R^0_\lambda(x - y)$ the equation

$$R^0_\lambda(x - y + ib - i0) + R^0_\lambda(x - y - ib + i0) - \lambda R^0_\lambda(x - y) = \delta(x - y). \tag{5.9}$$

Indeed, setting $y = 0$ and using (5.6), (5.8), and the regularity of $R^0_\lambda(x)$ at $x = 0$, we have

$$R^0_\lambda(x + ib - i0) + R^0_\lambda(x - ib + i0) - \lambda R^0_\lambda(x)$$

$$= \frac{i}{2b\sinh(2\pi bk)}\big[f_-(x + ib, k)\theta_b(-x + i0) + f_+(x + ib, k)\theta_b(x - i0)$$

$$+ f_-(x - ib, k)\theta_b(-x - i0) + f_+(x - ib, k)\theta_b(x + i0)\big]$$

$$- \frac{i}{b}\coth(2\pi bk)(f_-(x, k)\theta_b(-x) + f_+(x, k)\theta_b(x))$$

$$= \frac{1}{4\sinh(2\pi bk)}\big[f_-(ib, k) - f_+(ib, k) - f_-(-ib, k) + f_+(-ib, k)\big]\delta(x) = \delta(x).$$

By using the representation (5.7) and equation (5.9), it is easy to verify directly that for $\lambda \in \mathbb{C} \setminus [2, \infty)$ the integral operator (5.3) is the inverse of the operator $H_0 - \lambda I$ (see §§ 3 and 4).

5.3. The operator H. Here we consider the equation

$$\psi(x + ib - i0) + \psi(x - ib + i0) + e^{2\pi bx}\psi(x) = 2\cosh(2\pi bk)\psi(x), \tag{5.10}$$

which is the q-analogue of the equation

$$-\psi'' + e^{2x}\psi = k^2\psi$$

for the Bessel functions. As is well known, the last equation has a solution that is decreasing as $x \to \infty$, the modified Bessel function of the second kind $K_{ik}(e^x)$ given by the inverse Mellin transform of the product of two gamma functions. The equation (5.10) also has a solution that is decreasing as $x \to \infty$, the Fourier transform of a product involving another wonderful special function, Faddeev's quantum dilogarithm. This function was introduced by Faddeev in [16] and has the integral representation[15]

$$\Phi_b(z) = \exp\left\{\frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{2itz}}{\sinh bt\,\sinh b^{-1}t}\frac{dt}{t}\right\}, \tag{5.11}$$

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[15]The function $\Phi_b(z)$ has an interesting history (see [43], where the notation $\gamma(z)$ was used).

where the contour of integration passes above the singularity at $t = 0$. The representation (5.11) is valid for $|\operatorname{Im} z| < c_b = (b + b^{-1})/2$ and defines a meromorphic function with poles $z = -ic_b - mib - nib^{-1}$ for integers $m, n \geqslant 0$, which satisfies the functional equations

$$\Phi_b(z + ib) = (1 + q^{-1}e^{-2\pi bz})\Phi_b(z), \qquad q = e^{\pi ib^2},$$

$$\Phi_b(z + ib^{-1}) = (1 + \tilde{q}^{-1}e^{-2\pi b^{-1}z})\Phi_b(z), \qquad \tilde{q} = e^{\pi ib^{-2}}.$$

Let

$$\hat{\varphi}(p, k) = \exp\{-i\beta - \pi ik^2 - \pi i(p - ic_b)^2\}\Phi_b(p - k - ic_b)\Phi_b(p + k - ic_b),$$

where $\beta = \dfrac{\pi}{12}(b^2 + b^{-2})$. By using the analytic properties of $\Phi_b(z)$ (see [43], for example) it is easy to verify that the function

$$\varphi(x, k) = \int_{-\infty}^{\infty} \hat{\varphi}(p, k)e^{2\pi ipx}\, dp$$

is a solution of (5.10), where the contour of integration passes above the singularities at $p = \pm k$. Namely, the following statements hold.

1. For real k the function $\varphi(x, k)$ is an even real-valued function of k, having the asymptotics

$$\varphi(x, k) = M(k)e^{2\pi ikx} + M(-k)e^{-2\pi ikx} + o(1)$$

as real $x \to -\infty$, where

$$M(k) = \exp\left\{i\left(\beta + \frac{\pi}{4}\right) - 2\pi ik(k - ic_b)\right\}\Phi_b(2k - ic_b), \quad \overline{M(k)} = M(-k),$$

and

$$\frac{1}{|M(k)|^2} = 4\sinh(2\pi bk)\sinh(2\pi b^{-1}k).$$

Thus, $\varphi(x, k)$ is a scattering solution for (5.10).

2. For real x the function $\varphi(x, k)$ admits analytic continuation into the strip $0 < \operatorname{Im} k \leqslant 1/(2b)$ and satisfies the reality condition

$$\overline{\varphi(x, k)} = \varphi(x, -\overline{k}).$$

3. For fixed k in the physical strip, the function $\varphi(x, k)$ extends to an entire function of the complex variable x and satisfies (5.10).

4. The following estimates hold:

$$|\varphi(x, k)| \leqslant C\exp\{-2\pi \operatorname{Im} k\, x\}$$

uniformly for $-\infty < x \leqslant a$, and

$$|\varphi(x, k)| \leqslant C\exp\{-\pi(b + b^{-1})x\}, \quad |\varphi(x \pm ib, k)| \leqslant C\exp\{\pi(b - b^{-1})x\}$$

uniformly for $a \leqslant x < \infty$.

As $x \to -\infty$, equation (5.10) takes on the free form (5.6), so it is natural to assume that (5.10) has Jost solutions, that is, solutions $f_\pm(x, k)$ with the asymptotics

$$f_\pm(x, k) = e^{\pm 2\pi i k x} + o(1) \quad \text{as } x \to -\infty. \tag{5.12}$$

Namely, let

$$f_+(x, k) = \frac{1}{4 \sinh(2\pi b^{-1} k) M(k)}$$
$$\times \left(\varphi(x - ib^{-1}, k) - \varphi(x + ib^{-1}, k) + 2 \sinh(2\pi b^{-1} k) \varphi(x, k)\right)$$

and $f_-(x, k) = f_+(x, -k)$. From properties **1** and **3** of the function $\varphi(x, k)$ we immediately get that for real x the functions $f_\pm(x, k)$ are solutions of (5.10) and

$$\varphi(x, k) = M(k) f_+(x, k) + M(-k) f_-(x, k). \tag{5.13}$$

From the properties of the solution $\varphi(x, k)$ listed above it is not difficult to derive the following properties of the Jost solutions.

1′. For real x and k the functions $f_\pm(x, k)$ have the asymptotics (5.12).

