Russian Math. Surveys **75**:1 1–87 *Uspekhi Mat. Nauk* **75**:1 3–94

DOI: https://doi.org/10.1070/RM9900

To the memory of Mark Vishik

Attractors of nonlinear Hamiltonian partial differential equations

A. I. Komech and E. A. Kopylova

Abstract. This is a survey of the theory of attractors of nonlinear Hamiltonian partial differential equations since its appearance in 1990. Included are results on global attraction to stationary states, to solitons, and to stationary orbits, together with results on adiabatic effective dynamics of solitons and their asymptotic stability, and also results on numerical simulation. The results obtained are generalized in the formulation of a new general conjecture on attractors of G-invariant nonlinear Hamiltonian partial differential equations. This conjecture suggests a novel dynamical interpretation of basic quantum phenomena: Bohr transitions between quantum stationary states, de Broglie's wave-particle duality, and Born's probabilistic interpretation.

Bibliography: 212 titles.

Keywords: Hamiltonian equations, nonlinear partial differential equations, wave equation, Maxwell equations, Klein–Gordon equation, limiting amplitude principle, limiting absorption principle, attractor, steady states, soliton, stationary orbits, adiabatic effective dynamics, symmetry group, Lie group, Schrödinger equation, quantum transitions, wave-particle duality.

Contents

1. Introduction 3
2. Global attraction to stationary states 8
2.1. 1D nonlinear wave equations 9
2.2. A string coupled to a nonlinear oscillator 11
2.3. Wave-particle system 12
2.3.1. Liénard–Wiechert asymptotics 15
2.3.2. The free wave equation 16
2.3.3. Scattering of energy to infinity 17
2.3.4. A convolution representation and relaxation of acceleration and velocity 18

The first author was supported by the Austrian Science Fund (FWF) under project no. P28152-N35, and the second author by the Russian Foundation for Basic Research (grant no. 18-01-00524).

AMS 2010 *Mathematics Subject Classification.* Primary 35B41; Secondary 35B40, 35C08.

© 2020 Russian Academy of Sciences (DoM), London Mathematical Society, Turpion Ltd.

2.3.5. A compact attracting set 19
2.3.6. Global attraction 20
2.4. The Maxwell–Lorentz equations: radiation damping 20
2.5. The wave equation with concentrated nonlinearities 21
2.6. Remarks 28
3. Global attraction to solitons 28
3.1. A translation-invariant 'wave-particle' system 29
3.1.1. A canonical transformation and a reduced system 31
3.1.2. Orbital stability of solitons 33
3.1.3. The strong Huygens principle and soliton asymptotics 34
3.2. Invariance of symplectic structure 37
3.3. The translation-invariant Maxwell–Lorentz system 38
3.4. The case of weak interaction 38
4. The adiabatic effective dynamics of solitons 39
4.1. A 'wave-particle' system with a slowly varying external potential 39
4.2. Mass-energy equivalence 41
5. Global attraction to stationary orbits 41
5.1. Method of omega-limit trajectories 44
5.2. The spectral representation and the limiting absorption principle 46
5.3. A nonlinear analogue of Kato's theorem 47
5.4. Splitting into dispersion and bound components 49
5.5. Omega-compactness 50
5.6. Reduction of spectrum of omega-limit trajectories to a spectral gap 51
5.7. Reduction of spectrum of omega-limit trajectories to a single point 52
5.7.1. Equation for omega-limit trajectories and spectral inclusion 52
5.7.2. Titchmarsh convolution theorem 52
5.8. Remarks on dispersion radiation and nonlinear energy transfer 53
6. Asymptotic stability of stationary orbits and solitons 56
6.1. Asymptotic stability of stationary orbits. Orthogonal projection 57
6.2. Asymptotic stability of solitons. Symplectic projection 58
6.3. Generalizations and applications 62
6.4. Further generalizations 63
6.5. Linear dispersion 64
7. Numerical simulation of soliton asymptotics 66
7.1. Kinks of relativistically invariant Ginzburg–Landau equations 66
7.2. Numerical observation of soliton asymptotics 70
7.3. Adiabatic effective dynamics of relativistic solitons 70
8. Appendix: attractors and quantum mechanics 72
8.1. On dynamical interpretation of quantum jumps 73
8.2. Bohr postulates via perturbation theory 75
8.3. Conclusion 75
Bibliography 76

1. Introduction

This paper is a survey of results since 1990 on long-time behaviour and attractors for solutions of nonlinear Hamiltonian partial differential equations.

The theory of attractors for nonlinear PDEs began in Landau's 1944 seminal paper [156], where he proposed the first mathematical interpretation of the onset of turbulence as the growth of the dimension of attractors of the Navier–Stokes equations when the Reynolds number increases.

The foundation for the corresponding mathematical theory was laid in 1951 by Hopf, who first established the existence of global solutions of the 3D Navier–Stokes equations [71]. He introduced the 'method of compactness', which is a nonlinear version of Faedo–Galerkin approximations. This method is based on a priori estimates and Sobolev embedding theorems and has had an essential influence on the development of the theory of nonlinear PDEs (see [161]).

The modern development of the theory of attractors for general *dissipative systems*, that is, systems with friction, originated in 1975–1985 in publications by Foiaş, Ghidaglia, Hale, Henry, and Temam, and was developed further by Vishik, Babin, Chepyzhov, Ilyin, Pata, Titi, Zelik, and others. A typical property of dissipative systems is global convergence to stationary states in the absence of external excitation: any finite-energy solution of a dissipative *autonomous* equation in a region $\Omega \subset \mathbb{R}^n$ converges to a stationary state:

$$\psi(x,t) \to S(x), \qquad t \to +\infty, \tag{1.1}$$

where as a rule the convergence holds in the $L^2(\Omega)$-metric. In particular, the relaxation to an equilibrium regime in chemical reactions is due to energy dissipation.

The results obtained concern a wide class of nonlinear dissipative PDEs, including fundamental equations of applied and mathematical physics: the Navier–Stokes equations, nonlinear parabolic equations, reaction-diffusion equations, wave equations with friction, integro-differential equations, equations with delay, with memory, and so on. Very clever techniques of functional analysis of nonlinear PDEs were developed for the study of the structure of attractors, their smoothness and their fractal and Hausdorff dimensions, dependence on parameters, on averaging, and so on. An essential part of the theory up to 2000 was covered in the monographs [8], [28], [53], [64], [67], [70], and [202].

The development of a similar theory for *Hamiltonian PDEs* seemed at first to be unmotivated and even impossible in view of energy conservation and time reversal for these equations. However, it turned out that such a theory is possible, and its basic directions were suggested by a novel mathematical interpretation of fundamental postulates of quantum theory:

I. Transitions between quantum stationary orbits (Bohr, 1913).

II. Wave-particle duality (de Broglie, 1924).

III. Probabilistic interpretation (Born, 1927).

Namely, postulate I can be interpreted as the global attraction (1.8) of all quantum trajectories to an attractor formed by stationary orbits (see § 8), and postulate II can be interpreted as decay into solitons (1.7). The probabilistic interpretation can also be justified by the asymptotics (1.7). More details can be found in [102].

Results obtained in 1990–2019 suggest that such long-time asymptotics of solutions are in fact typical for nonlinear Hamiltonian PDEs. These results are presented in this survey. The theory is only at an initial stage of development and cannot be compared with the theory of attractors of dissipative PDEs with regard to richness and diversity of results. For Hamiltonian PDEs it differs significantly from the case of dissipative systems, where the attraction to stationary states is connected with energy dissipation due to friction. For Hamiltonian equations the friction and energy dissipation are absent, and the attraction is caused by radiation which irreversibly carries energy to infinity.

The modern development of the theory of nonlinear Hamiltonian equations dates back to Jörgens [85], who established the existence of global solutions for nonlinear wave equations of the form

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) + F(\psi(x,t)), \qquad x \in \mathbb{R}^n, \tag{1.2}$$

by developing the Hopf method of compactness. The subsequent studies in this direction were well presented by Lions in [161].

The first results on the long-time asymptotics of solutions of nonlinear Hamiltonian PDEs were obtained by Segal ([187], [188]) and Morawetz and Strauss ([172], [173], [196]). In these papers *local energy decay* was proved for solutions of equations (1.2) with *defocusing type* nonlinearities $F(\psi) = -m^2\psi - \kappa|\psi|^p\psi$, where $m^2 \geqslant 0$, $\kappa > 0$, and $p > 1$. Namely, for sufficiently smooth and small initial data,

$$\int_{|x|<R} \left[|\dot{\psi}(x,t)|^2 + |\nabla\psi(x,t)|^2 + |\psi(x,t)|^2\right] dx \to 0, \qquad t \to \pm\infty, \tag{1.3}$$

for any finite $R > 0$. Moreover, the corresponding nonlinear wave and scattering operators were constructed in these papers. In [198] and [199] Strauss established the completeness of the scattering operators for small solutions of more general equations. The decay (1.3) means that the energy escapes each bounded region for large times.

For convenience, characteristic properties of all finite-energy solutions of an equation will be referred to as *global*, in order to distinguish them from the corresponding *local* properties of the solutions with initial data sufficiently close to an attractor.

All the above-mentioned results on local energy decay (1.3) mean that the corresponding *local attractor* of solutions with small initial states consists of only the zero point. The first results on *global attraction* for nonlinear Hamiltonian PDEs were obtained by one of the present authors in 1991–1995 for 1D models ([93], [95], [96]), and were later extended to nD equations. Note that global attraction to a (proper) attractor is impossible for all finite-dimensional Hamiltonian systems, because of energy conservation.

Global attraction for Hamiltonian PDEs is derived from an analysis of the irreversible energy radiation to infinity, which plays the role of dissipation. Such an analysis requires subtle methods of harmonic analysis: the Wiener Tauberian theorem, the Titchmarsh convolution theorem, the theory of quasi-measures, the Paley–Wiener estimates, eigenfunction expansions for non-selfadjoint Hamiltonian operators based on M. G. Krein's theory of J-selfadjoint operators, and others.

The results obtained so far indicate a certain dependence of long-time asymptotics of solutions on the symmetry group of the equation: for example, it may be the trivial group $G = \{e\}$, or the unitary group $G = \mathrm{U}(1)$, or the group of translations $G = \mathbb{R}^n$. Namely, the results suggest the conjecture that for 'generic' nonlinear Hamiltonian *autonomous* PDEs with a Lie symmetry group G, any finite-energy solution admits the asymptotics

$$\psi(x,t) \sim e^{g_\pm t}\psi_\pm(x), \qquad t \to \pm\infty. \tag{1.4}$$

Here, $e^{g_\pm t}$ is a representation of the one-parameter subgroup of G which corresponds to the generators $g_\pm$ in the corresponding Lie algebra, while the $\psi_\pm(x)$ are some 'scattering states' depending on the trajectory $\psi(x,t)$ considered. Both pairs (g_+,ψ_+) and (g_-,ψ_-) are solutions of the corresponding nonlinear eigenfunction problem.

In the case of the trivial symmetry group, the conjecture (1.4) means global attraction to the corresponding stationary states

$$\psi(x,t) \to S_\pm(x), \qquad t \to \pm\infty \tag{1.5}$$

(see Fig. 1), where the $S_\pm(x)$ depend on the trajectory $\psi(x,t)$ under consideration, and the convergence holds in local seminorms, that is, in norms of type $L^2(|x| < R)$ with any $R > 0$. The convergence (1.5) in global norms (that is, corresponding to $R = \infty$) cannot hold due to energy conservation.

In particular, the asymptotics (1.5) can easily be proved for the d'Alembert equation (see (2.1)–(2.7)). In this example the convergence (1.5) in global norms obviously fails due to the presence of travelling waves $f(x \pm t)$. Similarly, a solution of the 3D wave equation with unit velocity of propagation is concentrated in spherical layers $|t| - R < |x| < |t| + R$ if the initial data have support in the ball $|x| \leqslant R$. Therefore, the solution converges to zero as $t \to \pm\infty$, although its energy remains constant. This convergence corresponds to the well-known *strong Huygens principle*. Thus, attraction to stationary states (1.5) is a generalization of the strong Huygens principle to nonlinear equations. The difference is that for a linear wave equation the limit is always zero, while for nonlinear equations the limit can be any stationary solution.

Further, in the case of the symmetry group of translations $G = \mathbb{R}^n$ the asymptotics (1.4) means global attraction to solitons (travelling waves)

$$\psi(x,t) \sim \psi_\pm(x - v_\pm t), \qquad t \to \pm\infty, \tag{1.6}$$

for solutions of the *generic* translation-invariant equation. In this case the convergence holds in local seminorms *in the comoving frame of reference*, that is, in $L^2(|x - v_\pm t| < R)$ for any $R > 0$. The validity of such local asymptotics in comoving reference systems suggests that there may be several such solitons which provide the refined asymptotics

$$\psi(x,t) \sim \sum_k \psi_\pm(x - v_\pm^k t) + w_\pm(x,t), \qquad t \to \pm\infty, \tag{1.7}$$

where the $w_\pm$ are some dispersion waves that are solutions of the corresponding free equation, and the convergence now holds in some *global norm*. A trivial example is given by the d'Alembert equation (2.1) with solutions $\psi(x,t) = f(x-t) + g(x+t)$.

Asymptotic expressions (1.7) with several solitons were first discovered in 1965 by Kruskal and Zabusky in numerical simulations of the Korteweg–de Vries (KdV) equation. Later on, global asymptotics of this type were proved for nonlinear *integrable* translation-invariant equations (KdV and others) by Ablowitz, Segur, Eckhaus, van Harten, and others, using the method of the *inverse scattering problem* [46].

Finally, for the unitary symmetry group $G = \mathrm{U}(1)$, the asymptotics (1.4) mean global attraction to 'stationary orbits' (or *'solitary waves'*)

$$\psi(x,t) \sim \psi_{\pm}(x)e^{-i\omega_{\pm}t}, \qquad t \to \pm\infty, \tag{1.8}$$

in the same local seminorms (see Fig. 3). These asymptotics were inspired by the Bohr postulate on transitions between quantum stationary states (see the Appendix (§ 8) for details). Our results confirm such asymptotics for generic U(1)-invariant nonlinear equations of type (5.4) and (5.16)–(5.18). More precisely, we have proved global attraction *to the manifold of all stationary orbits*, though attraction to a particular stationary orbit, with fixed frequencies $\omega_{\pm}$, is still an open problem.

