Russian Math. Surveys **75**:1 89–146 *Uspekhi Mat. Nauk* **75**:1 95–154

DOI: https://doi.org/10.1070/RM9905

to Valery Vasilievich Kozlov
on the occasion of his 70th birthday

Extremal problems in hypergraph colourings

A. M. Raigorodskii and D. D. Cherkashin

Abstract. Extremal problems in hypergraph colouring originate implicitly from Hilbert's theorem on monochromatic affine cubes (1892) and van der Waerden's theorem on monochromatic arithmetic progressions (1927). Later, with the advent and elaboration of Ramsey theory, the variety of problems related to colouring of explicitly specified hypergraphs widened rapidly. However, a systematic study of extremal problems on hypergraph colouring was initiated only in the works of Erdős and Hajnal in the 1960s. This paper is devoted to problems of finding edge-minimum hypergraphs belonging to particular classes of hypergraphs, variations of these problems, and their applications. The central problem of this kind is the Erdős–Hajnal problem of finding the minimum number of edges in an n-uniform hypergraph with chromatic number at least three. The main purpose of this survey is to spotlight the progress in this area over the last several years.

Bibliography: 168 titles.

Keywords: extremal combinatorics, hypergraph colourings.

Contents

This work was supported by the Russian Science Foundation (project no. 16-11-10014).
AMS 2010 *Mathematics Subject Classification.* Primary 05C15, 05C35, 60C05.

1. Introduction

We start with basic definitions. A pair of sets $H := (V, E)$ is called *a hypergraph* if V is finite and $E \subset 2^V$. In this case V is called the set of vertices and E the set of (hyper)edges, and it is supposed that each vertex is incident to at least one edge. Note that an ordinary graph is a particular example of a hypergraph. A hypergraph is said to be *n-uniform* if all of its edges have cardinality n. Accordingly, any graph is a 2-uniform hypergraph. In most sections of this survey all the hypergraphs under consideration are uniform; for brevity we sometimes refer to them just as n-graphs.

A *colouring* of a hypergraph with r colours is a map $f\colon V \to \{1, \dots, r\}$. A colouring of a hypergraph with r colours is said to be *proper* if each edge $e \in E$ contains two vertices $v_1, v_2 \in e$ such that $f(v_1) \neq f(v_2)$. In other words, the existence of a proper colouring of a hypergraph with r colours means that the set of its vertices can be partitioned into r subsets, $V = V_1 \cup V_2 \cup \dots \cup V_r$, so that no edge is a subset of V_i. The minimum number r for which there exists a proper r-colouring of H is called the *chromatic number* of the hypergraph H.

There arises a natural problem, first formulated in 1961 by Erdős and Hajnal ([67], [68]): find the minimum number of edges in a 2-uncolourable n-uniform hypergraph. They also introduced the notation $m(n)$ for this quantity.

Let us explain why this problem is so pertinent. In some sense the chromatic number is indicative of the extent to which the hypergraph is non-trivial. Below we shall see that this agrees well with the fact that all examples of n-uniform hypergraphs with large chromatic numbers and few edges are constructed with the use of probabilistic techniques. Moreover, the first explicit method for constructing a somewhat comparable example appeared only as recently as 2013 [81]. It also turns out that the size of the hypergraph is characterized by the number of edges rather than the number of vertices, and furthermore it does not make sense to regard the number of vertices as an additional parameter.

The Erdős–Hajnal problem is considered in § 2. In the voluminous literature 2-colourability of hypergraphs is often referred to as property B. This term was introduced by Miller [126] in honor of Bernstein, who proved [33] that any countable family of infinite sets has property B. The Erdős–Hajnal problem and its generalizations were the subject of a survey by Raigorodskii and Shabanov [132].

Our survey does not cover the complexity aspects of hypergraph colourings. We note only that in contrast to graphs, even the problem of deciding the 2-colourability of a 3-uniform hypergraph is NP-complete [59]. In the case of a graph it is sufficient to suspend the graph by any one of its vertices and to make sure that there are no edges within each level, which can be realized in a time $O(|V| + |E|)$.

Nor do we address the structural theory of hypergraphs. The interested reader is referred to Berge's classical monograph [28] or the more recent survey [105] by Kostochka.

Section 3 is devoted to generalizations of the Erdős–Hajnal problem to various classes of hypergraphs. A generalization to simple (or linear) hypergraphs is considered in § 3.1. It turns out that the simplicity condition significantly increases the order of growth of the minimum number of edges in a hypergraph with a given chromatic number. The class of intersecting families appears in § 3.2, a class especially notable for two reasons: first, there is no way to effectively apply the probabilistic approach, and second, it turns out that any large intersecting family is 2-colourable. In § 3.3 we consider the problem with the uniformity condition replaced by a constraint on the minimum size of an edge. Surprisingly, dropping the uniformity condition results in a significantly larger gap between the best known upper and lower bounds.

Sections 4, 5, and 6 are devoted to variations of the problem with modified requirements on the colouring. In § 4 we consider colourings of hypergraphs with prescribed colours (listed colourings). It turns out that the list chromatic number of a simple graph increases with the average vertex degree. Section 5 is concerned with the situation which is in a certain sense opposite to the classical one, namely, a panchromatic colouring requires that each of the colours be used on every edge; thus, as the number of colours increases, this constraint strengthens rather than weakens. Section 6 is devoted to generalizations of a well-known theorem of Hajnal and Szemerédi on equitable colourings of graphs to the case of hypergraphs.

In § 7 we discuss the known problem of discrepancy. In the Erdős–Hajnal problem a proper colouring has discrepancy less than n. Surprisingly, all estimates of the size of a minimal (with respect to the number of edges) n-uniform hypergraph with a positive discrepancy depend only on number-theoretic properties of n.

In § 8 we gather explicit constructions and examples of various parts of the survey. Understandably, in the same section we consider small values of n.

Finally, § 9 is devoted to mathematical applications of the methods and theorems mentioned in this survey. Applications to other branches of science can be found in the monograph [40].

The authors thank Fedor Petrov for an endless list of important remarks. We are grateful to Margarita Akhmejanova, who kindly made us aware of several interesting manuscripts by Dmitrii Shabanov and his students. A lot of inaccuracies were noticed by Alexey Gordeev, Margarita Akhmejanova, Anatoly Kulikov, Dmitrii Shabanov, and Alexander Sidorenko.

2. The Erdős–Hajnal problem

We recall that the Erdős–Hajnal problem consists in finding the minimum number $m(n)$ such that there exists an n-uniform hypergraph with $m(n)$ edges which cannot be properly coloured with two colours.

2.1. Classical estimates. In 1961 Erdős and Hajnal proposed [67] the first upper bound for $m(n)$, which is attained on the set of all n-element subsets of a $(2n-1)$-element set:

$$m(n) \leqslant \binom{2n-1}{n} = \bigl(4+o(1)\bigr)^n. \qquad (2.1)$$

As early as 1963 Erdős derived the first non-trivial estimates for the quantity $m(n)$.

Theorem 2.1.1 (Erdős [63], [64], 1963). *For any* $n \geqslant 2$

$$2^{n-1} \leqslant m(n) \leqslant (1+o(1))\,\frac{e\log 2}{4}\,n^2 \cdot 2^n. \qquad (2.2)$$

The lower estimate was improved by Schmidt [141] in 1964 by about a factor of two. It is noteworthy that both estimates were derived using a simple probabilistic approach, whereas deterministic methods still give no results with comparable asymptotics: it was only in 2013 that Gebauer [81] constructed an explicit example of an n-uniform hypergraph with $2^{n+O(n^{2/3})}$ edges and chromatic number 3; up to that moment the smallest base of the exponential was $\sqrt{7}$. These examples, as well as the solution of the problem for small values of n, are presented in § 8.

Proof of Theorem 2.1.1. We start by deriving the *lower bound*. Consider an arbitrary n-graph $H = (V, E)$ with $|E| < 2^{n-1}$ and let us show that it admits a proper 2-colouring.

It turns out to be sufficient to colour each vertex blue or red with probability 1/2 independently of all the other vertices. In this case the probability that the edge e is monochromatic equals 2^{1-n}, and the probability that there exists a monochromatic edge clearly does not exceed the sum over all edges $e \in E$ of the probability that e is monochromatic. Since there are fewer than 2^{n-1} edges, this sum is strictly less than 1. Thus, the random colouring is proper with positive probability, which proves the existence of a proper colouring.

The proof of the *upper bound* is not that trivial. To avoid rounding in calculations, assume that n is even. Consider a set of vertices of cardinality $v = n^2/2$, and choose m random edges uniformly and independently. The number m will be specified later.

Easy calculations show that for any fixed colouring C the probability that a randomly chosen edge is monochromatic equals

$$p := \frac{\dbinom{v_1}{n} + \dbinom{v_2}{n}}{\dbinom{v}{n}},$$

where v_1 and v_2 denote the numbers of vertices of the first and second colour, respectively. Hence, since the edges are chosen independently, the probability is $(1-p)^m$ that after choosing m random independent edges the colouring C is proper. Let

$$q := \frac{2\dbinom{v/2}{n}}{\dbinom{v}{n}} = 2\,\frac{\dbinom{n^2/4}{n}}{\dbinom{n^2/2}{n}} = (1 + o(1))\frac{2e}{2^n}.$$

Note that $p \geqslant q$ by virtue of the convexity of the sequence $\left\{\dbinom{v}{n}\right\}_{v \geqslant 0}$. Since the total number of colourings is $2^{n^2/2}$, it suffices to verify the inequality

$$2^{n^2/2}(1-q)^m \leqslant e^{\log 2 \cdot n^2/2 - qm} < 1, \tag{2.3}$$

which holds for an appropriately chosen value of m of the form

$$m = (1 + o(1))\frac{e \log 2}{4}\, n^2 \cdot 2^n. \tag{2.4}$$

Thus, for the chosen value of m the probability is positive that none of the colourings are proper. This means that we have established the existence of a hypergraph with the required number of edges and chromatic number at least 3. $\square$

Moreover, it follows from our arguments that a random hypergraph on m edges is *almost surely* not 2-colourable if m is chosen in accordance with (2.4) but with another choice of $o(1)$.

Such a concentration of probability is one of the reasons why the upper bound for a fixed number of colours still remains unimproved. Even for slightly weaker problems (for example, see § 3.3) we still have no other ways of proving upper bounds. The situation with lower bounds is quite different.

2.2. Improvements of the lower bound. In their famous 1973 paper [70], Erdős and Lovász put forward the conjecture that

$$\frac{m(n)}{2^n} \to \infty.$$

In the 1970s this conjecture was proved in a series of papers by Beck [23], [24] and Spencer [160], who brought Beck's calculations to optimal form. In these papers the following estimate was established:

$$m(n) \geqslant c\biggl(\frac{n}{\log n}\biggr)^{1/3} \cdot 2^n.$$

We note that the lower bound in Theorem 2.1.1 was proved by substituting a random colouring which is completely independent of the hypergraph. In the above papers this problem was solved using the *method of recolouring*. In a nutshell, this method consists in the following: suppose that the hypergraph contains $k \cdot 2^{n-1}$ edges; then for a random colouring an average of k edges are monochromatic. If we now randomly recolour the vertices belonging to monochromatic edges, then we can succeed even for $k > 1$.

In 2000 Radhakrishnan and Srinivasan [131] modified the Beck–Spencer method of recolouring and demonstrated that $m(n) \geqslant c\sqrt{n/\log n} \cdot 2^n$. In 2009 Pluhár [130] proposed a very simple and completely new greedy method and derived the inequality $m(n) > cn^{1/4} \cdot 2^n$, which, however, is weaker even than the Beck–Spencer estimates, but is remarkable for its simplicity. Finally, in 2015 Cherkashin and Kozik simplified [50] the arguments of Radhakrishnan and Srinivasan on the basis of Pluhár's ideas. We present these in the next subsection.

2.2.1. *Greedy approach.* Pluhár's argument [130] consists in the following. Instead of a random colouring consider a random ordering of vertices (we denote this order by π).

Definition 2.2.1. An ordered pair of edges is called a 2-*chain* if these edges have exactly one vertex in common. A 2-chain (e_1, e_2) is said to be *ordered* according to the order π if $\pi(v_1) \leqslant \pi(v_2)$ for any two vertices $v_1 \in e_1$, $v_2 \in e_2$.

Lemma 2.2.2 (Pluhár [130], 2009). *A hypergraph admits a proper* 2*-colouring if and only if there exists a vertex order* π *containing no ordered* 2*-chains.*

Proof. If there exists a proper 2-colouring, then any order in which all vertices of one colour precede all vertices of the other colour obviously contains no ordered 2-chains.

Assume that there exists a vertex order π containing no ordered 2-chains. Consider the vertices one by one according to the order π and colour each vertex with the minimal colour that does not form monochromatic edges on the already coloured vertices. If a vertex v cannot be coloured, then there exists an edge $e_2 \ni v$ such that all other vertices of this edge already have the colour 2. Consider the π-least vertex $w \in e_2$. Since w cannot be coloured with the first colour, there exists an edge $e_1 \ni w$ in which all other vertices have already been coloured with the first colour. This is a contradiction, since (e_1, e_2) is a π-ordered 2-chain. Hence, each vertex can be coloured in this way, and by construction the colouring is proper. □

Remark 2.2.3. There is a simpler way to prove the 'if' part (however, it cannot be generalized to more colours). Let us colour the π-least vertex in each edge with the first colour, and the π-maximal vertex with the second colour. Note that none of the vertices will be coloured with both colours, since the hypergraph contains no π-ordered 2-chains.

Now consider the random order. The probability that a particular 2-chain is ordered equals

$$p := \frac{[(n-1)!]^2}{(2n-1)!}\,.$$

Note that the number of 2-chains is not greater than $|E|^2$, hence for $p|E|^2 < 1$ the hypergraph is 2-colourable with a positive probability. With the use of Stirling's formula we obtain the following result.

Theorem 2.2.4 (Pluhár [130], 2009). *There exists a constant $c > 0$ such that for any n*

$$m(n) \geqslant cn^{1/4} \cdot 2^n.$$

In particular, Lemma 2.2.2 shows that a hypergraph whose chromatic number is greater than r should contain many r-chains. This observation is sometimes referred to as the Lovász criterion, and further investigations for $r = 2$ were carried out in [73].

Remark 2.2.5. With use of the greedy approach one can also derive the Caro–Tuza bound [44] on the independence number. If $H = (V, E)$ is an n-graph, then

$$\alpha(H) \geqslant \sum_{v \in V} \binom{\deg(v) + 1/(n-1)}{\deg(v)}^{-1},$$

where $\deg(v)$ is the degree of the vertex v.

2.2.2. *Mixed approach.* We start by presenting the Radhakrishnan–Srinivasan algorithm, which gives the estimate

$$m(n) \geqslant c\sqrt{\frac{n}{\log n}} \cdot 2^n. \tag{2.5}$$

In the first step we colour all vertices independently with either colour with probability $1/2$. Then we order the vertices randomly and consider them in accordance with this order. Suppose that the current vertex v appears in a monochromatic edge e (with respect to the initial colouring), and that no other vertex of the edge e has been recoloured so far. Then we change the colour of v with probability $p := (\log n)/(2n)$.

In the next subsection we prove the correctness of the Cherkashin–Kozik algorithm, which gives the result (2.5) with the same constant c in the case of two colours. In the first step of the algorithm each vertex is coloured independently with probability $(1-p)/2$ with either colour (correspondingly, the vertex remains uncoloured with probability p). In the second step the remaining vertices are coloured in accordance with Pluhár's algorithm.

We conclude this subsection with the Erdős conjecture on the correct order of growth of $m(n)$.

Conjecture 2.2.6 (Erdős [64], 1964).

$$m(n) = (1+o(1))n \cdot 2^n.$$

2.3. The case of $r > 2$ colours. The problem of finding the minimum number $m(n,r)$ of edges in an n-uniform hypergraph that admits no proper colouring with r colours was posed by Herzog and Schönheim [93]. They also proposed direct generalizations of Theorem 2.1.1 and presented some easy observations. For example, the inequalities in Theorem 2.1.1 turn into the inequalities

$$r^{n-1} \leqslant m(n,r) \leqslant \left(1 + O\left(\frac{1}{n}\right)\right) \frac{e}{2}\, n^2 (\log r)(r-1) r^{n-1}. \tag{2.6}$$

In this subsection we consider the case of a fixed r and increasing n. In this case no upper estimates better than (2.6) are known as yet.

The lower bound, in turn, has already been improved several times in the 21st century. We start with an algorithm due to Kostochka [104] which yields the following result.

Theorem 2.3.1 (Kostochka [104], 2004). *If* $r < \sqrt{\dfrac{1}{8} \log \dfrac{\log n}{2}}$, *then for* $a = \lfloor \log_2 r \rfloor$

$$m(n,r) > e^{-4r^2} \left(\frac{n}{\log n}\right)^{a/(a+1)} r^n.$$

Now let us present a generalization of Pluhár's algorithm.

Theorem 2.3.2 (Pluhár [130] and Shabanov [148], 2009). *For any* $n \geqslant 2$ *and* $r \geqslant 2$

$$m(n,r) \geqslant c n^{1/2 - 1/(2r)} r^n.$$

Proof. A sequence of edges $a_1, \dots, a_r$ is called an *r-chain* if $|a_i \cap a_j| = 1$ for $|i-j| = 1$ and $a_i \cap a_j = \varnothing$ otherwise, and an r-chain is called an *ordered r-chain* if it follows from $i < j$ that no vertex of a_i exceeds any vertex of a_j (with respect to the fixed total order on V).

Pluhár's theorem [130] states that the existence of a total order on V having no ordered r-chains is equivalent to r-colourability of the hypergraph $H = (V, E)$ (the statement and the proof are completely analogous to those of Lemma 2.2.2). Consider a random order on the vertex set V. Note that the probability of any r-chain being ordered is

$$\frac{[(n-1)!]^2[(n-2)!]^{r-2}}{((n-1)r+1)!}.$$

On the other hand, the number of r-chains does not exceed $2|E|^r/r!$ (any family of r edges gives at most two r-chains). Therefore, the inequality

$$2\frac{|E|^r}{r!}\,\frac{[(n-1)!]^2[(n-2)!]^{r-2}}{((n-1)r+1)!} < 1$$

guarantees that there exists a proper r-colouring of H. Some technical details complete the proof of the theorem. □

Now we turn to the generalizations announced in the previous subsection. First of all, a generalization of the Radhakrishnan–Srinivasan algorithm (Shabanov [147]) yields the estimate

$$m(n,r) \geqslant c\sqrt{\frac{n}{\log n}}\, r^{n-1}$$

for all $n, r \geqslant 2$. Three years later this result was slightly improved.

Theorem 2.3.3 (Shabanov [151], 2012). *For any $n \geqslant 3$ and $r \geqslant 3$*

$$m(n,r) \geqslant \frac{1}{2}\sqrt{n}\, r^{n-1}.$$

In the general case the result of Cherkashin and Kozik is formulated as follows.

Theorem 2.3.4 (Cherkashin–Kozik [50], 2015). *For any fixed integer $r \geqslant 2$*

$$m(n,r) \geqslant c\left(\frac{n}{\log n}\right)^{(r-1)/r} r^{n-1}. \tag{2.7}$$

Proof. Let (V, E) be an n-uniform hypergraph with kr^{n-2} edges, with k to be specified later. To each vertex we assign a weight, a random real number between 0 and 1 (chosen uniformly and independently), and denote it by w. Put $p := (2\log n)/n$ and call an edge *short* if the weights of all its vertices lie in an interval of length less than $(1-p)/r$. The expected number of short edges does not exceed

$$kr^{n-2}n\left(\frac{1-p}{r}\right)^{n-1} \approx \frac{k}{rn},$$

since we sum up the expectations of the indicator functions of the events $A(e, v)$ corresponding to the edge e being short and the vertex $v \in e$ having the smallest weight in e for all e and v.

Let us estimate the probability of occurrence of an ordered r-chain without short edges. Suppose that the edges $e_1, \dots, e_r$ form such an r-chain and $e_i \cap e_{i+1} = \{v_i\}$.