2′. For real x the functions $f_\pm(x, k)$ admit analytic continuation to the physical strip $0 < \operatorname{Im} k \leqslant 1/(2b)$ and satisfy the condition

$$\overline{f_\pm(x, k)} = f_\pm(x, -\bar{k}).$$

3′. For fixed k in the physical strip, the functions $f_\pm(x, k)$ are entire functions of the variable x and satisfy equation (5.10) and condition (5.13). Moreover, the asymptotics in **1′** remain valid for $0 \leqslant \operatorname{Im} x \leqslant b$.

4′. The estimates

$$|f_\pm(x, k)| \leqslant C e^{\mp 2\pi \operatorname{Im} k \, x}$$

hold uniformly for $-\infty < x \leqslant a$, and

$$|f_\pm(x, k)| \leqslant C e^{\pi(b^{-1} - b)x}, \qquad |f_\pm(x + ib, k)| \leqslant C e^{\pi(b + b^{-1})x}$$

uniformly for $a \leqslant x < \infty$.

Using these analytic properties and the Phragmén–Lindelöf theorem, one can prove that the Casorati determinant of the Jost solutions does not depend on x, and therefore

$$C(f_-, f_+)(x, k) = 2 \sinh(2\pi b k).$$

Arguing as in case of the free operator H_0, from this we get that for $\lambda \in \mathbb{C} \setminus [2, \infty)$ the integral operator R_λ acting in $L^2(\mathbb{R})$ with the symmetric kernel

$$R_\lambda(x, y) = \frac{i}{2b \sinh(2\pi b k) M(k)}$$
$$\times \left(f_-(x, k) \varphi(y, k) \theta_b(y - x) + f_-(y, k) \varphi(x, k) \theta_b(x - y)\right) \tag{5.14}$$

is the resolvent of the operator H. Indeed, since the functions $\varphi(x, k)$ and $f_-(x, k)$ satisfy (5.10), we get from (5.8) the equation

$$R_\lambda(x + ib - i0, y) + R_\lambda(x - ib + i0, y) + (e^{2\pi b x} - \lambda) R_\lambda(x, y) = \delta(x - y), \tag{5.15}$$

and we get from the analytic properties of these functions the estimate

$$|R_\lambda(x,y)| \leqslant Ce^{-2\pi \operatorname{Im} k\,|x-y|},$$

so that for $\lambda \in \mathbb{C} \setminus [2,\infty)$ the operator R_λ is bounded on $L^2(\mathbb{R})$. Using (5.15) and the identity

$$C(f_-,\varphi)(x,k) = 2\sinh(2\pi bk)M(k),$$

we obtain the desired statement that $R_\lambda = (H - \lambda I)^{-1}$.

Finally, the eigenfunction expansion theorem for the operator H is obtained from (2.3). Namely, by computing the jump of the resolvent kernel $R_\lambda(x,y)$ on the branch cut $[2,\infty)$ using (5.13), we get that the operator $\mathscr{U}$ given by the formula

$$(\mathscr{U}\psi)(k) = \int_{-\infty}^{\infty} \psi(x)\varphi(x,k)\,dx, \qquad \psi(x) \in L^2(\mathbb{R}),$$

maps $L^2(\mathbb{R})$ isometrically onto the Hilbert space $\mathscr{H}_0 = L^2([0,\infty), |M(k)|^{-2}\,dk)$, that is,

$$\mathscr{U}^*\mathscr{U} = I \quad \text{and} \quad \mathscr{U}\mathscr{U}^* = I_0,$$

where I_0 is the identity operator on $\mathscr{H}_0$. Moreover, the operator $\mathscr{U}H\mathscr{U}^{-1}$ is the operator of multiplication by the function $2\cosh(2\pi bk)$ on $\mathscr{H}_0$, so H has a simple absolutely continuous spectrum filling $[2,\infty)$. As was noted in [43], the eigenfunction expansion theorem for H is a q-analogue of the classical Kontorovich–Lebedev transform in the theory of special functions.

5.4. Operators for mirror curves. In [1] a remarkable connection was found between the functional-difference operators constructed from the Weyl operators U and V and the quantization of algebraic curves that are the images of toric Calabi–Yau threefolds under the mirror symmetry. A typical example of such varieties is the total space of the canonical bundle of a toric del Pezzo surface S. The spectral properties of such operators were studied in [23]. In the simplest case, when S is the Hirzebruch surface $S = \mathbb{P}^1 \times \mathbb{P}^1$, we obtain the operator

$$H(\zeta) = U + U^{-1} + V + \zeta V^{-1},$$

where the parameter $\zeta > 0$ plays the role of a physical mass. Thus, the operator H considered above corresponds to the massless case $H = H(0)$. When $S = \mathbb{P}(1,m,n)$ is a weighted projective space with $m,n \in \mathbb{N}$, the corresponding operator has the form

$$H_{m,n} = U + V + q^{-mn}U^{-m}V^{-n}.$$

In [30] the spectral properties of the self-adjoint operators $H(\zeta)$ and $H_{m,n}$ in $L^2(\mathbb{R})$ were investigated. In particular, there it was proved that these operators have a purely discrete spectrum, and an asymptotic expression for the eigenvalues was obtained which implies that $H(\zeta)^{-1}$ and $H_{m,n}^{-1}$ are trace-class operators. Furthermore, an analogue of Weyl's asymptotic law was obtained for the eigenvalue-counting function $N(\lambda)$: it was proved that

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\log^2 \lambda} = \frac{1}{(\pi b)^2}$$

for the operator $H(\zeta)$ and

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\log^2 \lambda} = \frac{c_{m,n}}{(2\pi b)^2}, \quad \text{where} \quad c_{m,n} = \frac{(m+n+1)^2}{2mn},$$

for $H_{m,n}$. Hence for the eigenvalues λ_k we get that

$$\lambda_k = e^{\alpha \sqrt{k}} \left(1 + o(1)\right)$$

as $k \to \infty$, where $\alpha = \pi b$ for $H(\zeta)$ and $\alpha = 2\pi b/\sqrt{c_{m,n}}$ for $H_{m,n}$ (see (3.2) in §3.1).

A detailed proof of these formulae was given in [30]. We note here that it would be quite interesting to obtain more accurate asymptotic formulae for the eigenvalues of the operators $H(\zeta)$ and $H_{m,n}$. As we saw in §3.2, it is very instructive to compare the asymptotics of the eigenvalues with the asymptotics of the Fredholm determinants of the operators $H(\zeta)^{-1}$ and $H_{m,n}^{-1}$. In [23] a remarkable connection was pointed out between these determinants and the enumerative invariants of the corresponding Calabi–Yau manifolds. We leave it to the reader to reflect on these intriguing connections and associations.