The existence of stationary orbits $\psi(x)e^{i\omega t}$ for a broad class of U(1)-invariant nonlinear wave equations (1.2) was extensively studied in the 1960s–1980s. The most complete results were obtained by Strauss, Berestycki, and Lions [17], [18], [196]. Moreover, Esteban, Georgiev, and Séré [51] constructed stationary orbits for relativistically invariant nonlinear Maxwell–Dirac equations (8.5). The key role in these papers was played by the Lusternik–Schnirelmann theory [163], [164].

The orbital stability of stationary orbits was studied by Grillakis, Shatah, Strauss, and others [62], [63].

Let us emphasize that we are conjecturing the asymptotics (1.8) for *generic* U(1)-invariant equations. This means that the long-time behaviour of solutions may be quite different for U(1)-invariant equations of 'positive codimension'. In particular, for solutions of the linear Schrödinger equation

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t), \qquad x \in \mathbb{R}^n,$$

the asymptotics (1.8) generally fail. Namely, any finite-energy solution admits the spectral representation

$$\psi(x,t) = \sum C_k\psi_k(x)e^{-i\omega_k t} + \int_0^{\infty} C(\omega)\psi(\omega,x)e^{-i\omega t}\,d\omega,$$

where ψ_k and $\psi(\omega,\cdot)$ are the corresponding eigenfunctions of the discrete and continuous spectrum, respectively. The last integral is a dispersion wave, which decays to zero in the norms $L^2(|x| < R)$ with any $R > 0$ (under appropriate conditions on the potential $V(x)$). Correspondingly, the attractor is the linear span of the eigenfunctions ψ_k. Thus, the long-time asymptotics does not reduce to a single term like (1.8), so the linear case is degenerate in this sense. Note that our results for equations (5.4) and (5.16)–(5.18) are established for a *strictly nonlinear case*: see the condition (5.12) below, which eliminates linear equations.

For more sophisticated symmetry groups $G = \mathrm{U}(N)$, the asymptotics (1.4) mean the attraction to N-frequency trajectories, which can be quasi-periodic. In particular, the symmetry groups SU(2), SU(3) and others were suggested in 1961 by

Gell-Mann and Ne'eman for strong interaction of baryons [58], [175]. This suggestion is based on the parallelism discovered between the empirical data for baryons and the 'Dynkin scheme' of the Lie algebra su(3) with eight generators (the famous 'eightfold way'). This theory resulted in the scheme of quarks and in the development of quantum chromodynamics [3], [65], and also in the prediction of a new baryon with prescribed values of its mass and decay products. This particle, the Ω^--hyperon, was promptly discovered experimentally [12].

This empirical correspondence between Lie algebra generators and elementary particles presumably gives evidence in favour of the general conjecture (1.4) for equations with Lie symmetry groups.

Note that our conjecture (1.4) specifies the notion of 'localized solution / coherent structures' from the 'Grande Conjecture' and the 'Petite Conjecture' of Soffer (see [190], p. 460) in the context of G-invariant equations. The Grande Conjecture was proved in [124] for a 1D wave equation coupled to a nonlinear oscillator (2.20). Moreover, suitable versions of the Grande Conjecture were also proved in [78], [79] for the 3D wave, Klein–Gordon, and Maxwell equations coupled to a relativistic particle with sufficiently small charge (3.34) (see Remark 3.12). Finally, for any matrix symmetry group G, (1.4) implies the Petite Conjecture, since then the localized solutions $e^{g_\pm t}\psi_\pm(x)$ are quasi-periodic.

Below we dwell upon available results on the asymptotics (1.5)–(1.8). In §§ 2 and 3 we review results on global attraction to stationary states and to solitons, respectively. Section 4.1 concerns adiabatic effective dynamics of solitons, and § 4.2 concerns the mass-energy equivalence. In § 5 we give a concise complete proof of the attraction to stationary orbits. Sections 6.1 and 6.2 concern the asymptotic stability of stationary orbits and solitons, and § 6.3 is devoted to various generalizations. In § 7 we present results on numerical simulation of soliton asymptotics for relativistically invariant equations. In the Appendix (§ 8) we comment on the relationship between the general conjecture (1.4) and the Bohr postulates in quantum mechanics.

In conclusion let us comment on previous related surveys in this area. The survey [98] presents results only for 1D equations. The results on asymptotic stability of solitons were described in detail in [74] for linear equations coupled to a particle, and in [135] for the relativistically invariant Ginzburg–Landau equations. In the present article we give only a short statement of these results (§§ 2.1, 2.2, and 6.3). Finally, our survey gives much more information on our methods than [101]. Our main novelties are as follows.

(i) Streamlined and simplified proofs of the results in [128]–[130] on global attraction to stationary states and to solitons for systems of a relativistic particle coupled to a scalar wave equation and to the Maxwell equation. These results give the first rigorous justification of the famous *radiation damping* in classical electrodynamics. We omit unessential technical details, but we carefully explain our approach, which relies on the Wiener Tauberian theorem, in §§ 2.3, 2.4, and 3.1.

(ii) The complete proof of the nonlinear analogue of the Kato theorem on the absence of embedded eigenvalues (§ 5.3), which is a crucial point in the proof of global attraction to stationary orbits for U(1)-invariant equations in [99], [103]–[108], [92], [136], [142], [143], [30], [31].

(iii) Informal arguments on the dispersion radiation and the nonlinear spreading of the spectrum (§ 5.8), which mean a nonlinear energy transfer from lower to higher harmonics and lie behind our application of the Titchmarsh convolution theorem.

(iv) Recent results in [136], [137], [142], [143] on global attractors for nonlinear wave, Klein–Gordon, and Dirac equations with concentrated nonlinearities. We give a detailed survey of the methods and results in § 2.5.

These methods and ideas are being presented here for the first time in a survey.

The authors express their deep gratitude to H. Spohn and B. Vainberg for the longtime collaboration with them on attractors of Hamiltonian PDEs, as well as to A. Shnirelman for many useful discussions. We are also grateful to V. Imaykin and A. A. Komech for their collaboration over many years.

2. Global attraction to stationary states

Here we review the results on global attraction to stationary states (1.5) that were obtained in 1991–1999 for nonlinear Hamiltonian PDEs. The first results of this type were obtained for one-dimensional nonlinear wave equations [93], [95]–[98]. Later on these results were extended to three-dimensional wave equations and Maxwell's equations coupled to a charged relativistic particle [130], [129], and also to three-dimensional wave equations with concentrated nonlinearity [137]. In [45], [204] the attraction (1.5) was established for finite systems of oscillators coupled to an infinite-dimensional thermostat.

The global attraction (1.5) can easily be demonstrated using the trivial (but instructive) example of the d'Alembert equation

$$\ddot{\psi}(x,t) = \psi''(x,t), \qquad x \in \mathbb{R}. \tag{2.1}$$

All derivatives here and below are understood in the sense of distributions. This equation is formally equivalent to the Hamiltonian system

$$\dot{\psi}(t) = D_\pi \mathscr{H}, \quad \dot{\pi}(t) = -D_\psi \mathscr{H} \tag{2.2}$$

with Hamiltonian

$$\mathscr{H}(\psi,\pi) = \frac{1}{2}\int [|\pi(x)|^2 + |\psi'(x)|^2]\,dx, \qquad (\psi,\pi) \in \mathscr{E}_c := H^1_c(\mathbb{R}) \oplus [L^2(\mathbb{R}) \cap L^1(\mathbb{R})], \tag{2.3}$$

where $H^1_c(\mathbb{R})$ is the space of continuous functions $\psi(x)$ with finite norm

$$\|\psi\|_{H^1_c(\mathbb{R})} := \|\psi'\|_{L^2(\mathbb{R})} + |\psi(0)|. \tag{2.4}$$

Furthermore, let

$$\psi(x) \to C_\pm, \qquad x \to \pm\infty. \tag{2.5}$$

For such initial data $(\psi(x,0), \dot{\psi}(x,0)) = (\psi(x), \pi(x)) \in \mathscr{E}_c$ the d'Alembert formula gives

$$\psi(x,t) \to S_\pm(x) = \frac{C_+ + C_-}{2} \pm \frac{1}{2}\int_{-\infty}^{\infty} \pi(y)\,dy, \qquad t \to \pm\infty, \tag{2.6}$$

where the convergence is uniform on every finite interval $|x| < R$. Moreover,

$$\dot\psi(x,t) = \frac{\psi'(x+t) - \psi'(x-t)}{2} + \frac{\pi(x+t) + \pi(x-t)}{2} \to 0, \qquad t \to \pm\infty, \quad (2.7)$$

where the convergence holds in $L^2(-R,R)$ for each $R > 0$. Thus, the set of states $(\psi(x), \pi(x)) = (C, 0)$, where $C \in \mathbb{R}$ is any constant, is an attractor. Note that for positive and negative times the limits (2.6) may be different.

2.1. 1D nonlinear wave equations. In [97], global attraction to stationary states was proved for nonlinear wave equations of the type

$$\ddot\psi(x,t) = \psi''(x,t) + \chi(x)F(\psi(x,t)), \qquad x \in \mathbb{R}, \quad (2.8)$$

where

$$\begin{gathered} \chi \in C_0^\infty(\mathbb{R}), \quad \chi(x) \geqslant 0, \quad \chi(x) \not\equiv 0, \\ F(\psi) = -\nabla U(\psi), \qquad \psi \in \mathbb{R}^N, \quad U(\psi) \in C^2(\mathbb{R}^N). \end{gathered} \quad (2.9)$$

The equation (2.8) can be formally written as the Hamiltonian system (2.2) with Hamiltonian

$$\mathscr{H}(\psi,\pi) = \frac{1}{2}\int [|\pi(x)|^2 + |\psi'(x)|^2 + \chi(x)U(\psi(x,t))]\,dx, \qquad (\psi,\pi) \in \mathscr{E}_{\mathrm{c}}^N = \mathscr{E}_{\mathrm{c}} \otimes \mathbb{R}^N.$$

We assume that the potential is *confining*, that is,

$$U(\psi) \to \infty, \qquad |\psi| \to \infty. \quad (2.10)$$

In this case it is easy to prove that a finite-energy solution $Y(t) = (\psi(t), \pi(t)) \in C(\mathbb{R}, \mathscr{E}_{\mathrm{c}}^N)$ exists and is unique for any initial state $Y(0) \in \mathscr{E}_{\mathrm{c}}^N$, and that the energy is conserved:

$$\mathscr{H}(Y(t)) = \mathrm{const}, \qquad t \in \mathbb{R}. \quad (2.11)$$

Definition 2.1. (i) $\mathscr{E}_F^N$ denotes the space $\mathscr{E}_{\mathrm{c}}^N$ endowed with the seminorms

$$\|(\psi,\pi)\|_{\mathscr{E}_{\mathrm{c}}^N,R} = \|\psi'\|_R + |\psi(0)| + \|\pi\|_R, \qquad R = 1, 2, \dots, \quad (2.12)$$

where $\|\cdot\|_R$ denotes the norm in $L_R^2 := L^2(-R,R)$.

(ii) Convergence in $\mathscr{E}_F^N$ is defined as convergence in every seminorm (2.12).

The space $\mathscr{E}_F^N$ is not complete, and convergence in $\mathscr{E}_F^N$ is equivalent to convergence in the metric

$$\operatorname{dist}(Y_1, Y_2) = \sum_{R=1}^\infty 2^{-R} \frac{\|Y_1 - Y_2\|_{\mathscr{E}_{\mathrm{c}}^N,R}}{1 + \|Y_1 - Y_2\|_{\mathscr{E}_{\mathrm{c}}^N,R}}, \qquad Y_1, Y_2 \in \mathscr{E}_{\mathrm{c}}^N. \quad (2.13)$$

The main result in [97] is the following theorem, which is illustrated by Fig. 1. Denote by $\mathscr{S}$ the set of stationary states $(\psi(x), 0) \in \mathscr{E}_{\mathrm{c}}^N$, where $\psi(x)$ is the solution of the stationary equation

$$\psi''(x) + \chi(x)F(\psi(x)) = 0, \qquad x \in \mathbb{R}.$$

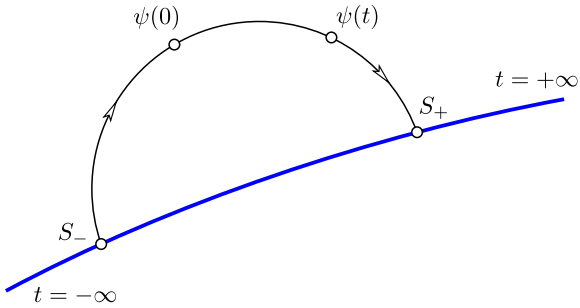

Figure 1. Convergence to stationary states.

Theorem 2.2. (i) *Let the conditions* (2.9) *and* (2.10) *hold. Then any finite-energy solution* $Y(t) = (\psi(t), \pi(t)) \in C(\mathbb{R}, \mathscr{E}_c^N)$ *is attracted to* $\mathscr{S}$:

$$Y(t) \xrightarrow{\mathscr{E}_F^N} \mathscr{S}, \qquad t \to \pm\infty, \tag{2.14}$$

in the metric (2.13). *This means that*

$$\operatorname{dist}(Y(t), \mathscr{S}) := \inf_{S \in \mathscr{S}} \operatorname{dist}(Y(t), S) \to 0, \qquad t \to \pm\infty. \tag{2.15}$$

(ii) *Suppose additionally that the function* $F(\psi)$ *is real-analytic for* $\psi \in \mathbb{R}^N$. *Then* $\mathscr{S}$ *is a discrete subset of* $\mathscr{E}_c^N$, *and for any finite-energy solution* $Y(t) = (\psi(t), \pi(t)) \in C(\mathbb{R}, \mathscr{E}_c^N)$

$$Y(t) \xrightarrow{\mathscr{E}_F^N} S_\pm \in \mathscr{S}, \qquad t \to \pm\infty. \tag{2.16}$$

Sketch of the proof. It suffices to consider only the case where $t \to \infty$. Our proof of (2.14) and (2.16) in [97] was based on the new method of *omega-limit trajectories*, which is a development of the method of omega-limit points used in [96]. Subsequently this method played an essential role in the theory of global attractors for U(1)-invariant PDEs ([99], [103]–[108], [92], [136], [142], [143], [30], [31]).

