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BRIEF COMMUNICATIONS

Volume preserving diffeomorphisms as Poincaré maps for volume preserving flows

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All the objects below will be C^∞-smooth. Let M be a compact manifold with $\dim M = m$, and let ν be a volume form on M. Here we mean that ν is a nowhere vanishing differential m-form such that $\int_M \nu > 0$. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z} = \{t \bmod 1\}$. Then $\omega = dt \wedge \nu$ is a volume form on the manifold $\mathbb{T} \times M$.

Let $\pi_{\mathbb{T}} \colon \mathbb{T} \times M \to \mathbb{T}$ and let $\pi_M \colon \mathbb{T} \times M \to M$ be the natural projections

$$\mathbb{T} \times M \ni (t,x) \mapsto \pi_{\mathbb{T}}(t,x) = t, \qquad (t,x) \mapsto \pi_M(t,x) = x.$$

Consider a vector field v on $\mathbb{T} \times M$. We assume that

(A) the first component of v is positive: $D\pi_{\mathbb{T}}\, v = v_{\mathbb{T}} > 0$,

(B) v preserves the form ω: $L_v\omega = 0$, where L_v is the Lie derivative.

Let g_v^s be the flow generated by the vector field v on $\mathbb{T} \times M$. Condition (A) implies that the Poincaré map (the first-return map) $P_v \colon \{0\} \times M \to \{0\} \times M$ is well defined. The map P_v preserves the volume form $\lambda = \imath_v\omega|_{\{t=0\}}$ on $\{0\} \times M$.

Theorem 1. *Let P_v be the diffeomorphism defined above and let $Q \colon \{0\} \times M \to \{0\} \times M$ be another diffeomorphism which preserves λ. Assume that Q is (smoothly) isotopic to P_v in the group of λ-preserving self-maps of $\{0\} \times M$. Then there exists an ω-preserving vector field u on $\mathbb{T} \times M$ such that $D\pi_{\mathbb{T}}\, u > 0$ and $Q = P_u$.*

This result, probably interesting on itself, was obtained as a tool required in the proof of the main result of [1]. Now we sketch the proof of Theorem 1.

(a) Consider an s-smooth family of diffeomorphisms $\mathscr{T}_M(s) \colon M \to M$ of the manifold M into itself, $s \in \mathbb{R}$. We extend $\mathscr{T}_M$ to a family of diffeomorphisms $\mathscr{T}^s$ of $\mathbb{R} \times M$ by defining

$$\mathbb{R} \times M \ni (t,x) \mapsto \mathscr{T}^s(t,x) = \bigl(t+s, \mathscr{T}_M(t+s) \circ \mathscr{T}_M^{-1}(t)(x)\bigr). \tag{1}$$

A direct computation shows that $\mathscr{T}^s$ is a flow, that is, $\mathscr{T}^0 = \mathrm{id}$ and $\mathscr{T}^{s_2} \circ \mathscr{T}^{s_1} = \mathscr{T}^{s_1+s_2}$ for any $s_1, s_2 \in \mathbb{R}$. The flow $\mathscr{T}^s$ generates a vector field $\mathscr{U}$ on $\mathbb{R} \times M$:

$$\mathscr{U} = \biggl(\frac{d}{ds}\bigg|_{s=0} \mathscr{T}^s\biggr) \circ \mathscr{T}^{-s}, \qquad D\pi_{\mathbb{R}}(\mathscr{U}) = 1, \tag{2}$$

where $\pi_{\mathbb{R}} \colon \mathbb{R} \times M \to \mathbb{R}$ is the natural projection.

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(b) We put $\widehat{v} = v/v_{\mathbb{T}}$. Then $D\pi_{\mathbb{T}}\widehat{v} = 1$. Let $(t,x) \mapsto g^s_{\widehat{v}}(t,x)$, $(t,x) \in \mathbb{T}\times M$, be the flow of the vector field $\widehat{v}$. Then $g^s_{\widehat{v}}$ preserves the form $v_{\mathbb{T}}\omega$. The Poincaré maps P_v and $P_{\widehat{v}}$ coincide. Hence $g^1_{\widehat{v}}(0,x) = (1, P_v(x))$ for any $x \in M$.

Let G^s be the lift of the flow $g^s_{\widehat{v}}$ to the covering space $\mathbb{R}\times M$. We define a family $\sigma_s\colon M \to M$ by $G^s(0,x) = (s, \sigma_s(x))$, $s \in \mathbb{R}$.