6. Laplace operator on a fundamental domain of a discrete group on the Lobachevsky plane

Let $\mathbb{H} = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$ be the Poincaré model of the Lobachevsky plane with the metric $ds^2 = \dfrac{|dz|^2}{y^2}$ and the area form $d\mu(z) = \dfrac{dx \wedge dy}{y^2}$, $z = x + iy$. The group of motions of the Lobachevsky plane is the Lie group $G = \operatorname{PSL}(2, \mathbb{R})$, which acts on $\mathbb{H}$ by linear-fractional transformations,

$$\mathbb{H} \ni z \mapsto gz = \frac{az+b}{cz+d} \in \mathbb{H}, \quad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Denote by $\mathcal{A}$ the Laplace operator of the Poincaré metric,

$$\mathcal{A} = -y^2 \Delta, \quad \text{where} \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{6.1}$$

defined on the space $C_0^\infty(\mathbb{H})$ of smooth functions with compact support. The operator $\mathcal{A}$ commutes with the action of G on $\mathbb{H}$ and is essentially self-adjoint on the Hilbert space $\mathscr{H}_0 = L^2(\mathbb{H}, d\mu)$. Its closure, a self-adjoint operator $A_0 = \overline{\mathcal{A}}$, has an absolutely continuous spectrum of infinite multiplicity, filling the interval $[1/4, \infty)$. It is convenient to use the parametrization $\lambda = s(1-s)$, in which the resolvent set $\mathbb{C} \setminus [1/4, \infty)$ corresponds to the half-plane $\operatorname{Re} s > 1/2$.

The operator A_0 is invariant under the action of G on $\mathbb{H}$, so its resolvent $R_\lambda^0 = (A_0 - s(1-s)I)^{-1}$ is the integral operator with kernel

$$R_\lambda^0(z, z') = r_0(z, z'; s) = \varphi(u(z, z'), s),$$

where

$$u(z, z') = \frac{|z - z'|^2}{4yy'}$$

is a point-pair which is invariant in the Lobachevsky geometry ($u(gz, gz') = u(z, z')$ for all $g \in G$), and $\varphi(u, s)$ is given by the classical integral

$$\varphi(u, s) = \frac{1}{4\pi} \int_0^1 [t(1 - t)]^{s-1} (t + u)^{-s} \, dt$$

and can be expressed explicitly in terms of a hypergeometric function. For fixed s the function $\varphi(u, s)$ has the asymptotics

$$\varphi(u, s) = -\frac{1}{4\pi} \log u + O(1) \quad \text{as } u \to 0 \tag{6.2}$$

and

$$\varphi(u, s) = O(u^{-\sigma}) \quad \text{as } u \to \infty \quad (\text{here } \sigma = \operatorname{Re} s). \tag{6.3}$$

Moreover,

$$\overline{r_0(z, z'; s)} = r_0(z, z'; \bar{s}). \tag{6.4}$$

Using these formulae, it is easy to check directly that if $f \in \mathscr{H}_0$, then

$$(R_\lambda^0 f)(z) = \iint_{\mathbb{H}} r_0(z, z'; s) f(z') \, d\mu(z') \in D(A_0), \quad \text{where} \quad \lambda = s(1 - s),$$

and $(A_0 - \lambda I) R_\lambda^0 f = f$ (see [14] and also the monographs [26] and [29]).

6.1. The resolvent and the eigenfunction expansion. Let Γ denote a Fuchsian group of the first kind, that is, a discrete subgroup of $G = \operatorname{PSL}(2, \mathbb{R})$ such that the quotient $\Gamma \backslash \mathbb{H}$ has finite area

$$\mu(F) = \iint_F \frac{dx \, dy}{y^2} < \infty,$$

where F is a fundamental domain of Γ in $\mathbb{H}$. Recall that F is an open subset of $\mathbb{H}$ such that $\gamma_1 F \cap \gamma_2 \overline{F} = \varnothing$ when $\gamma_1 \neq \gamma_2$, and the union $\bigcup \gamma \overline{F}$ over all $\gamma \in \Gamma$ is $\mathbb{H}$. Equivalently, a Fuchsian group of the first kind is a discrete subgroup Γ of G which is finitely generated by hyperbolic generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$, elliptic generators $\sigma_1, \ldots, \sigma_m$ of orders $k_1, \ldots, k_m \geqslant 2$, and parabolic generators $\tau_1, \ldots, \tau_n$. They satisfy the relations

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \sigma_1 \cdots \sigma_m \tau_1 \cdots \tau_n = 1 \quad \text{and} \quad \sigma_1^{k_1} = \cdots = \sigma_m^{k_m} = 1,$$

where $[\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}$, and the condition

$$\chi(\Gamma) = 2 - 2g - n - \sum_{j=1}^m \left(1 - \frac{1}{k_j}\right) < 0,$$

wherein $\mu(F) = -2\pi \chi(\Gamma)$. In the case $n > 0$ the closure $\overline{F}$ of the fundamental domain F is non-compact in $\mathbb{H}$ and contains n cusps, fixed points of the parabolic

transformations $\tau_1, \ldots, \tau_n$, which lie on $\mathbb{R} \cup \{\infty\}$. The simplest example (which was already known to Gauss; see [28]), is the fundamental domain F of the modular group $\mathrm{PSL}(2, \mathbb{Z})$,

$$F = \left\{ z \in \mathbb{H} \colon |x| < \frac{1}{2} \text{ and } x^2 + y^2 > 1 \right\}.$$

We have $\mathrm{PSL}(2, \mathbb{Z}) \backslash \mathbb{H} \simeq \widetilde{F}$, where $\widetilde{F}$ is the so-called modular figure

$$\widetilde{F} = \left\{ z \in \mathbb{H} \colon -\frac{1}{2} < x < 0 \text{ and } x^2 + y^2 > 1 \right\} \cup \left\{ z \in \mathbb{H} \colon 0 \leqslant x \leqslant \frac{1}{2} \text{ and } x^2 + y^2 \geqslant 1 \right\}.$$

A measurable function f on $\mathbb{H}$ is said to be Γ-automorphic if

$$f(\gamma z) = f(z)$$

for all $z \in \mathbb{H}$ and $\gamma \in \Gamma$. Let $\mathscr{H} = L^2(F, d\mu)$ be the Hilbert space of Γ-automorphic functions that are square integrable on F with respect to the measure $d\mu$, with

$$\|f\|^2 = \iint_{\mathbb{H}} |f(z)|^2 \, \frac{dx \, dy}{y^2} < \infty.$$

It is not difficult to show that the differential expression (6.1), defined on the space $C_0^\infty(F)$ of smooth functions on F with compact support, is an essentially self-adjoint operator in $\mathscr{H}$. Denote its closure by A.