First we note that the finiteness of the energy radiated from the segment $[-a, a] \supset \operatorname{supp} \chi$ implies the finiteness of the 'dissipation integral':

$$\int_0^\infty \left[|\dot\psi(-a,t)|^2 + |\psi'(-a,t)|^2 + |\dot\psi(a,t)|^2 + |\psi'(a,t)|^2\right] dt < \infty$$

(see [97], (6.3)). This means, roughly, that

$$\psi(\pm a, t) \sim C_\pm, \quad \psi'(\pm a, t) \sim 0, \qquad t \to \infty. \tag{2.17}$$

More precisely, the functions $\psi(\pm a, t)$ and $\psi'(\pm a, t)$ are slowly varying for large times, so their shifts form *omega-compact families*. Namely, from an arbitrary sequence $s_k \to \infty$, one can choose a subsequence $s_{k'} \to \infty$ for which

$$\psi(\pm a, t + s_{k'}) \to C_\pm, \qquad k' \to \infty, \tag{2.18}$$

where the constants $C_\pm$ depend on the subsequence, and the convergence holds in $C[0,T]$ *for any* $T>0$. It remains to prove that

$$\psi(x,t+s_{k'}) \to S_+(x) \in \mathscr{S}, \qquad k' \to \infty, \tag{2.19}$$

in $C([0,T];H^1[-a,a])$ for any $T>0$. In other words, *each omega limit trajectory is a stationary state.*

Roughly speaking, we need to justify the well-posedness of the boundary value problem for a nonlinear differential equation (2.8) in the half-strip $-a \leqslant x \leqslant a$, $t>0$, with the Cauchy boundary conditions (2.17) on the sides $x=\pm a$. Then the convergence (2.18) of boundary values implies the convergence (2.19) of the solution inside the strip.

Our main idea is to use the evident symmetry of the wave equation with respect to interchange of the variables x and t with simultaneous change of the sign of the potential U. However, in this equation with 'time' x the condition (2.10) makes the new potential $-U$ unbounded below! Consequently, this dynamics with x as the time variable is not correct on the interval $|x| \leqslant a$. For example, in the case $U(\psi)=\psi^4$, the equation (2.8) for solutions of type $\psi(x,t)=\psi(x)$ is $\psi''(x)-4\psi^3(x)=0$. Solutions of this ordinary differential equation with finite Cauchy initial data at $x=-a$ can become infinite at any point $x \in (-a,a)$. However, in our situation local correctness is sufficient due to *a priori bounds* which follow from the energy conservation (2.11) in view of the conditions (2.9) and (2.10).

Remark 2.3. (i) The energy of the limit states $S_\pm$ may be less than the conserved energy of the corresponding solution. This limit jump of energy is similar to the well-known property of the norm for *weak convergence* of a sequence in the Hilbert space.

(ii) The discreteness of the set $\mathscr{S}$ is essential for the asymptotics (2.16). For example, the convergence (2.16) fails for the solution

$$\psi(x,t)=\sin[\log(|x-t|+2)]$$

in the case when $F(\psi)=0$ for $|\psi| \leqslant 1$.

2.2. A string coupled to a nonlinear oscillator.

I. The first results on global attraction to stationary states (2.14) and (2.16) were established in [93], [95], and [124] for the case of a point nonlinearity (the 'Lamb system'):

$$(1+m\delta(x))\ddot{\psi}(x,t)=\psi''(x,t)+\delta(x)F(\psi(0,t)), \qquad x \in \mathbb{R}. \tag{2.20}$$

This equation describes transversal oscillations of a string with vector displacements $\psi(t) \in \mathbb{R}^N$ coupled to an oscillator attached at $x=0$ and acting on the string with a force $F(\psi(0,t))$ orthogonal to the string; $m>0$ is the mass of a particle attached to the string at the point $x=0$. For a linear force function $F(\psi)=-k\psi$ such a system was first considered by Lamb [155].

The conserved energy is

$$\mathscr{H}(\psi,\pi,p)=\frac{1}{2}\int\left[|\pi(x)|^2+|\psi'(x)|^2\right]dx+\frac{mp^2}{2}+U(\psi(0)). \tag{2.21}$$

We write $Z := \{z \in \mathbb{R}^N \colon F(z) = 0\}$. Obviously, every finite-energy stationary solution of the equation (2.20) is a constant function $\psi_z(x) = z \in Z$. Let $\mathscr{S}$ denote the manifold of all finite-energy stationary states,

$$\mathscr{S} := \{S_z = (\psi_z, 0) \colon z \in Z\}.$$

This set is discrete in $\mathscr{E}_c$ if Z is discrete in $\mathbb{R}^N$. The proof of the attraction (2.14) and (2.16) is now based on the reduced equation for the oscillator

$$m\ddot{y}(t) = F(y(t)) - 2\dot{y}(t) + 2\dot{w}_{\rm in}(t), \qquad t > 0,$$

where $\dot{w}_{\rm in} \in L^2(0, \infty)$. This equation follows from the d'Alembert representation for the solution $\psi(x, t)$ at $x > 0$ and $x < 0$.

In [124] stronger asymptotics in the global norm of the Hilbert space $\mathscr{E}_c$ were obtained instead of the asymptotics (2.16) in local seminorms. This was achieved by identifying the corresponding d'Alembert outgoing and incoming waves. In [125] and [126] the asymptotic completeness of the corresponding nonlinear scattering operators was proved.

II. In [96] we extended the results in [93] and [95] on global attraction to stationary states, to the case of a string with several oscillators:

$$\ddot{\psi}(x, t) = \psi''(x, t) + \sum_{k=1}^{M} \delta(x - x_k) F_k(\psi(x_k, t)).$$

This equation reduces to a system of M ordinary differential equations with delay. Its study required a new approach based on a special analysis of *omega-limit points* of trajectories.

We remark that detailed proofs of all results in [93] and [95]–[97] are available in the survey [98].

2.3. Wave-particle system. In [130] the first result on global attraction to stationary states (1.5) was obtained for a three-dimensional real scalar wave field coupled to a relativistic particle. The scalar field satisfies the 3D wave equation

$$\ddot{\psi}(x, t) = \Delta\psi(x, t) - \rho(x - q(t)), \qquad x \in \mathbb{R}^3, \tag{2.22}$$

where $\rho \in C_0^\infty(\mathbb{R}^3)$ is a fixed function representing the charge density of the particle, and $q(t) \in \mathbb{R}^3$ is the particle position. The particle motion obeys the Hamiltonian equation with relativistic kinetic energy $\sqrt{1 + p^2}$:

$$\dot{q}(t) = \frac{p(t)}{\sqrt{1 + p^2(t)}}, \quad \dot{p}(t) = -\nabla V(q(t)) - \int \nabla\psi(x, t)\rho(x - q(t))\, dx. \tag{2.23}$$

Here $-\nabla V(q)$ is the external force corresponding to the real potential $V(q)$, and the integral term is a self-force.

Thus, the wave function ψ is generated by the charged particle and plays the role of a potential acting on the particle, along with the external potential $V(q)$.

The system (2.22), (2.23) can be formally represented in the Hamiltonian form

$$\dot{\psi} = D_\pi \mathscr{H}, \quad \dot{\pi} = -D_\psi \mathscr{H}, \quad \dot{q}(t) = D_p \mathscr{H}, \quad \dot{p} = -D_q \mathscr{H} \tag{2.24}$$

with Hamiltonian (energy)

$$\mathscr{H}(\psi,\pi,q,p)=\frac{1}{2}\int\left[|\pi(x)|^2+|\nabla\psi(x)|^2\right]dx+\int\psi(x)\rho(x-q)\,dx+\sqrt{1+p^2}+V(q). \tag{2.25}$$

By $\|\cdot\|$ we denote the norm in the Hilbert space $L^2:=L^2(\mathbb{R}^3)$, and by $\|\cdot\|_R$ the norm in $L^2(B_R)$, where B_R is the ball $|x|\leqslant R$. Let $\mathring{H}^1:=\mathring{H}^1(\mathbb{R}^3)$ be the completion of the space $C_0^\infty(\mathbb{R}^3)$ in the norm $\|\nabla\psi(x)\|$.

Definition 2.4. (i) $\mathscr{E}:=\mathring{H}^1\oplus L^2\oplus\mathbb{R}^3\oplus\mathbb{R}^3$ is the Hilbert phase space of quadruples (ψ,π,q,p) with finite norm

$$\|(\psi,\pi,q,p)\|_{\mathscr{E}}=\|\nabla\psi\|+\|\pi\|+|q|+|p|.$$

(ii) $\mathscr{E}_\sigma$ with $\sigma\in\mathbb{R}$ is the space of quadruples $Y=(\psi,\pi,q,p)\in\mathscr{E}$ with $\psi\in C^2(\mathbb{R}^3)$ and $\pi\in C^1(\mathbb{R}^3)$ satisfying the estimate

$$|\nabla\psi(x)|+|\pi(x)|+|x|(|\nabla\nabla\psi(x)|+|\nabla\pi(x)|)=\mathscr{O}(|x|^{-\sigma}),\qquad |x|\to\infty. \tag{2.26}$$

(iii) $\mathscr{E}_F$ is the space $\mathscr{E}$ with metric of the type (2.13), where the corresponding seminorms are defined by

$$\|(\psi,\pi,q,p)\|_{\mathscr{E},R}=\|\nabla\psi\|_R+\|\psi\|_R+\|\pi\|_R+|q|+|p|. \tag{2.27}$$

Obviously, the energy (2.25) is a continuous functional on $\mathscr{E}$, and $\mathscr{E}_\sigma\subset\mathscr{E}$ for $\sigma>3/2$, and moreover, convergence in $\mathscr{E}_F$ is equivalent to convergence in every seminorm (2.27). We assume that the external potential is confining:

$$V(q)\to\infty,\qquad |q|\to\infty. \tag{2.28}$$

In this case the Hamiltonian (2.25) is bounded below:

$$\inf_{Y\in\mathscr{E}}\mathscr{H}(Y)=V_0+\frac{1}{2}(\rho,\Delta^{-1}\rho), \tag{2.29}$$

where

$$V_0:=\inf_{q\in\mathbb{R}^3}V(q)>-\infty. \tag{2.30}$$

The following lemma was proved in [130], Lemma 2.1.

Lemma 2.5. *Let $V(q)\in C^2(\mathbb{R}^3)$ satisfy the condition (2.30). Then for any initial state $Y(0)\in\mathscr{E}$ there is a unique finite-energy solution $Y(t)=(\psi(t),\pi(t),q(t),p(t))\in C(\mathbb{R},\mathscr{E})$ such that*:

(i) *for every $t\in\mathbb{R}$ the map $W_t\colon Y_0\mapsto Y(t)$ is continuous both in the space $\mathscr{E}$ and in $\mathscr{E}_F$*;

(ii) *the energy $\mathscr{H}(Y(t))$ is conserved, that is*,

$$\mathscr{H}(Y(t))=\mathscr{H}(Y_0),\qquad t\in\mathbb{R}; \tag{2.31}$$

(iii) *there are a priori estimates*

$$\sup_{t\in\mathbb{R}}[\|\nabla\psi(t)\| + \|\pi(t)\|] < \infty, \quad \sup_{t\in\mathbb{R}}|\dot q(t)| = \overline{v} < 1; \tag{2.32}$$

(iv) *if* (2.28) *holds, then also*

$$\sup_{t\in\mathbb{R}}|q(t)| = \overline{q}_0 < \infty. \tag{2.33}$$

Remark 2.6. In the case of a point particle $\rho(x) = \delta(x)$, the system (2.22), (2.23) is incorrect, since in this case any solution of the wave equation (2.22) is singular at the point $x = q(t)$, and, accordingly, the integral in (2.23) is not defined. The energy functional (2.25) in this case is not bounded below, because the integral in (2.29) diverges and is equal to $-\infty$. Indeed, in the Fourier transform this integral has the form

$$(\rho, \Delta^{-1}\rho) = -\int \frac{|\widehat\rho(k)|^2}{k^2}\,dk,$$

where $\widehat\rho(k) \equiv 1$. This is the famous 'ultraviolet divergence'. Thus, the self-energy of the point charge is infinite, which prompted Abraham to introduce the model of an 'extended electron' with continuous charge density $\rho(x)$.

Let $Z = \{q \in \mathbb{R}^3 \colon \nabla V(q) = 0\}$. It is easy to verify that stationary states of the system (2.22), (2.23) have the form $S_q = (\psi_q, 0, q, 0)$, where $q \in Z$ and $\Delta\psi_q(x) = \rho(x-q)$. Therefore, $\psi_q(x)$ is the Coulomb potential

$$\psi_q(x) := -\frac{1}{4\pi}\int \frac{\rho(y-q)\,dy}{|x-y|}\,.$$

Correspondingly, the set of all stationary states of this system is

$$\mathscr{S} := \{S_q \colon q \in Z\}.$$

If the set $Z \subset \mathbb{R}^N$ is discrete, then the set $\mathscr{S}$ is also discrete in $\mathscr{E}$ and in $\mathscr{E}_F$. Finally, assume that the 'form-factor' ρ satisfies the *Wiener condition*

$$\widehat\rho(k) := \int e^{ikx}\rho(x)\,dx \neq 0, \qquad k \in \mathbb{R}^3. \tag{2.34}$$

Remark 2.7. The Wiener condition means a strong coupling of the scalar wave field $\psi(x)$ to the particle. It is a corresponding version of the 'Fermi Golden Rule' for the system (2.22), (2.23): the perturbation $\rho(x-q)$ is not orthogonal to eigenfunctions of the continuous spectrum of the Laplacian Δ.

For simplicity we assume that

$$\rho \in C_0^\infty(\mathbb{R}^3); \qquad \rho(x) = 0 \quad \text{for } |x| \geqslant R_\rho; \qquad \rho(x) = \rho_r(|x|). \tag{2.35}$$

The main result in [130] is as follows.