Then since the chain contains no short edges, each $w(v_i)$ $(1 \leqslant i \leqslant r-1)$ lies in the interval $\left[\dfrac{i-ip}{r}, \dfrac{i+(r-i)p}{r}\right]$ (otherwise one of the remaining edges of the chain does not fit on the left or on the right). The probability that all the w_i lie in the required intervals is p^{r-1}, since each interval has length p. Let $w_0 := 0$, $w_r := 1$, and $w_i = w(v_i)$. Then all other vertices of the edge e_i should fall in the interval $[w_{i-1}, w_i]$. The probability of this event is

$$\prod_{i=0}^{r-1}(w_{i+1} - w_i)^{n-2} \leqslant r^{-r(n-2)},$$

where the estimate follows from the inequality between the arithmetic and geometric means. Accordingly, the expected number of chains in question does not exceed

$$\frac{2|E|^r}{r!}\, p^{r-1} r^{-r(n-2)} = \frac{2}{r!}\, k^r \left(\frac{2\log n}{n}\right)^{r-1}.$$

For

$$k < cr\left(\frac{n}{\log n}\right)^{(r-1)/r}$$

the expected number is less than $1/2$ for an appropriately chosen constant $c > 0$. The same condition on k is sufficient for the expected number of short edges to tend to zero with increasing n. Consequently, with a positive probability the algorithm returns a proper colouring of the graph. $\square$

2.4. The case of a large number of colours. Consider the situation where the number of colours r is much greater than n. In the case of graphs the problem is trivial, since for any colouring of the vertices of a graph G with $\chi(G)$ colours there should be at least one edge between any two colours. Thus,

$$m(2,r) \geqslant \binom{r+1}{2}.$$

On the other hand, the complete graph on $r+1$ vertices cannot be coloured with r colours, and therefore, $m(2,r) = \dbinom{r+1}{2}$.

2.4.1. *Upper estimates.* Similarly, the complete n-uniform hypergraph on $r(n-1)+1$ vertices cannot be properly coloured with r colours. Consequently,

$$m(n,r) \leqslant \binom{r(n-1)+1}{n}.$$

Note that for large values of r this example is already better than the main probabilistic bound in (2.6). Erdős posed the conjecture [66] that for any n there exists an integer $r_0(n)$ such that for $r > r_0$ the best possible example is the complete hypergraph on $r(n-1)+1$ vertices. In other words,

$$m(n,r) = \binom{r(n-1)+1}{n} \quad \text{for } r > r_0.$$

This conjecture was disproved by Alon [10] with the use of the Turán numbers. The *Turán number* $T(v,b,n)$ is the minimum number of edges in an n-uniform hypergraph on v vertices such that any b-element subset of vertices contains an edge of the hypergraph. It follows immediately from the pigeonhole principle that

$$m(n,r) \leqslant \min_{b\geqslant n} T\bigl(r(b-1)+1,b,n\bigr).$$

With the use of this inequality Alon derived the estimates

$$m(n,r) \leqslant \binom{rn}{n} \frac{\log n}{\log n - 1} \frac{1}{\lfloor n/\log n \rfloor}$$

and

$$m(n,r) \leqslant cn^2(\log n)\left(\frac{3e}{4}\right)^n r^n.$$

Note that for $n \geqslant 13$ the first of Alon's estimates disproves Erdős' conjecture (for $n \geqslant 4$, substituting other estimates of Turán numbers also disproves this conjecture). More information about the Turán numbers can be found in the survey [158] by Sidorenko (an overview of a more general problem of Turán is presented in [97]).

Alon also conjectured [10] that for any n the sequence $a_n := m(n,r)/r^n$ has a limit. This conjecture was affirmed by Cherkashin and Petrov [52]. We give a sketch of their proof in § 2.4.3.

With the use of the best estimates on the Turán numbers presented in [159] for $b=n^2$ (it should be noted that Alon substituted $b=n+1$ and $b=1.5n$) Akolzin and Shabanov obtained in [9] the current best upper estimate

$$m(n,r) \leqslant cn^3 \cdot \log n \cdot r^n.$$

In conclusion we note that for $n=3$ Erdős' conjecture still remains open; this case is discussed in § 2.4.4.

2.4.2. *Lower bounds.* Alon [10] invented the following trick, which employs the so-called *alterations method* (see Chap. 3 in [15]): colour an n-graph randomly with $a<r$ colours, and then recolour dangerous edges. The expected number of dangerous edges (see end of § 3.1.1) is

$$|E| \cdot a^{1-n}.$$

We note that there remain $r-a$ unused colours, and with each of these we can colour an arbitrary assembly of at most $n-1$ vertices in such a way that there are no monochromatic edges. Thus, for

$$|E| < a^{n-1}(r-a)(n-1)$$

the hypergraph can be coloured properly with r colours. Substituting $a=\left\lfloor \frac{n-1}{n} r \right\rfloor$, we get that

$$m(n,r) \geqslant (n-1)\left\lceil \frac{r}{n} \right\rceil \left\lfloor \frac{n-1}{n} r \right\rfloor^{n-1}. \tag{2.8}$$

Note that Theorem 2.3.2 in our case yields an estimate of the form $m(n,r) \geqslant c\sqrt{n}\,r^n$. Combining the methods of Alon and Cherkashin–Kozik, Akolzin and Shabanov [9] derived the estimate

$$m(n,r) \geqslant c\,\frac{n}{\log n}\,r^n.$$

This result required refining the estimate in Theorem 2.3.4 up to r^n for large values of r, which can be done by carrying out a more precise analysis instead of using the inequality between the means.

2.4.3. *Regularity of the chromatic number.* In this subsection we give a sketch of the proof of the Alon conjecture.

Theorem 2.4.1 (Cherkashin–Petrov [52], 2018). *The sequence $a_r := m(n,r)/r^n$ has a limit for any fixed n.*

The main idea of the proof is to consider the inverse function and employ subadditivity-like inequalities. Let $f(N)$ denote the maximum chromatic number of an n-uniform hypergraph with N edges (we extend the definition to $f(0) := 1$). Obviously, f is a weakly increasing function and

$$m(n,r) = \min\{N \mid f(N) > r\}. \tag{2.9}$$

Therefore, $m(n,r) \sim Cr^n$ is equivalent to $f(N) \sim (N/C)^{1/n}$.

Lemma 2.4.2. *For any $N > 0$ and any positive integer p*

$$f(N) \leqslant \max_{a_1+a_2+\dots+a_p \leqslant N/p^{n-1}} \bigl(f(a_1)+f(a_2)+\dots+f(a_p)\bigr).$$

Proof. Let $H = (V,E)$ be an n-uniform hypergraph with $E = N$. We colour the vertices of the hypergraph randomly and independently with auxiliary colours $\eta(v) \in \{1,2,\dots,p\}$. Let $V_i = \eta^{-1}(\{i\})$, and let H_i be the hypergraph induced by H on V_i. Suppose that H_i has a_i edges. The expected sum $\sum_i a_i$ is $|E|/p^{n-1}$ (since each edge $e \in E$ belongs to each hypergraph H_i with probability $1/p^n$). Hence, there exists an auxiliary colouring η with $\sum a_i \leqslant N/p^{n-1}$. Fix η and colour each H_i using $f(a_i)$ colours in such a way that no colour is used for more than one i. Thus, we have used a total of $\sum f(a_i)$ colours to properly colour the hypergraph H. $\square$

Various problems and results of this kind were considered in the survey [39].

The rest of the proof consists of a purely analytic argument: it remains to show that for any function $f(N)$ satisfying the hypotheses of Lemma 2.4.2 the function $g(N) := f(N)N^{-1/n}$ has a limit as $N \to \infty$.

The question of whether $m(n,r)$ is regular with respect to the first argument remains open: is it true that

$$\lim_{n\to\infty} \frac{m(n+1,r)}{m(n,r)} = r?$$

2.4.4. *The case of* 3*-graphs.* We are interested in estimating the limit of the ratio $m(3,r)/r^3$. In the previous subsection we showed that such a limit exists. Denote it by L.

Let us compare the applicability of the above methods in this case. We first note that the Erdős conjecture implies that $L = 4/3$, and the example of a complete 3-graph does give the estimate $L \leqslant 4/3$.

Now we turn to lower bounds. First of all, Alon's bound in (2.8) yields $L \geqslant 8/27 = 0.296\dots$. Pluhár's method in this case gives the estimate $L \geqslant 4/e^3 = 0.199\dots$. Akolzin and Shabanov presented no specific calculations of constants, but their method gives the estimate $L \geqslant 0.205\dots$, as demonstrated in [49].

The equality (2.9) shows that the asymptotic upper bounds on $f(N)$ and lower bounds on L are equivalent. The paper [52] contains the following lemma.

Lemma 2.4.3. *For $c_n := \lceil (1-2^{1/n-1})^{-n} \rceil$ and any $N \geqslant M > 0$*

$$f(N) \leqslant N^{1/n} \max_{M \leqslant a < c_n M} f(a)\, a^{-1/n}.$$

It is known that $f(0) = 1$, $f(1) = \dots = f(6) = 2$ (this is shown below in § 8.1), and $f(7) = \dots = f(26) = 3$ (see [8]). Taking Lemmas 2.4.2 and 2.4.3 into account and carrying out some machine calculations, we obtain the inequality

$$m(3,r) \geqslant 0.324\ldots \cdot r^3$$

for $r > r_0$.

Finally, we note that the number of r-chains in the graph can be estimated more accurately than by $|E|^r/r!$.

Proposition 2.4.4 (Cherkashin [49], 2019). *Any hypergraph $H = (V,E)$ contains at most*

$$\frac{|E|}{2}\left(\frac{|E|}{r-1}\right)^{r-1}$$

r-chains.

The proof is based on a study of the induced r-paths in the auxiliary graph $G = (E,F)$. The vertices of G are edges of the original hypergraph H, and a pair $(e_1,e_2) \in E \times E$ of vertices in G is joined by an edge in G if $|e_1 \cap e_2| = 1$. It is clear that the number of r-chains in H is estimated from above by the number of r-chains in G.

It remains to apply an estimate obtained by Pippenger and Golumbic [129] for the number of such paths. This made it possible to improve Pluhár's bound to $L \geqslant 0.54\dots$.

2.5. The Lovász. local lemma The well-known Brooks' theorem states that if a connected graph with vertex degrees at most d is neither a cycle of odd length nor a complete graph, then it can be properly coloured with d colours.

In the process of solving a similar local problem for hypergraphs the Lovász local lemma was invented. This lemma is widely used in the most diverse areas of mathematics, from combinatorics and probability theory to Diophantine approximations and analytic number theory. This is one of the few general ways to pass from local assertions to global assertions.

Theorem 2.5.1 (Erdős–Lovász [70], 1973). *Let $A_1, \dots, A_m$ be events in a probability space, and let $J(1), \dots, J(m)$ be subsets of $\{1, \dots, m\}$ such that $i \notin J(i)$. Also let x_i be real numbers with $0 < x_i < 1$ for $1 \leqslant i \leqslant m$. Suppose that the following conditions are valid*:

(a) *for any i the event A_i is independent of the algebra generated by the events $\{A_j, j \notin J(i) \sqcup \{i\}\}$*;

(b) *for each i*

$$\Pr(A_i) \leqslant x_i \prod_{j \in J(i)} (1 - x_j),$$

where $\Pr(A_i)$ is the probability of the event A_i.

Then with positive probability none of the events A_i occur.

Sometimes the local lemma is employed just in this form. In the case where all events have the same probability estimates, we obtain the following result.

Theorem 2.5.2 (symmetric version of the local lemma). *Assume that $ep(d+1) \leqslant 1$, each event A_i occurs with probability at most p, and $|J(i)| \leqslant d$ for all i. Then with positive probability none of the events A_i occur.*

Proof. Take $x_i = x = 1/(d+1)$. Then $(1-x)^d \geqslant 1/e$: for example, this follows from the definition of the quantity e. Consequently, $p \leqslant x(1-x)^d$, and the hypotheses of the local lemma is satisfied. □

Theorem 2.5.2 directly implies the following non-trivial result.

Theorem 2.5.3 (Erdős–Lovász [70], 1973). *If each edge of an n-uniform hypergraph meets at most $2^{n-3}/n$ other edges, then the graph can be coloured properly with two colours.*

The next version of the local lemma was obtained by Kozik [113]. We will use it in § 3.1. More specific versions were used by Beck [25] and Szabó [162].

Lemma 2.5.4. *Let $X_1, \dots, X_m$ be independent random variables, and let $\mathscr{A}$ be the set of events determined by these variables. For $A \in \mathscr{A}$ let $\operatorname{vbl}(A)$ denote the minimum set of variables that determine A. For each X_i define the polynomial*

$$w_{X_i}(z) = \sum_{A \in \mathscr{A} : X_i \in \operatorname{vbl}(A)} \Pr(A) z^{|\operatorname{vbl}(A)|}.$$

Suppose that there exists a function $w(z)$ such that for any $z > 1$ and any X_i

$$w(z) > w_{X_i}(z).$$

If there exists a $\tau \in (0,1)$ such that

$$w\biggl(\frac{1}{1-\tau}\biggr) \leqslant \tau,$$

then with positive probability none of the events $\mathscr{A}$ occur.

Proof. We apply Lemma 2.5.1 with the parameters

$$J(i) := \{A_j \mid \mathrm{vbl}(A_i) \cap \mathrm{vbl}(A_j) \neq \varnothing\}$$

and $x_i := (1-\tau)^{-|\,\mathrm{vbl}(A_i)|}\,\Pr(A_i)$. It remains only to verify the condition

$$\begin{aligned} x_i \prod_{A_j \in J(i)} (1-x_j) &\geqslant x_i \prod_{X \in \mathrm{vbl}(A_i)} \prod_{j: X \in \mathrm{vbl}(A_j)} (1-x_j) \\ &\geqslant x_i \prod_{X \in \mathrm{vbl}(A_i)} \biggl(1 - \sum_{j: X \in \mathrm{vbl}(A_j)} x_j\biggr) \\ &\geqslant x_i \prod_{X \in \mathrm{vbl}(A_i)} \biggl(1 - w_X\biggl(\frac{1}{1-\tau}\biggr)\biggr) \\ &\geqslant x_i \biggl(1 - w\biggl(\frac{1}{1-\tau}\biggr)\biggr)^{|\,\mathrm{vbl}(A_i)|} = \Pr(A_i). \qquad \square \end{aligned}$$

The local version of Theorem 2.3.1 was obtained by Kostochka, Rödl, and Kumbhat [111]. Local versions of Theorems 2.2.4, 2.3.3, 2.3.4, as well as of the Radhakrishnan–Srinivasan estimate, were obtained in the same papers as the original theorems. The estimate (2.8) admits no local version.

Further development of the combinatorial local theory can be found in [31], for example.

2.6. Critical hypergraphs. A hypergraph is said to be *critical* (*edge-critical*), if the removal of any of its edges decreases the chromatic number. The following theorem proved to be rather important, for it appears in different areas of combinatorics.

Theorem 2.6.1 (Lovász [119], 1970; Woodall [168], 1972; Seymour [143], 1974; Burshtein [43], 1976). *A critical hypergraph $H = (V, E)$ without vertices of zero degree and with chromatic number greater than two satisfies the inequality $|E| \geqslant |V|$.*

It should be noted that this theorem is often formulated without the condition that the hypergraph be edge-critical. Such a formulation is clearly erroneous, since one can take any hypergraph with chromatic number greater than two and add a sufficiently large edge to it.

More details about critical hypergraphs can be found in the survey [105] by Kostochka.

3. Other classes of hypergraphs

3.1. Simple (linear) and *b*-simple hypergraphs. A hypergraph is said to be *simple* (sometimes, *linear*) if no two different edges share more than one vertex. By analogy with the quantity $m(n, r)$ we define the quantity $s(n, r)$ as the minimum number of edges in a simple n-graph that admits no proper r-colouring. Correspondingly, we denote the local version (with respect to the *edge* degree) by $d(n, r)$.

3.1.1. *Lower bounds.* First of all, let us show that the local version of the lower bound for a simple hypergraph yields a significantly stronger estimate for the number of edges. This argument is due to Erdős and Lovász [70]. Consider an n-uniform simple hypergraph $H=(V,E)$ that admits no proper colouring with r colours. We remove a vertex of maximum degree from each edge and thereby obtain a simple $(n-1)$-graph $H_1=(V_1,E_1)$ (this is called *trimming* the graph H). Clearly, it is still not r-colourable. Hence $d(H_1)>d(n-1,r)$, which means that there exists a vertex $v\in V_1$ of degree at least $d(n-1,r)/n$. However, each of the edges $e_1,\dots,e_t\in E_1$ $(t>d(n-1,r)/n)$ containing v was obtained by removing a vertex $v_i\in V$ $(i=1,\dots,t)$ of higher degree, and due to the simplicity of H all these vertices are distinct. Note that by taking the total sum of the degrees of the vertices v_i we count each edge at most n times, which immediately implies that

$$s(n,r)\geqslant\frac{[d(n-1,r)]^2}{n^3}\,.$$

The best asymptotic lower bounds of the quantity $s(n,r)$ for fixed r follow from this inequality, and for this reason we estimate only the quantity $d(n,r)$ in this subsection.

Kostochka [105] showed how to improve the argument that involves trimming. Above we saw that a simple n-graph $H=(V,E)$ with chromatic number larger than r contains at least $d(n-1,r)/n$ vertices of degree at least $d(n-1,r)/n$. We sort the vertices of H in descending order of their degrees, and then we remove the vertices one by one; in removing a vertex we also remove all the edges incident to it. It follows from the simplicity of the graph that along with the vertex v_i we remove at least $\deg(v_i)-(i-1)$ edges. Hence, in the first $\lfloor d(n-1,r)/n\rfloor$ steps we remove at least

$$\biggl\lfloor\frac{d(n-1,r)}{n}\biggr\rfloor+\biggl\lfloor\frac{d(n-1,r)}{n}-1\biggr\rfloor+\dots+1\geqslant c\biggl\lfloor\frac{d(n-1,r)}{n}\biggr\rfloor^2$$

edges, which gives the estimate

$$s(n,r)\geqslant c\,\frac{\lfloor d(n-1,r)\rfloor^2}{n^2}\,.$$

Lower bounds of the form $d(n,r)\geqslant cn^{1-\varepsilon(n)}r^{n-1}$ were derived for various functions $\varepsilon(n)$ tending to zero in papers of Szabó [162], Kostochka and Kumbhat [106], Shabanov [152], Kozik [113], and Kupavskii and Shabanov [115]. Finally, Kozik and Shabanov obtained a lower estimate without $\varepsilon(n)$.

Theorem 3.1.1 (Kozik–Shabanov [114], 2016). *For any $r\geqslant 2$ and $n\geqslant 3$*

$$d(n,r)\geqslant cnr^{n-1}.$$

We sketch the proof of this theorem. The colouring algorithm is amazingly simple. Fix a value of the parameter $p\in[0,1]$ and choose a cyclic order on r colours. Consider a random (uniformly and independently distributed) colouring of vertices with r colours and assign to each vertex a weight, a randomly (uniformly and independently) chosen number in the interval $[0,1]$. As long as there are monochromatic edges we recolour the vertex having the lowest weight among those that have not yet been recoloured with the next consecutive colour, provided that its weight is less than p (a vertex with such weight is said to be *free*).

Since each vertex is recoloured at most once, the algorithm eventually stops.

Now let us give a sketch of the proof that the algorithm returns a proper colouring with positive probability. We call an edge *degenerate* if it contains at least $n/2$ free vertices, and *dangerous* if after the initial colouring and the assigning of weights there is a possibility that this edge becomes monochromatic (that is, all non-free vertices have colour i, whereas the free vertices have colour i or $i-1$).

If the algorithm does not work, then for this order and the precolouring there exists the structure of a subhypertree with certain properties (exact formulations of the properties are rather cumbersome, and the proofs are almost tautological). Taking $p := (5\log n)/n$ and applying the Lovász local lemma in the form of Lemma 2.5.4 completes the proof.

3.1.2. *b-simple hypergraphs and upper estimates.* First of all, the general Theorem 3.1.4 yields the estimate

$$s(n,r) \leqslant 1600n^4 r^{2(n+1)}.$$

It turns out that most of the upper bounds that follow can be established in a more general form.

A hypergraph is said to be *b-simple* if any two of its edges share at most b vertices. Sometimes ([128], [133], [109]) a b-simple n-graph is called a *partial Steiner $(n,b+1)$-system*. The case $b=1$ was considered in the previous subsection, but now we consider the case of arbitrary b. The quantities $s(n,r,b)$ and $d(n,r,b)$ are defined by analogy with $s(n,r)$ and $d(n,r)$.

The Kostochka–Kumbhat lower estimates [106] can be generalized to b-simple hypergraphs. A generalization of Theorem 3.1.1 to b-simple hypergraphs was obtained by Akhmejanova and Shabanov.

Theorem 3.1.2 (Akhmejanova–Shabanov [5], 2017, [6], 2019). *For any $b \geqslant 1$, $r \geqslant 2$, and $n \geqslant n_0(b)$*

$$d(n,r,b) \geqslant \frac{1}{16e^4}\, nr^{n-b}.$$

In 2009 Kostochka and Kumbhat [106] obtained an upper bound for $s(n,r,b)$, which was improved by Kostochka and Rödl a year later.

Theorem 3.1.3 (Kostochka–Rödl [110], 2010). *For any $r \geqslant 2$, $b \geqslant 1$, and sufficiently large n*

$$s(n,r,b) \leqslant (4e^r)^b (n\log r)^{1+1/b} r^{n+n/b}.$$

3.1.3. *Hypergraphs with large girth. A cycle of length s* in a hypergraph $H=(V,E)$ is a sequence

$$(A_0, v_0, \dots, A_{s-1}, v_{s-1}, A_s),$$

where $A_0,\dots,A_{s-1}$ are distinct edges of H, the edge A_s coincides with A_0, and $v_0,\dots,v_{s-1}$ are distinct vertices of H such that $v_i \in A_i \cap A_{i+1}$ for $i=0,1,\dots,s-1$. The *girth* of the hypergraph H is the length $g(H)$ of its shortest cycle. Note that according to this definition simple hypergraphs are hypergraphs with $g(H)>2$.