Since the Laplace operator in the spaces $\mathscr{H}_0$ and $\mathscr{H}$ is given by the same differential expression (6.1), it is natural to assume that the resolvent of A in $\mathscr{H}$, that is, the operator

$$R_\lambda = (A - s(1-s)I)^{-1}, \quad \text{where} \quad \lambda = s(1-s),$$

is still an integral operator with the integral kernel $R_\lambda(z, z') = r(z, z'; s)$ obtained from $r_0(z, z'; s)$ by the classical method of images. Using a simple criterion for the convergence over a discrete group (see [14] and [29]) and the estimate (6.3), we can easily prove that if $z \neq \gamma z'$ for all $\gamma \in \Gamma$, then for $\sigma > 1$ the series

$$r(z, z'; s) = \sum_{\gamma \in \Gamma} r_0(z, \gamma z'; s) \tag{6.5}$$

is absolutely convergent, uniformly with respect to z, z' on every compact subset, and it satisfies the reality condition (6.4). The further analysis depends essentially on whether the closure $\overline{F}$ of the fundamental domain in $\mathbb{H}$ is compact (the case $n = 0$) or non-compact (the case $n \geqslant 1$).

The case $n = 0$ is elementary. Indeed, for $\sigma > 1$ the kernel $r(z, z'; s)$ has a weak singularity on the diagonal in $F \times F$ and defines a compact operator on $\mathscr{H}$, the resolvent R_λ of the operator A. The eigenfunction expansion theorem immediately follows from the first Hilbert identity

$$R_\lambda - R_\mu = (\lambda - \mu) R_\mu R_\lambda$$

(see (2.1)) and the Hilbert–Schmidt decomposition for the compact operator.

Indeed, choose $\varkappa > 1$ and put $R = R_\mu$, where $\mu = \varkappa(1-\varkappa) < 0$. The self-adjoint compact operator R is positive, so

$$R = \sum_{n=1}^{\infty} \mu_n \Pi_n,$$

where the Π_n are the orthogonal projections from $\mathscr{H}$ onto the finite-dimensional eigenspaces of R corresponding to the eigenvalues $\mu_n > 0$, and $\mu_1 = -\mu^{-1}$. Here

$$\sum_{n=1}^{\infty} \Pi_n = I \quad \text{and} \quad \lim_{n\to\infty} \mu_n = 0.$$

Rewriting the Hilbert identity as an equation for R_λ,

$$(I - (\lambda - \mu)R)R_\lambda = R, \tag{6.6}$$

we obtain the eigenfunction expansion theorem for A:

$$R_\lambda = \sum_{n=1}^{\infty} \frac{\Pi_n}{\lambda_n - \lambda}, \quad \text{where} \quad \lambda_n = \mu + \frac{1}{\mu_n}.$$

In the case $n \geqslant 1$ the derivation of the eigenfunction expansion theorem for A is rather complicated. Namely, the spectrum of the Laplace operator now consists of the n-fold absolutely continuous spectrum $[1/4, \infty)$ and finite-multiplicity eigenvalues lying on $0 \leqslant \lambda < \infty$ without accumulation points on a finite interval. Moreover, the so-called Eisenstein–Maass series defined by series over cosets of Γ which are absolutely convergent for $\sigma = \operatorname{Re} s > 1$ admit meromorphic continuation to the whole of the complex s-plane, with poles for $\sigma < 1/2$ and on the interval $[1/2, 1]$, and the eigenfunctions of the continuous spectrum of A are given by analytic continuation of the Eisenstein–Maass series to the line $\sigma = 1/2$. These results were announced[16] in Selberg's famous paper [38], and were first proved by Faddeev [14]. The monograph [29] of Lang is devoted to a detailed presentation of Faddeev's method.

It is easy to show [14] that the series (6.5) still converges for $\sigma > 1$, and for $\sigma > 2$ it is the integral kernel of the resolvent R_λ of A. Nevertheless, equation (6.6) is no longer suitable for an investigation of the resolvent R_λ for all $\lambda = s(1-s) \in \mathbb{C}\backslash[1/4, \infty)$. The fact is that for $\mu = \varkappa(1-\varkappa)$ with $\varkappa > 2$ the operator $R = R_\mu$ is no longer compact, but rather has an absolutely continuous spectrum. A remarkable observation made by Faddeev in [14] is that the main part of R generating this spectrum can be identified and explicitly inverted! The paper [14] is based on a virtuoso use of the resolvent technique, the spectral theory of Sturm–Liouville operators, and the Fredholm theory. Here we present only the main steps of the algebraic scheme of calculations; a detailed derivation of all the necessary estimates can be found in [14] and [29].

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[16]A proof based on potential theory was presented in Selberg's then unpublished 1954 lectures at the University of Göttingen.

In particular, for simplicity consider the case of one cusp[17] $i\infty$ and choose a fundamental domain F in the form

$$F = F_0 \cup F_1,$$

where $\overline{F_0}$ is compact and $\overline{F_1}$ is a strip $\{z = x + iy\colon -1/2 \leqslant x \leqslant 1/2,\ y \geqslant a\}$ for some $a > 0$. Denote by P_0 and $P_1 = I - P_0$ the orthogonal projection operators on $\mathscr{H}$ corresponding to multiplication by the indicator functions of the regions F_0 and F_1, and write the operator R as

$$R = R_{00} + R_{10} + R_{01} + R_{11},$$

where $R_{00} = P_0 R P_0$, $R_{01} = P_0 R P_1$, $R_{10} = P_1 R P_0$, and $R_{11} = P_1 R P_1$.

By using (6.5) it is not difficult to show that for $\varkappa > 2$ the operators R_{00}, R_{01}, and R_{10} are compact. It follows from the representation (6.5) that the 'cusp part' of R, that is, the operator $R_{11} = P_1 R P_1$, is the integral operator with kernel

$$R_{11}(z, z') = \sum_{\gamma \in \Gamma_\infty} R_\varkappa^0(z, \gamma z'),$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, n \in \mathbb{Z} \right\}$ is the stabilizer of the cusp $i\infty$ in the group Γ. Rewriting this formula as

$$R_{11}(z, z') = \sum_{n=-\infty}^{\infty} \varphi\left(\frac{|z - z' - n|^2}{4yy'}, \varkappa \right),$$

we see that the kernel $R_{11}(z, z')$ is an even periodic function of the variable $x - x'$ with period 1, and it can be expanded in a Fourier series

$$R_{11}(z, z') = t_0(y, y') + \sum_{m=1}^{\infty} t_m(y, y') \cos\big(2\pi m(x - x')\big). \tag{6.7}$$

The constant term, that is, the function $t_0(y, y')$, is easily computable:

$$t_0(y, y') = \frac{1}{2\varkappa - 1} \begin{cases} y^\varkappa y'^{1-\varkappa}, & y \leqslant y', \\ y^{1-\varkappa} y'^\varkappa, & y > y', \end{cases}$$

and for the functions $t_m(y, y')$ it is not difficult to obtain expressions in terms of the modified Bessel functions.