Theorem 2.8. (i) *Let the conditions* (2.28) *and* (2.34) *hold, and let* $\sigma > 3/2$. *Then for any initial state* $Y(0) = (\psi_0, \pi_0, q_0, p_0) \in \mathscr{E}_\sigma$ *the corresponding solution* $Y(t) = (\psi(t), \pi(t), q(t), p(t)) \in C(\mathbb{R}, \mathscr{E})$ *of the system* (2.22), (2.23) *is attracted to the set of stationary states*:

$$Y(t) \xrightarrow{\mathscr{E}_F} \mathscr{S}, \qquad t \to \pm\infty, \tag{2.36}$$

where the attraction holds in the metric (2.13) *defined by the seminorms* (2.27).

(ii) *In addition, let the set* Z *be discrete in* $\mathbb{R}^3$. *Then*

$$Y(t) \xrightarrow{\mathscr{E}_F} S_\pm \in \mathscr{S}, \qquad t \to \pm\infty. \tag{2.37}$$

The key point in the proof of this is the relaxation of the acceleration

$$\ddot{q}(t) \to 0, \qquad t \to \pm\infty. \tag{2.38}$$

This relaxation has long been known in classical electrodynamics as 'radiation damping'. Namely, the Liénard–Wiechert formulae for retarded potentials suggest that a particle with a non-zero acceleration radiates energy to infinity. The radiation cannot last forever, because the total energy of the solution is finite. These arguments result in the conclusion (2.38) that can be found in any textbook on classical electrodynamics.

However, a rigorous proof is not so obvious, and a proof was first given in [130]. It is based on a calculation of the total energy radiated to infinity using the Liénard–Wiechert formulae. The central point is the representation of this radiated energy as a convolution and the subsequent application of Wiener's Tauberian theorem.

Below we give a streamlined version of this proof for $t \to +\infty$.

Remark 2.9. (i) The condition (2.28) is not necessary for the relaxation (2.38). Relaxation also takes place under the condition (2.30) (see Remark 2.12).

(ii) The Wiener condition (2.34) is not necessary for the relaxation (2.38) either. For example, (2.38) obviously holds in the case when $V(x) \equiv 0$ and $\rho(x) \equiv 0$. More generally, such relaxation also holds when $V(x) \equiv 0$ and the norm $\|\rho\|$ is sufficiently small (see (3.34)).

2.3.1. *Liénard–Wiechert asymptotics.* We recall the long-range asymptotics of the Liénard–Wiechert potentials [130], [129]. Denote by $\psi_r(x,t)$ the retarded potential

$$\psi_r(x,t) = -\frac{1}{4\pi}\int \frac{d^3y\,\theta(t-|x-y|)}{|x-y|}\rho(y - q(t-|x-y|)), \tag{2.39}$$

let $\pi_r(x,t) = \dot{\psi}_r(x,t)$, and let $T_r := \overline{q}_0 + R_\rho$.

Lemma 2.10. *The following asymptotics hold*:

$$\begin{cases} \pi_r(x, |x|+t) = \overline{\pi}(\omega(x), t)|x|^{-1} + \mathscr{O}(|x|^{-2}), \\ \nabla\psi_r(x, |x|+t) = -\omega(x)\overline{\pi}(\omega(x), t)|x|^{-1} + \mathscr{O}(|x|^{-2}), \end{cases} \qquad |x| \to \infty, \tag{2.40}$$

uniformly with respect to $t \in [T_r, T]$ *for any* $T > T_r$. *Here* $\omega(x) = x/|x|$, *and* $\overline{\pi}(\omega(x), t)$ *is given in* (2.42).

Proof. The integrand in (2.39) vanishes for $|y| > T_r$. Then $|x-y| \leqslant t$ for $t-|x| > T_r$, and (2.39) implies that

$$\nabla\psi_r(x,t) = \int \frac{d^3y}{4\pi|x-y|} n\nabla\rho(y - q(t-|x-y|)) \cdot \dot q(t-|x-y|) + \mathscr{O}(|x|^{-2})$$
$$= -\omega(x)\pi_r(x,t) + \mathscr{O}(|x|^{-2}), \qquad t-|x| > T_r,$$

because $n = (x-y)/|x-y| = \omega(x) + \mathscr{O}(|x|^{-1})$ for bounded $|y|$. Hence, it suffices to prove only the asymptotics (2.40) for π_r. We have

$$\pi_r(x,t) = -\int d^3y \frac{1}{4\pi|x-y|} \nabla\rho(y-q(\tau)) \cdot \dot q(\tau), \qquad \tau := t-|x-y|. \tag{2.41}$$

Replacing t by $|x|+t$ in the definition of τ, we obtain

$$\tau = |x| + t - |x-y| = t + \omega(x)\cdot y + \mathscr{O}(|x|^{-1}) = \overline{\tau} + \mathscr{O}(|x|^{-1}), \qquad \overline{\tau} = t + \omega\cdot y,$$

since

$$|x| - |x-y| = |x| - \sqrt{|x|^2 - 2x\cdot y + |y|^2} \sim |x|\left(\frac{x\cdot y}{|x|^2} - \frac{|y|^2}{2|x|^2}\right) = \omega(x)\cdot y + \mathscr{O}(|x|^{-1}).$$

Hence (2.41) implies (2.40) with

$$\overline{\pi}(\omega,t) := -\frac{1}{4\pi}\int d^3y \nabla\rho(y-q(\overline{\tau}))\cdot\dot q(\overline{\tau}). \qquad \square \tag{2.42}$$

2.3.2. *The free wave equation.* Consider now the solution $\psi_K(x,t)$ of the free wave equation with the initial conditions

$$\psi_K(x,0) = \psi_0(x), \quad \dot\psi_K(x,0) = \pi_0(x), \qquad x\in\mathbb{R}^3. \tag{2.43}$$

The Kirchhoff formula gives us that

$$\psi_K(x,t) = \frac{1}{4\pi t}\int_{S_t(x)} d^2y\pi_0(y) + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{S_t(x)} d^2y\psi_0(y)\right), \tag{2.44}$$

where $S_t(x)$ is the sphere $\{y\colon |y-x| = t\}$. Let $\pi_K(x,t) = \dot\psi_K(x,t)$.

Lemma 2.11. *Let $Y_0 \in \mathscr{E}_\sigma$. Then for any $R>0$ and any $T_2 > T_1 \geqslant 0$*

$$\int_{R+T_1}^{R+T_2} dt \int_{\partial B_R} d^2x(|\pi_K(x,t)|^2 + |\nabla\psi_K(x,t)|^2) \leqslant I_0 < \infty. \tag{2.45}$$

Proof. The formula (2.44) implies that

$$\nabla\psi_K(x,t) = \frac{t}{4\pi}\int_{S_1} d^2z\nabla\pi_0(x+tz) + \frac{1}{4\pi}\int_{S_1} d^2z\nabla\psi_0(x+tz)$$
$$+ \frac{t}{4\pi}\int_{S_1} d^2z\nabla_x(\nabla\psi_0(x+tz)\cdot z).$$

Here $S_1 := S_1(0)$. From (2.26) it follows that

$$|\nabla\psi_K(x,t)| \leqslant C\sum_{s=0}^{1} t^s \int_{S_1} d^2z|x+tz|^{-\sigma-1-s}$$
$$= C\sum_{s=0}^{1} \frac{2\pi t^{s-1}}{(\sigma+s-1)|x|}\big((t-|x|)^{-\sigma-s+1} - (t+|x|)^{-\sigma-s+1}\big).$$

Therefore,

$$\int_{R+T_1}^{R+T_2} dt \int_{\partial B_R} d^2x\,|\nabla\psi_k(x,t)|^2$$
$$\leqslant C\int_{R+T_1}^{R+T_2}\left[\frac{(t+R)^{2-2\sigma}+(t-R)^{2-2\sigma}}{t^2} + (t-R)^{-2\sigma}\right]dt$$
$$\leqslant C_1\int_{R+T_1}^{R+T_2} dt\left[\left(1+\frac{R}{t}\right)^2 + \left(1-\frac{R}{t}\right)^2 + 1\right](t-R)^{-2\sigma} \leqslant I_0 < \infty.$$

The integral with $\nabla\pi_K(x,t)$ can be estimated similarly. □

2.3.3. *Scattering of energy to infinity.* We now obtain a bound on the total energy radiated to infinity, which we will represent as a 'radiation integral'. This integral has to be bounded a priori in view of (2.32). Indeed, the energy $\mathscr{H}_R(t)$ at time $t \in \mathbb{R}$ in the ball B_R is defined by

$$\mathscr{H}_R(t) = \frac{1}{2}\int_{B_R} d^3x\big(|\pi(x,t)|^2 + |\nabla\psi(x,t)|^2\big) + \sqrt{1+p^2(t)} + V(q(t))$$
$$+ \int d^3x\,\psi(x,t)\rho(x-q(t)).$$

Consider the energy $I_R(T_1,T_2)$ radiated from the ball B_R during the time interval $[T_1,T_2]$ with $T_2 > T_1 > 0$:

$$I_R(T_1,T_2) = \mathscr{H}_R(T_1) - \mathscr{H}_R(T_2).$$

This energy is bounded a priori, because by (2.32) the energy $\mathscr{H}_R(T_1)$ is bounded above, while $\mathscr{H}_R(T_2)$ is bounded below. Thus,

$$I_R(T_1,T_2) \leqslant I < \infty, \tag{2.46}$$

where I does not depend on T_1, T_2, or R. Further, one has

$$\frac{d}{dt}\mathscr{H}_R(t) = \int_{\partial B_R} d^2x\,\omega(x)\cdot\pi(x,t)\nabla\psi(x,t), \qquad t > R.$$

Hence, (2.46) implies that

$$\int_{R+T_1}^{R+T_2} dt\int_{\partial B_R} d^2x\,\omega(x)\cdot\pi(x,t)\nabla\psi(x,t) \leqslant I.$$

The solution admits the splitting $\pi = \pi_r + \pi_K$, $\psi = \psi_r + \psi_K$, and hence

$$\int_{R+T_1}^{R+T_2} dt \int_{\partial B_R} d^2x\, \omega(x) \cdot (\pi_r \nabla\psi_r + \pi_K \nabla\psi_r + \pi_r \nabla\psi_K + \pi_K \nabla\psi_K) \leqslant I.$$

Lemmas 2.10 and 2.11 together with the Cauchy–Schwarz inequality imply that

$$\int_{T_r}^{T} dt \int_{S_1} d^2\omega |\overline{\pi}(\omega, t)|^2 \leqslant I_1 + T\mathscr{O}(R^{-1}), \qquad T > T_r,$$

where $I_1 < \infty$ does not depend on T or R. Taking the limit as $R \to \infty$ and then as $T \to \infty$, we obtain the finiteness of the energy radiated to infinity:

$$\int_0^\infty dt \int_{S_1} d^2\omega\, |\overline{\pi}(\omega, t)|^2 < \infty. \tag{2.47}$$

2.3.4. *A convolution representation and relaxation of acceleration and velocity.* Applying integration by parts in (2.42), we obtain

$$\begin{aligned}
\overline{\pi}(\omega, t) &= \int d^3y\, \nabla\rho(y - q(\overline{\tau})) \cdot \dot{q}(\overline{\tau}) = \int d^3y\, \nabla_y \rho(y - q(\overline{\tau})) \cdot \dot{q}(\overline{\tau})\, \frac{1}{1 - \omega \cdot \dot{q}(\overline{\tau})} \\
&= -\int d^3y\, \rho(y - q(\overline{\tau}))\, \frac{\partial}{\partial y_\alpha}\, \frac{\dot{q}_\alpha(\overline{\tau})}{1 - \omega \cdot \dot{q}(\overline{\tau})} \\
&= \frac{1}{4\pi} \int d^3y\, \rho(y - q(\overline{\tau}))\, \frac{\omega \cdot \ddot{q}(\tau)}{(1 - \omega \cdot \dot{q}(\tau))^2}\,.
\end{aligned} \tag{2.48}$$

The function $\overline{\pi}(\omega, t)$ is globally Lipschitz-continuous in ω and t in view of (2.32). Hence, (2.47) implies that

$$\lim_{t\to\infty} \overline{\pi}(\omega, t) = 0 \tag{2.49}$$

uniformly for $\omega \in S_1$. Let

$$r(t) = \omega \cdot q(t), \quad s = \omega \cdot y, \quad \widetilde{\rho}(q_3) = \int dq_1\, dq_2\, \rho(q_1, q_2, q_3)$$

and decompose the y-integration in (2.48) into integration along ω and transversal to it. Then we obtain the convolution

$$\begin{aligned}
\overline{\pi}(\omega, t) &= \int ds\, \widetilde{\rho}(s - r(t + s))\, \frac{\ddot{r}(t + s)}{(1 - \dot{r}(t + s))^2} \\
&= \int d\tau\, \widetilde{\rho}(t - (\tau - r(\tau)))\, \frac{\ddot{r}(\tau)}{(1 - \dot{r}(\tau))^2} = \int d\theta\, \widetilde{\rho}(t - \theta) g_\omega(\theta) = \widetilde{\rho} * g_\omega(t).
\end{aligned}$$

Here $\theta = \theta(\tau) = \tau - r(\tau)$ is a non-degenerate diffeomorphism of $\mathbb{R}$ since $\dot{r} \leqslant \overline{r} < 1$ due to (2.32), and

$$g_\omega(\theta) = \frac{\ddot{r}(\tau(\theta))}{(1 - \dot{r}(\tau(\theta)))^3}\,. \tag{2.50}$$

Let us extend $q(t)$ to be 0 for $t < 0$. Then $\widetilde{\rho} * g_\omega\,(t)$ is defined for all t and coincides with $\overline{\pi}(\omega, t)$ for sufficiently large t. Hence (2.49) is the convolution limit

$$\lim_{t\to\infty} \widetilde{\rho} * g_\omega(t) = 0. \tag{2.51}$$

Moreover, $g'_\omega(\theta)$ is bounded in view of (2.32). Therefore, (2.51) and the Wiener condition (2.34) imply that

$$\lim_{\theta\to\infty} g_\omega(\theta) = 0, \qquad \omega \in S_1, \tag{2.52}$$

by Pitt's extension of Wiener's Tauberian theorem ([185], Theorem 9.7, (b)). Hence (2.50) implies that

$$\lim_{t\to\infty} \ddot{q}(t) = 0 \tag{2.53}$$

since $\theta(t) \to \infty$ as $t \to \infty$. Finally,

$$\lim_{t\to\infty} \dot{q}(t) = 0, \tag{2.54}$$

because $|q(t)| \leqslant \overline{q}_0$ in view of (2.32).