Erdős and Lovász proposed a far-reaching generalization of the well-known theorem of Erdős [62] that there exist graphs with arbitrarily large chromatic number and girth.

Theorem 3.1.4 (Erdős–Lovász [70], 1973). *For given positive integers $s \geqslant 2$, $n \geqslant 2$, and $r \geqslant 2$ define the quantities*

$$v := 4 \cdot 20^{s-1} n^{3s-2} r^{sn-n+s}, \quad m := 4 \cdot 20^{s} n^{3s-2} r^{s(n+1)}, \quad \text{and} \quad d := 20 n^2 r^{n-1}.$$

Then there exists an n-uniform hypergraph H on v vertices with at most m edges and with vertex degrees at most d such that $g(H) > s$ and $\chi(H) > r$.

Let $\Delta(n, r, g)$ denote the minimum number d such that there exists an n-uniform hypergraph with chromatic number larger than r, girth g, and maximum vertex degree d. It follows from Theorem 3.1.4 that for $g \geqslant 3$

$$\Delta(n, r, g) \leqslant 20 n^2 r^{n+1}.$$

This estimate was improved by Kostochka and Rödl.

Theorem 3.1.5 (Kostochka–Rödl [110], 2010). *For all $n, r \geqslant 2$ and $g \geqslant 3$*

$$\Delta(n, r, g) \leqslant n r^{n-1} \log r.$$

Let us briefly mention the known results for graphs. Kim [100] showed that $\Delta(2, r, 5) > (r + o(r)) \log r$ for sufficiently large r. On the other hand, Kostochka and Mazurova [107], and also Bollobás [38], demonstrated that $\Delta(2, r, g) \leqslant 2r \log r$ for all g. Tashkinov [163] proved that $\Delta(2, 3, g) \leqslant 6$ for all g.

Ajtai, Komlós, Pintz, Spencer, and Szemerédi [4] proved that any n-uniform hypergraph $H = (V, E)$ with girth at least 5 and maximum vertex degree d contains an independent set of size at least

$$|V| \left(\frac{\log d}{d} \right)^{1/(n-1)}.$$

Spencer conjectured that it suffices to assume only that the hypergraph is simple, and this was shown by Duke, Lefmann, and Rödl [60]. Frieze and Mubayi strengthened that theorem and showed that not only is there an independent set (say, α) of the indicated size, but also the hypergraph can be coloured with $O(|V|/\alpha)$ colours.

Theorem 3.1.6 (Frieze–Mubayi [80], 2013). *Let H be an n-uniform simple hypergraph with maximum vertex degree d. Then*

$$\chi(H) \leqslant \left(\frac{d}{\log d} \right)^{1/(n-1)}.$$

As an immediate corollary we get that any n-uniform hypergraph with vertex degree at most

$$c(n) r^{n-1} \log r$$

can be coloured with r colours.

3.1.4. *The case of a large number of colours.* In this subsection we discuss estimates in the situation where r is much larger than n. With the use of random designs Grable, Phelps, and Rödl [84] improved the Erdős–Lovász estimate. They showed that for infinitely many values of r with $r > r_0(n)$ the following inequality holds:

$$s(n,r) \leqslant c \cdot 4^n n^2 r^{2n-2} \log^2 r.$$

Kostochka, Mubayi, Rödl, and Tetali [108] derived the following bounds for the class of b-simple hypergraphs: for given n and b

$$c(n,b)(r^{n-1}\log r)^{1+1/b} \leqslant s(n,r,b) \leqslant C(n,b)(r^{n-1}\log r)^{1+1/b}, \tag{3.1}$$

and moreover, for given b the constant $C(n,b)$ is polynomial in n.

In the same paper [108] they obtained for even r the estimate

$$s(n,r,b) \geqslant \frac{n-b}{n}\,\frac{1}{(2^{n-1}ne)^{b/(b-1)}}\,r^{(n-1)(b+1)/b}.$$

In the case of a simple hypergraph the last estimate was later strengthened.

Theorem 3.1.7 (Shabanov [152], 2012). *For $n \geqslant 3$ and any even $r \geqslant 4$*

$$s(n,r,b) \geqslant c\,\frac{n}{n^2 \cdot 2^{2n}}\,r^{2n-2}.$$

3.1.5. *F-free hypergraphs.* Let $\mathscr{F}$ be a family of n-graphs. Define the quantity $m(r,\mathscr{F})$ as the minimum number of edges in an n-graph that contains no n-graph $F \in \mathscr{F}$ as a subgraph and has chromatic number at least $r+1$. Obviously, the class of b-simple n-graphs is obtained through forbidding $(n-b-1)$ particular n-graphs. With the use of this fact Bohman, Frieze, and Mubayi [37] strengthened the lower bound (3.1) for $s(n,r,b)$.

Even for graphs and rather simple families F the problem is fairly difficult. Gimbel and Thomassen [82] demonstrated that the minimum number of edges in a triangle-free graph with chromatic number r has order of growth $r^3 \log^2 r$. At the same time, no asymptotic expression is known for the cases of forbidden complete graph on four vertices or forbidden cycle on four vertices. Bohman, Frieze, and Mubayi showed that the cases of graphs and hypergraphs differ essentially.

Theorem 3.1.8 (Bohman–Frieze–Mubayi [37], 2010). *Let $k > n \geqslant 3$. Then the minimum number of edges in a K_k^n-free hypergraph with chromatic number r has order of growth*

$$r^{n+o(1)},$$

where K_k^n is the complete n-graph on k vertices. On the other hand, for any $s \geqslant 3$ there exists an $\varepsilon(s) > 0$ such that the minimum number of edges in a K_s-free graph with chromatic number r has order of growth at least $r^{2+\varepsilon}$.

Conjecture 3.1.9 (Bohman–Frieze–Mubayi [37], 2010). *There is a simple 3-graph H such that the minimum number of edges in an H-free hypergraph with chromatic number r has order of growth*

$$r^{3+o(1)}.$$

In the same paper the authors posed the problem of describing the class of 3-graphs H such that the minimum number of edges in an H-free hypergraph with chromatic number r has order of growth $r^{3+o(1)}$.

Bohman, Frieze, and Mubayi [37] showed that any n-graph with chromatic number at least $2(n-1)(t-1)+2$ contains a copy of any n-tree with t edges. They also conjectured that their statement is far from optimal, and this was later proved by Loh.

Theorem 3.1.10 (Loh [118], 2009). *Let $H=(V,E)$ be an n-uniform hypergraph with chromatic number larger than r. Then H contains a copy of any n-uniform hypertree with r edges.*

Subsequently, Gyárfás and Lehel [87] demonstrated that this theorem follows directly from the greedy approach of Pluhár.

3.1.6. *Steiner systems.* A *Steiner system* with parameters (v,n,l) is an n-graph on v vertices in which any collection of l vertices is contained in exactly one edge (as a subset).

Papers by Grable, Phelps, and Rödl [84] and Phelps and Rödl [128] have described the asymptotic behaviour (up to a constant multiplicative factor) of the minimum independence number for Steiner $(n,k,2)$- and $(n,k,3)$-systems for a given k as n tends to infinity. This asymptotic expression and the inequality $\chi(H)\geqslant |V(H)|/\alpha(H)$, where $H=(V,E)$, yield estimates for the corresponding chromatic numbers.

The chromatic numbers for Steiner systems have also been studied in the case of small parameters. For example, Horak [96] showed that any Steiner $(25,3,2)$-system has chromatic number 3 or 4. The authors of [55] studied various properties of vertex and edge colourings of $(19,3,2)$-systems.

3.2. Intersecting (cliques) and cross-intersecting families. In this subsection we show that the gap between the upper and lower bounds increases considerably in the case where the probabilistic method is not applicable.

Definition 3.2.1. An intersecting family is a hypergraph $H=(V,E)$ such that $e\cap f\neq\varnothing$ for any $e,f\in E$.

Intersecting families were introduced in combinatorics in the paper [69] by Erdős, Ko, and Rado, where they found the largest number of elements in an n-uniform intersecting family on a given set of vertices.

Definition 3.2.2. A *cross-intersecting family* is a hypergraph $H=(V,E)$ equipped with a (not necessarily disjoint) covering $E=A\cup B$ by non-empty sets A and B of edges such that any $a\in A$ intersects any $b\in B$. With a slight abuse of notation, we write both $H=(V,E)$ and $H=(V,A,B)$ interchangeably.

Cross-intersecting families appeared in the study of maximal and almost maximal intersecting families (the corresponding notation was introduced in [125]). In the Hilton–Milner theorem [95] this notion is used to determine the maximum number of edges in an n-uniform intersecting family with empty intersection on a given set of vertices. Frankl's theorem [75] refines the Hilton–Milner theorem in the case where the maximum vertex degree is bounded. Recently a general approach to the study of the indicated problems was proposed by Kupavskii and Zakharov [116] (which the reader may also consult for a survey). An even wider range of related problems was considered in [79].

Erdős and Lovász [70] carried over the classical problems of hypergraph colouring to the subclass of intersecting families (cliques, in their terminology). Unfortunately, no probabilistic methods are known so far which would enable one to effectively construct an intersecting family. For example, it is much more difficult to derive a probabilistic upper bound in the case of a specific class of hypergraphs than in the general case.

As shown by Cherkashin in [47], it is somewhat simpler to solve the same problems for cross-intersecting families.

3.2.1. *Chromatic number.* We formulate an obvious corollary of Pluhár's algorithm (this assertion is also not hard to establish directly).

Corollary 3.2.3. *Let $H = (V, E)$ be an intersecting family. Then*:

(i) *for any vertex $v \in V$ there exists a proper colouring of H with three colours such that one of the colours is used only for the vertex v*;

(ii) *if any two edges share at least two vertices*, *then H can be properly coloured with two colours.*

Below we shall see that intersecting families are in a certain sense divided into the categories of 'simple', or 2-colourable, families and 'complex' families, having chromatic number 3.

The situation with cross-intersecting families is much more interesting. First, we observe that the chromatic number of a cross-intersecting family can be arbitrarily large. Take an arbitrary positive integer $r > 1$ and consider a hypergraph $H_0 = (V_0, E_0)$ with chromatic number r. Let $A := E_0$ and $B := \{V_0\}$. It is clear that $H := (V_0, A, B)$ is a cross-intersecting family with chromatic number r. However, under quite natural conditions (which obviously hold for n-uniform hypergraphs) the chromatic number of a cross-intersecting family is bounded.

Proposition 3.2.4. *Let $H = (V, A, B)$ be a cross-intersecting family. Suppose that both A and B contain minimal elements of E, that is, there are $a \in A$ and $b \in B$ that contain no subedges of H. Then $\chi(H) \leqslant 4$.*

Proof. We colour $a \cap b$ with colour 1, $a \setminus b$ with colour 2, $b \setminus a$ with colour 3, and all other vertices with colour 4. It is easily seen that this is a proper colouring, because a and b contain no subedges. □

It turns out that if there is no pair of edges $e_1, e_2 \in E$ such that $e_1 \subset e_2$, and each edge has size at least 3, then the cross-intersecting family has chromatic number 2 or 3. Moreover, the following theorem is valid.

Theorem 3.2.5 (Cherkashin [47], 2018). *Let $H = (V, A, B)$ be a cross-intersecting family such that there is no pair $e_1, e_2 \in A \cup B$ with $e_1 \subset e_2$ (that is, (V, E) is a Sperner system). Then either $\chi(H) \leqslant 3$ or $V = \{v_1, \dots, v_m, u_1, \dots, u_l\}$, $B = \{\{v_1, \dots, v_m\}, \{u_1, \dots, u_l\}\}$, and $A = \{\{v_i, u_j\}$ for all $i, j\}$ (modulo A-B-symmetry), where $m, l \geqslant 2$.*

We shall prove the following corollary.

Corollary 3.2.6. *Let H satisfy the hypotheses of Theorem 3.2.5 and assume that* $\min(|A|,|B|) \geqslant 3$. *Then* $\chi(H) \leqslant 3$.

Proof. Consider a pair $a \in A$, $b \in B$ on which the minimum of $|a \cup b|$ is realized. Take arbitrary vertices $v_a \in a \setminus b$ and $v_b \in b \setminus a$ and colour v_a and v_b with colour 1, $a \cup b \setminus \{v_a, v_b\}$ with colour 2, and all other vertices with colour 3.

We show that this colouring is proper. Since any edge has size at least 3, there are no monochromatic edges of colour 1. Any edge intersects either a or b, which means that there are no monochromatic edges of colour 3. Assume that there is a monochromatic edge e of colour 2. Without loss of generality it may be supposed that $e \in A$. Then $e \subset |a \cup b \setminus \{v_a\}|$, and hence $|e \cup b| < |a \cup b|$, a contradiction. □

3.2.2. *Maximum number of edges.* For such classes of hypergraphs it is also surprisingly interesting to study the problem which is in a certain sense opposite in its statement to most problems discussed in this survey: find the *maximum* number of edges in a 'non-trivial' hypergraph. In the case of intersecting families there are two conventional ways to formalize the notion of a non-trivial hypergraph in relation to hypergraph colourings. The first way is to call a hypergraph non-trivial if $\chi(H) \geqslant 3$ (the corresponding maximum will be denoted by $M(n)$). According to the second approach, H is non-trivial if and only if $\tau(H) = n$ (the corresponding maximum will be denoted by $r(n)$), where $\tau(H)$ is defined below.

Definition 3.2.7. Let $H = (V, E)$ be a hypergraph. The *covering number* $\tau(H)$ (also called the *transversal number* or the *blocking number*) of the hypergraph H is defined to be the cardinality of the smallest set $A \subset V$ such that any $e \in E$ intersects A.

Although this definition is not directly related to colourings, it turns out that, first, $M(n) \leqslant r(n)$ (since for an n-graph H it follows from the estimate $\tau(H) < n$ that $\chi(H) = 2$), and second, it is not yet known whether these quantities can take different values. For this reason, in what follows we use the authors' notation.

Erdős and Lovász derived the first bounds for $M(n)$.

Theorem 3.2.8 (Erdős–Lovász [70], 1973). *The following inequalities hold*:

$$\lfloor (e-1)n! \rfloor \leqslant M(n) \leqslant n^n.$$

The upper bound in Theorem 3.2.8 follows from Lemma 3.2.12 below. This bound was improved in [46], [51], and [17]). The current best upper bound is due to Frankl: $r(n) \leqslant n^n e^{-n^{1/4}/6}$ (see [76]).

The lower bound established in Theorem 3.2.8 is attained in Example 3.2.13. Lovász conjectured [120] that the lower bound is sharp. However, this conjecture was disproved by Frankl, Ota, and Tokushige [78]. They gave an explicit example of an n-uniform hypergraph H with $\tau(H) = n$ and at least

$$c\left(\frac{n}{2}\right)^{n-1} \tag{3.2}$$

edges.

It is remarkable that a very similar bound can also be found for cross-intersecting families. We introduce the notion of a non-trivial cross-intersecting family.

Definition 3.2.9. A cross-intersecting family $H = (V, A, B)$ is said to be *critical* if:

(a) for any edge $a \in A$ and any vertex $v \in a$ there is a $b \in B$ such that $a \cap b = \{v\}$;

(b) for any edge $b \in B$ and any vertex $v \in b$ there is an $a \in A$ such that $a \cap b = \{v\}$.

Note that if an n-uniform intersecting family $H = (V, E)$ has $\tau(H) = n$, then (V, E, E) is a critical cross-intersecting family.

Theorem 3.2.10 (Cherkashin [47], 2018). *For a critical cross-intersecting family $H = (V, A, B)$ let*

$$n := \max_{e \in A \cup B} |e|.$$

Then

$$\max(|A|, |B|) \leqslant n^n.$$

Proof. We need the following definition.

Definition 3.2.11. Let $H = (V, E)$ be a hypergraph, let W be a subset of V, and let

$$H_W := (V \setminus W, \{e \setminus W \mid e \in E\}).$$

If $\tau(H_W) \geqslant k$, then the hypergraph H is called a flower with k petals and core W.

Lemma 3.2.12 (Håstad–Jukna–Pudlák [89], 1995). *Let $H = (V, E)$ be a hypergraph and let $n := \max_{e \in E} |e|$. If $|E| > (k-1)^n$, then H contains a flower with k petals.*

Proof. We use induction on n. The *basis* $n = 1$ is trivial.

Induction step. Supposing that the statement is true for $n-1$, we prove it for n. If $\tau(H) \geqslant k$, then H itself is a flower with k petals (and an empty core). Otherwise, some set of size $k-1$ intersects all the edges of H, and hence at least $|E|/(k-1)$ edges must contain some vertex x. The hypergraph $H_{\{x\}} = (V_{\{x\}}, E_{\{x\}})$ has

$$|E_{\{x\}}| \geqslant \frac{|E|}{k-1} > (k-1)^{n-1}$$

edges, each of cardinality at most $n-1$. By the induction hypothesis, $H_{\{x\}}$ contains a flower with k petals and some core Y. Adding the element x back to the sets in this flower, we obtain a flower in H with the same number of petals and core $Y \cup \{x\}$. $\square$

We turn to the proof of the theorem. Suppose the contrary: without loss of generality assume that $|A| \geqslant n^n + 1$. Then by Lemma 3.2.12 the hypergraph contains a flower with $n+1$ petals. This means that every set $b \in B$ intersects the core of the flower, which means that H is not a critical family, a contradiction. $\square$

For cross-intersecting families Example 3.2.15 presented in the next subsection shows that the estimate established in Theorem 3.2.10 is sharp.

3.2.3. *Examples.* It is evident that for $v \leqslant 2n-1$ the set $\binom{[v]}{n}$ is an intersecting family (even with chromatic number 3 for $v = 2n-1$). Now we present an example where the lower estimate in Theorem 3.2.8 is attained.

Example 3.2.13. We construct a series of examples $H_n = (V_n, E_n)$ using induction on n.

The basis: take a complete graph on three vertices as H_2.

The step. Consider the hypergraph $H_{n-1} = (V_{n-1}, E_{n-1})$ and an n-element set T that does not intersect V_{n-1}. Define

$$V_n := V_{n-1} \cup T \quad \text{and} \quad E_n := \{T\} \cup \{e \cup \{t\} \mid e \in E_{n-1},\ t \in T\}.$$

It is clear that H_n is an n-uniform intersecting family with chromatic number 3. Moreover,

$$|E_n| = n|E_{n-1}| + 1 = n\lfloor (e-1)(n-1)! \rfloor + 1 = \lfloor (e-1)n! \rfloor.$$

We recall that an example with a larger number of edges was constructed by Frankl, Ota, and Tokushige [78]. We present their construction in the case of an even k.

Example 3.2.14. Let $k = 2a+2$, $a \geqslant 1$. The set of vertices consists of a distinguished vertex x and $2a+1$ disjoint $(a+2)$-element sets A_i, $i = 0, 1, \dots, 2a$, none of which contain x. The edge set consists of edges of $2a+2$ types: for $i = 0, 1, \dots, 2a$ define

$$E_i := \{e \colon |e| = k,\ A_i \subset e,\ |e \cap A_j| = 1,\ j = i+1, \dots, i+a \pmod{2a+1}\},$$

and also define the distinguished type

$$F := \{e \colon |e| = k,\ x \in e,\ |e \cap A_i| = 1,\ i = 0, 1, \dots, 2a\}.$$

This is an intersecting set with size

$$(a+2)^{k-1} + (k-1)(a+2)^a = (1 + o(1))e\left(\frac{k}{2}\right)^{k-1}.$$

Already for $k = 4$ there are examples with 42 edges [78], [122] which disprove the Lovász conjecture, since in Example 3.2.13 we are dealing with 41 edges.

Example 3.2.15. Consider an arbitrary $n > 2$ and let

$$V := \{v_{ij} \mid 1 \leqslant i, j \leqslant n\}, \quad A := \bigl\{\{v_{i1}, \dots, v_{in}\} \mid 1 \leqslant i \leqslant n\bigr\},$$
$$B := \bigl\{\{v_{1i_1}, v_{2i_2}, \dots, v_{ni_n}\} \mid 1 \leqslant i_1, i_2, \dots, i_n \leqslant n\bigr\}.$$

Note that $|A| = n$ and $|B| = n^n$. Obviously, $H := (V, A, B)$ is a cross-intersecting family and $\chi(H) = 3$.

3.2.4. *Open questions.* We mention several unsolved problems which do not reduce to a direct improvement of the estimates for the quantities defined above.