The operator R_{11} acts in the Hilbert space $P_1 \mathscr{H}$, which has a natural subspace $L^2([a, \infty); y^{-2}\, dy)$ consisting of functions independent of x, and the orthogonal projection operator P from $P_1 \mathscr{H}$ onto $L^2([a, \infty); y^{-2}\, dy)$ is given by integration:

$$f(z) \mapsto P(f)(y) = \int_{-1/2}^{1/2} f(x + iy)\, dx, \qquad y \geqslant a. \tag{6.8}$$

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[17]The case of several cusps is considered in the same way.

We write the kernel $R_{11}(z, z')$ in the form

$$R_{11} = T + R'_{11},$$

where $T = PR_{11}P$ is the integral operator acting in $L^2([a, \infty); y^{-2} dy)$ with kernel $t_0(y, y')$, and R'_{11} is the integral operator with kernel $R_{11}(z, z') - t_0(y, y')$. Using the Fourier expansion (6.7) and standard estimates for the modified Bessel functions, we can easily prove (for details, see [29]) that the operator R'_{11} is compact. By recalling formulae in § 3.1 it is easy to verify that

$$T = (B - \varkappa(1 - \varkappa)I)^{-1},$$

where B is a self-adjoint operator acting in $L^2([a, \infty); y^{-2} dy)$ and given by the differential expression $\mathcal{B}\varphi = -y^2 d^2\varphi/dy^2$ and the boundary condition[18]

$$\varkappa\varphi(a) = a\varphi'(a).$$

For $\sigma > 1/2$ the resolvent $Q_\lambda = (B - s(1 - s)I)^{-1}$ is the integral operator with kernel

$$q(y, y'; s) = \frac{1}{2s - 1}\big(\varphi(y, s)y'^{1-s}\theta(y' - y) + y^{1-s}\varphi(y', s)\theta(y - y')\big),$$

where

$$\varphi(y, s) = y^s + a^{2s-1}\frac{s - \varkappa}{s + \varkappa - 1}y^{1-s}.$$

For $\sigma = 1/2$ the functions $\varphi(y, s)$ form a complete system of continuous-spectrum eigenfunctions for the operator B in the space $L^2([a, \infty); y^{-2} dy)$.

Summing up, for $\varkappa > 2$ we have

$$R = T + V,$$

where V is a compact operator. Therefore, A can be regarded as a perturbation of B with the same absolutely continuous spectrum! Namely, we now write (6.6) in the form

$$(I - (\lambda - \mu)T)R_\lambda = R + (\lambda - \mu)VR_\lambda, \tag{6.9}$$

where $\lambda = s(1 - s)$ and $\mu = \varkappa(1 - \varkappa)$. It follows from the first Hilbert identity for B that

$$I - (\lambda - \mu)T = (I + (\lambda - \mu)Q_\lambda)^{-1},$$

so (6.9) can be rewritten as

$$R_\lambda = (I + (\lambda - \mu)Q_\lambda)(T + V) + (\lambda - \mu)(I + (\lambda - \mu)Q_\lambda)VR_\lambda,$$

or

$$R_\lambda = Q_\lambda + (I + (\lambda - \mu)Q_\lambda)V + (\lambda - \mu)(I + (\lambda - \mu)Q_\lambda)VR_\lambda \tag{6.10}$$

---

[18]Here we correct the typing error in [14] after (3.7) and also in the corresponding place in [42], § 3.1.

if one uses the Hilbert identity once again. Putting

$$R_\lambda = Q_\lambda + \big(I + (\lambda - \mu)Q_\lambda\big)U_\lambda\big(I + (\lambda - \mu)Q_\lambda\big), \tag{6.11}$$

we obtain for U_λ the equation

$$U_\lambda = V + H_\lambda U_\lambda, \quad \text{where} \quad H_\lambda = (\lambda - \mu)V\big(I + (\lambda - \mu)Q_\lambda\big). \tag{6.12}$$

Equation (6.12), Faddeev's equation in the theory of automorphic functions, has the following remarkable properties [14].

1) The operator H_λ is a Fredholm operator acting in the Banach space $\mathfrak{B}$ of continuous functions $f(z)$ on F with the norm

$$\|f\|_{\mathfrak{B}} = \sup_{z \in F_0} |f(z)| + \sup_{z \in F_1} y|f(z)|$$

and depends analytically on s in the strip $0 < \sigma < 2$.

2) The singular points of the operator $I - H_\lambda$, that is, the values of s for which the homogeneous equation

$$v = H_\lambda v$$

has a non-trivial solution in the space $\mathfrak{B}$, are discrete in the strip $0 < \sigma < 2$.

3) The singular points with $\sigma \geqslant 1/2$, $s \neq 1/2$, correspond to non-negative eigenvalues $\lambda = s(1 - s)$ of A of finite multiplicity, so that $\sigma = 1/2$ or $1/2 < s \leqslant 1$. The corresponding eigenfunctions $\psi \in \mathscr{H}$ have the form

$$\psi = (I + (\lambda - \mu)Q_\lambda)v,$$

where $v \in \mathfrak{B}$ is a solution of the homogeneous equation. Eigenfunctions corresponding to the case $\sigma = 1/2$ are cusp forms,[19] that is, $P(\psi)(y) = 0$ for all $y > 0$.

4) The resolvent kernel $r(z, z'; s)$ of A for fixed $z \neq z'$ and $\sigma > 1$ admits meromorphic continuation to the strip $0 < \sigma < 2$, with discrete poles of finite multiplicity. For $\sigma \geqslant 1/2$ these poles lie only on the line $\sigma = 1/2$ and on the interval $1/2 \leqslant s \leqslant 1$ and are simple, with the possible exception of $s = 1/2$.

5) The resolvent $(A - \lambda I)^{-1}$ of A, where $\lambda = s(1 - s) \in \mathbb{C} \setminus [1/4, \infty)$ with non-singular s and $\sigma > 1/2$, is the operator R_λ in (6.11), constructed from the solution U_λ of Faddeev's equation (6.12). The operator R_λ is an integral operator acting in $\mathscr{H}$ with integral kernel $r(z, z'; s)$.

Equation (6.12) is also used for analytic continuation of the continuous-spectrum eigenfunctions of A. In particular, consider the decomposition $F = F_0 \cup F_1$ and define the function $\psi(z, s)$ on F by $\psi(z, s) = \varphi(z, s)$ for $z \in F_1$ and $\psi(z, s) = 0$ for $z \in F_0$. Clearly, if a is large enough, then $\psi(z, s)$ determines a piecewise smooth Γ-automorphic function on $\mathbb{H}$. We put

$$\Psi(z, s) = \big(I + (\lambda - \mu)(I + (\lambda - \mu)Q_\lambda)U_\lambda\big)\psi(z, s)$$

and list the properties of $\Psi(z, s)$ [14].