Remark 2.12. (i) We used the condition (2.28) in the proof of (2.46), but (2.30) is also sufficient at this point. Consequently, the relaxation (2.53) holds also under the condition (2.30).

(ii) For a point charge $\rho(x) = \delta(x)$, (2.51) implies (2.52) directly.

(iii) The condition (2.34) is necessary for the implication (2.52) $\Rightarrow$ (2.53). Indeed, if (2.34) is violated, then $\widehat{\rho}_a(\xi) = 0$ for some $\xi \in \mathbb{R}$, and with the choice $g(\theta) = \exp(i\xi\theta)$ we have $\rho_a * g(t) \equiv 0$, whereas g does not tend to zero.

2.3.5. *A compact attracting set.* Here we show that the set

$$\mathscr{A} = \{S_q \colon q \in \mathbb{R}^3,\ |q| \leqslant \overline{q}_0\} \tag{2.55}$$

is an attracting subset. It is compact in $\mathscr{E}_F$ since $\mathscr{A}$ is homeomorphic to a closed ball in $\mathbb{R}^3$.

Lemma 2.13. *The following attraction holds*:

$$Y(t) \xrightarrow{\mathscr{E}_F} \mathscr{A}, \qquad t \to \pm\infty. \tag{2.56}$$

Proof. We need to check that for every $R > 0$

$$\begin{aligned}\operatorname{dist}_R(Y(t), \mathscr{A}) &= |p(t)| + \|\pi(t)\|_R \\ &+ \inf_{S_q\in\mathscr{A}} \big(|q(t) - q| + \|\psi(t) - \psi_q\|_R + \|\nabla(\psi(t) - \psi_q)\|_R\big) \to 0, \qquad t \to \infty.\end{aligned} \tag{2.57}$$

We estimate each summand separately.

(i) $|p(t)| \to 0$ as $t \to \infty$ by (2.53).

(ii) $\inf_{|q|\leqslant\overline{q}_0} |q(t) - q| = 0$ for any $t \in \mathbb{R}$ by (2.32).

(iii) (2.39) implies that

$$|\pi_r(x,t)| \leqslant C \max_{t-R-T_r\leqslant\tau\leqslant t} |\dot{q}(\tau)| \int_{|y|<T_r} d^3y\, \frac{1}{|x-y|} |\nabla\rho(y - q(t - |x-y|))|$$

for $t > R + T_r$ and $|x| < R$. The integral on the right-hand side is bounded uniformly for $t > R + T_r$ and $x \in B_R$. Hence $\|\pi_r(t)\|_R \to 0$ as $t \to \infty$ by (2.54). Then also $\|\pi(t)\|_R \to 0$.

(iv) Obviously, we can replace q by $q(t)$ in the last summand in (2.57). Then for $t > R + T_r$ and $|x| < R$ one has

$$\psi_r(x,t) - \psi_{q(t)}(x) = -\int_{|y|<T_r} d^3y\, \frac{1}{4\pi|x-y|}\big[\rho(y - q(t-|x-y|)) - \rho(y-q(t))\big]$$

by (2.39). Moreover, $\rho(y - q(t-|x-y|)) - \rho(y-q(t)) \to 0$ as $t \to \infty$ uniformly for $x \in B_R$ in view of (2.54). Hence $\|\psi_r(t) - \psi_{q(t)}\|_R \to 0$ as $t \to \infty$. Then also $\|\psi(t) - \psi_{q(t)}\|_R \to 0$. Finally, $\|\nabla(\psi(t) - \psi_{q(t)})\|_R$ can be estimated similarly. □

2.3.6. *Global attraction.* Now we complete the proof of Theorem 2.8.

(i) Let $Y(t) \in C(\mathbb{R}, \mathscr{E})$ be any finite-energy solution of the system (2.22), (2.23). If the attraction (2.36) does not hold, then there is a sequence $t_k \to \infty$ for which

$$\operatorname{dist}(Y(t_k), \mathscr{S}) \geqslant \delta > 0, \qquad k = 1, 2, \dots . \tag{2.58}$$

Since $\mathscr{A}$ is a compact set in $\mathscr{E}_F$, (2.56) implies that

$$Y(t_{k'}) \xrightarrow{\mathscr{E}_F} \overline{Y} \in \mathscr{A}, \qquad k' \to \infty, \tag{2.59}$$

for some subsequence $k' \to \infty$. It remains to check that $\overline{Y} = S_{q_*} \in \mathscr{S}$ with some $q_* \in Z$, since this contradicts (2.58).

First, $\overline{Y} = S_q$ with some $|q| \leqslant \overline{q}_0$ by the definition (2.55). Similarly, by the continuity of the map W_t in $\mathscr{E}_F$,

$$W_t Y(t_{k'}) = Y(t_{k'} + t) \xrightarrow{\mathscr{E}_F} W_t \overline{Y} = S_{Q(t)}, \qquad k' \to \infty, \tag{2.60}$$

where $Q(\,\cdot\,) \in C^2(\mathbb{R}, \mathscr{E})$, since $W_t\overline{Y} \in C(\mathbb{R}, \mathscr{E})$ is a solution of the system (2.22), (2.23). Finally, for $S_{Q(t)}$ to be a solution of the system (2.22), (2.23), it is necessary that $\dot{Q}(t) \equiv 0$. Therefore, $Q(t) \equiv q_* \in Z$ and $\overline{Y} = S_{q_*} \in \mathscr{S}$.

(ii) If the set Z is discrete in $\mathbb{R}^3$, then the solitary manifold $\mathscr{S}$ is discrete in $\mathscr{E}_F$.

Theorem 2.8 is proved.

2.4. The Maxwell–Lorentz equations: radiation damping. In [129] global attraction to stationary states that is analogous to (2.36), (2.37) was extended to the Maxwell–Lorentz equations with a charged relativistic particle:

$$\begin{gathered}
\dot{E}(x,t) = \operatorname{rot} B(x,t) - \dot{q}\rho(x-q), \quad \dot{B}(x,t) = -\operatorname{rot} E(x,t),\\
\operatorname{div} E(x,t) = \rho(x-q), \quad \operatorname{div} B(x,t) = 0,\\
\dot{q}(t) = \frac{p(t)}{\sqrt{1+p^2(t)}},\\
\dot{p}(t) = \int \big[E(x,t) + E^{\mathrm{ext}}(x,t) + \dot{q}(t) \wedge (B(x,t) + B^{\mathrm{ext}}(x,t))\big]\rho(x-q(t))\,dx.
\end{gathered} \tag{2.61}$$

Here $\rho(x-q)$ is the particle charge density, $\dot{q}\rho(x-q)$ is the corresponding current density, and $E^{\mathrm{ext}} = -\nabla\phi^{\mathrm{ext}}(x)$ and $B^{\mathrm{ext}} = -\operatorname{rot} A^{\mathrm{ext}}(x)$ are the external static Maxwell fields. Similarly to (2.28), we assume that the *effective scalar potential* is confining:

$$V(q) := \int \phi^{\mathrm{ext}}(x)\rho(x-q)\,dx \to \infty, \qquad |q| \to \infty. \tag{2.62}$$

This system describes classical electrodynamics with the 'extended electron' introduced by Abraham [1], [2]. In the case of a point electron, when $\rho(x) = \delta(x)$, such a system is not well defined. Indeed, in this case any solutions $E(x,t)$ and $B(x,t)$ of Maxwell's equations (the first lines in (2.61)) are singular for $x = q(t)$, and, accordingly, the integral in the last equation in (2.61) does not exist.

This system may be formally represented in the Hamiltonian form if the fields are expressed in terms of the potentials: $E(x,t) = -\nabla\phi(x,t) - \dot{A}(x,t)$ and $B(x,t) = -\operatorname{rot} A(x,t)$ [76]. The corresponding Hamiltonian functional is

$$\begin{aligned}\mathscr{H} &= \frac{1}{2}[\langle E,E\rangle + \langle B,B\rangle] + V(q) + \sqrt{1+p^2}\\ &= \frac{1}{2}\int [E^2(x) + B^2(x)]\,dx + V(q) + \sqrt{1+p^2}\,. \end{aligned} \tag{2.63}$$

The Hilbert phase space of finite-energy states is defined as $\mathscr{E} := L^2 \oplus L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$. Under the condition (2.62) a finite-energy solution

$$Y(t) = (E(x,t), B(x,t), q(t), p(t)) \in C(\mathbb{R}, \mathscr{E})$$

exists and is unique for any initial state $Y(0) \in \mathscr{E}$.

The Hamiltonian (2.63) is conserved along solutions, and this provides *a priori estimates* that play an important role in proving attraction of the type (2.36), (2.37) in [129]. The key role in the proof of relaxation of acceleration is again played by (2.38), which is derived by a suitable generalization of our methods [130]: the expression of the energy radiated to infinity via Liénard–Wiechert retarded potentials, its representation in the form of a convolution, and the use of Wiener's Tauberian theorem.

In classical electrodynamics the relaxation (2.38) is known as *radiation damping*. It is traditionally derived from the Larmor and Liénard formulae for the radiation power of a point particle (see the formulae (14.22) and (14.24) in [83]), but this approach ignores the field feedback even though it plays the key role in relaxation. The main problem is that this reverse field reaction for point particles is infinite. The rigorous meaning of these classical calculations was first found in [130] and [129] for the Abraham model of an 'extended electron' under the Wiener condition (2.34). Details can be found in [195].

2.5. The wave equation with concentrated nonlinearities. Here we prove the result in [136] on global attraction to the solitary manifold for the 3D wave equation with point coupling to an U(1)-invariant nonlinear oscillator. This goal is inspired by the fundamental mathematical problem of interaction of point particles with fields.

Point interaction models were first considered beginning in 1933 in the papers of Wigner, Bethe and Peierls, Fermi, and others (see [6] for a detailed survey), and also of Dirac [40]. Rigorous mathematical results were obtained beginning in 1960 by Zeldovich, Berezin, Faddeev, Cornish, Yafaev, Zeidler, and others ([19], [32], [59], [209], [212]), and since 2000 by Noja, Posilicano, and others ([176], [210], [4]).

We consider a real wave field $\psi(x,t)$ coupled to a nonlinear oscillator

$$\begin{cases} \ddot{\psi}(x,t) = \Delta\psi(x,t) + \zeta(t)\delta(x), \\ \lim\limits_{x\to 0}(\psi(x,t) - \zeta(t)G(x)) = F(\zeta(t)), \end{cases} \qquad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \tag{2.64}$$

where $G(x) = 1/(4\pi|x|)$ is the Green's function of the operator $-\Delta$ in $\mathbb{R}^3$. The nonlinear function $F(\zeta)$ admits a potential:

$$F(\zeta) = U'(\zeta), \qquad \zeta \in \mathbb{R}, \quad U \in C^2(\mathbb{R}). \tag{2.65}$$

We assume that the potential is confining, that is,

$$U(\zeta) \to \infty, \qquad \zeta \to \pm\infty. \tag{2.66}$$

The system (2.64) has stationary solutions $\psi_q = qG(x) \in L^2_{\rm loc}(\mathbb{R}^3)$, where $q \in Q := \{q \in \mathbb{R}\colon F(q) = 0\}$. We assume that the set Q is non-empty and does not contain intervals, that is,

$$[a, b] \not\subset Q \tag{2.67}$$

for any $a < b$.

As before, $\|\cdot\|$ and $\|\cdot\|_R$ denote the norms in $L^2 = L^2(\mathbb{R}^3)$ and $L^2(B_R)$, respectively, and $\mathring{H}^1 = \mathring{H}^1(\mathbb{R}^3)$ is the completion of the space $C_0^\infty(\mathbb{R}^3)$ in the norm $\|\nabla\psi(x)\|$. Let

$$\mathring{H}^2 = \mathring{H}^2(\mathbb{R}^3) := \{f \in \mathring{H}^1\colon \Delta f \in L^2\}, \qquad t \in \mathbb{R}.$$

We define the sets of functions

$$D = \Big\{\psi \in L^2\colon \psi(x) = \psi_{\rm reg}(x) + \zeta G(x),\ \psi_{\rm reg} \in \mathring{H}^2,\ \zeta \in \mathbb{R},\ \lim_{x\to 0}\psi_{\rm reg}(x) = F(\zeta)\Big\}$$

and

$$\dot{D} = \{\pi \in L^2\colon \pi(x) = \pi_{\rm reg}(x) + \eta G(x),\ \pi_{\rm reg} \in \mathring{H}^1,\ \eta \in \mathbb{R}\}.$$

Obviously, $D \subset \dot{D}$.

Definition 2.14. $\mathscr{D}$ is the Hilbert manifold of states $\Psi = (\psi, \pi) \in D \times \dot{D}$.

First we prove global well-posedness for the system (2.64).

Theorem 2.15. *Assume the conditions* (2.65) *and* (2.66). *Then the following assertions hold.*

(i) *For arbitrary initial data* $\Psi_0 = (\psi_0, \pi_0) \in \mathscr{D}$ *the system* (2.64) *has a unique solution* $\Psi(t) = (\psi(t), \dot\psi(t)) \in C(\mathbb{R}, \mathscr{D})$.

(ii) *The energy is conserved*:

$$\mathscr{H}(\Psi(t)) := \frac{1}{2}\big(\|\dot\psi(t)\|^2 + \|\nabla\psi_{\rm reg}(t)\|^2\big) + U(\zeta(t)) = {\rm const}, \qquad t \in \mathbb{R}. \tag{2.68}$$

(iii) *There is an a priori bound*

$$|\zeta(t)| \leqslant C(\Psi_0), \qquad t \in \mathbb{R}. \tag{2.69}$$

Proof. It suffices to prove the theorem for $t \geqslant 0$.