I. *A lower bound for the number of edges in an intersecting family with chromatic number* 3. It is quite natural to suppose that this estimate is rather strong. Under such an assumption it would be interesting to establish a stronger lower bound than an estimate for $m(n)$, that is, a bound for an arbitrary hypergraph. It follows from results of Östergård [127] (namely, the uniqueness of the 4-graph on which the value $m(4) = 23$ is attained) that the minimum number of edges in a 4-uniform clique with chromatic number 3 is larger than $m(4)$. Unfortunately, no other results in this direction have been obtained so far.

II. *The set of cardinalities of pairwise intersections of edges*. For a hypergraph $H = (V, E)$ consider the set of pairwise intersections of edges:

$$Q(H) := \{|e_1 \cap e_2| \colon e_1, e_2 \in E\}.$$

It follows from Pluhár's algorithm (the Lovász criterion) that $Q(H)$ contains 1 if $\chi(H) > 2$.

Erdős and Lovász [70] showed with the use of a theorem of Deza [58] that for an n-uniform intersecting family H with $\chi(H) = 3$, where n is sufficiently large, the following estimate holds: $3 \leqslant |Q(H)|$. On the other hand, so far no example with $|Q(H)| < (n-1)/2$ is known (this bound is attained on the 'exponentiated' Fano plane; see § 8). At the same time, for cross-intersecting families there is a simple example with $|Q(H)| = 4$ (see [47]).

It also follows from Lemma 3.2.12 and the estimate $m(n) > 2^{n-1}$ that the cardinality of the maximum intersection of edges in an intersecting family with $\chi \geqslant 3$ cannot be less than $n/\log_2 n$. However, in all examples of non-trivial intersecting families there are pairs of edges that intersect in at least $n-2$ elements.

III. Problems on intersecting and cross-intersecting families which are not related to colourings are presented in [79].

3.3. Non-uniform hypergraphs. Let $H = (V, E)$ be a hypergraph. Define the quantity $q(H)$ by

$$q(H) := \sum_{e \in E} 2^{-|e|},$$

where $|e|$ is the size of the edge. As a natural generalization of the Erdős–Hajnal problem we are interested in the minimum value of $q(H)$ over all hypergraphs that are not 2-colourable and all of whose edges contain at least n vertices. Denote this minimum by $q(n)$. Theorem 2.1.1 immediately yields the estimate $q(n) \leqslant m(n) \cdot 2^{-n}$. Unfortunately, we are not aware of any better upper estimate.

Now we turn to lower bounds for $q(n)$. It is clear that arguments similar to the ones in Theorem 2.1.1 give $q(n) \geqslant 1/2$, but even the inequality $q(n) \geqslant 1$ is not that easy to establish. In 1978 Beck [24] proved the inequality $q(n) \geqslant \log^* n$, where $\log^*$ is the iterated logarithm. It should be noted that in 2008 Lu announced [121] significant progress, but in his argument he made a fundamental mistake, and it works only for *simple* hypergraphs (details are given below in this subsection). Finally, quite recently Duraj, Gutowski, and Kozik established an even stronger result.

Theorem 3.3.1 (Duraj–Gutowski–Kozik [61], 2018). *There is a constant $C > 0$ such that*

$$q(n) > C \log n. \tag{3.3}$$

We present here the algorithm without the corresponding calculations. In the *first step* we colour each vertex independently and uniformly. Then for each vertex we sample a weight, a real number chosen uniformly between 0 and 1.

In the *second step* consider the vertices one by one in the increasing order of weights. Suppose that the current vertex is v. If after the first step there is a monochromatic edge e such that v is its heaviest vertex and none of its vertices with smaller weight have been recoloured, then we change the colour of v. Note that in any edge which was monochromatic after the first step we have recoloured at least one vertex (that is, monochromatic edges appear in the final colouring after all vertices with one of the colours have been recoloured).

Non-uniform simple hypergraphs. Recall that in 2008 Lu [121] announced the estimate

$$q(n) \geqslant c\,\frac{\log n}{\log\log n}\,.$$

Unfortunately, the proof proposed in that paper works only for simple hypergraphs. This was noticed by Shabanov, who strengthened these estimates significantly.

We define $q_g(n)$ to be the corresponding minimum taken over n-graphs with girth at least g.

Theorem 3.3.2 (Shabanov [153], 2014). *For any $n \geqslant 3$*

$$q_4(n) \geqslant \frac{1}{2}\biggl(\frac{n}{\log n}\biggr)^{2/3}.$$

A year later Shabanov improved his own result.

Theorem 3.3.3 (Shabanov [154], 2015). *For any $n \geqslant 2$*

$$q_3(n) \geqslant c\sqrt{n}\,, \quad q_4(n) \geqslant cn.$$

Note that these theorems are proved for colourings with r colours.

4. List colourings of graphs and hypergraphs

4.1. List colourings. To each vertex v we assign a list $L(v)$ of colours which can be used for v. The *list chromatic number* $\operatorname{ch}(H)$ of the hypergraph H is the minimum number k such that for any lists $L(v)$ of length at least k there exists a proper colouring of the hypergraphs. List colourings of graphs and hypergraphs were introduced by Vizing [167] and by Erdős, Rubin, and Taylor [71].

Clearly, $\operatorname{ch}(H) \geqslant \chi(H)$, since all lists can be taken equal to $\{1,\dots,\operatorname{ch}(H)\}$. At the same time, the chromatic number and the list chromatic number are different, for example, for the graph $K_{3,3}$, for which $\operatorname{ch}(K_{3,3}) = 3$ (equality is attained for the lists $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ assigned to the vertices of each part).

Kostochka [104], [105] generalized the Erdős–Hajnal problem to list colourings. He proposed finding the quantity $m_l(n, r)$ defined as the minimum number of edges in a hypergraph H satisfying the condition $\mathrm{ch}(H) > r$. It is clear that

$$m_l(n, r) \leqslant m(n, r).$$

No *upper* estimates are known so far which do not follow immediately from this inequality. In particular, as in § 3.3, in the case $r = 2$ no one has succeeded in deriving an upper bound which would be better than the one provided by Theorem 2.1.1.

Let us turn to *lower* estimates for $m_l(n, r)$. First of all,

$$m_l(2, r) = m(2, r) = \binom{r+1}{2},$$

since the degree of each vertex should be at least r (and, hence, there should be at least $r + 1$ vertices).

Repeating the proof of the lower bound in Theorem 2.1.1, we derive the inequality

$$m_l(n, r) \geqslant r^{n-1}.$$

It was noted by Kostochka in [104] that the proof of the Radhakrishnan–Srinivasan estimate applies as well to list colourings, whereas the methods of Alon and Kostochka do not. Raigorodskii and Shabanov [132] noted that the estimates due to Pluhár and Shabanov also cannot be carried over directly to list colourings.

Theorem 4.1.1 (Rozovskaya–Shabanov [135], 2010). *For any $n \geqslant 3$ and $r \geqslant 2$*

$$m_l(n, r) \geqslant (\sqrt{3} - 1)\sqrt{\frac{n}{\log n}}\, r^{n-1}.$$

Proof. We present here another proof, without finding the precise constant (that is, with a constant c instead of $\sqrt{3} - 1$). Consider the 1-skeleton of the unit simplex Δ with $\left|\bigcup L(v)\right|$ vertices and with the induced metric, and fix a one-to-one correspondence f between the colours and the vertices of the simplex. Any vertex v is uniformly mapped onto the 1-skeleton of the subsimplex spanned by $f(L(v))$.

Fix a parameter $p \in [0, 1]$ and on each edge of the simplex distinguish a segment of length p which is equidistant from the vertices. A vertex v is said to be *free* if it lies in the union of the distinguished segments. If v is not free, then we colour v with the colour corresponding to the vertex of Δ closest to v. The free vertices will later be coloured with the colour corresponding to an endpoint of the simplex edge containing v.

Thus, at the current moment all vertices outside the distinguished segments are coloured. The expected number of completely coloured monochromatic edges at the moment does not exceed

$$|E| r \left(\frac{1-p}{r}\right)^n. \tag{4.1}$$

Now consider the set of edges that may become monochromatic after the free vertices are coloured. All the vertices of such an edge e that have been coloured have the same colour (say, the colour $q(e)$), and all the free vertices belong to edges containing $f(q)$. Then we try to colour its vertex that is farthest from $f(q)$ (denote it by v; with probability 1 it is unique) with the colour corresponding to the second vertex of the simplex edge containing v.

If the resulting colouring is internally consistent, that is, no vertex is coloured with two colours, then it is proper. Note that internal inconsistency means that there exists a pair of edges e_1, e_2 which colour v with colours q_1 and q_2 (that is, a 2-chain). Then none of the vertices of e_1 are closer to $f(q_1)$ than v, and none of the vertices of e_2 are closer to $f(q_2)$ than v. The probability of such an event is

$$\int_{(1-p)/2}^{(1+p)/2} \left(\frac{2a}{r}\right)^{n-1} \left(\frac{2-2a}{r}\right)^{n-1} da < pr^{2-2n}.$$

In combination with (4.1) this is completely similar to the probabilities occurring in the proof of Theorem 2.3.4 for $r = 2$. Hence, putting $p := (2\log n)/n$ completes the proof of the theorem. □

Theorem 2.3.3 can be carried over to list colourings and provides the current best result

$$m_l(n,r) \geqslant \frac{1}{4}\sqrt{n}\, r^{n-1}.$$

We turn to the case when the number of colours significantly exceeds the edge size. The following unpublished result was obtained by Sudakov.

Theorem 4.1.2. *For all $n \geqslant 2$ and $r \geqslant r_0(n)$*

$$m_l(n,r) \geqslant Cr^n.$$

Proof. We need to slightly modify Alon's method. Consider an n-graph $H = (V,E)$ with $|E| = cr^n$ and a sample of lists $\{L(v)\}$, $v \in V$. Let

$$\mathscr{L} := \bigcup_{v\in V(H)} L(v).$$

Consider a random partition of the set of colours into two classes: with probability $(n-1)/n$ (independently of each other) each colour occurs in the class $\mathscr{L}_1$ and with probability $1/n$ it occurs in $\mathscr{L}_2$. We note that $|V(H)| \leqslant Cnr^{n+1}$ and $|\mathscr{L}| \geqslant r$. Consequently, by the central limit theorem we can assume that for each $v \in V(H)$ the size of the set $L(v) \cap \mathscr{L}_1$ is almost surely $(1+o(1))(n-1)r/n$.

We turn to colouring. In the first step we assign to each vertex a random colour (uniformly and independently) from the list $\mathscr{L}_1$. In the second step we consider monochromatic edges one by one and change the colour of one arbitrarily chosen vertex in each monochromatic edge to any other colour in the list $L(v) \cap \mathscr{L}_2$ in such a way that no new monochromatic edges occur, if possible.

Let us show that with positive probability we obtain a proper colouring. The expected number of monochromatic edges after the first step is not greater than

$$|E|(1+o(1))\frac{(n-1)r}{n}\left((1+o(1))\frac{(n-1)r}{n}\right)^{-n} \leqslant c|E|r^{1-n}.$$

Thus, for a suitable constant C in the statement of the theorem at most $r/3$ vertices are recoloured in the second step. Assume that there is a vertex v that cannot be recoloured. This means that for any colour in $L(v) \cap \mathscr{L}_2$ there are already $n-1$ vertices which have been coloured with this colour. Hence, we have already changed the colour of at least $(n-1)|L(v) \cap \mathscr{L}_2|$ vertices, and this almost surely is greater than $r/3$. $\square$

In [52] Theorem 2.4.1 was also generalized to list colourings (that is, it was shown that the sequence $m_l(n,r)/r^n$ has a limit for a fixed n).

Finally, we remark that it would be interesting to describe the set of pairs n, r, for which

$$m_l(n,r) < m(n,r),$$

and in particular, to find out whether there is at least one such pair.

4.2. List colourings of arbitrary hypergraphs. It turns out that under some natural conditions the list chromatic number of any n-graph grows with the average degree. This was shown by Alon and Kostochka in [14]. For ordinary graphs it was established by Alon [12]). We present the current best numerical versions of this assertion. Here and below we assume that n remains constant, whereas the average degree of the n-graph increases.

Theorem 4.2.1 (Saxton–Thomason [138], 2015, [139], 2016). *Let n be a fixed integer and let H be an n-graph with average vertex degree d. Suppose that any set of $j \geqslant 2$ vertices is contained in at most $d^{(n-j)/(n-1)+o(1)}$ edges. Then*

$$\operatorname{ch}(H) \geqslant (1+o_d(1))\frac{1}{(n-1)^2}\log_n d.$$

If H is d-regular, then a stronger inequality is valid:

$$\operatorname{ch}(H) \geqslant (1+o_d(1))\frac{1}{n-1}\log_n d.$$

The bounds in this theorem are sharp up to a constant, since, as shown by Haxell and Verstraete [90],

$$\operatorname{ch}(K_{n\times m}) = (1+o(1))\log_n m,$$

where $K_{n\times m}$ denotes the complete n-partite n-graph with parts of size m. The proof of this theorem is very similar to that of Theorem 9.4.3.

In § 4.3 below we illustrate the idea of the proof with the following simplest example.

Theorem 4.2.2 (Saxton–Thomason [137], [140], 2012). *Let G be a simple d-regular n-graph. Then*

$$\operatorname{ch}(G) \geqslant \left(\frac{1}{(2n-1)\log n} + o(1)\right)\log d.$$

4.3. Containers. Since a proper colouring of a hypergraph is the same as a partition of its vertices into independent sets, examining the set $\mathscr{I}(H)$ of independent sets of the hypergraph H is a closely related problem.

4.3.1. *The method of containers for graphs.* The method of containers for graphs appeared in the papers [101] and [102] by Kleitman and Winston. We briefly describe this method. Consider a graph $G=(V,E)$.

It is clear that

$$2^{\alpha(G)} \leqslant i(G) \leqslant \sum_{m=0}^{\alpha(G)} \binom{|V(G)|}{m},$$

where $i(G)=|\mathscr{I}(G)|$. Obviously, if $\alpha(G)$ is not very close to $|V(G)|$, then the upper and lower estimates differ significantly. The method of containers was invented with the express aim of improving the upper estimate in this case. It is based on a quite natural idea of associating with each independent set a collection of vertices which almost completely determines it. Clearly, if we already have a set of vertices, then we cannot include adjacent vertices in the independent set. The idea is that, given a set, we choose such vertices greedily. Below we present the formal algorithm and the Kleitman–Winston lemmas.

Fix an order π on the vertices of the graph G. For each set $A\subset V(G)$ define an internal order on the vertices of A in the following way: the vertex v_i has the maximum degree in the induced graph formed from $A\setminus\{v_1,\dots,v_{i-1}\}$ (if there are several ways to choose v_i, then we choose the smallest with respect to π). Now we describe the algorithm: the input consists of an independent set I and a number $q<|I|$. Let $A=V(G)$ and $S=\varnothing$, and carry out the following steps for $s=1,\dots,q$:

1) consider the internal order $(v_1,\dots,v_{|A|})$ on the vertices of A;
2) consider the minimum index j_s such that v_{j_s} belongs to I;
3) move v_{j_s} from A to S;
4) remove $v_1,\dots,v_{j_{s-1}}$ from A;
5) remove from A all the vertices adjacent to v_{j_s}.

The algorithm returns the sequence $(j_1,\dots,j_q)$ and $A\cap I$.

We note that the sets A and S can be uniquely reconstructed from $(j_1,\dots,j_q)$, since the algorithm can be executed again. On the other hand,

$$I=S(j_1,\dots,j_q)\cup(A(j_1,\dots,j_q)\cap I).$$

Lemma 4.3.1. *Let G be a graph on n vertices, let q be a positive integer, and let the real numbers R and $\beta\in[0,1]$ be such that*

$$R>e^{\beta q}n. \tag{4.2}$$

Suppose that for any set $U\subset V(G)$ such that $|U|>R$ the following inequality is valid:

$$e[G(U)]>\beta\binom{|U|}{2}.$$

Then for any integer $m>q$

$$i(G,m)\leqslant\binom{n}{q}\binom{R}{m-q},$$

where $i(G,m)$ is the number of independent sets of size m in the graph G.

Proof. Since we remove at least j_s vertices from A in step s,

$$j_1+\dots+j_q \leqslant |V(G)|-|A(j_1,\dots,j_q)|.$$

Further,

$$i(G,m)\leqslant \sum_{(j_s)} i(G[A(j_1,\dots,j_q)],m-q)\leqslant \sum_{(j_s)} \binom{|A(j_1,\dots,j_q)|}{m-q}$$

for any $m>q$.

Note that we have exactly $\binom{n}{q}$ sequences $(j_1,\dots,j_q)$ satisfying the condition $j_1+\dots+j_q \leqslant n$ and such that $j_s\geqslant 1$ for each s. Correspondingly, there are at most $\binom{n}{q}$ terms in the sum. Thus, it suffices to show that for each sequence $(j_1,\dots,j_q)$ returned by the algorithm the set $A(j_1,\dots,j_q)$ contains at most R elements. Assume the opposite: for a certain output $(j_1,\dots,j_q)$ the set $A\setminus\{v_1,\dots,v_{j_{s-1}}\}$ contains more than R elements for each s with $1\leqslant s\leqslant q$. Then the subgraph $G[A(j_1,\dots,j_s)]$ has edge density at least β. Hence, each iteration reduces $|A|$ by a factor of at least $(1-\beta)^{-1}$, and we obtain

$$|A(j_1,\dots,j_q)|\leqslant (1-\beta)^q n\leqslant e^{\beta q}n\leqslant R,$$

which is a contradiction. □

Examples of this method in various combinatorial and additive-combinatorial problems can be found in the remarkable survey [136].

4.3.2. *Containers for hypergraphs.* The method of containers for hypergraphs was independently proposed by Saxton and Thomason and by Balogh, Morris, and Samotij as a far-reaching (and technically sophisticated) extension of the method for graphs.

We give an informal definition of a container. Let $H=(V,E)$ be an n-graph. A family $\mathscr{C}\subset 2^{V(H)}$ of subsets is called a *container* if the following conditions are satisfied:

(a) any independent set I is contained in some $C\in\mathscr{C}$;

(b) there exists a constant q such that the 'cardinality' of each container $C\in\mathscr{C}$ is at least q times less than that of $V(H)$;

(c) $|\mathscr{C}|\leqslant 2^{\alpha|V(H)|}$ for a certain α.

In different formulations 'cardinality' means the number of elements, the total degree of the elements, or the number of edges entirely contained in the set.

At the present moment there are a lot of theorems that establish the existence of containers under certain conditions on H, which are usually rather general. The most general theorems were presented in the fundamental papers [138] and [21]. An improvement (and a simplification) of the probabilistic algorithm in [138] was proposed in [139]. Versions of these theorems for simple hypergraphs are presented in [140]. A short proof by induction can be found in [32]. Wide applications of the method of containers to problems in various areas of combinatorics were described in the survey [22].

4.3.3. *List colourings of arbitrary simple hypergraphs.* In this subsection we give a sketch of how Theorem 4.2.2 is derived from the container theorem in the form of Theorem 4.3.2 below. Let $H=(V,E)$, let $C\subset V$, and let $E[C]$ denote the set of edges of H that are contained in C.

Theorem 4.3.2 (Saxton–Thomason [140], [137]). *Let $H=(V,E)$ be a simple d-regular n-graph, and let $0<\delta<1$. Then there exists a family $\mathscr{C}\subset 2^{V(H)}$ of subsets such that*:

(a) *any independent set I is contained in some $C\in\mathscr{C}$*;

(b) $|E[C]|\leqslant\delta|E(H)|$ *for any $C\in\mathscr{C}$*;

(c) $|\mathscr{C}|\leqslant 2^{\alpha|V(H)|}$, *where $\alpha=d^{-1/(2n-1)}$*.

Let $\mathscr{C}\subset 2^{V(H)}$. A colouring is said to be *$\mathscr{C}$-compatible* if for any colour the set of all vertices coloured with this colour is contained in some $C\in\mathscr{C}$.

Theorem 4.3.3 (Saxton–Thomason [137]). *Fix a $c>0$ and let $k_0(c)<k<|V(H)|$. If a family $\mathscr{C}\subset 2^{V(H)}$ satisfies the conditions*

(a) $|\mathscr{C}|\leqslant 2^{|V(H)|/k}$,

(b) $|C|\leqslant(1-c)|V(H)|$ *for all $C\in\mathscr{C}$*,

then there exists a collection of lists, each of size

$$(1+o(1))\frac{\log k}{-\log c}\,,$$

which does not admit a $\mathscr{C}$-compatible colouring.

Let

$$l:=\left\lfloor(1-\varepsilon\frac{\log k}{-\log c}\right\rfloor\quad\text{and}\quad t:=\left\lfloor\frac{2l^2}{c}\right\rfloor,\quad\varepsilon>0.$$

It turns out that for a sufficiently small ε a random (uniform and independent) choice of l-element subsets of a t-element set as lists satisfies the hypotheses of the theorem with positive probability.