---

[19]In general, the space $\mathscr{H}^{(0)}$ of cusp forms is an invariant subspace of $\mathscr{H}$ consisting of functions with zero integrals over all horocycles in $\Gamma \backslash \mathbb{H}$. It is not difficult to show [20] that the spectrum of A in $\mathscr{H}^{(0)}$ is discrete.

(i) For fixed z the function $\Psi(z,s)$ is analytic in the strip $0 < \sigma < 2$, except for the singular points for which $\sigma < 1/2$ or $1/2 \leqslant s \leqslant 1$, and $\Psi(z,s)$ is analytic in a neighbourhood of the line $\sigma = 1/2$, with the possible exception of $s = 1/2$.

(ii) For non-singular s in the strip $0 < \sigma < 2$ the function $\Psi(z,s)$ is a smooth Γ-automorphic function on $\mathbb{H}$ satisfying the equation

$$-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi(z,s) = s(1-s)\Psi(z,s). \tag{6.13}$$

For $\sigma > 1$ the solution of (6.13) can be found 'explicitly' as the Eisenstein–Maass series $E(z,s)$:

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y^s(\gamma z). \tag{6.14}$$

Namely, it is easy to show that for $\sigma > 1$ the series converges absolutely and uniformly on compact subsets of $\mathbb{H}$, and defines a Γ-automorphic function satisfying (6.13). For $\sigma > 1$ it is not difficult to prove the equality $\Psi(z,s) = E(z,s)$, which gives a meromorphic continuation of $E(z,s)$ to the strip $0 < \sigma \leqslant 1$, and on the line $\sigma = 1/2$ the function $E(z,s)$ has no singularities, except, possibly, at the point $s = 1/2$.

Finally, the eigenfunction expansion theorem for A is obtained from the above results using (2.3). The reader can find detailed proofs in Faddeev's paper [14], the indicated book by Lang [29], and Venkov's monograph [44], which generalizes Faddeev's method to vector-valued functions. The characteristic determinant of the operator A is defined using an appropriate regularization of the formula (2.5) and M. G. Krein's method of the spectral shift function. Moreover, the characteristic determinant of A is expressed in terms of the Selberg zeta-function of the Fuchsian group Γ, and the calculation of the regularized trace in (2.5) reduces to the famous Selberg trace formula! We refer the reader to [45] for details of these non-trivial calculations. This completes our exposition of Faddeev's method.

As an interesting example, consider the case of the modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. The corresponding Eisenstein–Maass series $E(z,s)$ admits a simple expression in terms of the Epstein zeta-function of the positive-definite binary quadratic form $Q(m,n) = am^2 + bmn + cn^2$ with discriminant $b^2 - 4ac = d < 0$, where $a = 1$, $b = -2x$, and $c = x^2 + y^2$, so that $d = -4y^2$. Furthermore, $z = x + iy \in \mathbb{H}$ is the root of Q,

$$z = \frac{-b + \sqrt{d}}{2a}.$$

In particular, from (6.14) we easily obtain

$$2\zeta(2s)y^{-s}E(z,s) = \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{Q(m,n)^s}, \tag{6.15}$$

where $\zeta(s)$ is the Riemann zeta-function, and the prime on the summation sign indicates that the term with $m = n = 0$ is omitted. The Fourier series expansion

of the function $E(z, s)$ is given by the beautiful formula[20]

$$E(z, s) = y^s + c(s)y^{1-s} + \frac{4\sqrt{y}}{\xi(2s)} \sum_{n=1}^{\infty} \sigma_{1-2s}(n)n^{s-1/2}K_{s-1/2}(2\pi ny) \cos(2\pi nx), \quad (6.16)$$

where $K_s(y)$ is the modified Bessel function of the second kind and

$$\sigma_s(n) = \sum_{d|n} d^s, \quad c(s) = \frac{\xi(2s-1)}{\xi(2s)}, \quad \xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

It follows from Faddeev's method described above that for fixed z the Eisenstein–Maass series $E(z, s)$ is a holomorphic function on the 'physical sheet' $\sigma = \operatorname{Re} s > 1/2$, and it is regular on the line $\sigma = 1/2$. From this it immediately follows that the zeta-function $\zeta(s)$ does not vanish on the line $\sigma = 1$, which implies the asymptotic law of prime numbers! However, this method does not give any information about the poles of $E(z, s)$ on the 'non-physical sheet' $\sigma < 1/2$. One can only say that the non-trivial zeros of $\zeta(s)$ are related to the so-called resonances of the Laplace operator on the modular figure.

6.2. Pseudo-cusp forms and zeros of L-series. In 1977, in H. Hass's diploma work at the University of Heidelberg under the direction of H. Neuenhöffer, several of the first eigenvalues of the discrete spectrum of the Laplace operator on the modular figure were calculated. Stark and Hejhal soon noticed that if one writes $\lambda_k = 1/4 + t_k^2$, then the values $s_k = 1/2 + it_k$ correspond to the first non-trivial zeros of the Riemann zeta-function and the Dirichlet L-series $L(s, \chi)$ with the quadratic character modulo 3! This unexpected observation caused a sensation and was actively discussed in correspondence between Cartier and Weil in 1979 [6], as well as by Venkov, A. I. Vinogradov, Faddeev, and the author at the Leningrad Branch of the Steklov Mathematical Institute of the USSR Academy of Sciences. Hejhal decided to independently verify Haas's calculations and did not find these zeros among the eigenvalues of the Laplace operator.