(c) Let γ_s be an s-smooth isotopy from the conditions of Theorem 1: for any $s \in [0,1]$ the map γ_s is a λ-preserving diffeomorphism of $M \cong \{0\}\times M$. Then $\gamma_0 = P_v$ and $\gamma_1 = Q$. By smoothly changing the parametrization on γ_s we can assume that $\gamma_s = P_v$ if s takes values close to 0, and $\gamma_s = Q$ if s is close to 1. We extend γ_s to the whole axis $\mathbb{R} = \{s\}$, for example, by putting $\gamma_s = P_v$ for $s < 0$ and $\gamma_s = Q$ for $s > 0$.

(d) Consider the family of maps $\mathscr{T}_M(s)\colon M \to M$ with $\mathscr{T}_M(s) = \sigma_s \circ P^{-1}_{\widehat{v}} \circ \gamma_s$, $s \in \mathbb{R}$. Then $\mathscr{T}_M(0) = \mathrm{id}$ and $\mathscr{T}_M(1) = Q$, hence both $\mathscr{T}_M(0)$ and $\mathscr{T}_M(1)$ preserve the form λ.

Let $\mathscr{T}^s$ be the flow (1) on $\mathbb{R}\times M$ generated by the family $\mathscr{T}_M(s)$, and let $\mathscr{U}$ be the corresponding vector field on $\mathbb{R}\times M$. Then by (1)

$$D\pi_{\mathbb{T}}\,\mathscr{U} = 1, \quad \mathscr{T}^0 = \mathrm{id}_{\mathbb{T}\times M}, \quad \mathscr{T}^1(0,x) = (1, Q(x)).$$

By (1) and (2)

$$\mathscr{U} = \left(1, \frac{d}{ds}\bigg|_{s=0} (\sigma_{t+s}\circ P^{-1}_{\widehat{v}} \circ \gamma_{t+s}) \circ (\sigma_t \circ P^{-1}_{\widehat{v}} \circ \gamma_t)^{-1}\right) = \widehat{v} + \mathbf{w},$$
$$\widehat{v} = \left(1, \left(\frac{d}{dt}\sigma_t\right)\circ \sigma_t^{-1}\right), \qquad \mathbf{w} = \left(0, D(\sigma_t \circ P^{-1}_{\widehat{v}})\left(\frac{d}{dt}\gamma_t\right)\circ \gamma_t^{-1} \circ P_{\widehat{v}} \circ \sigma_t^{-1}\right).$$

Near the points $t = 0$ and $t = 1$ we have: $d\gamma_t/dt = 0$, and therefore $\mathscr{U} = \widehat{v}$. Hence $\mathscr{U}|_{s\in[0,1]}$ can be extended to an s-periodic vector field $\widehat{\mathscr{U}}$ on $\mathbb{R}\times M$. Let $\widehat{\vartheta}^s$ be the corresponding flow on $\mathbb{R}\times M$. Since $\widehat{\mathscr{U}}$ is periodic, projections of $\widehat{\vartheta}^s$ and $\widehat{\mathscr{U}}$ on a flow ϑ^s and a vector field U on $\mathbb{T}\times M$ are well defined.

(e) Let $\mathbf{1}$ be the vector field on $\mathbb{R}\times M$ defined by the equalities $D\pi_{\mathbb{T}}\mathbf{1} = 1$ and $D\pi_M\mathbf{1} = 0$. The flow $\widehat{\vartheta}^s$ preserves some volume form Ω on $\mathbb{R}\times M$ which can be chosen so that $\imath_{\mathbf{1}}\Omega|_{t=0} = \imath_{\mathbf{1}}\Omega|_{t=1} = \lambda$.

Any volume form on $\mathbb{R}\times M$ equals $\hat\rho\omega$, where $\hat\rho\colon \mathbb{R}\times M \to \mathbb{R}$ is a positive function. Therefore, $\Omega = \hat\rho\omega$, where $\hat\rho|_{t=0} = \hat\rho|_{t=1} = \rho_0$ and $\lambda = \rho_0\nu$.

These equalities and the periodicity of $\widehat{\mathscr{U}}$ imply that $\hat\rho$ is 1-periodic in t. Hence there exists a function $\rho\colon \mathbb{T}\times M \to \mathbb{R}$ such that $\hat\rho = \rho\circ\pi$, where $\pi\colon \mathbb{R}\times M \to \mathbb{T}\times M$ is the canonical projection. The flow ϑ^s preserves the volume form $\rho\omega$.

The vector field $u = \rho U$ preserves ω. It remains to note that $P_u = P_U = Q$.

Corollary. *The vector field* $\mathbf{w}$ *is small if the isotopy* γ_s *is close to the identity. Moreover, in this case* U *is close to* $\widehat{v}$*, and therefore* ρ *is close to* $v_{\mathbb{T}}$ *and then* u *is a small perturbation of* v*.*

Bibliography

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