Consider the set $\mathscr{C}$ in Theorem 4.3.2. We show that the inequality $|E[C]|\leqslant\delta|E(H)|$ implies that $|C|\leqslant(1-1/n+\delta/n)|V(H)|$. Indeed, the sum of the degrees of the vertices in C satisfies the estimate

$$d|C|\leqslant n|E[C]|+(n-1)(|E(H)|-|E[C]|),$$

since each edge in $E(H)\setminus E[C]$ intersects C in at most $n-1$ vertices. If $|E[C]|\leqslant\delta|E(H)|$, then

$$d|C|\leqslant(n-1+\delta)|E(H)|,$$

which is equivalent to the desired inequality $|C|\leqslant(1-1/n+\delta/n)|V(H)|$, since H is d-regular.

Thus, we can apply Theorem 4.3.3 to the family $\mathscr{C}$ and find lists which do not admit a $\mathscr{C}$-compatible colouring, and hence (by the first property of containers) not a proper colouring. In this case we get that $k=d^{1/(2n-1)}$, $c=(1-\delta)/n$, and

$$\chi(H)\geqslant(1+o(1))\frac{\log k}{-\log c}=\left(\frac{1}{(2n-1)\log n}+o(1)\right)\log d.$$

5. Panchromatic colourings

An r-colouring of vertices of a hypergraph is said to be *panchromatic* if every edge contains vertices of every colour. This definition, as well as an analogue of Theorem 2.5.3, appeared in the paper [70] by Erdős and Lovász. Kostochka [103] posed the problem of finding the minimum number of edges $p(n,r)$ in an n-uniform hypergraph admitting no panchromatic r-colouring (clearly, $m(n,2) = p(n,2)$). Kostochka also presented the estimates

$$\frac{1}{r}e^{cn/r} \leqslant p(n,r) \leqslant re^{Cn/r}, \tag{5.1}$$

where $c < 1$ and $C \geqslant 4$ are positive constants. This result follows from Theorem 9.4.1 and Alon's estimates [11] on the list chromatic number of a graph. Later both the upper and lower estimates were improved several times.

A sufficient condition for panchromatic n-colourability was obtained by Kostochka and Woodall in terms of the Hall ratio.

Theorem 5.0.1 (Kostochka–Woodall [112], 2001). *Let $H = (V,E)$ be an n-uniform graph with $n \neq 3,5$ such that for any non-empty subset F of edges the following inequalities hold*:

$$\biggl|\bigcup_{f\in F} f\biggr| \geqslant \begin{cases} \dfrac{(n^2-2n+2)|F|+n-1}{n}, & n \neq 3,5, \\ \dfrac{(n^2-2n+2)|F|+n}{n}, & n = 3,5. \end{cases}$$

Then H is panchromatically n-colourable.

In the same paper they conjectured that there was no need to strengthen the hypotheses of the theorem for the case $n \in \{3,5\}$.

5.1. Upper bounds. By the pigeonhole principle any r-colouring contains a colour of size at most $\lfloor (1/r)|V| \rfloor$. The complement of this colour has size at least

$$|V| - \left\lfloor \frac{1}{r}\,|V| \right\rfloor = \left\lceil \frac{r-1}{r}\,|V| \right\rceil.$$

Consequently, $p(n,r) \leqslant p'(n,r)$. In spirit this argument resembles the standard estimate for the chromatic number of a graph in terms of the number of vertices and the independence number.

In [149] and [150] Shabanov derived the following upper estimates:

$$p(n,r) \leqslant c\,\frac{n^2\log r}{r^2}\left(\frac{r}{r-1}\right)^n \quad \text{if } 3 \leqslant r = O(\sqrt{n}\,) \text{ and } n > n_0; \tag{5.2}$$

$$p(n,r) \leqslant c\,\frac{n^{3/2}\log r}{r}\left(\frac{r}{r-1}\right)^n \quad \text{if } r = O(n^{2/3}) \text{ and } n_0 < n = O(r^2); \tag{5.3}$$

$$p(n,r) \leqslant c\max\left(\frac{n^2}{r}\,, n^{3/2}\right)\log r\left(\frac{r}{r-1}\right)^n \quad \text{for all } n,r \geqslant 2. \tag{5.4}$$

It is clear that (5.2)–(5.4) give estimates of the form (5.1) with $C = 1$ under the condition $r \leqslant cn/\log n$.

We present the idea that underlies the derivation of most of the upper estimates. Let $p'(n,r)$ be the minimum number of edges in an n-uniform hypergraph $H = (V,E)$ such that any subset $V' \subset V$ of vertices of size $|V'| \geqslant \left\lceil \dfrac{r-1}{r}\,|V| \right\rceil$ contains an edge. In fact, $p'(n,r)$ coincides with

$$\min_{|V|} T\biggl(|V|, \frac{r-1}{r}\,|V|, n\biggr)$$

(recall that $T(a,b,c)$ denotes the Turán number; see § 2.4.1).

Erdős' proof of the upper bound in Theorem 2.1.1 can be generalized as follows.

Theorem 5.1.1 (Cherkashin [48], 2018). *For all $n \geqslant 2$ and $r \geqslant 2$*

$$p(n,r) \leqslant c\,\frac{n^2\log r}{r}\biggl(\frac{r}{r-1}\biggr)^n.$$

5.2. Lower bounds. We remark that standard probabilistic arguments yield the estimate

$$p(n,r) \geqslant \frac{1}{r}\biggl(\frac{r}{r-1}\biggr)^n$$

(that is, the lower bound in (5.1), where $c = 1$). It was significantly improved by Shabanov [149]:

$$p(n,r) \geqslant c\,\frac{1}{r^2}\biggl(\frac{n}{\log n}\biggr)^{1/3}\biggl(\frac{r}{r-1}\biggr)^n \quad \text{for} \quad n,r \geqslant 2, \quad r < n.$$

Later Rozovskaya and Shabanov [135] showed that

$$p(n,r) \geqslant c\,\frac{1}{r^2}\sqrt{\frac{n}{\log n}}\,\biggl(\frac{r}{r-1}\biggr)^n \quad \text{for} \quad n,r \geqslant 2, \quad r \leqslant \frac{n}{2\log n}\,.$$

The proof of the next theorem involves the so-called method of small alterations (see [15], Chap. 3, and § 2.4.2). The result of this theorem is the best known estimate for $r \geqslant c\sqrt{n}\,$.

Theorem 5.2.1 (Cherkashin [48], 2018). *For $n \geqslant r \geqslant 2$*

$$p(n,r) \geqslant e^{-1}\,\frac{r-1}{n-1}\,e^{(n-1)/(r-1)}.$$

Proof. Consider a uniform independent random colouring of the set of vertices with $a > r$ colours. Then the expected number of pairs (e,q) such that the edge $e \in E$ does not contain the colour q is $|E|a\Bigl(\dfrac{a-1}{a}\Bigr)^n$. Therefore, if $|E|a\Bigl(\dfrac{a-1}{a}\Bigr)^n$ $< a - r$, then with positive probability there are r colours contained in each edge. Taking $a = \dfrac{n-1}{n-r}\,r$, we get the existence of a panchromatic colouring for

$$|E| \leqslant \frac{r-1}{n-1}\biggl(\frac{nr-r}{nr-n}\biggr)^n \leqslant e^{-1}\,\frac{r-1}{n-1}\,e^{(n-1)/(r-1)}. \qquad \square$$

Theorem 5.2.1 does not admit a local version. We shall prove a somewhat weaker estimate based on a geometric interpretation of ideas due to Pluhár [130] which can be combined with the Lovász local lemma. Consider an $(r-1)$-dimensional unit simplex with measure which is continuously and uniformly distributed over the 1-skeleton (edges of the simplex) and the induced metric on this skeleton, and fix a one-to-one correspondence f between the colours and the vertices of the simplex.

Let us construct a random map of the set of vertices of H onto the 1-skeleton independently and in accordance with the uniform measure. We try to colour the hypergraph in the following way: for each edge e and each colour i we assign the colour i to the vertex of e which is closest to the simplex vertex $f(i)$ (with probability 1 such a vertex is unique). If this method is internally consistent (that is, each vertex is coloured at most once), then the colouring is obviously panchromatic. Now we estimate the probability of internal inconsistency: if the number of edges is bounded above, then this probability is less than 1, which proves the following theorem.

Theorem 5.2.2 (Cherkashin [48], 2018). *For $n \geqslant r \geqslant 2$ such that $r \leqslant cn/\log n$*

$$p(n,r) \geqslant c \max\left(\frac{n^{1/4}}{r\sqrt{r}}, \frac{1}{\sqrt{n}}\right)\left(\frac{r}{r-1}\right)^n.$$

5.3. The case of small n/r. Consider the case where the ratio n/r is small (that is, $r > cn/\log n$); $n/r = \text{const}$ is a good example. The best of the upper bounds mentioned above in the case $n/r = O(\log n)$ is the bound $re^{cn/r}$ (see (5.1)), where $c \geqslant 4$ is a constant. With the use of the following theorem we establish an estimate that depends only on n/r.

Proposition 5.3.1 (Cherkasin [48], 2018). *For any positive integers m, n, and r*

$$p(mn, mr) \leqslant p'(n,r).$$

As a corollary of Theorem 5.3.1 and the obvious inequality

$$\max(p(n,r), p(n+1,r+1)) \leqslant p(n+1,r),$$

we obtain a bound which is best in the case where n/r is small.

Corollary 5.3.2. *For any positive integer $k \leqslant r$*

$$p(n,r) \leqslant p'\left(\left\lceil\frac{n}{r-k+1}\right\rceil k, k\right).$$

In particular, if $n < r^2$, then one can take $k := \alpha n/r$ and obtain

$$p(n,r) \leqslant c\left(\frac{n}{r}\right)^2 \log\frac{n}{r} \cdot e^{n/r}.$$

Theorem 5.2.1 provides a non-trivial lower estimate even in the case of small n/r, but at the same time it should be noted that there exists an elementary greedy algorithm.

Proposition 5.3.3. *For all positive integers $n \geqslant r$*

$$p(n,r) \geqslant \left\lfloor \frac{n}{r} \right\rfloor.$$

Proof of Proposition 5.3.3. Consider a hypergraph $H = (V,E)$ such that $|E| \leqslant \lfloor n/r \rfloor$. We take an arbitrary edge $e \in E$ and colour any r of its vertices with different colours. Next we remove from H the edge e and all the vertices that have been coloured. The remaining hypergraph has $|E|-1$ edges and the size of each edge is at least $n-r$. Therefore, we can repeat this operation $\lfloor n/r \rfloor$ times. □

6. Equitable colourings

A vertex colouring is said to be *equitable* if the cardinalities of the colours differ by at most 1. Erdős posed the following conjecture, which was later proved by Hajnal and Szemerédi.

Theorem 6.0.1 (Hajnal–Szemerédi [88]). *Let G be a graph all of whose vertices have degree at most d. Then it can be equitably coloured with $d+1$ colours.*

A considerably simpler proof is due to Kierstead and Kostochka [99]. They also formulated a similar result in terms of edge degrees.

Theorem 6.0.2 (Kierstead–Kostochka [98], 2008). *Let the graph G be such that for each edge xy the inequality $d(x)+d(y) \leqslant 2r+1$ is satisfied. Then G has an equitable $(r+1)$-colouring.*

The problem was generalized to n-graphs in [29] and [161].

The following theorem was proved in [155].

Theorem 6.0.3 (Shabanov [155], 2015). *Let $H = (V,E)$ be an n-graph with $|V| \geqslant n^2 \cdot 2^n$. If*

$$d(H) \leqslant \frac{1}{64}\,\frac{2^n}{\sqrt{n \log n}}\,,$$

where $d(H)$ is the maximum vertex degree in H, then H has an equitable 2-colouring.

Recently this theorem was significantly strengthened.

Theorem 6.0.4 (Akhmejanova–Shabanov [7], 2019). *For sufficiently large n and for $r \leqslant \sqrt[5]{\log n}$ an n-graph $H = (V,E)$ is equitably r-colourable if*

$$|E| \leqslant 0.01\left(\frac{n}{\log n}\right)^{(r-1)/r} r^{n-1} \quad \textit{and } |V| \textit{ is divisible by } r.$$

Equitable colourings of simple hypergraphs. Shabanov also considered the problem in the classes of simple and b-simple hypergraphs. We present a corollary to the main theorem in [155].

Theorem 6.0.5 (Shabanov [155], 2015). *Let $H = (V,E)$ be a simple n-graph. If*

$$d(H) \leqslant c\,\frac{2^n}{\sqrt{n \log n}}\,,$$

where $d(H)$ is the maximum vertex degree in H, then H has an equitable 2-colouring.

The case of non-uniform simple hypergraphs. Shirgazina proved ([156], [157]) that if $H=(V,E)$ is a simple hypergraph with minimum edge-cardinality n,

$$\sum_{e\in E} r^{1-|e|} \leqslant c\sqrt{k}, \quad \text{and } |V| \text{ is divisible by } r,$$

then H has an equitable r-colouring.

7. Discrepancy

The *discrepancy of a 2-colouring* of a hypergraph with red and blue colours is the maximum absolute value of the difference between the number of red and blue vertices in an edge, taken over all edges of the hypergraph. The *discrepancy of a hypergraph* is the minimum discrepancy over all colourings of the hypergraph. Discrepancy problems were studied in the monographs [124] and [45].

7.1. Hadamard matrices. A Hadamard matrix H is a square matrix of order n with entries $+1$ and -1 whose columns are mutually orthogonal. In other words,

$$H\cdot H^{\mathrm{T}} = nE_n,$$

where E_n is the identity matrix of size n and H^{T} is the transpose of the matrix H.

There are a lot of explicit constructions for Hadamard matrices. It can easily be shown that the order n of a Hadamard matrix should be either two or a multiple of four. The famous Hadamard conjecture states that such matrices exist for all values of n divisible by four. Details can be found in the fundamental survey [54].

Theorem 7.1.1. *Suppose that there exists a Hadamard matrix of order n. Then there is a family of n sets on n vertices with discrepancy at least $\sqrt{n}/2$.*

Proof. Consider the Hadamard matrix $H=\{h_{ij}\}$ of order n such that all the entries in the first row and the first column are 1s (any Hadamard matrix can be reduced to such a form by multiplying its rows and columns by -1).

Let J be the $n\times n$ matrix of 1s. Note that the entries of $(H+J)/2$ are 0s and 1s. The supports of its columns constitute a desired family.

With a colouring we associate a vector $v=(v_1,\dots,v_n)$ with $v_i=\pm1$ (the red colour corresponds to 1s, the blue to minus 1s). Then

$$Hv = v_1c_1+\dots+v_nc_n,$$

where c_i denotes the ith column of H. Put $Hv=(L_1,\dots,L_n)$ and let $|c|$ denote the Euclidean norm. Then

$$L_1^2+\dots+L_n^2 = |Hv|^2 = v_1^2|c_1|^2+\dots+v_n^2|c_n|^2 = n+\dots+n = n^2,$$

since the c_i are pairwise orthogonal. Note also that

$$L_1+\dots+L_n = \sum_{i,j=1}^{n} v_jh_{ij} = \sum_{j=1}^{n} v_j \sum_{i=1}^{n} h_{ij} = nv_1 = \pm n.$$

Let $\lambda = v_1 + \dots + v_n$. Then λ is an even number, and $Jv = (\lambda, \dots, \lambda)$. We have

$$\frac{H+J}{2}\,v = \left(\frac{L_1+\lambda}{2}, \dots, \frac{L_n+\lambda}{2}\right),$$

$$\begin{aligned}\sum_{i=1}^{n}(L_i+\lambda)^2 &= \sum_{i=1}^{n} L_i^2 + 2\lambda\sum_{i=1}^{n} L_i + n\lambda^2 = n^2 \pm 2n\lambda + n\lambda^2 \\ &= n^2 - n + n(x\pm 1)^2 \geqslant n^2. \end{aligned} \tag{7.1}$$

Hence, for some i

$$\left|\frac{L_i+\lambda}{2}\right| \geqslant \frac{\sqrt{n}}{2}. \qquad \square$$

7.2. Local setting. Beck and Fiala [26] estimated the discrepancy from above in terms of the maximum vertex degree $\deg(H)$ of the hypergraph H.

Theorem 7.2.1 (Beck–Fiala [26], 1981). *Let H be a hypergraph with* $\deg(H) = t$. *Then*

$$\operatorname{disc}(H) \leqslant 2t - 1. \tag{7.2}$$

They also conjectured that Theorem 7.2.1 can be significantly strengthened.

Conjecture 7.2.2. *There exists a constant C such that for any hypergraph H with* $\deg(H) = t$

$$\operatorname{disc}(H) \leqslant C\sqrt{t}\,. \tag{7.3}$$

However, despite the fact that this conjecture is widely known, the best result $2t - \log^* t$ (here $\log^* t$ denotes the iterated logarithm) was obtained only in 2016, by Bukh [42] (intermediate estimates were obtained in [27] and [92]).

7.3. Uniform hypergraphs with positive discrepancy. Let $f(n)$ denote the minimum number of edges in an n-uniform hypergraph with positive discrepancy. Quite unexpectedly, all the existing estimates of $f(n)$ depend only on the smallest non-divisor of n (denoted below by $\operatorname{snd}(n)$).

Obviously, if $2 \nmid n$, then $f(n) = 1$, whereas if $2 \mid n$ but $4 \nmid n$, then $f(n) = 3$. Erdős and Sós were concerned with the question of whether the function $f(n)$ is unbounded. Alon, Kleitman, Pomerance, Saks, and Seymour [13] proved the following theorem, which shows, in particular, that $f(n)$ is unbounded.

Theorem 7.3.1 (Alon–Kleitman–Pomerance–Saks–Seymour [13], 1987). *Let n be a positive integer such that $4 \mid n$. Then*

$$c_1\,\frac{\log \operatorname{snd}(n/2)}{\log\log \operatorname{snd}(n/2)} \leqslant f(n) \leqslant c_2\,\frac{\log^3 \operatorname{snd}(n/2)}{\log\log \operatorname{snd}(n/2)}\,. \tag{7.4}$$

To prove the upper bound the authors introduced several definitions. Let $\mathscr{M}$ denote the set of all matrices M with entries in $\{0,1\}$ such that the equation $Mx = e$ has exactly one non-negative solution (here e is the vector with all components equal to 1). This solution is denoted by x^M. Let $z(M)$ be the least positive integer such that $z(M)x^M$ is an integer and let $y^M = z(M)x^M$. For each positive integer n, let $t(n)$ be the least r for which there is a matrix $M \in \mathscr{M}$ with r rows such that

$z(M) = n$ (it is clear that $t(n) \leqslant n+1$, because $z(J_{n+1} - I_{n+1}) = n$, where J_{n+1} is the $(n+1) \times (n+1)$ matrix of 1s and I_{n+1} is the $(n+1) \times (n+1)$ identity matrix). The upper bound in (7.4) follows from the inequality $f(n) \leqslant t(m)$ proved in [13] for values of m with $\lfloor n/m \rfloor$ odd.

Alon and Vũ [16] later showed that

$$t(m) \leqslant (2 + o(1)) \frac{\log m}{\log \log m}$$

for infinitely many values of m. However, they noted that the validity of the inequality $t(m) \leqslant c \log m$ for arbitrary m was not obvious. This gap was filled by Cherkashin and Petrov.

Theorem 7.3.2 (Cherkashin–Petrov [53], 2019). *Let n be a positive integer. Then*

$$f(n) \leqslant c \log \operatorname{snd}(n) \tag{7.5}$$

for some constant $c > 0$.

Corollary 7.3.3. *Let n be a positive integer. Then*

$$f(n) \leqslant c \log \log n$$

for some constant $c > 0$.

The main idea of the proof is to find a matrix with small entries and determinant $\operatorname{snd}(n)$ which satisfies some technical conditions. We conclude this subsection with several conjectures.

Conjecture 7.3.4 (Alon–Kleitman–Pomerance–Saks–Seymour [13]). *There exists a function g such that*

$$f(n) = g(\operatorname{snd}(n)).$$

Conjecture 7.3.5 (Alon–Kleitman–Pomerance–Saks–Seymour [13]). *There exists a function Θ such that*

$$f(n) = \Theta\left(\frac{\log \operatorname{snd}(n)}{\log \log \operatorname{snd}(n)}\right).$$

7.4. Discrepancy of uniform hypergraphs. Consider an n-uniform hypergraph H and all possible colourings of its vertices with two colours. It is clear that the existence of a colouring such that each edge contains at least k vertices of each colour is equivalent to the existence of a colouring with discrepancy at most $n - 2k$. In this connection it is natural to introduce the quantity $m_k(n)$ equal to the minimum number of edges in an n-uniform hypergraph such that for any 2-colouring of its vertices there is an edge that contains at most $k-1$ vertices of one colour.