Step (i). First we consider the free wave equation with initial data in $\mathscr{D}$:

$$\ddot\psi_f(x,t) = \Delta\psi_f(x,t), \quad (\psi_f(0), \dot\psi_f(0)) = (\psi_0, \pi_0) = (\psi_{0,\rm reg}, \pi_{0,\rm reg}) + (\zeta_0 G, \eta_0 G) \in \mathscr{D}, \tag{2.70}$$

where $(\psi_{0,\rm reg}, \pi_{0,\rm reg}) \in \mathring{H}^2 \oplus \mathring{H}^1$.

Lemma 2.16. *There exists a unique solution* $\psi_f(t) \in C([0;\infty), L^2_{\mathrm{loc}})$ *of the problem* (2.70). *Moreover, for any* $t > 0$ *there exists the limit*

$$\lambda(t) := \lim_{x\to 0} \psi_f(x,t) \in C[0,\infty),$$

and

$$\dot\lambda(t) \in L^2_{\mathrm{loc}}[0,\infty). \tag{2.71}$$

Proof. We split $\psi_f(x,t)$ into two terms

$$\psi_f(x,t) = \psi_{f,\mathrm{reg}}(x,t) + g(x,t),$$

where $\psi_{f,\mathrm{reg}}$ and g solve the free wave equation with initial data $(\psi_{0,\mathrm{reg}}, \pi_{0,\mathrm{reg}})$ and $(\zeta_0 G, \eta_0 G)$, respectively. We note that

$$\psi_{f,\mathrm{reg}} \in C([0,\infty), \mathring{H}^2)$$

by energy conservation. Hence, the limit $\lim_{x\to 0} \psi_{f,\mathrm{reg}}(x,t)$ exists for any $t \geqslant 0$ since $\mathring{H}^2(\mathbb{R}^3) \subset C(\mathbb{R}^3)$.

Let us obtain an explicit formula for g. Note that the function $h(x,t) = g(x,t) - (\zeta_0 + \eta_0 t)G(x)$ satisfies the equations

$$\ddot h(x,t) = \Delta h(x,t) - (\zeta_0 + \eta_0 t)\delta(x), \qquad h(x,0) = 0, \quad \dot h(x,0) = 0. \tag{2.72}$$

The unique solution of this Cauchy problem is the spherical wave

$$h(x,t) = -\frac{\theta(t-|x|)}{4\pi|x|}(\zeta_0 + \eta_0(t-|x|)), \qquad t \geqslant 0. \tag{2.73}$$

Here θ is the Heaviside function. Hence,

$$\begin{aligned} g(x,t) &= h(x,t) + (\zeta_0 + \eta_0 t)G(x) \\ &= -\frac{\theta(t-|x|)(\zeta_0 + \eta_0(t-|x|))}{4\pi|x|} + \frac{\zeta_0 + \eta_0 t}{4\pi|x|} \in C([0,\infty), L^2_{\mathrm{loc}}(\mathbb{R}^3)), \end{aligned}$$

and then

$$\lim_{x\to 0} g(x,t) = \frac{\eta_0}{4\pi}, \qquad t > 0.$$

Finally, $\dot\psi_{f,\mathrm{reg}}(0,t) \in L^2_{\mathrm{loc}}[0,\infty)$ by [176], Lemma 3.4. Then (2.71) follows. □

Step (ii). Now we prove local well-posedness. We modify the nonlinearity F so that it becomes globally Lipschitz-continuous. Define

$$\Lambda(\Psi_0) = \sup\{|\zeta| \colon \zeta \in \mathbb{R}, U(\zeta) \leqslant \mathscr{H}(\Psi_0)\}.$$

We may pick a modified potential function $\widetilde U(\zeta) \in C^2(\mathbb{R})$ so that

$$\begin{cases} \widetilde U(\zeta) = U(\zeta), & |\zeta| \leqslant \Lambda(\Psi_0), \\ \widetilde U(\zeta) > \mathscr{H}(\Psi_0), & |\zeta| > \Lambda(\Psi_0), \end{cases} \tag{2.74}$$

and the function $\widetilde F(\zeta) = \widetilde U'(\zeta)$ is Lipschitz-continuous:

$$|\widetilde F(\zeta_1) - \widetilde F(\zeta_2)| \leqslant C|\zeta_1 - \zeta_2|, \qquad \zeta_1, \zeta_2 \in \mathbb{R}.$$

The following lemma is trivial.

Lemma 2.17. *For small $\tau > 0$ the Cauchy problem*

$$\frac{1}{4\pi}\dot\zeta(t) + \widetilde F(\zeta(t)) = \lambda(t), \qquad \zeta(0) = \zeta_0, \tag{2.75}$$

has a unique solution $\zeta \in C^1[0,\tau]$.

Let

$$\psi_S(t,x) := \frac{\theta(t-|x|)}{4\pi|x|}\zeta(t-|x|), \qquad t \in [0,\tau],$$

with ζ in Lemma 2.17.

Lemma 2.18. *The function $\psi(x,t) := \psi_f(x,t) + \psi_S(x,t)$ is the unique solution of the system*

$$\begin{cases} \ddot\psi(x,t) = \Delta\psi(x,t) + \zeta(t)\delta(x), \\ \lim\limits_{x\to 0}(\psi(x,t) - \zeta(t)G(x)) = \widetilde F(\zeta(t)), \qquad x \in \mathbb{R}^3, \quad t \in [0,\tau], \\ \psi(x,0) = \psi_0(x), \quad \dot\psi(x,0) = \pi_0(x), \end{cases} \tag{2.76}$$

satisfying the condition

$$(\psi(t), \dot\psi(t)) \in \mathscr{D}, \qquad t \in [0,\tau]. \tag{2.77}$$

Proof. The initial conditions in (2.76) follow from (2.70). Further,

$$\begin{aligned} \lim_{x\to 0}(\psi(t,x) - \zeta(t)G(x)) &= \lambda(t) + \lim_{x\to 0}\left(\frac{\theta(t-|x|)\zeta(t-|x|)}{4\pi|x|} - \frac{\zeta(t)}{4\pi|x|}\right) \\ &= \lambda(t) - \frac{1}{4\pi}\dot\zeta(t) = \widetilde F(\zeta(t)). \end{aligned}$$

Thus, the second equation in (2.76) is satisfied. Finally,

$$\ddot\psi = \ddot\psi_f + \ddot\psi_S = \Delta\psi_f + \Delta\psi_S + \zeta\delta = \Delta\psi + \zeta\delta,$$

and then ψ solves the first equation in (2.76).

It remains to check (2.77). The function $\varphi_{\rm reg}(x,t) = \psi(x,t) - \zeta(t)G_1(x) = \psi_{\rm reg}(x,t) + \zeta(t)(G(x) - G_1(x))$, where $G_1(x) = G(x)e^{-|x|}$, satisfies the equation

$$\ddot\varphi_{\rm reg}(x,t) = \Delta\varphi_{\rm reg}(x,t) + (\zeta(t) - \ddot\zeta(t))G_1(x)$$

with initial data in $H^2 \oplus H^1$. Moreover, (2.71) and (2.75) imply that $\ddot\zeta \in L^2[0,\tau]$. Consequently,

$$(\varphi_{\rm reg}(x,t), \dot\varphi_{\rm reg}(x,t)) \in H^2 \oplus H^1, \qquad t \in [0,\tau],$$

by Lemma 3.2 in [176]. Therefore, the function

$$\psi_{\rm reg}(x,t) = \psi(x,t) - \zeta(t)G(x) = \varphi_{\rm reg}(x,t) + \zeta(t)(G_1(x) - G(x))$$

satisfies $(\psi_{\rm reg}(t), \dot\psi_{\rm reg}(t)) \in \mathring H^2 \oplus \mathring H^1$, $t \in [0,\tau]$, and then (2.77) holds.

It remains to prove uniqueness. Suppose now that there exists another solution $\tilde{\psi} = \tilde{\psi}_{\mathrm{reg}} + \tilde{\zeta}G$ of the system (2.76) with $(\tilde{\psi}, \dot{\tilde{\psi}}) \in \mathscr{D}$. Then by a reversal of the above argument, the second equation in (2.76) implies that $\tilde{\zeta}$ solves the Cauchy problem (2.75). The uniqueness of the solution of (2.75) implies that $\tilde{\zeta} = \zeta$. Then, defining

$$\psi_S(t,x) := \frac{\theta(t-|x|)}{4\pi|x|}\zeta(t-|x|), \qquad t \in [0,\tau],$$

we get for the difference $\tilde{\psi}_f = \tilde{\psi} - \psi_S$ that

$$\ddot{\tilde{\psi}}_f = \ddot{\tilde{\psi}} - \ddot{\psi}_S = \Delta\tilde{\psi}_{\mathrm{reg}} - (\Delta\psi_S + \zeta\delta) = \Delta(\tilde{\psi}_{\mathrm{reg}} - (\psi_S - \zeta G)) = \Delta\tilde{\psi}_f,$$

that is, $\tilde{\psi}_f$ solves the Cauchy problem (2.70). Hence, $\tilde{\psi}_f = \psi_f$ by the uniqueness of the solution of (2.70), and thus $\tilde{\psi} = \psi$. □

Step (iii). We are now able to prove global well-posedness. According to [176], Lemma 3.7,

$$\mathscr{H}_{\widetilde{F}}(\Psi(t)) = \|\dot{\psi}(t)\|^2 + \|\nabla\psi_{\mathrm{reg}}(t)\|^2 + \widetilde{U}(\zeta(t)) = \mathrm{const}, \qquad t \in [0,\tau]. \tag{2.78}$$

First, note that

$$\widetilde{U}(\zeta(t)) = U(\zeta(t)), \qquad t \in [0,\tau]. \tag{2.79}$$

Indeed, $\mathscr{H}_F(\Psi_0) \geqslant U(\zeta_0)$ by the definition of the energy in (2.68). Therefore, $|\zeta_0| \leqslant \Lambda(\Psi_0)$, and then $\widetilde{U}(\zeta_0) = U(\zeta_0)$ and $\mathscr{H}_{\widetilde{F}}(\Psi_0) = \mathscr{H}_F(\Psi_0)$. Further,

$$\mathscr{H}_F(\Psi_0) = \mathscr{H}_{\widetilde{F}}(\Psi(t)) \geqslant \widetilde{U}(\zeta(t)), \qquad t \in [0,\tau],$$

and (2.74) implies that

$$|\zeta(t)| \leqslant \Lambda(\Psi_0), \qquad t \in [0,\tau]. \tag{2.80}$$

Now we can replace $\widetilde{F}$ by F in Lemma 2.18 and in (2.78). The solution $\Psi(t) = (\psi(t), \dot{\psi}(t)) \in \mathscr{D}$ constructed in Lemma 2.18 exists for $0 \leqslant t \leqslant \tau$, where the time τ in Lemma 2.17 depends only on $\Lambda(\Psi_0)$. Hence, the estimate (2.80) at $t = \tau$ allows us to extend the solution Ψ to the time interval $[\tau, 2\tau]$. We proceed by induction to obtain a solution for all $t \geqslant 0$. Theorem 2.15 is proved. □

The main result in [136] is as follows.

Theorem 2.19. *Let $\Psi(x,t) = (\psi(x,t), \dot{\psi}(x,t))$ be a solution of (2.64) with initial data in $\mathscr{D}$. Then*

$$\Psi(x,t) \to (\psi_{q_\pm}, 0), \qquad t \to \pm\infty,$$

where $q_\pm \in Q$ and the convergence holds in $L^2_{\mathrm{loc}}(\mathbb{R}^3) \oplus L^2_{\mathrm{loc}}(\mathbb{R}^3)$.

Proof. It suffices to prove this theorem only for $t \to +\infty$. By Lemma 2.18, the solution $\psi(x,t)$ of (2.64) with initial data $(\psi_0, \pi_0) \in \mathscr{D}$ can be represented as the sum

$$\psi(x,t) := \psi_f(x,t) + \psi_S(x,t), \qquad t \geqslant 0, \tag{2.81}$$

where the *dispersion component* $\psi_f(x,t)$ is the unique solution of (2.70) and the *singular component* $\psi_S(x,t)$ is the unique solution of the Cauchy problem

$$\ddot{\psi}_S(x,t) = \Delta\psi_S(x,t) + \zeta(t)\delta(x), \qquad \psi_S(x,0) = 0, \quad \dot{\psi}_S(x,0) = 0. \tag{2.82}$$

Here $\zeta(t) \in C_b^1[0,\infty)$ is the unique solution of the Cauchy problem

$$\frac{1}{4\pi}\dot{\zeta}(t) + F(\zeta(t)) = \lambda(t), \qquad \zeta(0) = \zeta_0. \tag{2.83}$$

We can now prove the local decay of $\psi_f(x,t)$.

Lemma 2.20. *For any $R > 0$,*

$$\|(\psi_f(t), \dot{\psi}_f(t))\|_{H^2(B_R)\oplus H^1(B_R)} \to 0, \qquad t \to \infty, \tag{2.84}$$

where B_R is the ball of radius R.

Proof. We represent the initial data

$$(\psi_0, \pi_0) = (\psi_{0,\mathrm{reg}}, \pi_{0,\mathrm{reg}}) + (\zeta_0 G, \eta_0 G) \in \mathscr{D}$$

as the sum

$$(\psi_0, \pi_0) = (\varphi_0, p_0) + (\zeta_0\chi G, \eta_0\chi G),$$

where the cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$ satisfies

$$\chi(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2. \end{cases} \tag{2.85}$$

Let us show that

$$(\varphi_0, p_0) \in H^2 \oplus H^1.$$

Indeed,

$$(\varphi_0, p_0) = (\psi_0 - \zeta_0\chi G, \pi_0 - \eta_0\chi G) \in L^2 \oplus L^2.$$

On the other hand,

$$(\varphi_0, p_0) = (\psi_{0,\mathrm{reg}} + \zeta_0(1-\chi)G, \pi_{0,\mathrm{reg}} + \eta_0(1-\chi)G) \in \mathring{H}^2 \oplus \mathring{H}^1.$$

Now we split the dispersion component $\psi_f(x,t)$ into the two terms

$$\psi_f(x,t) = \varphi(x,t) + \varphi_G(x,t), \qquad t \geqslant 0,$$

where φ and φ_G are defined as the solutions of the free wave equation with the initial data (φ_0, p_0) and $(\zeta_0\chi G, \eta_0\chi G)$, respectively, and we study the decay properties of φ_G and φ.