What was the reason for this discrepancy? As Hejhal explained in [25], Haas was using the standard collocation method for the Neumann problem on the modular figure, and he did not notice the appearance of a logarithmic singularity at the corners $z = \rho$ and $z = i$ of the modular figure, where $\rho = (1 + \sqrt{-3})/2$. In particular, the function $f(z) = r(z, z_0; s)$ for $z \neq z_0$ satisfies the equation

$$\mathcal{A}f = \lambda f, \quad \text{where} \quad \lambda = s(1 - s), \quad (6.17)$$

and if $z_0 = \rho$ or $z_0 = i$, then with discrete approximation it is easy to miss the logarithmic singularity (6.2) at $z \to z_0$. As $y \to \infty$ for fixed z_0, the resolvent kernel has the asymptotics [17]

$$r(z, z_0; s) = \frac{y^{1-s}}{2s - 1}E(z_0, s) + O(e^{-2\pi y}),$$

---

[20] (6.16) is sometimes called the Selberg–Chowla formula.

and therefore if $E(z_0, s) = 0$, then $f(z) \in L^2(F, d\mu)$. It is remarkable that for $z_0 = i$ and $z_0 = \rho$ the function $\zeta(2s)E(z_0, s)$ is proportional to the Dedekind zeta-function of the imaginary quadratic fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, so $\lambda = s(1-s)$ can be expressed in terms of zeros of $\zeta(s)$ and the corresponding L-series. However, $f(z)$ is not an eigenfunction of the Laplace operator, since it is not a cusp form. Namely, the condition

$$\int_{-1/2}^{1/2} f(x + iy)\, dx = 0$$

holds only for $y > \operatorname{Im} z_0$; such functions are called pseudo-cusp forms. Moreover, for z_0 one can take any point on the modular figure, for example, $z_0 = \sqrt{-5}$. It is well known that the function $\zeta(2s)E(\sqrt{-5}, s)$ has zeros outside the line $\operatorname{Re} s = 1/2$, so the corresponding values of λ will not even be real. Thus, the pseudo-cusp forms have no relation to the discrete spectrum of the Laplace operator, and equation (6.17) does not impose any restrictions on λ.

Indeed, if $f \in L^2(F, d\mu)$ satisfies (6.17) and $f \in D(A)$, then from the self-adjoint property of the operator A we obtain

$$(\lambda - \overline{\lambda})\|f\|^2 = (Af, f) - (f, Af) = 0.$$

However, although $f(z) = r(z, z_0; s) \in L^2(F, d\mu)$ when $\zeta(2s)E(z_0, s) = 0$, $f \notin D(A)$ and the integral (Af, f) is divergent, hence the previous argument does not apply. Specifically,

$$(Af, f) = \iint_F \mathcal{A}f(z)\overline{f(z)}\, d\mu(z) = \lambda\|f\|^2 + r(z_0, z_0; s),$$

where the second term is obviously divergent. By using the reality condition (6.4), the difference $(Af, f) - (f, Af)$ can be defined as a limit, which one easily computes via the first Hilbert identity:

$$\lim_{z \to z_0} \left(r(z, z_0; s) - r(z, z_0; \overline{s}) \right) = (\lambda - \overline{\lambda})\|f\|^2.$$

The last formula does not impose any restriction on $\lambda = s(1-s)$, except for the assumption that $\zeta(2s)E(z_0, s) = 0$.

Nevertheless, it makes sense to consider Hilbert spaces of pseudo-cusp forms

$$\mathscr{H}_a = \left\{ f \in \mathscr{H} : \int_{-1/2}^{1/2} f(x + iy)\, dx = 0 \text{ for } y \geqslant a \right\}$$

for a fixed $a > 0$. In particular, denote by Δ_a the Friedrichs extension of the operator Δ restricted to the subspace of smooth functions with compact support in $\mathscr{H}_a$. Lax and Philips proved [32] that the self-adjoint operator Δ_a in $\mathscr{H}_a$ has a purely discrete spectrum, which was studied by Colin de Verdière [7], [8]. Furthermore, it was suggested in [8] that the discrete spectrum of the operator Δ_a for $a = \sqrt{3}/2$ is related to zeros of the Dedekind zeta-function of the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$.

6.3. Heegner points and Linnik asymptotics. The formula (6.15) provides an explicit expression for the Dedekind zeta-function $\zeta_K(s)$ of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ of the fundamental discriminant $d < 0$ in terms of the Eisenstein–Maass series. As is well known (for example, see [4]), the ideal class group of the field K is isomorphic to the group of classes of properly equivalent primitive, positive-definite, integral binary quadratic forms with discriminant d. Each such quadratic form can be written as $Q(m, n) = am^2 + bmn + cn^2$ with integer coefficients a, b, and c satisfying the conditions

$$a > 0, \quad (a, b, c) = 1, \quad \text{and} \quad b^2 - 4ac = d.$$

The root z_Q of the quadratic form Q is given by

$$z_Q = \frac{-b + \sqrt{d}}{2a} \in \mathbb{H},$$

and the proper equivalence class of Q is completely determined by the condition $z_Q \in \widetilde{F}$, where $\widetilde{F}$ is the modular figure. Points $z_Q \in \widetilde{F}$ are called Heegner points for the discriminant d. From this we obtain[21]

$$\zeta_K(s) = \frac{2}{w_d} \left(\frac{|d|}{4} \right)^{-s/2} \zeta(2s) \sum_{i=1}^{h(d)} E(z_i, s), \tag{6.18}$$

where $h(d)$ is the ideal class number of the field K, w_d is the number of units in K, and the z_i run over all Heegner points z_Q of the discriminant d.

Siegel's celebrated theorem [40] states that for any $0 < \varepsilon < 1/2$

$$h(d) > c(\varepsilon)|d|^{1/2-\varepsilon}$$

with a non-explicit constant $c(\varepsilon) > 0$. Before the classical paper [40], the only known result was Hecke's theorem that the generalized Riemann hypothesis for all Dirichlet L-series with quadratic characters implies that $h(d) \to \infty$ as $d \to -\infty$. Surprisingly, in 1933 Deuring [9] proved an unexpected result that the condition $h(d) = 1$ for infinitely many negative fundamental discriminants implies the Riemann hypothesis!

Indeed, if $h(d) = 1$, then

$$z_Q = \begin{cases} \dfrac{1 + \sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4, \\[2mm] \sqrt{D} & \text{if } d = 4D \text{ and } D \equiv 2, 3 \pmod 4. \end{cases}$$

In the latter case, we get immediately from (6.16) and (6.18) that for such d

$$\zeta_K(s) = |D|^{-s/2}\zeta(2s)E(\sqrt{D}, s) = \zeta(2s)(1 + c(s)|D|^{1/2-s}) + O(e^{-2\pi|D|}). \tag{6.19}$$

Suppose now that $\zeta(\rho) = 0$ and $\operatorname{Re}\rho > 1/2$. Because $\zeta_K(s) = \zeta(s)L(s, \chi_d)$, where χ_d is a quadratic character modulo d given by the Kronecker symbol, by passing to

---

[21]For details see [48], for example, where zeta-functions of orders in imaginary quadratic fields are also considered.

the limit $d \to -\infty$ in (6.19) we get that $\zeta(2\rho) = 0$, a contradiction. The case $d \equiv 1$ (mod 4) is considered similarly.

Mordell [36] soon generalized Deuring's result and proved that if the class number takes a fixed value for infinitely many fundamental negative discriminants, then the Riemann hypothesis is true. His proof also used the formulae (6.16) and (6.18). Finally, Heilbronn [24] used the same assumption to deduce the generalized Riemann hypothesis for all Dirichlet L-series with quadratic characters. From this and the aforementioned theorem of Hecke, it follows that $h(d) \to \infty$ as $d \to -\infty$.