The quantity $m_k(n)$ was first introduced by Raigorodskii (a very similar problem with the same notation was considered in [117]). Using the same arguments as in the proof of Theorem 2.1.1, one can derive the estimate

$$m_k(n) \geqslant \frac{2^{n-1}}{\sum_{i=0}^{k-1} \binom{n}{i}}.$$

In [144]–[146] Shabanov adapted various algorithms and proved that for $k \leqslant c\log n$

$$m_k(n) \geqslant c\left(\frac{n}{\log n}\right)^{1/2} \frac{(4e)^{k/2}}{\sqrt{k}} \frac{2^{n-1}}{\binom{n}{k-1}}.$$

In [134] Rozovskaya applied a generalization of Pluhár's method and showed that for $k \leqslant c\sqrt{n}$

$$m_k(n) \geqslant cn^{1/4} \frac{2^{n-1}}{\binom{n}{k-1}}.$$

In [144] and [145] Shabanov obtained a result corresponding to Theorem 2.1.1 for $k < cn/\log n$:

$$m_k(n) \leqslant cn^2 \frac{2^n}{\sum_{i=0}^{k-1}\binom{n}{i}}.$$

Finally, Demidovich proved the following theorem.

Theorem 7.4.1 (Demidovich [57], 2019). *For any $n \geqslant 30$ and $k \geqslant 2$ such that*

$$k \leqslant \sqrt{\frac{n}{\log n}},$$

the following inequality holds:

$$m_k(n) \geqslant \sqrt{\frac{n}{k\log n}} \frac{2^{n-1}}{\binom{n}{k-1}}.$$

8. Explicit constructions and small values of variables

8.1. Small parameters. Recall that $m(2,r) = \dbinom{r+1}{2}$, in particular, $m(2) = 3$.

8.1.1. *Fano plane.* Let us show that $m(3) = 7$ and that the only example is the Fano plane (the projective plane over $\mathbb{F}_2$).

We define the operation of *merging vertices*. Let v_1 and v_2 be vertices of a hypergraph $H = (V, E)$. Then the result of merging v_1 and v_2 is the hypergraph $H_{v_1v_2}$ in which v_1 and v_2 are replaced by a single vertex v and the edges of the new hypergraph are obtained from those of the original one by replacing v_1 and (or) v_2 by v.

Consider an n-uniform hypergraph H_0 which is not 2-colourable. As long as there is a pair of vertices not contained in any one edge, we merge the pair. After such an operation the graph remains n-uniform and two-uncolourable, and the number of edges does not increase. As a result, we obtain an n-uniform two-uncolourable hypergraph $H = (V, E)$ such that any two of its vertices are adjacent. Hence

$$|E| \geqslant \left\lceil \frac{|V|}{n} \left\lceil \frac{|V|-1}{n-1} \right\rceil \right\rceil. \tag{8.1}$$

On the other hand, if the number of vertices is small, then it is much more efficient to use an equal number of colours. A randomly chosen equitable colouring provides the estimate

$$m(n) \geqslant \left\lceil \binom{v}{\lceil v/2 \rceil} \left(\binom{v-n}{\lceil v/2 \rceil - n} + \binom{v-n}{\lfloor v/2 \rfloor - n} \right)^{-1} \right\rceil. \tag{8.2}$$

Now consider an arbitrary hypergraph H_0 that realizes $m(3)$, and apply the operation of merging until we obtain an unmergeable hypergraph $H = (V, E)$. If $|V| \leqslant 6$, then the estimate (8.2) implies that $m(3) \geqslant 10$, which is not true, since we already have an example with 7 edges. In the case $|V| \geqslant 7$ the estimate (8.1) yields $|E| \geqslant 7$, with equality only when $|V| = 7$ and any pair of vertices belongs to exactly one edge. However, this precisely determines the finite projective plane, since:

(i) through any two points there is exactly one line incident to both of them;

(ii) any two lines intersect in exactly one point incident to both of them (this is so because by (i) they share at most one point, and if they do not intersect, then the remaining line passes through the remaining point x_7, but the degree of x_7 is 3, a contradiction);

(iii) there are four points such that no line contains more than two of them (this follows from (ii), since the complement of any line cannot contain a line).

It is well known (and easily established by exhaustion) that the projective plane of order 2 is unique. It remains to note that application of the inverse of the merging operation to the Fano plane immediately reduces the chromatic number.

The Erdős–Hajnal problem was considered in [65] in the case of arbitrary n and a number of vertices which is linear in n.

The complete 3-graph on seven vertices realizes the estimate $m(3, 3) \leqslant 35$. The inequality $m(3, 3) \geqslant 27$ was established by Akolzin [8]. It is noteworthy that the same inequality served as a starting point for the paper [108] but was omitted from its final version.

8.1.2. *$m(4)$ and larger values of n.* Seymour [142] and Toft [165] independently showed that $m(4) \leqslant 23$. They used the example of a hypergraph on 11 vertices with the following edges:

$$\begin{gathered}
\{1,2,9,10\}, \quad \{3,4,9,10\}, \quad \{5,6,9,10\}, \quad \{7,8,9,10\}, \\
\{1,2,9,11\}, \quad \{3,4,9,11\}, \quad \{5,6,9,11\}, \quad \{7,8,9,11\}, \\
\{1,2,10,11\}, \quad \{3,4,10,11\}, \quad \{5,6,10,11\}, \quad \{7,8,10,11\}, \\
\{1,3,5,8\}, \quad \{1,3,6,7\}, \quad \{1,4,5,7\}, \quad \{1,4,6,7\}, \quad \{1,4,6,8\}, \\
\{2,3,5,7\}, \quad \{2,3,6,7\}, \quad \{2,3,6,8\}, \quad \{2,4,5,7\}, \quad \{2,4,5,8\}, \quad \{2,4,6,8\}.
\end{gathered}$$

Toft obtained this example as a particular case of a construction yielding the relation (8.5). Östergård [127] proved by a computer search that $m(4) = 23$, and the example on 11 vertices is unique.

For $m(5)$ the range is rather wide — the best estimates are $29 \leqslant m(5) \leqslant 51$, the lower bound was derived in [3], and the upper was established by the construction for (8.4).

8.2. Recurrence relations. The following proposition summarizes various recurrence relations. Some other relations can be found in [3] and [123]. They are all established by explicit constructions.

Proposition 8.2.1 (Abbott–Moser–Hanson–Toft [2], [1], [165]). *For any positive $a, b, n \geqslant 2$ the following inequalities hold*:

$$m(ab) \leqslant m(a)[m(b)]^a; \tag{8.3}$$

$$m(n) \leqslant m(n-2)n + 2^{n-1} \qquad \textit{if } n \textit{ is odd}; \tag{8.4}$$

$$m(n) \leqslant m(n-2)n + 2^{n-1} + \frac{1}{2}\binom{n}{n/2} \qquad \textit{if } n \textit{ is even}; \tag{8.5}$$

$$m(n) \leqslant (2n-1)(m(n-2)+1). \tag{8.6}$$

Let us prove the inequality (8.3). Suppose that H_a and H_b are hypergraphs on which the values of $m(a)$ and $m(b)$ are attained. We shall use them to construct a hypergraph H that gives the necessary estimate. We replace each vertex of H_a by a copy of H_b. Thus,

$$V(H) := \bigsqcup_{v \in V(H_a)} V(H_b^v).$$

Now from each edge of H_a we make an edge of H by replacing the vertex of H_a by an edge of the corresponding copy of H_b in all possible ways:

$$E(H) := \biggl\{ \bigsqcup_{1 \leqslant i \leqslant a} e_{v_i} \mid (v_1, \dots, v_a) \in E(H_a),\ e_{v_i} \in E(H_b^{v_i}) \biggr\}.$$

Clearly, $|E(H)| = m(a)[m(b)]^a$. Since H_b is not 2-colourable, for any 2-colouring of H each copy of H_b contains a monochromatic edge. Since H_a is not 2-colourable, out of these edges we can assemble a monochromatic edge of H, which means that H is not 2-colourable, as required.

8.3. Asymptotic explicit constructions. First of all, note that all n-element subsets of a $(2n-1)$-element set form a hypergraph with chromatic number 3 and with $\binom{2n-1}{n} = (4+o(1))^n$ edges.

Substituting various explicit constructions into recurrence relations makes it possible to obtain hypergraphs with at least $(2.65\ldots + o(1))^n$ edges. This number is attained on the hypergraph obtained by repeated application of (8.3) to Fano planes (note that the resulting hypergraph is an intersecting set).

In 2013 Gebauer constructed a hypergraph with $2^{n+O(n^{2/3})}$ edges which admits no proper 2-colouring (and a similar example with $(r+o(1))^n$ edges for r colours). We present the latter construction. For simplicity, below we ignore divisibility problems.

Theorem 8.3.1 (Gebauer [81], 2013). *For any r there exists an explicit construction of an n-uniform hypergraph $H = (V, E)$ admitting no proper r-colouring and such that*

$$|E(H)| = (r+o(1))^n.$$

Proof. Let

$$V := \{(i,j) \mid 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant rt\} = [k] \times [rt]$$

for some positive integer t and $k = r^t n/t$. We define the set of edges in the following way:

$$E := \bigcup_{\substack{A\subset[rt]\\|A|=t}} \bigcup_{\substack{0\leqslant i_\alpha<k\\ \alpha\in A}} \bigcup_{\substack{B\subset[k]\\|B|=n/t}} \{((\beta+i_\alpha) \mod k,\ \alpha) \mid \alpha\in A,\ \beta\in B\}.$$

Note that

$$|E| \leqslant \binom{rt}{t} k^t \binom{k}{n/t} \leqslant (rt)^t \left(\frac{r^t n}{t}\right)^t (er^t)^{n/t} (rn)^t r^{t^2} e^{n/t} r^n.$$

Let

$$t := \left(\frac{n}{\log r}\right)^{1/3}.$$

Note that $(rn)^t \leqslant n^{2t} = e^{2t\log n} = e^{o(n)}$. Moreover, $r^{t^2} = r^{o(n)}$ and $e^{n/t} = e^{o(n)}$. As a result, $|E(H)| = (r+o(1))^n$.

Now we show that H admits no proper colouring. Assume the opposite and consider a proper colouring. We call a colour q a *major colour* for a line $[k]\times\{i\}$ if the line has at least $\lfloor k/r\rfloor$ vertices. By the pigeonhole principle every line $[k]\times\{i\}$ has at least one major colour. Similarly, we obtain a set $A\subset[rt]$ of lines with the same major colour q such that $|A|\geqslant t$. Next, for any fixed β the proportion of $\{i_\alpha\}_{\alpha\in A}$ such that $\{((\beta+i_\alpha) \mod k,\ \alpha) \mid \alpha\in A\}$ is contained in q is at least $(1/r)^t$. By the linearity of the mathematical expectation there is a choice of $\{i_\alpha\}_{\alpha\in A}$ such that at least $k(1/r)^t = n/t$ indices $\beta\in B$ give q-free sets $\{((\beta+i_\alpha) \mod k,\ \alpha) \mid \alpha\in A\}$. Therefore, for any colouring there is an edge with colour q, which is a contradiction. □

The same example carries over to the case of panchromatic colourings.

Theorem 8.3.2 (Cherkashin [48], 2018). *Let $r = o(\sqrt{n/\log n})$. Then there exists an explicit construction of an n-uniform hypergraph $H=(V,E)$ admitting no panchromatic r-colouring and such that*

$$|E(H)| = \left(\frac{r}{r-1}+o(1)\right)^n.$$

Proof. We construct a hypergraph $H_1=(V_1,E_1)$ in the following way. Fix an integer $t\mid n$ and put

$$k := \left\lceil \left(\frac{r}{r-1}\right)^t \right\rceil \frac{n}{t}.$$

Then $V := \{(i,j) \mid 1\leqslant i\leqslant k,\ 1\leqslant j\leqslant rt\} = [k]\times[rt]$. We define the set of edges as follows:

$$E := \bigcup_{\substack{A\subset[rt]\\|A|=t}} \bigcup_{\substack{0\leqslant i_\alpha<k\\ \alpha\in A}} \bigcup_{\substack{B\subset[k]\\|B|=n/t}} \{((\beta+i_\alpha) \mod k,\ \alpha) \mid \alpha\in A,\ \beta\in B\}.$$

Note that

$$|E| \leqslant \binom{rt}{t} k^t \binom{k}{n/t} \leqslant (rt)^t \biggl(\biggl(\frac{r}{r-1}\biggr)^t \frac{n}{t}\biggr)^t \biggl(e\biggl(\frac{r}{r-1}\biggr)^t\biggr)^{n/t}$$
$$\leqslant (rn)^t \biggl(\frac{r}{r-1}\biggr)^{t^2} e^{n/t} \biggl(\frac{r}{r-1}\biggr)^{n}.$$

Let $t := \sqrt{n/\log n}$. Since $r = o(\sqrt{n/\log n}\,)$, we can see that $(rn)^t \leqslant n^{2t} = e^{2t\log n} = e^{o(n/r)}$. Moreover,

$$\biggl(\frac{r}{r-1}\biggr)^{t^2} = \biggl(\frac{r}{r-1}\biggr)^{o(n)} \quad \text{and} \quad e^{n/t} = e^{o(n/r)}.$$

As a result, $|E(H)| = (r/(r-1)+o(1))^n$.

Now we show that H has no panchromatic colouring. Assume the opposite and consider a panchromatic colouring. We call a colour q a *minor colour* for a line $[k]\times\{i\}$, if it has at most $\lfloor k/r\rfloor$ vertices. By the pigeonhole principle every line $[k]\times\{i\}$ has a minor colour. Again, there is a set $A\subset[rt]$ of lines with the same minor colour q and $|A|\geqslant t$. Next, for any fixed β the proportion of $\{i_\alpha\}_{\alpha\in A}$ such that $\{((\beta+i_\alpha) \bmod k,\ \alpha) \mid \alpha\in A\}$ has no colour q is at least $((r-1)/r)^t$. By the linearity of the mathematical expectation there is a choice of $\{i_\alpha\}_{\alpha\in A}$ such that at least $k((r-1)/r)^t = n/t$ indices $\beta\in B$ give q-free sets $\{((\beta+i_\alpha) \bmod k,\ \alpha) \mid \alpha\in A\}$. Thus, there is an edge without the colour q, which is a contradiction. $\square$

9. Applications

9.1. Hilbert's monochromatic cubes. The pioneering result related to colourings of hypergraphs is in fact Hilbert's theorem [94] on monochromatic affine cubes. An affine cube for a set $A\subset\mathbb{Z}$ and a number x is the set of numbers

$$HC(A,x) := \{x+\Sigma(B) \mid B\subset A\},$$

where $\Sigma(B)$ denotes the sum of the elements of the set B. The dimension of the cube is $|A|$. We note that sometimes hypercubes are also considered for multisets A.

Hilbert's theorem states that for any n and r there exists an N such that for any colouring of the set $[N]$ with r colours it contains a monochromatic affine n-cube. Denote the minimum value of such N by $h(n,r)$. Hilbert's proof gives the estimate

$$h(n,r)\leqslant r^{((3+\sqrt{5}\,)/2)^n}.$$

The asymptotic behaviour of the quantity $h(2,r)$ was established by Brown, Erdös, Chung, and Graham [41]:

$$h(2,r) = (1+o(1))r^2.$$

Conlon, Fox, and Sudakov [56] improved a result due to Hegyvári [91] and showed that

$$h(n,r)\geqslant r^{cn^2}.$$

In [86] and [85] Gunderson, Rödl, and Sidorenko derived the estimate

$$h(n,r) \geqslant r^{(2^{n-1}/n)(1-o(1))}.$$

Finally, quite recently Balogh, Lavrov, Shakan, and Wagner [20] demonstrated that for any constant $\varepsilon > 0$

$$h(n,r) \geqslant \min(W(c(\varepsilon)k^2, 2), 2^{k^{2.5-\varepsilon}}),$$

where W is the van der Waerden number, which we introduce in the next subsection.

9.2. Van der Waerden function. The well-known theorem of van der Waerden [166] states that if a sufficiently long segment $[N]$ of positive integers is coloured with r colours, then it contains a monochromatic arithmetic progression of prescribed length. If the length of the desired progression is denoted by n, then the function $W(n,r)$ returns the smallest integer N for which the theorem is valid.

The problem of finding the value of $W(n,r)$ can be reformulated in the following way: find the smallest integer N, for which the hypergraph $H(N,n) = ([N], \mathrm{AP}(N,n))$ is not r-colourable, where $\mathrm{AP}(N,n)$ denotes the set of arithmetic progressions of length n composed of integers from 1 through N.

We note that this hypergraph is 'almost' simple in the sense that most pairs of edges share at most one vertex. Modifications of methods employed in colouring simple hypergraphs give the best lower estimates which do not depend on the number-theoretic properties of n (the best estimates in the case of a prime $n-1$ were presented in [30] and [34]). These modifications were presented in almost all the papers cited in § 3.1. Correspondingly, the current best lower estimate is established by a modification of the proof of Theorem 3.1.1.

Theorem 9.2.1 (Kozik–Shabanov [114], 2016). *For any $r \geqslant 2$ and $n \geqslant 3$*

$$W(n,r) \geqslant cr^{n-1}.$$

At the same time, the best known upper estimate for an arbitrary n involves an exponentiation tower.

Theorem 9.2.2 (Gowers [83], 2001). *For any $r \geqslant 2$ and $n \geqslant 3$*

$$W(n,r) \leqslant 2^{2^{r^{2^{2^{n+9}}}}}.$$

For $n = 3$ Bloom [35] recently proved that

$$W(3,r) \leqslant 2^{cr(\log r)^4}.$$

It should also be mentioned that Graham has offered \$ 1000 for a proof or disproof of the inequality

$$W(n,2) < 2^{n^2}.$$

9.3. Explicit estimates in Folkman's theorem. For a set A of integers define the set of its partial sums as

$$S_A := \{\Sigma(B) \mid B \subset A,\ B \neq \varnothing\},$$

where $\Sigma(B)$ is the sum of the elements of the set B. Take the hypergraph $H(N,k)$ whose vertices are the integers from 1 through N and whose edges are the sets S_A for all $A \subset [N]$ of size k such that $S_A \subset [N]$ (that is, the sum of the numbers in A does not exceed N). Clearly, the size of any edge does not exceed $2^k - 1$. Define $F(k)$ as the smallest number N such that the hypergraph $H(N,k)$ is not 2-colourable. Folkman's famous theorem states that such a number $F(k)$ exists for any k.

Erdős and Spencer [72] noted that the size of an edge cannot be less than $k(k+1)/2$ (this value is attained on $A = \{1,\dots,k\}$) and that at most $(kN)^{\log u} u^{2k}$ edges have size at most u. Then they randomly (uniformly and independently) coloured the vertices with two colours, estimated the probability of success, and derived the estimate

$$F(k) \geqslant 2^{ck^2/\log k}.$$

Quite recently this result was significantly improved by a group of authors.

Theorem 9.3.1 (Balogh–Eberhard–Narayanan–Treglown–Wagner [19], 2017). *The following estimate holds*:

$$F(k) \geqslant 2^{2^{k-1}k^{-1}}.$$

Proof. Consider the class of 2-colourings in which n and $2n$ have different colours for any n. Such colourings are uniquely determined by an arbitrary colouring of the odd numbers. The key observation is that in such a colouring there are no monochromatic edges of non-maximal size. Indeed, assume that for a set A there are two equal representatives of $S(A)$. Then

$$\Sigma(B) = \Sigma(C)$$

for B, $C \subset A$, which is equivalent to

$$\Sigma(B \setminus C) = \Sigma(C \setminus B).$$

However, in that case

$$2\Sigma(B \setminus C) = \Sigma(B \setminus C \cup C \setminus B),$$

which gives elements of different colours in $S(A)$.

Note that if $|S(A)| = 2^k - 1$, then $S(A)$ contains exactly 2^{k-1} odd elements. Then the probability that $S(A)$ is monochromatic in a randomly (uniformly and independently) chosen 2-colouring of odd numbers is equal to $2^{1-2^{k-1}}$. Hence, the expected number of monochromatic edges in $H(N,k)$ is at most

$$\binom{N}{k} \cdot 2^{1-2^{k-1}},$$

which is less than 1 for $N < 2^{2^{k-1}k^{-1}}$. Thus, with positive probability there exists a proper 2-colouring. $\square$

One can try to improve this estimate by replacing the simplest method of random colouring with a more progressive one, but the improvement is expected to be rather minor, whereas the current best upper estimate also involves an exponentiation tower [164]:

$$F(k) \leqslant 2^{2^{3^{2^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}}$$

with total height $4k-3$.

9.4. Theorems of Erdős–Rubin–Taylor type. Let $N(n,r)$ denote the minimum number of vertices in an n-partite graph with list chromatic number larger than r.

Theorem 9.4.1 (Erdős, Rubin, Taylor [71], 1979). *For any r*

$$m(r) \leqslant N(2,r) \leqslant 2m(r).$$

Kostochka proposed two generalizations of Theorem 9.4.1 (recall that $p(n,2) = m(n,2) = m(n)$ by definition). We present the proof of one of them.