First, by the strong Huygens principle,

$$\varphi_G(x,t) = 0 \quad \text{for } t \geqslant |x| + 2.$$

Indeed, $\varphi_G(x,t) = \zeta_0\dot{\psi}_G(x,t) + \eta_0\psi_G(x,t)$, where $\psi_G(x,t)$ is the solution of the free wave equation with initial data $(0, \chi G) \in H^1 \oplus L^2$, and $\psi_G(x,t)$ satisfies the strong Huygens principle by [180], Theorem XI.87.

It remains to check that

$$\|(\varphi(t),\dot\varphi(t))\|_{H^2(B_R)\oplus H^1(B_R)}\to 0,\qquad t\to\infty,\quad \forall\, R>0. \tag{2.86}$$

For $r\geqslant 1$ let $\chi_r=\chi(x/r)$, where $\chi(x)$ is the cut-off function (2.85), and let $\phi_0=(\varphi_0,\pi_0)$. Let $u_r(t)$ and $v_r(t)$ be solutions of the free wave equations with the initial data $\chi_r\phi_0$ and $(1-\chi_r)\phi_0$, respectively, so that $\varphi(t)=u_r(t)+v_r(t)$. By the strong Huygens principle,

$$u_r(x,t)=0\quad \text{for } t\geqslant |x|+2r.$$

To obtain (2.86), it remains to note that

$$\begin{aligned}\|(v_r(t),\dot v_r(t))\|_{H^2(B_R)\oplus H^1(B_R)}&\leqslant C(R)\|(v_r(t),\dot v_r(t))\|_{\mathring H^2\oplus H^1}\\ &=C(R)\|(1-\chi_r)\phi_0\|_{\mathring H^2\oplus H^1}\\ &\leqslant C(R)\|(1-\chi_r)\phi_0\|_{H^2\oplus H^1}\end{aligned} \tag{2.87}$$

by the energy conservation for the free wave equation. We also use the embedding $\mathring H^1(\mathbb{R}^3)\subset L^6(\mathbb{R}^3)$. The right-hand side of (2.87) can be made arbitrarily small if $r\geqslant 1$ is sufficiently large. □

By (2.81) and (2.84), to prove Theorem 2.19 it suffices to verify the convergence of $\psi_S(x,t)$ to stationary states.

Lemma 2.21. *Let $\psi_S(x,t)$ and $\zeta(t)$ be solutions of (2.82) and (2.83), respectively. Then*

$$(\psi_S(t),\dot\psi_S(t))\to(\psi_{q_\pm},0),\qquad t\to\infty,$$

where $q_\pm\in Q$ and the convergence holds in $L^2_{\mathrm{loc}}(\mathbb{R}^3)\oplus L^2_{\mathrm{loc}}(\mathbb{R}^3)$.

Proof. The unique solution of (2.82) is the spherical wave

$$\psi_S(x,t)=\frac{\theta(t-|x|)}{4\pi|x|}\zeta(t-|x|),\qquad t\geqslant 0 \tag{2.88}$$

(cf. (2.72) and (2.73)). Then the a priori bound (2.69) and equation (2.83) imply that

$$(\psi_S(t),\dot\psi_S(t))\in L^2(B_R)\oplus L^2(B_R),\qquad 0\leqslant R<t.$$

First we prove the convergence of $\zeta(t)$. From (2.69) it follows that $\zeta(t)$ has upper and lower limits:

$$\varliminf_{t\to\infty}\zeta(t)=a\quad\text{and}\quad\varlimsup_{t\to\infty}\zeta(t)=b. \tag{2.89}$$

Suppose that $a<b$. Then the trajectory $\zeta(t)$ oscillates between a and b. The assumption (2.67) implies that $F(\zeta_0)\neq 0$ for some $\zeta_0\in(a,b)$. Assume for definiteness that $F(\zeta_0)>0$. The convergence (2.84) implies that

$$\lambda(t)=\psi_f(0,t)\to 0,\qquad t\to\infty. \tag{2.90}$$

Hence, for sufficiently large T we have

$$-F(\zeta_0)+\lambda(t)<0,\qquad t\geqslant T.$$

Then for $t \geqslant T$ the transition of the trajectory from left to right through the point ζ_0 is impossible by (2.83). Therefore, $a = b = q_+$, where $q_+ \in Q$ since $F(q_+) = 0$ by (2.83). Hence (2.89) implies that

$$\zeta(t) \to q_+, \qquad t \to \infty. \tag{2.91}$$

Further,

$$\theta(t - |x|) \to 1, \qquad t \to \infty, \tag{2.92}$$

uniformly for $|x| \leqslant R$. Then (2.88) and (2.91) imply that

$$\psi_S(t) \to q_+ G, \qquad t \to \infty,$$

where the convergence holds in $L^2_{\rm loc}(\mathbb{R}^3)$. It remains to verify the convergence of $\dot\psi_S(t)$. Differentiating (2.88), we have

$$\dot\psi_S(x,t) = \frac{\theta(t-|x|)}{4\pi|x|}\dot\zeta(t-|x|), \qquad |x| < t.$$

From (2.83), (2.90), and (2.91) it follows that $\dot\zeta(t) \to 0$ as $t \to \infty$. Then

$$\dot\psi_S(t) \to 0, \qquad t \to \infty,$$

in $L^2_{\rm loc}(\mathbb{R}^3)$ by (2.92). □

This completes the proof of Theorem 2.19. □

2.6. Remarks. All the above results on global attraction to stationary states refer to 'generic' systems with trivial symmetry group. These systems are characterized by a suitable discreteness of attractors, by the Wiener condition, and so on.

The global attraction to stationary states (1.5) resembles the analogous asymptotics (1.1) for dissipative systems. However, there are a number of fundamental differences.

I. In dissipative systems an attractor always consists of *stationary states*, the attraction (1.1) *holds only as* $t \to +\infty$, and this attraction is associated with absorption of energy and can be in global norms. Such an attraction also *holds for all finite-dimensional dissipative systems.*

II. On the other hand, in Hamiltonian systems an *attractor may differ from the set of stationary states*, as will be seen below. In addition, energy absorption in these systems is absent, and the attraction (1.5) to stationary states is due to *radiation of energy to infinity*, which plays the role of energy absorption. This attraction takes place both *as* $t \to \infty$, *and as* $t \to -\infty$, and it holds *only in local seminorms.* Finally, it *cannot hold for any finite-dimensional Hamiltonian system* (except in the case when the Hamiltonian is identically constant).

3. Global attraction to solitons

As already mentioned in the Introduction, soliton asymptotic expressions (1.7) with several solitons were first discovered numerically in 1965 by Kruskal and

Zabusky for KdV. Such asymptotics were later proved by Ablowitz, Segur, Eckhaus, van Harten, and others using the method of the *inverse scattering problem* for nonlinear translation-invariant *integrable* Hamiltonian equations (see [46]).

Here we present results on global attraction to one soliton (1.6) for nonlinear translation-invariant *non-integrable* Hamiltonian equations. Such an attraction was first proved in [128] and in [76] for a charged relativistic particle coupled to a scalar wave field and to the Maxwell field, respectively.

3.1. A translation-invariant 'wave-particle' system. In [128] the system (2.22), (2.23) was considered in the case of zero potential $V(x) \equiv 0$:

$$\begin{aligned} &\ddot{\psi}(x,t) = \Delta\psi(x,t) - \rho(x-q), \qquad x \in \mathbb{R}^3; \\ &\dot{q} = \frac{p}{\sqrt{1+p^2}}\,, \qquad \dot{p} = -\int \nabla\psi(x,t)\rho(x-q)\,dx. \end{aligned} \tag{3.1}$$

This system can be written in the Hamiltonian form (2.24). Its Hamiltonian is given by (2.25) with $V = 0$, and it is conserved along trajectories. By Lemma 2.5 with $V(x) \equiv 0$, global solutions exist for all initial data $Y(0) \in \mathscr{E}$, and there are a priori estimates (2.32).

The system is translation-invariant, so the corresponding total momentum

$$P = p - \int \pi(x)\nabla\psi(x)\,dx \tag{3.2}$$

is also conserved. Correspondingly, the system (3.1) admits traveling-wave type solutions (solitons)

$$\psi_v(x-a-vt), \qquad q(t) = a + vt, \quad p_v = \frac{v}{\sqrt{1-v^2}}\,, \tag{3.3}$$

where $v, a \in \mathbb{R}^3$ and $|v| < 1$. These functions are easily determined: for $|v| < 1$ there is a unique function ψ_v which makes (3.3) a solution of (3.1),

$$\psi_v(x) = -\int d^3y\,(4\pi|(y-x)_\parallel + \lambda(y-x)_\perp|)^{-1}\rho(y), \tag{3.4}$$

where $\lambda = \sqrt{1-v^2}$ and $x = x_\parallel + x_\perp$, with $x_\parallel \parallel v$ and $x_\perp \perp v$ for $x \in \mathbb{R}^3$. Indeed, substituting (3.3) into the wave equation in (3.1), we get the stationary equation

$$(v\cdot\nabla)^2\psi_v(x) = \Delta\psi_v(x) - \rho(x). \tag{3.5}$$

In terms of the Fourier transform,

$$\widehat{\psi}_v(k) = -\frac{\widehat{\rho}(k)}{k^2 - (v\cdot k)^2}\,, \tag{3.6}$$

which implies (3.4). The set of all solitons forms a 6-dimensional *soliton submanifold* in the Hilbert phase space $\mathscr{E}$:

$$\mathscr{S} = \{S_{v,a} = (\psi_v(x-a), \pi_v(x-a), a, p_v)\colon v, a \in \mathbb{R}^3,\ |v| < 1\}, \tag{3.7}$$

where $\pi_v := -v\nabla\psi_v$. Recall that the spaces $\mathscr{E}$ and $\mathscr{E}_\sigma$ were defined in Definition 2.4. The following theorem is the main result in [128].

Theorem 3.1. *Let the Wiener condition* (2.34) *hold and let* $\sigma > 3/2$. *Then for any initial state* $Y(0) \in \mathscr{E}_\sigma$ *the corresponding solution* $Y(t) = (\psi(t), \pi(t), q(t), p(t))$ *of the system* (3.1) *converges to the solitary manifold* $\mathscr{S}$ *in the sense that*

$$\ddot{q}(t) \to 0, \quad \dot{q}(t) \to v_\pm, \qquad t \to \pm\infty, \tag{3.8}$$

$$(\psi(x,t), \dot{\psi}(x,t)) = (\psi_{v_\pm}(x - q(t)), \pi_{v_\pm}(x - q(t))) + (r_\pm(x,t), s_\pm(x,t)), \tag{3.9}$$

where the remainder tends to zero locally in the comoving frame: *for each* $R > 0$

$$\|\nabla r_\pm(q(t) + x, t)\|_R + \|r_\pm(q(t) + x, t)\|_R + \|s_\pm(q(t) + x, t)\|_R \to 0, \qquad t \to \pm\infty. \tag{3.10}$$

This theorem means that, in particular,

$$\psi(x,t) \sim \psi_v(x - v_\pm t + \varphi_\pm(t)), \quad \text{where} \quad \dot{\varphi}_\pm(t) \to 0, \quad t \to \pm\infty. \tag{3.11}$$

The proof in [128] is based on (a) relaxation of acceleration (2.38) in the case $V = 0$ (see Remark 2.12, (i)), and (b) a *canonical change of variables* to the comoving frame. The key role is played by the fact that the soliton $S_{v,a}$ minimizes the Hamiltonian (2.25) (in the case $V = 0$) with fixed total momentum (3.2), which implies *orbital stability of the solitons* [62], [63]. In addition, the proof essentially relies on the *strong Huygens principle* for the three-dimensional wave equation.

Before beginning a more precise and technical discussion of the proof, it may be useful to give the general idea of our strategy. As we mentioned above, the total momentum (3.2) is conserved because of translation invariance. We transform the system (3.1) to the new variables

$$(\Psi(x), \Pi(x), Q, P) = (\psi(q + x), \pi(q + x), q, P(\psi, q, \pi, p)).$$

The key role in our strategy is played by the fact that this transformation is canonical, as is proved in § 3.2. Through this canonical transformation one obtains the new Hamiltonian

$$\mathscr{H}_P(\Psi, \Pi) = \mathscr{H}(\psi, \pi, q, p) = \int d^3x \left(\frac{1}{2}|\Pi(x)|^2 + \frac{1}{2}|\nabla\Psi(x)|^2 + \Psi(x)\rho(x) \right) + \left[1 + \left(P + \int d^3x\, \Pi(x)\nabla\Psi(x) \right)^2 \right]^{1/2}.$$

Since Q is a cyclic coordinate (that is, the Hamiltonian $\mathscr{H}_P$ does not depend on Q), we may regard P as a fixed parameter and consider only the reduced system for (Ψ, Π). Let us define

$$\pi_v(x) = -v \cdot \nabla\psi_v(x), \qquad P(v) = p_v + \int d^3x\, v \cdot \nabla\psi_v(x) \nabla\psi_v(x), \quad p_v = \frac{v}{(1 - v^2)^{1/2}}. \tag{3.12}$$

We will prove that (ψ_v, π_v) is the unique critical point and global minimum of $\mathscr{H}_{P(v)}$. Thus, if the initial data are close to (ψ_v, π_v), then the corresponding solution must always remain close by conservation of energy, which means the orbital stability of the solitons. Here we follow the ideas from the paper [11] by Bambusi and Galgani,

where the orbital stability of solitons for the Maxwell–Lorentz system was first proved. For general nonlinear wave equations with symmetries such an approach to orbital stability of solitons was developed in the well-known papers [62], [63].

However, orbital stability itself is not enough. It only ensures that initial states close to a soliton remain so always; it not only fails to yield the convergence of $\dot{q}(t)$ in (3.8), but even less so the asymptotics (3.9) and (3.10). Thus, we need an additional (far from obvious) argument which combines the relaxation (2.38) with the orbital stability, in order to establish the soliton-like asymptotics (3.8)–(3.10). As one essential input we will use the strong Huygens principle for the wave equation.