However, in the same year of 1934, Siegel proved his famous theorem, which naturally moved the Deuring–Mordell–Heilbronn method to the background. It was only in the 1960s that some of their arguments were used to solve the celebrated tenth discriminant problem of Gauss, in which Heegner points played a prominent role.

Since $h(d) \to \infty$ as $d \to -\infty$, the question arises as to the distribution of the Heegner points on the modular figure. The equivalent problem of the distribution of the integer points on the reduction domain of a two-sheeted hyperboloid $b^2 - 4ac = d < 0$ was solved by Linnik [34] using his ergodic method. Here the condition

$$\left(\frac{d}{p}\right) = 1$$

was also assumed for some prime p, where $\left(\dfrac{n}{p}\right)$ is the Legendre symbol. In particular, Linnik proved that as $d \to -\infty$ the Heegner points are uniformly distributed on the modular figure with respect to the measure

$$d\mu^* = \frac{3}{\pi} \frac{dx\,dy}{y^2}, \quad \text{so that} \quad \mu^*(F) = 1.$$

In [47] the uniform distribution was proved on average over d, that is, for those values of d for which so-called Salié sums admit a good estimate. Finally, Duke [11] proved the uniform distribution of the Heegner points as $d \to -\infty$ using a non-trivial estimate for the Fourier coefficients of modular forms of half-integer weight obtained by Iwaniec [27]. Specifically, let Ω be a convex domain with piecewise smooth boundary in the modular figure, and let $N(\Omega)$ be the number of Heegner points in Ω. Then the Linnik asymptotic expression

$$\frac{N(\Omega)}{h(d)} = \mu^*(\Omega) + O(|d|^{-\delta}) \tag{6.20}$$

is valid for some $\delta > 0$ (possibly dependent on Ω).

We now return to the representation (6.18) for $\zeta_K(s)$ and, as proposed in [47], we use the uniform distribution of the Heegner points on $\widetilde{F}_d$, the modular figure $\widetilde{F}$ with the restriction $\operatorname{Im} z \leqslant \sqrt{|d|}/2$. More precisely, assuming that the $\delta > 0$ in (6.20) does not depend on the domain Ω, we replace the sum in (6.18) by an integral! As a result, as $d \to -\infty$ we get that

$$\zeta_K(s) = \left(\frac{|d|}{4}\right)^{-s/2} \zeta(2s) h(d) \iint_{\widetilde{F}_d} E(z, s)\, d\mu^*(z) + O(h(d)|d|^{-\delta-\sigma/2}), \tag{6.21}$$

where $\sigma = \operatorname{Re} s$. The integral in (6.21) can be evaluated explicitly. Namely, by using (6.13), the integral Green's formula, the invariance of $E(z, s)$ with respect to the modular group, and the Fourier expansion (6.16), we obtain

$$
\iint_{\widetilde{F}_d} E(z, s) \, d\mu(z) = \frac{1}{s(s-1)} \iint_{\widetilde{F}_d} \Delta E(z, s) \, dx \, dy
$$

$$
= \frac{1}{s(s-1)} \int_{-1/2}^{1/2} \frac{\partial E}{\partial y}(z, s) \Big|_{y=\sqrt{|d|}/2} \, dx
$$

$$
= \frac{1}{s-1} \left(\frac{|d|}{4} \right)^{(s-1)/2} - \frac{c(s)}{s} \left(\frac{|d|}{4} \right)^{-s/2}.
$$

Thus, for fixed s we have

$$
\zeta_K(s) = \frac{6h(d)}{\pi\sqrt{|d|}} \zeta(2s) \left(\frac{1}{s-1} - \frac{c(s)}{s} \left(\frac{|d|}{4} \right)^{1/2-s} \right) + O(h(d)|d|^{-\delta-\sigma/2}). \quad (6.22)
$$

Suppose now that for some sequence of fundamental discriminants d, we can choose $\delta = 1/4 + \varepsilon$ with arbitrary $\varepsilon > 0$ in the Linnik asymptotics (see the corresponding arguments in [47]). Let $\zeta(\rho) = 0$, where $\operatorname{Re} \rho > 1/2$. Since $\zeta_K(s) = \zeta(s)L(s, \chi_d)$, by letting $d \to -\infty$ we get from (6.22) that $\zeta(2\rho) = 0$. This contradiction 'proves' the Riemann hypothesis, as in the Deuring–Mordell–Heilbronn approach.

Of course, the starting formula (6.21) needs to be proved, since for domains Ω lying on the very 'top' of the truncated modular figure $\widetilde{F}_d$, the Linnik asymptotic expression (6.20) loses its meaning, and the Heegner points are no longer uniformly distributed as $d \to -\infty$. The easiest way to see this is to average the representation (6.18) of a zeta-function $\zeta_D(s)$ of order $\mathfrak{D}_D$ with discriminant $D = df^2$ in the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$, over all $-D \leqslant X$. The corresponding formula, an analogue of the classical Vinogradov–Gauss formula [46] in the critical strip, was obtained in [48] and has the form

$$
\sum_{-D\leqslant X} \left(\frac{|D|}{4} \right)^{s/2} \zeta_D(s) = \zeta(2s)(\Phi(s)X^{1+s/2} + c(s)\Phi(1-s)X^{1+(1-s)/2}) + R_s(X).
$$

$$(6.23)$$

Here

$$
\Phi(s) = \frac{2^{-s}\zeta(s)}{(s+2)\zeta(s+2)} \quad \text{and} \quad R_s(X) = O\left(\frac{X^{3/4} \log^3 X}{|s-1|(|s-1/2| + \log^{-1} X)} \right),
$$

and the estimate of the remainder is uniform with respect to s on compact subsets of the critical strip. The leading term of the asymptotics in (6.23) is proportional to $\zeta(s)$, which makes the previous argument inapplicable. Thus if Linnik's asymptotic expression can hold up to the very top of the truncated modular figure $\widetilde{F}_d$, then it is only for very special values of d.

In the above arguments the formula (6.18) played the key role. It was also used by Zagier [49] to construct a non-trivial representation of $SL(2, \mathbb{R})$ connected

with the zeros of the Riemann zeta-function. We should also mention recent works of Bombieri and Garrett on the spectrum of the Laplace operator on the space of pseudo-cusp forms in connection with the zeros of $\zeta_K(s)$ (see the talks [3] and [18]). We leave the reader alone with these intriguing works and the literature cited there.

Note added in proof: results of the talks [3] and [18] have now appeared in the preprint of Enrico Bombieri and Paul Garrett, "Designed pseudo-Laplacians", https://arxiv.org/abs/2002.07929.

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