Theorem 9.4.2 (Kostochka [103], 2002). *For all $r, n \geqslant 2$*

$$p(r,n) \leqslant N(n,r) \leqslant np(r,n).$$

Proof. Let $H = (V,E)$ be an r-graph with the edge set $E = \{e_1, \dots, e_{p(r,n)}\}$ which admits no panchromatic n-colouring. Consider the complete n-partite graph $G = (W,A)$ with parts $W_1, \dots, W_n$, where $W_i = \{w_{i,1}, \dots, w_{i,|E|}\}$ for $1 \leqslant i \leqslant n$. The ground set for the lists is V. Recall that each e_i is an r-subset of V. For any $i = 1, \dots, n$ and $j = 1, \dots, |E|$ define $L(w_{i,j}) := e_j$.

Assume that f is a proper colouring of G which corresponds to these lists. Since G is a complete n-partite graph, each colour is used on at most one part. Then f induces an n-colouring g_f on V: $g_f(v)$ has the colour i such that v equals $f(w_{i,j})$ for some j or equals 1 if there is no such $w_{i,j}$ at all. Since for each j all the vertices in $\{w_{1,j}, w_{2,j}, \dots, w_{n,j}\}$ must get different colours in f, the colouring g_f is a panchromatic n-colouring of H, which is a contradiction. Thus, $N(n,r) \leqslant np(r,n)$.

Now consider a complete n-partite graph $G = (W,A)$ with parts $W_1, \dots, W_n$ and $|W| < p(r,n)$. Let L be an arbitrary assignment of r-lists for W. Let $H = (V,E)$ be the hypergraph with

$$V := \bigcup_{w \in W} L(w) \quad \text{and} \quad E := \{L(w) \mid w \in W\}.$$

Since $|E| = |W| < p(r,n)$, there exists a panchromatic n-colouring g of the hypergraph H. We define the colouring f_g of the vertices of W as follows: if $w \in W_i$, then in the edge $L(w)$ of H we choose any vertex v with $g(v) = i$ and we let $f_g(w) = v$. Then vertices in different parts cannot get the same colour, and f is a proper colouring of G which is consistent with the lists. This proves the inequality $N(n,r) \geqslant p(r,n)$. $\square$

Let $Q(n,r)$ denote the minimum number of edges in an n-uniform n-partite hypergraph with list chromatic number larger than r.

Theorem 9.4.3 (Kostochka [103], 2002). *For all* $r, n \geqslant 2$

$$m(r,n) \leqslant Q(n,r) \leqslant nm(r,n).$$

9.5. Colourings of generalized Kneser graphs. Let $K(n,k,s)$ denote the *generalized Kneser graph*, that is, the graph with vertex set $\binom{[n]}{k}$ and with edges connecting pairs of vertices whenever they correspond to sets that intersect in s or fewer elements, where $[n] = \{1, \dots, n\}$. The original Kneser graphs correspond to $s = 1$.

Lovász's famous theorem (a positive solution of the Kneser conjecture that had remained open for thirty years) states that $\chi[K(n,k,1)] = n - 2k + 2$ for $n \geqslant 2k$. For fixed k and s the chromatic numbers of the generalized Kneser graphs were studied by Frankl and Füredi [74], [77]. We are interested in the case where k is close to half of n and s is small.

Theorem 9.5.1 (Bobu–Kupriyanov [36], 2016). *For all* $s < n/2$

$$s + 2 \leqslant \chi\left[K\left(n, \frac{n}{2}, s\right)\right] \leqslant 2\binom{2s-1}{s}.$$

For $s \leqslant s'\sqrt{n}$ *there exists a constant* $c = c(s')$ *such that*

$$\chi\left[K\left(n, \frac{n}{2}, s\right)\right] \leqslant cn.$$

The lower bound follows immediately from Lovász's theorem. The first of the upper estimates is a particular case of the next lemma with

$$H = \left([2s-1], \binom{[2s-1]}{s}\right),$$

that is, H is a complete s-graph on $2s-1$ vertices.

Lemma 9.5.2 (Balogh–Cherkashin–Kiselev [18], 2019). *Let* $H = (V,E)$ *be a hypergraph with discrepancy at least* s *and with* $|V| \leqslant n$. *Then*

$$\chi\left[K\left(n, \frac{n}{2} - t, s\right)\right] \leqslant 2|E|.$$

Proof. We embed the graph H in $[n]$. For any hyperedge $e \in E$ we define the colours 1_e and 2_e in the following way (where $\overline{e}$ denotes the complement of e):

$$1_e := \left\{A \in V\left(K\left(n, \frac{n}{2} - t, s\right)\right) \;\middle|\; A \cap e| \geqslant \frac{|e| + s}{2}\right\};$$
$$2_e := \left\{A \in V\left(K\left(n, \frac{n}{2} - t, s\right)\right) \;\middle|\; A \cap \overline{e}| \geqslant \frac{|\overline{e}| + s}{2}\right\}.$$

Note that these colours form independent subsets of our graph, since vertices of the same colour intersect in at least s points.

Any subset $A \subset [n]$ of size $n/2$ can be viewed as a colouring of the vertices of H with two colours as follows: vertices in $V(H) \cap A$ are considered blue and vertices in $V(H)\setminus A$ are red. By the condition imposed on the discrepancy of H there exists a hyperedge $e \in E$ such that

$$\big|\,|A \cap e| - |\overline{A} \cap e|\,\big| \geqslant s.$$

Since $|A \cap e| + |\overline{A} \cap e| = |e|$, we have

$$|A \cap e| \geqslant \frac{|e| + s}{2} \quad \text{or} \quad |A \cap e| \leqslant \frac{|e| - s}{2}\,.$$

Hence, either $A \in 1_e$ or $A \in 2_e$, respectively, because $|A \cap \overline{e}| \geqslant (|\overline{e}| + s)/2$ is equivalent to $|A \cap e| \leqslant (|e| - s)/2$. Thus, all vertices are coloured with one of $2|E|$ colours. $\square$

Theorem 7.1.1 and Lemma 9.5.2 immediately yield the following assertion.

Theorem 9.5.3 (Balogh–Cherkashin–Kiselev [18], 2019). *Let $n \geqslant m > 4s^2$ and suppose that there exists a Hadamard matrix of order m. Then*

$$\chi\left[K\left(n, \frac{n}{2}, s\right)\right] \leqslant 2m.$$

9.6. Euclidean Ramsey theory. Alon and Kostochka applied some results on the growth of the list chromatic number of a hypergraph with an increase in its degree to obtain the following results of Ramsey type.

Theorem 9.6.1 (Alon–Kostochka [14], 2011). *For any finite set X in the Euclidean plane and any positive integer s there is an assignment of a list of size s to every point in the plane such that whenever the points in the plane are coloured with colours from the corresponding lists, there is a monochromatic isometric copy of X.*

Proof. Put $n = |X|$ and let us show that for any d there is a d-regular simple n-graph whose vertex set is a finite set of points in $\mathbb{R}^2$ such that the vertices of each edge form an isometric copy of X.

Consider n arbitrary points $\{v_{11}, v_{12}, \dots, v_{1n}\} =: X_0 \subset \mathbb{R}^2$ that form a copy of X. We choose $d-1$ rotations of the set X_0 in a generic way and denote the resulting points by $\{v_{i1}, v_{i2}, \dots, v_{in}\}$, $2 \leqslant i \leqslant d$. Now we construct the desired hypergraph $H = (V, E)$:

$$V := \{v_{1j_1} + v_{2j_2} + \dots + v_{dj_d} \mid 1 \leqslant j_t \leqslant n \text{ for all } t\},$$

and each edge is defined by fixing all the terms but one,

$$E := \big\{\{v_{1j_1} + v_{2j_2} + \dots + v_{dj_d} \mid 1 \leqslant j_t \leqslant n\} \,\big|\, 1 \leqslant t \leqslant d,$$
$$1 \leqslant v_{1j_1}, \dots, v_{t-1,j_{t-1}}, v_{t+1,j_{t+1}}, \dots, v_{dj_d} \leqslant n\big\}.$$

We chose rotations in a generic way, hence $|V| = n^d$, $|E| = dn^{d-1}$, and all edges have size n. Also, H is d-regular and simple by construction. Theorem 4.2.2 states that $\mathrm{ch}(H) > s$ under the condition $d > d_0(s)$. $\square$

Bibliography

[1] H. L. Abbott and D. Hanson, “On a combinatorial problem of Erdős”, *Canad. Math. Bull.* **12**:6 (1969), 823–829.

[2] H. L. Abbott and L. Moser, “On a combinatorial problem of Erdős and Hajnal”, *Canad. Math. Bull.* **7**:2 (1964), 177–181.

[3] S. Aglave, V. A. Amarnath, S. Shannigrahi, and S. Singh, “Improved bounds for uniform hypergraphs without property B”, *Australas. J. Combin.* **76**:1 (2020), 73–86.

[4] M. Ajtai, J. Komlós, J. Pintz, J. Spencer, and E. Szemerédi, “Extremal uncrowded hypergraphs”, *J. Combin. Theory Ser. A* **32**:3 (1982), 321–335.

[5] M. Akhmejanova and D. Shabanov, “Colorings of b-simple hypergraphs”, *The European conference on combinatorics, graph theory and applications, EuroComb* 2017 (Vienna 2017), Electron. Notes Discrete Math., vol. 61, Elsevier Science B.V., Amsterdam 2017, pp. 29–35.

[6] M. Akhmejanova and D. Shabanov, “Coloring hypergraphs with bounded cardinalities of edge intersections”, *Discrete Math.*, Publ. online 2019, 111692, 11 pp.

[7] M. B. Akhmejanova and D. A. Shabanov, “Equitable colorings of hypergraphs with few edges”, *Discrete Appl. Math.*, Publ. online 2019, 1–11.

[8] И. А. Акользин, “О раскрасках 3-однородных гиперграфов в три цвета”, *Геометрия и механика*, Итоги науки и техн. Сер. Соврем. матем. и ее прил. Темат. обз., **150**, ВИНИТИ РАН, М. 2018, с. 26–39. [I. A. Akolzin, “On 3-homogeneous hypergraph colourings in 3 colours”, *Geometry and mechanics*, Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz., vol. 150, VINITI, Moscow 2018, pp. 26-39.]

[9] I. Akolzin and D. Shabanov, “Colorings of hypergraphs with large number of colors”, *Discrete Math.* **339**:12 (2016), 3020–3031.

[10] N. Alon, “Hypergraphs with high chromatic number”, *Graphs Combin.* **1** (1985), 387–389.

[11] N. Alon, “Choice numbers of graphs: a probabilistic approach”, *Combin. Probab. Comput.* **1**:2 (1992), 107–114.

[12] N. Alon, “Restricted colorings of graphs”, *Surveys in combinatorics*, 1993 (Keele), London Math. Soc. Lecture Note Ser., vol. 187, Cambridge Univ. Press, Cambridge 1993, pp. 1–33.

[13] N. Alon, D. J. Kleitman, C. Pomerance, M. Saks, and P. Seymour, “The smallest n-uniform hypergraph with positive discrepancy”, *Combinatorica* **7**:2 (1987), 151–160.

[14] N. Alon and A. Kostochka, “Hypergraph list coloring and Euclidean Ramsey theory”, *Random Structures Algorithms* **39**:3 (2011), 377–390.

[15] N. Alon and J. H. Spencer, *The probabilistic method*, 4th ed., Wiley Ser. Discrete Math. Optim., John Wiley & Sons, Inc., Hoboken, NJ 2016, xiv+375 pp.

[16] N. Alon and V. H. Vũ, “Anti-Hadamard matrices, coin weighing, threshold gates, and indecomposable hypergraphs”, *J. Combin. Theory Ser. A* **79**:1 (1997), 133–160.

[17] A. Arman and T. Retter, “An upper bound for the size of a k-uniform intersecting family with covering number k”, *J. Combin. Theory Ser. A* **147** (2017), 18–26.

[18] J. Balogh, D. Cherkashin, and S. Kiselev, “Coloring general Kneser graphs and hypergraphs via high-discrepancy hypergraphs”, *European J. Combin.* **79** (2019), 228–236.

[19] J. Balogh, S. Eberhard, B. Narayanan, A. Treglown, and A. Z. Wagner, "An improved lower bound for Folkman's theorem", *Bull. Lond. Math. Soc.* **49**:4 (2017), 745–747.

[20] J. Balogh, M. Lavrov, G. Shakan, and A. Z. Wagner, "Monochromatic Hilbert cubes and arithmetic progressions", *Electron. J. Combin.* **26**:2 (2019), 2.22, 15 pp.

[21] J. Balogh, R. Morris, and W. Samotij, "Independent sets in hypergraphs", *J. Amer. Math. Soc.* **28**:3 (2015), 669–709.

[22] J. Balogh, R. Morris, and W. Samotij, "The method of hypergraph containers", *Proceedings of the* 2018 *International Congress of Mathematicians*, vol. IV (Rio de Janeiro), World Sci. Publ., Hackensack, NJ 2018, pp. 3059–3092.

[23] J. Beck, "On a combinatorial problem of P. Erdős and L. Lovász", *Discrete Math.* **17**:2 (1977), 127–131.

[24] J. Beck, "On 3-chromatic hypergraphs", *Discrete Math.* **24**:2 (1978), 127–137.

[25] J. Beck, "A remark concerning arithmetic progressions", *J. Combin. Theory Ser. A* **29**:3 (1980), 376–379.

[26] J. Beck and T. Fiala, " 'Integer-making' theorems", *Discrete Appl. Math.* **3**:1 (1981), 1–8.

[27] D. Bednarchak and M. Helm, "A note on the Beck–Fiala theorem", *Combinatorica* **17**:1 (1997), 147–149.

[28] C. Berge, *Hypergraphs. Combinatorics of finite sets*, North-Holland Math. Library, vol. 45, North-Holland Publishing Co., Amsterdam 1989, x+255 pp.

[29] C. Berge and F. Sterboul, "Equipartite colorings in graphs and hypergraphs", *J. Combin. Theory Ser. B* **22**:2 (1977), 97–113.

[30] E. R. Berlekamp, "A construction for partitions which avoid long arithmetic progressions", *Canad. Math. Bull.* **11**:3 (1968), 409–414.

[31] A. Bernshteyn, "Measurable versions of the Lovász Local Lemma and measurable graph colorings", *Adv. Math.* **353** (2019), 153–223.

[32] A. Bernshteyn, M. Delcourt, H. Towsner, and A. Tserunyan, "A short nonalgorithmic proof of the containers theorem for hypergraphs", *Proc. Amer. Math. Soc.* **147**:4 (2019), 1739–1749.

[33] F. Bernstein, "Zur Theorie der trigonometrischen Reihe", *J. Reine Angew. Math.* **1907**:132 (1907), 270–278.

[34] T. Blankenship, J. Cummings, and V. Taranchuk, "A new lower bound for van der Waerden numbers", *European J. Combin.* **69** (2018), 163–168.

[35] T. F. Bloom, "A quantitative improvement for Roth's theorem on arithmetic progressions", *J. Lond. Math. Soc.* (2) **93**:3 (2016), 643–663.

[36] А. В. Бобу, А. Э. Куприянов, "О хроматических числах дистанционных графов, близких к кнезеровским", *Пробл. передачи информ.* **52**:4 (2016), 64–83; English transl., A. V. Bobu and A. E. Kupriyanov, "On chromatic numbers of close-to-Kneser distance graphs", *Problems Inform. Transmission* **52**:4 (2016), 373–390.

[37] T. Bohman, A. Frieze, and D. Mubayi, "Coloring H-free hypergraphs", *Random Structures Algorithms* **36**:1 (2010), 11–25.

[38] B. Bollobás, "Chromatic number, girth and maximal degree", *Discrete Math.* **24**:3 (1978), 311–314.

[39] B. Bollobás and A. D. Scott, "Problems and results on judicious partitions", *Random Structures Algorithms* **21**:3-4 (2002), 414–430.

[40] A. Bretto, *Hypergraph theory. An introduction*, Math. Eng., Springer, Cham 2013, xiv+119 pp.

[41] T. C. Brown, P. Erdös, F. R. K. Chung, and R. L. Graham, “Quantitative forms of a theorem of Hilbert”, *J. Combin. Theory Ser. A* **38**:2 (1985), 210–216.

[42] B. Bukh, “An improvement of the Beck–Fiala theorem”, *Combin. Probab. Comput.* **25**:3 (2016), 380–398.

[43] М. И. Бурштейн, “Критические гиперграфы с минимальным числом ребер”, *Сообщ. АН ГССР* **83**:2 (1976), 285–288. [M. I. Burshtein, “Critical hypergraphs with the minimal number of edges”, *Sakharth. SSR Mecn. Akad. Moambe* **83**:2 (1976), 285–288.]

[44] Y. Caro and Z. Tuza, “Improved lower bounds on k-independence”, *J. Graph Theory* **15**:1 (1991), 99–107.

[45] W. Chen, A. Srivastav, and G. Travaglini (eds.), *A panorama of discrepancy theory*, Lecture Notes in Math., vol. 2107, Springer, Cham 2014, xvi+695 pp.

[46] D. Cherkashin, “On hypergraph cliques with chromatic number 3”, *Mosc. J. Comb. Number Theory* **1**:3 (2011), 3–11.

[47] D. Cherkashin, “Coloring cross-intersecting families”, *Electron. J. Combin.* **25**:1 (2018), 1.47, 9 pp.

[48] D. Cherkashin, “A note on panchromatic colorings”, *Discrete Math.* **341**:3 (2018), 652–657.

[49] Д. Д. Черкашин, “Задача Эрдёша–Хайнала для 3-графов”, *Комбинаторика и теория графов*. XI, Зап. науч. сем. ПОМИ, **488**, ПОМИ, СПб. 2019, с. 168–176; English transl., D. Cherkashin, *On the Erdős–Hajnal problem in the case of 3-graphs*, 2019, 6 pp., arXiv: 1905.02893.

[50] D. D. Cherkashin and J. Kozik, “A note on random greedy coloring of uniform hypergraphs”, *Random Structures Algorithms* **47**:3 (2015), 407–413.

[51] D. D. Cherkashin, A. B. Kulikov, and A. M. Raigorodskii, “On hypergraph cliques with chromatic number 3 and a given number of vertices”, *Труды МФТИ* **4**:1 (2012), 151–156.

[52] D. Cherkashin and F. Petrov, *Regular behaviour of the maximal hypergraph chromatic number*, 2019 (v1, 2018), 8 pp., arXiv: 1808.01482.

[53] D. Cherkashin and F. Petrov, “On small n-uniform hypergraphs with positive discrepancy”, *J. Combin. Theory Ser. B* **139** (2019), 353–359.

[54] C. J. Colbourn and J. H. Dinitz (eds.), *Handbook of combinatorial designs*, 2nd ed., Discrete Math. Appl. (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL 2007, xxii+984 pp.

[55] C. J. Colbourn, A. D. Forbes, M. J. Grannell, T. S. Griggs, P. Kaski, P. R. J. Östergård, D. A. Pike, and O. Pottonen, “Properties of the Steiner triple systems of order 19”, *Electron. J. Combin.* **17**:1 (2010), 98, 30 pp.

[56] D. Conlon, J. Fox, and B. Sudakov, “Short proofs of some extremal results”, *Combin. Probab. Comput.* **23**:1 (2014), 8–28.

[57] Yu. A. Demidovich, *On some generalizations of the property B-problem of an n-uniform hypergraph*, 2019, 28 pp., arXiv: 1903.11708.

[58] M. Deza, “Solution d’un problème de Erdős–Lovász”, *J. Combin. Theory Ser. B* **16**:2 (1974), 166–167.

[59] I. Dinur, O. Regev, and C. Smyth, “The hardness of 3-uniform hypergraph coloring”, *Combinatorica* **25**:5 (2005), 519–535.

[60] R. A. Duke, H. Lefmann, and V. Rödl, “On uncrowded hypergraphs”, *Random Structures Algorithms* **6**:2-3 (1995), 209–212.

[61] L. Duraj, G. Gutowski, and J. Kozik, “A note on two-colorability of nonuniform hypergraphs”, *45th international colloquium on automata, languages, and programming*, LIPIcs. Leibniz Int. Proc. Inform., vol. 107, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern 2018, 46, 13 pp.

[62] P. Erdős, “Graph theory and probability”, *Canad. J. Math.* **11** (1959), 34–38.

[63] P. Erdős, “On a combinatorial problem”, *Nordisk Mat. Tidskr.* **11** (1963), 5–10.

[64] P. Erdős, “On a combinatorial problem. II”, *Acta Math. Acad. Sci. Hungar.* **15** (1964), 445–447.

[65] P. Erdős, “On a combinatorial problem. III”, *Canad. Math. Bull.* **12**:4 (1969), 413–416.

[66] P. Erdős, “Some old and new problems in various branches of combinatorics”, *Proceedings of the 10th southeastern conference on combinatorics, graph theory and computing* (Florida Atlantic Univ., Boca Raton, FL 1979), Congress. Numer., vol. XXIII–XXIV, Utilitas Math., Winnipeg, MB 1979, pp. 19–37.

[67] P. Erdős and A. Hajnal, “On a property of families of sets”, *Acta Math. Acad. Sci. Hungar.* **12** (1961), 87–123.

[68] P. Erdős and A. Hajnal, “On chromatic number of graphs and set-systems”, *Acta Math. Acad. Sci. Hungar.* **17** (1966), 61–99.

[69] P. Erdős, C. Ko, and R. Rado, “Intersection theorems for systems of finite sets”, *Quart. J. Math. Oxford Ser.* (2) **12** (1961), 313–320.

[70] P. Erdős and L. Lovász, “Problems and results on 3-chromatic hypergraphs and some related questions”, *Infinite and finite sets*, vol. 2 (Keszthely 1973), Colloq. Math. Soc. János Bolyai, vol. 10, North-Holland, Amsterdam 1975, pp. 609–627.

[71] P. Erdős, A. L. Rubin, and H. Taylor, “Choosability in graphs”, *Proceedings of the west coast conference on combinatorics, graph theory and computing* (Humboldt State Univ., Arcata, CA 1979), Congress. Numer., vol. 26, Utilitas Math., Winnipeg, MB 1980, pp. 125–157.

[72] P. Erdős and J. H. Spencer, “Monochromatic sumsets”, *J. Combin. Theory Ser. A* **50**:1 (1989), 162–163.

[73] A. Ferber and A. Shapira, *A quantitative Lovász criterion for Property B*, 2019, 4 pp., arXiv: 1903.04968.

[74] P. Frankl, “On the chromatic number of the general Kneser-graph”, *J. Graph Theory* **9**:2 (1985), 217–220.

[75] P. Frankl, “Erdős–Ko–Rado theorem with conditions on the maximal degree”, *J. Combin. Theory Ser. A* **46**:2 (1987), 252–263.

[76] P. Frankl, “A near-exponential improvement of a bound of Erdős and Lovász on maximal intersecting families”, *Combin. Probab. Comput.* **28**:5 (2019), 733–739.

[77] P. Frankl and Z. Füredi, “Extremal problems concerning Kneser graphs”, *J. Combin. Theory Ser. B* **40**:3 (1986), 270–284.

[78] P. Frankl, K. Ota, and N. Tokushige, “Covers in uniform intersecting families and a counterexample to a conjecture of Lovász”, *J. Combin. Theory Ser. A* **74**:1 (1996), 33–42.

[79] P. Frankl and N. Tokushige, “Invitation to intersection problems for finite sets”, *J. Combin. Theory Ser. A* **144** (2016), 157–211.

[80] A. Frieze and D. Mubayi, “Coloring simple hypergraphs”, *J. Combin. Theory Ser. B* **103**:6 (2013), 767–794.

[81] H. Gebauer, “On the construction of 3-chromatic hypergraphs with few edges”, *J. Combin. Theory Ser. A* **120**:7 (2013), 1483–1490.

[82] J. Gimbel and C. Thomassen, “Coloring triangle-free graphs with fixed size”, *Discrete Math.* **219**:1-3 (2000), 275–277.

[83] W. T. Gowers, “A new proof of Szemerédi’s theorem”, *Geom. Funct. Anal.* **11**:3 (2001), 465–588.

[84] D. A. Grable, K. T. Phelps, and V. Rödl, “The minimum independence number for designs”, *Combinatorica* **15**:2 (1995), 175–185.

[85] D. S. Gunderson and V. Rödl, "Extremal problems for affine cubes of integers", *Combin. Probab. Comput.* **7**:1 (1998), 65–79.

[86] D. S. Gunderson, V. Rödl, and A. Sidorenko, "Extremal problems for sets forming Boolean algebras and complete partite hypergraphs", *J. Combin. Theory Ser. A* **88**:2 (1999), 342–367.

[87] A. Gyárfás and J. Lehel, "Trees in greedy colorings of hypergraphs", *Discrete Math.* **311**:2-3 (2011), 208–209.

[88] A. Hajnal and E. Szemerédi, "Proof of a conjecture of P. Erdős", *Combinatorial theory and its applications*, vol. 2 (Balatonfüred 1969), North-Holland, Amsterdam 1970, pp. 601–623.

[89] J. Håstad, S. Jukna, and P. Pudlák, "Top-down lower bounds for depth-three circuits", *Comput. Complexity* **5**:2 (1995), 99–112.

[90] P. Haxell and J. Verstraete, "List coloring hypergraphs", *Electron. J. Combin.* **17**:1 (2010), 129, 12 pp.

[91] N. Hegyvári, "On the dimension of the Hilbert cubes", *J. Number Theory* **77**:2 (1999), 326–330.

[92] M. Helm, "On the Beck–Fiala theorem", *Discrete Math.* **207**:1-3 (1999), 73–87.

[93] M. Herzog and J. Schönheim, "The B_r property and chromatic numbers of generalized graphs", *J. Combin. Theory Ser. B* **12**:1 (1972), 41–49.

[94] D. Hilbert, "Ueber die Irreducibilität ganzer rationaler Functionen mit ganzzahligen Coefficienten", *J. Reine Angew. Math.* **1892**:110 (1892), 104–129.

[95] A. J. W. Hilton and E. C. Milner, "Some intersection theorems for systems of finite sets", *Quart. J. Math. Oxford Ser.* (2) **18**:1 (1967), 369–384.

[96] P. Horak, "On the chromatic number of Steiner triple systems of order 25", *Discrete Math.* **299**:1-3 (2005), 120–128.

[97] P. Keevash, "Hypergraph Turán problems", *Surveys in combinatorics* 2011 (Exeter, UK), London Math. Soc. Lecture Note Ser., vol. 392, Cambridge Univ. Press, Cambridge 2011, pp. 83–139.

[98] H. A. Kierstead and A. V. Kostochka, "An Ore-type theorem on equitable coloring", *J. Combin. Theory Ser. B* **98**:1 (2008), 226–234.

[99] H. A. Kierstead and A. V. Kostochka, "A short proof of the Hajnal–Szemerédi theorem on equitable colouring", *Combin. Probab. Comput.* **17**:2 (2008), 265–270.

[100] J. H. Kim, "On Brooks' theorem for sparse graphs", *Combin. Probab. Comput.* **4**:2 (1995), 97–132.

[101] D. J. Kleitman and K. J. Winston, "The asymptotic number of lattices", *Combinatorical mathematics, optimal designs and their applications*, Ann. Discrete Math., vol. 6, North-Holland Publishing Co., Amsterdam 1980, pp. 243–249.

[102] D. J. Kleitman and K. J. Winston, "On the number of graphs without 4-cycles", *Discrete Math.* **41**:2 (1982), 167–172.

[103] A. Kostochka, "On a theorem of Erdős, Rubin, and Taylor on choosability of complete bipartite graphs", *Electron. J. Combin.* **9**:1 (2002), 9, 4 pp.

[104] A. Kostochka, "Coloring uniform hypergraphs with few colors", *Random Structures Algorithms* **24**:1 (2004), 1–10.

[105] A. Kostochka, "Color-critical graphs and hypergraphs with few edges: a survey", *More sets, graphs and numbers*, Bolyai Soc. Math. Stud., vol. 15, Springer, Berlin 2006, pp. 175–197.

[106] A. V. Kostochka and M. Kumbhat, "Coloring uniform hypergraphs with few edges", *Random Structures Algorithms* **35**:3 (2009), 348–368.

[107] А. В. Косточка, Н. П. Мазурова, “Одна оценка в теории раскраски графов”, Методы дискретного анализа в решении комбинаторных задач, **30**, Ин-т матем. СО АН СССР, Новосибирск 1977, с. 23–29. [A. V. Kostochka and N. P. Mazurova, “An estimate in the theory of graph colouring”, Metody Diskretn. Anal., vol. 30, Institute of Mathematics, Siberian Branch of the Academy of Sciences of the USSR, Novosibirsk 1977, pp. 23–29.]

[108] A. Kostochka, D. Mubayi, V. Rödl, and P. Tetali, “On the chromatic number of set systems”, *Random Structures Algorithms* **19**:2 (2001), 87–98.

[109] A. V. Kostochka and V. Rödl, “Partial Steiner systems and matchings in hypergraphs”, *Random Structures Algorithms* **13**:3-4 (1998), 335–347.

[110] A. V. Kostochka and V. Rödl, “Constructions of sparse uniform hypergraphs with high chromatic number”, *Random Structures Algorithms* **36**:1 (2010), 46–56.

[111] A. V. Kostochka, V. Rödl, and M. Kumbhat, “Coloring uniform hypergraphs with small edge degrees”, *Fete of combinatorics and computer science*, Bolyai Soc. Math. Stud., vol. 20, János Bolyai Math. Soc., Budapest 2010, pp. 213–238.

[112] A. V. Kostochka and D. R. Woodall, “Density conditions for panchromatic colourings of hypergraphs”, *Combinatorica* **21**:4 (2001), 515–541.

[113] J. Kozik, “Multipass greedy coloring of simple uniform hypergraphs”, *Random Structures Algorithms* **48**:1 (2016), 125–146.

[114] J. Kozik and D. Shabanov, “Improved algorithms for colorings of simple hypergraphs and applications”, *J. Combin. Theory Ser. B* **116** (2016), 312–332.

[115] A. Kupavskii and D. Shabanov, “Colourings of uniform hypergraphs with large girth and applications”, *Combin. Probab. Comput.* **27**:2 (2018), 245–273.

[116] A. Kupavskii and D. Zakharov, “Regular bipartite graphs and intersecting families”, *J. Combin. Theory Ser. A* **155** (2018), 180–189.

[117] H. Levinson and R. Silverman, “Generalizations of property B”, *2nd international conference on combinatorial mathematics* (New York 1978), Ann. New York Acad. Sci., vol. 319, no. 1, 1979, pp. 349–353.

[118] P.-S. Loh, “A note on embedding hypertrees”, *Electron. J. Combin.* **16**:1 (2009), 18, 4 pp.

[119] L. Lovász, “A generalization of Kónig's theorem”, *Acta Math. Acad. Sci. Hungar.* **21**:3-4 (1970), 443–446.

[120] L. Lovász, “On minimax theorems of combinatorics”, *Mat. Lapok* **26**:3-4 (1975), 209–264.

[121] L. Lu, *On a problem of Erdős and Lovász on coloring non-uniform hypergraphs*, 2008, 15 pp., `www.people.math.sc.edu/lu/papers/propertyB.pdf`.

[122] K. Majumder and S. Mukherjee, “A note on a series of families constructed over the cyclic graph”, *J. Combin. Theory Ser. A* **157** (2018), 41–48.

[123] J. Mathews, M. K. Panda, and S. Shannigrahi, “On the construction of non-2-colorable uniform hypergraphs”, *Discrete Appl. Math.* **180** (2015), 181–187.

[124] J. Matoušek, *Geometric discrepancy. An illustrated guide*, Algorithms Combin., vol. 18, Springer-Verlag, Berlin 1999, xii+288 pp.

[125] M. Matsumoto and N. Tokushige, “The exact bound in the Erdős–Ko–Rado theorem for cross-intersecting families”, *J. Combin. Theory Ser. A* **52**:1 (1989), 90–97.

[126] E. W. Miller, “On a property of families of sets”, *C. R. Soc. Sci. Varsovie* **30** (1937), 31–38.

[127] P. R. J. Östergård, “On the minimum size of 4-uniform hypergraphs without property B”, *Discrete Appl. Math.* **163**, Part 2 (2014), 199–204.

[128] K. T. Phelps and V. Rödl, “Steiner triple systems with minimum independence number”, *Ars Combin.* **21** (1986), 167–172.

[129] N. Pippenger and M. C. Golumbic, “The inducibility of graphs”, *J. Combin. Theory Ser. B* **19**:3 (1975), 189–203.

[130] A. Pluhár, “Greedy colorings of uniform hypergraphs”, *Random Structures Algorithms* **35**:2 (2009), 216–221.

[131] J. Radhakrishnan and A. Srinivasan, “Improved bounds and algorithms for hypergraph 2-coloring”, *Random Structures Algorithms* **16**:1 (2000), 4–32.

[132] А. М. Райгородский, Д. А. Шабанов, “Задача Эрдеша–Хайнала о раскрасках гиперграфов, ее обобщения и смежные проблемы”, *УМН* **66**:5(401) (2011), 109–182; English transl., A. M. Raigorodskii and D. A. Shabanov, “The Erdős–Hajnal problem of hypergraph colouring, its generalizations, and related problems”, *Russian Math. Surveys* **66**:5 (2011), 933–1002.

[133] V. Rödl and E. Šinajová, “Note on independent sets in Steiner systems”, *Random Structures Algorithms* **5**:1 (1994), 183–190.

[134] А. П. Розовская, “О двухцветных раскрасках общего вида для равномерных гиперграфов”, *Докл. РАН* **429**:3 (2009), 309–311; English transl., A. P. Rozovskaya, “On general two-colorings of uniform hypergraphs”, *Dokl. Math.* **80**:3 (2009), 837–839.

[135] А. П. Розовская, Д. А. Шабанов, “О правильных раскрасках гиперграфов в предписанные цвета”, *Дискрет. матем.* **22**:3 (2010), 94–109; English transl., A. P. Rozovskaya and D. A. Shabanov, “On proper colourings of hypergraphs using prescribed colours”, *Discrete Math. Appl.* **20**:4 (2010), 391–409.

[136] W. Samotij, “Counting independent sets in graphs”, *European J. Combin.* **48** (2015), 5–18.

[137] D. Saxton and A. Thomason, “List colourings of regular hypergraphs”, *Combin. Probab. Comput.* **21**:1-2 (2012), 315–322.

[138] D. Saxton and A. Thomason, “Hypergraph containers”, *Invent. Math.* **201**:3 (2015), 925–992.

[139] D. Saxton and A. Thomason, “Online containers for hypergraphs, with applications to linear equations”, *J. Combin. Theory Ser. B* **121** (2016), 248–283.

[140] D. Saxton and A. Thomason, “Simple containers for simple hypergraphs”, *Combin. Probab. Comput.* **25**:3 (2016), 448–459.

[141] W. M. Schmidt, “Ein kombinatorisches Problem von P. Erdős und A. Hajnal”, *Acta Math. Acad. Sci. Hungar.* **15**:3 (1964), 373–374.

[142] P. D. Seymour, “A note on a combinatorial problem of Erdős and Hajnal”, *J. Lond. Math. Soc.* (2) **8**:4 (1974), 681–682.

[143] P. D. Seymour, “On the two-colouring of hypergraphs”, *Quart. J. Math. Oxford Ser.* (2) **25**:1 (1974), 303–311.

[144] Д. А. Шабанов, “Об одной комбинаторной задаче Эрдёша”, *Докл. РАН* **396**:2 (2004), 166–169; English transl., “On one combinatorial problem of Erdős”, *Dokl. Math.* **69**:3 (2004), 359–362.

[145] Д. А. Шабанов, “Экстремальные задачи для раскрасок равномерных гиперграфов”, *Изв. РАН. Сер. матем.* **71**:6 (2007), 183–222; English transl., D. A. Shabanov, “Extremal problems for colourings of uniform hypergraphs”, *Izv. Math.* **71**:6 (2007), 1253–1290.

[146] Д. А. Шабанов, “Рандомизированные алгоритмы раскрасок гиперграфов”, *Матем. сб.* **199**:7 (2008), 139–160; English transl., D. A. Shabanov, “Randomized algorithms for colourings of hypergraphs”, *Sb. Math.* **199**:7 (2008), 1089–1110.

[147] Д. А. Шабанов, “О хроматическом числе конечных систем подмножеств”, *Матем. заметки* **85**:6 (2009), 951–954; English transl., D. A. Shabanov, “On the chromatic number of finite systems of subsets”, *Math. Notes* **85**:6 (2009), 902–905.
[148] Д. А. Шабанов, “Об улучшении нижней оценки в комбинаторной задаче Эрдёша–Хайнала”, *Докл. РАН* **426**:2 (2009), 177–178; English transl., D. A. Shabanov, “Improvement of the lower bound in the Erdős–Hajnal combinatorial problem”, *Dokl. Math.* **79**:3 (2009), 349–350.
[149] Д. А. Шабанов, “О существовании полноцветных раскрасок для равномерных гиперграфов”, *Матем. сб.* **201**:4 (2010), 137–160; English transl., D. A. Shabanov, “The existence of panchromatic colourings for uniform hypergraphs”, *Sb. Math.* **201**:4 (2010), 607–630.
[150] D. A. Shabanov, “On a generalization of Rubin's theorem”, *J. Graph Theory* **67**:3 (2011), 226–234.
[151] D. A. Shabanov, “On r-chromatic hypergraphs”, *Discrete Math.* **312**:2 (2012), 441–458.
[152] D. A. Shabanov, “Random coloring method in the combinatorial problem of Erdős and Lovász”, *Random Structures Algorithms* **40**:2 (2012), 227–253.
[153] D. A. Shabanov, “Coloring non-uniform hypergraphs without short cycles”, *Graphs Combin.* **30**:5 (2014), 1249–1260.
[154] D. A. Shabanov, “Around Erdős–Lovász problem on colorings of non-uniform hypergraphs”, *Discrete Math.* **338**:11 (2015), 1976–1981.
[155] D. A. Shabanov, “Equitable two-colorings of uniform hypergraphs”, *European J. Combin.* **43** (2015), 185–203.
[156] I. Shirgazina, “Equitable colorings of non-uniform simple hypergraphs”, *The eight European conference on combinatorics, graph theory and applications, EuroComb* 2015 (Bergen 2015), Electron. Notes Discrete Math., vol. 49, Elsevier Science B.V., Amsterdam 2015, pp. 699–703.
[157] И. Р. Ширгазина, “Справедливые раскраски неоднородных гиперграфов”, *Матем. заметки* **99**:3 (2016), 441–456; English transl., I. R. Shirgazina, “Equitable colorings of nonuniform hypergraphs”, *Math. Notes* **99**:3 (2016), 444–456.
[158] A. Sidorenko, “What we know and what we do not know about Turán numbers”, *Graphs Combin.* **11**:2 (1995), 179–199.
[159] A. Sidorenko, “Upper bounds for Turán numbers”, *J. Combin. Theory Ser. A* **77**:1 (1997), 134–147.
[160] J. Spencer, “Coloring n-sets red and blue”, *J. Combin. Theory Ser. A* **30**:1 (1981), 112–113.
[161] F. Sterboul, “An extremal problem in hypergraph coloring”, *J. Combin. Theory Ser. B* **22**:2 (1977), 159–164.
[162] Z. Szabó, “An application of Lovász' local lemma — a new lower bound for the van der Waerden number”, *Random Structures Algorithms* **1**:3 (1990), 343–360.
[163] В. А. Ташкинов, “О нижней границе для хроматического числа графов с заданными максимальной степенью вершин и обхватом”, *Сиб. электрон. матем. изв.* **1** (2004), 99–109. [V. A. Tashkinov, “On the lower bound for the chromatic number of graphs with given maximal degree and girth”, *Sib.Èlektron. Mat. Izv.* **1** (2004), 99–109.]
[164] A. D. Taylor, “Bounds for the disjoint unions theorem”, *J. Combin. Theory Ser. A* **30**:3 (1981), 339–344.
[165] B. Toft, “On color-critical hypergraphs”, *Infinite and finite sets*, vol. III (Keszthely 1973), Colloq. Math. Soc. János Bolyai, vol. 10, North Holland, Amsterdam 1975, pp. 1445–1457.

[166] B. L. van der Waerden, "Beweis einer Baudetschen Vermutung", *Nieuw Arch. Wisk.* **15** (1927), 212–216.

[167] В. Г. Визинг, "Раскраска вершин графа в предписанные цвета", Методы дискретного анализа в теории кодов и схем, **29**, Ин-т матем. СО АН СССР, Новосибирск 1976, с. 3–10. [V. G. Vizing, "Colouring the vertices of a graph in prescribed colors", Metody Diskretn. Anal., vol. 29, Institute of Mathematics, Siberian Branch of the Academy of Sciences of the USSR, Novosibirsk 1976, pp. 3–10.]

[168] D. R. Woodall, "Property B and the four-color problem", *Combinatorics* (Math. Inst., Oxford 1972), Inst. Math. Appl., Southend-on-Sea 1972, pp. 322–340.

Andrei M. Raigorodskii
Moscow Institute of Physics and Technology
(National Research University);
Lomonosov Moscow State University;
Caucasus Mathematical Center,
Adyghe State University;
Banzarov Buryat State University,
Institute of Mathematics and Computer Science
E-mail: mraigor@yandex.ru

Received 24/JUL/19
Translated by A. PANKRAT'EV

Danila D. Cherkashin
Chebyshev Laboratory,
St. Petersburg State University;
Laboratory of Advanced Combinatorics
and Network Applications,
Moscow Institute of Physics and Technology
(National Research University);
National Research University
Higher School of Economics
(St. Petersburg campus)
E-mail: matelk@mail.ru