3.1.1. *A canonical transformation and a reduced system.* Since the total momentum is conserved, it is natural to use P as a new coordinate. To maintain the symplectic structure we have to complete this coordinate to a canonical transformation of the Hilbert phase space $\mathscr{E}$.

Definition 3.2. Let the transformation $T\colon \mathscr{E} \to \mathscr{E}$ be defined by

$$T\colon Y = (\psi, \pi, q, p) \mapsto Y^T = (\Psi(x), \Pi(x), Q, P) = (\psi(q+x), \pi(q+x), q, P(\psi, q, \pi, p)), \tag{3.13}$$

where $P(\psi, q, \pi, p)$ is the total momentum (3.2).

Remark 3.3. (i) T is continuous on $\mathscr{E}$ and Fréchet-differentiable at the points $Y = (\psi, q, \pi, p)$ with sufficiently smooth functions $\psi(x)$ and $\pi(x)$, but it is not everywhere differentiable.

(ii) In the T-coordinates the solitons

$$Y_{v,a}(t) = (\psi_v(x - a - vt), \pi_v(x - a - vt), q = a + vt, p_v)$$

are stationary except for the coordinate Q:

$$TY_{v,a}(t) = (\psi_v(x), \pi_v(x), a + vt, P(v)), \tag{3.14}$$

with the total momentum $P(v)$ of the soliton defined in (3.12).

Let $\mathscr{H}^T(Y) = \mathscr{H}(T^{-1}Y)$ for $Y = (\Psi, \Pi, Q, P) \in \mathscr{E}$. Then

$$\begin{aligned}
\mathscr{H}^T(\Psi, \Pi, Q, P) &= \mathscr{H}_P(\Psi, \Pi) \\
&= \mathscr{H}\left(\Psi(x - Q), \Pi(x - Q), Q, P + \int d^3x\, \Pi(x)\nabla\Psi(x)\right) \\
&= \int d^3x \left[\frac{1}{2}|\Pi(x)|^2 + \frac{1}{2}|\nabla\Psi(x)|^2 + \Psi(x)\rho(x)\right] \\
&\quad + \left(1 + \left[P + \int d^3x\, \Pi(x)\nabla\Psi(x)\right]^2\right)^{1/2}.
\end{aligned}$$

The functionals $\mathscr{H}^T$ and $\mathscr{H}$ are Fréchet-differentiable on the phase space $\mathscr{E}$.

Proposition 3.4. *Let $Y(t) \in C(\mathbb{R}, \mathscr{E})$ be a solution of the system (3.1). Then*

$$Y^T(t) := TY(t) = (\Psi(t), \Pi(t), q(t), p(t)) \in C(\mathbb{R}, \mathscr{E})$$

is a solution of the Hamiltonian system

$$\begin{cases} \dot\Psi = D_\Pi \mathscr{H}^T, & \dot\Pi = -D_\psi \mathscr{H}^T, \\ \dot Q = D_P \mathscr{H}^T, & \dot P = -D_Q \mathscr{H}^T. \end{cases} \tag{3.15}$$

Proof. The equations for $\dot\Psi$, $\dot\Pi$, and $\dot Q$ can be checked by direct computation, while the one for $\dot P$ follows from conservation of the total momentum (3.2) since the Hamiltonian $\mathscr{H}^T$ does not depend on Q. □

Remark 3.5. Formally, Proposition 3.4 follows from the fact that T is a canonical transformation (see § 3.2).

Recall that Q is a cyclic coordinate. Consequently, the system (3.15) is equivalent to a reduced Hamiltonian system for Ψ and Π only, which can be written as

$$\dot\Psi = D_\Pi \mathscr{H}_P, \quad \dot\Pi = -D_\psi \mathscr{H}_P. \tag{3.16}$$

By (3.14), the soliton (ψ_v, π_v) is a stationary solution of (3.16) with $P = P(v)$. Moreover, for every $P \in \mathbb{R}^3$ the functional $\mathscr{H}_P$ is Fréchet-differentiable on the Hilbert space $\mathscr{F} = \mathring{H}^1 \oplus L^2$. Hence, (3.16) implies that the soliton is a critical point of $\mathscr{H}_{P(v)}$ on $\mathscr{F}$. The next lemma demonstrates that (ψ_v, π_v) is a global minimum of $\mathscr{H}_{P(v)}$ on $\mathscr{F}$.

Lemma 3.6. (i) *For every $v \in \mathbb{R}^3$ with $|v| < 1$ the functional $\mathscr{H}_{P(v)}$ has the lower bound*

$$\mathscr{H}_{P(v)}(\Psi, \Pi) - \mathscr{H}_{P(v)}(\psi_v, \pi_v) \geqslant \frac{1 - |v|}{2}(\|\Psi - \psi_v\|^2 + \|\Pi - \pi_v\|^2), \qquad (\Psi, \Pi) \in \mathscr{F}. \tag{3.17}$$

(ii) *$\mathscr{H}_{P(v)}$ has no other critical points on $\mathscr{F}$ except for the point (ψ_v, π_v).*

Proof. (i) Letting $\Psi - \psi_v = \psi$ and $\Pi - \pi_v = \pi$, we have

$$\begin{aligned} &\mathscr{H}_{P(v)}(\psi_v + \psi, \pi_v + \pi) - \mathscr{H}_{P(v)}(\psi_v, \pi_v) \\ &\quad = \int d^3x \, (\pi_v(x)\pi(x) + \nabla\psi_v(x) \cdot \nabla\psi(x) + \rho(x)\psi(x)) \\ &\qquad + \frac{1}{2}\int d^3x \, (|\pi(x)|^2 + |\nabla\psi(x)|^2) + (1 + (p_v + m)^2)^{1/2} - (1 + p_v^2)^{1/2}, \end{aligned} \tag{3.18}$$

where $p_v = P(v) + \int d^3x \, \pi_v(x)\nabla\psi_v(x)$, and

$$m = \int d^3x \, (\pi(x)\nabla\psi_v(x) + \pi_v(x)\nabla\psi(x) + \pi(x)\nabla\psi(x)).$$

Taking into account that $v = (1 + p_v^2)^{-1/2} p_v$, we obtain

$$\begin{aligned} &\mathscr{H}_{P(v)}(\psi_v + \psi, \pi_v + \pi) - \mathscr{H}_{P(v)}(\psi_v, \pi_v) \\ &\quad = \frac{1}{2}\int d^3x (|\pi(x)|^2 + |\nabla\psi(x)|^2) + (1 + p_v^2)^{-1/2} \int d^3x \, \pi(x) \, p_v \cdot \nabla\psi(x) \\ &\qquad - (1 + p_v^2)^{-1/2} p_v \cdot m + (1 + (p_v + m)^2)^{1/2} - (1 + p_v^2)^{1/2}. \end{aligned}$$

It is easy to check that the expression in the third line is non-negative. Then the lower bound in (3.17) follows because $|(1+p_v^2)^{-1/2}p_v| = |v|$.

(ii) If $(\Psi,\Pi)\in\mathscr{F}$ is a critical point for $\mathscr{H}_{P(v)}$, then it satisfies the equations

$$0=\Pi(x)+(1+\widetilde{p}^2)^{-1/2}\widetilde{p}\cdot\nabla\Psi(x),\qquad 0=-\Delta\Psi(x)+\rho(x)-(1+\widetilde{p}^2)^{-1/2}\widetilde{p}\cdot\nabla\Pi(x),$$

where $\widetilde{p}=P(v)+\int d^3x\,\Pi(x)\nabla\Phi(x)$. This system is equivalent to the equation (3.5) for solitons in the case of the velocity $\widetilde{v}=(1+\widetilde{p}^2)^{-1/2}\widetilde{p}$. Hence $\Psi=\psi_{\widetilde{v}}$, $\Pi=\pi_{\widetilde{v}}$, and $P(\widetilde{v})=P(v)$.

It remains to check that $\widetilde{v}=v$. Indeed, for the total momentum $P(v)$ of the soliton-like solution (3.3), the Parseval identity and (3.6) imply that

$$P(v)=p_v+\int d^3x\,v\cdot\nabla\psi_v(x)\nabla\psi_v(x)=\frac{v}{\sqrt{1-v^2}}+\frac{1}{(2\pi)^3}\int d^3k\,\frac{(v\cdot k)\widehat{\rho}(k)\overline{k\widehat{\rho}(k)}}{(k^2-(v\cdot k)^2)^2}\,.$$

Hence, $P(v)=\kappa(|v|)v$ with $\kappa(|v|)\geqslant 0$, and for $v\neq 0$ one has

$$|P(v)|=\frac{|v|}{\sqrt{1-v^2}}+\frac{1}{(2\pi)^3|v|}\int d^3k\,\frac{|(v\cdot k)\widehat{\rho}(k)|^2}{(k^2-(v\cdot k)^2)^2}\,.$$

Since $|P(v)|=\kappa(|v|)|v|$ is a monotone increasing function of $|v|\in[0,1)$, we conclude that $v=\widetilde{v}$. □

Remark 3.7. Proposition 3.4 is not really needed for the proof of Theorem 3.1. However, this proposition, together with (3.14) and (3.16), shows that (ψ_v,π_v) is a critical point and suggests an investigation of stability through a lower bound as in (3.17). In § 3.2 we sketch the derivation of Proposition 3.4 for sufficiently smooth solutions based only on the invariance of the symplectic structure. We expect that a similar proposition holds for other translation-invariant systems like (3.1).

3.1.2. *Orbital stability of solitons.* We follow [11] in deducing orbital stability from conservation of the Hamiltonian $\mathscr{H}_P$ together with its lower bound in (3.17). For $|v|<1$ let

$$\delta=\delta(v)=\|\psi^0(x)-\psi_v(x-q^0)\|+\|\pi^0(x)-\pi_v(x-q^0)\|+|p^0-p_v|. \tag{3.19}$$

Lemma 3.8. *Let $Y(t)=(\psi(t),\pi(t),q(t),p(t))\in C(\mathbb{R},\mathscr{E})$ be a solution of (3.1) with initial state $Y(0)=Y^0=(\psi^0,\pi^0,q^0,p^0)\in\mathscr{E}$. Then for any $\varepsilon>0$ there exists a $\delta_\varepsilon>0$ such that*

$$\|\psi(q(t)+x,t)-\psi_v(x)\|+\|\pi(q(t)+x,t)-\pi_v(x)\|+|p(t)-p_v|\leqslant\varepsilon,\quad t\in\mathbb{R}, \tag{3.20}$$

for $\delta\leqslant\delta_\varepsilon$.

Proof. Denote by P^0 the total momentum of the solution $Y(t)$ under consideration. There exists a soliton-like solution (3.3) corresponding to some velocity $\widetilde{v}$ with the same total momentum $P(\widetilde{v})=P^0$. Then (3.19) implies that $|P^0-P(v)|=|P(\widetilde{v})-P(v)|=\mathscr{O}(\delta)$. Hence also $|\widetilde{v}-v|=\mathscr{O}(\delta)$ and

$$\|\psi^0(x)-\psi_{\widetilde{v}}(x-q^0)\|+\|\pi^0(x)-\pi_{\widetilde{v}}(x-q^0)\|+|p^0-p_{\widetilde{v}}|=\mathscr{O}(\delta).$$

Therefore, with the notation $(\Psi^0, Q^0, \Pi^0, P^0) = TY^0$ we have

$$\mathscr{H}_{P(\widetilde{v})}(\Psi^0, \Pi^0) - \mathscr{H}_{P(\widetilde{v})}(\psi_{\widetilde{v}}, p_{\widetilde{v}}) = \mathscr{O}(\delta^2). \tag{3.21}$$

For $(\Psi(t), Q(t), \Pi(t), P^0) = TY(t)$ it follows from conservation of the total momentum and energy that

$$\mathscr{H}_{P(\widetilde{v})}(\Psi(t), \Pi(t)) = \mathscr{H}(TY(t)) = \mathscr{H}_{P(\widetilde{v})}(\Psi^0, \Pi^0), \qquad t \in \mathbb{R}.$$

Hence (3.21) and (3.17) with $\widetilde{v}$ instead of v imply that

$$\|\Psi(t) - \psi_{\widetilde{v}}\| + \|\Pi(t) - \pi_{\widetilde{v}}\| = \mathscr{O}(\delta) \tag{3.22}$$

uniformly for $t \in \mathbb{R}$. On the other hand, conservation of the total momentum implies that

$$p(t) = P(\widetilde{v}) + \langle \Pi(t), \nabla\Psi(t)\rangle, \qquad t \in \mathbb{R}.$$

Therefore, (3.22) leads to

$$|p(t) - p_{\widetilde{v}}| = \mathscr{O}(\delta) \tag{3.23}$$

uniformly for $t \in \mathbb{R}$. Finally, (3.22) and (3.23) together imply (3.20) because $|\widetilde{v} - v| = \mathscr{O}(\delta)$. □

3.1.3. *The strong Huygens principle and soliton asymptotics.* We combine the relaxation of the acceleration and orbital stability with the strong Huygens principle to prove Theorem 3.1.

Proposition 3.9. *Assume the conditions of Theorem 3.1. Then for any $\delta > 0$ there exist a $t_* = t_*(\delta)$ and a solution*

$$Y_*(t) = (\psi_*(x,t), \pi_*(x,t), q_*(t), p_*(t)) \in C([t_*, \infty), \mathscr{E})$$

of the system (3.1) such that:

(i) *$Y_*(t)$ coincides with $Y(t)$ in the future cone,*

$$q_*(t) = q(t) \qquad \text{for} \quad t \geqslant t_*, \tag{3.24}$$

$$\psi_*(x,t) = \psi(x,t) \quad \text{for} \quad |x - q(t_*)| < t - t_*; \tag{3.25}$$

(ii) *$Y_*(t_*)$ is close to a soliton $Y_{v,a}$ with some v and a,*

$$\|Y_*(t_*) - Y_{v,a}\|_{\mathscr{E}} \leqslant \delta. \tag{3.26}$$

Proof. The Kirchhoff formula gives

$$\psi(x,t) = \psi_r(x,t) + \psi_0(x,t), \qquad x \in \mathbb{R}^3, \quad t > 0,$$

where

$$\psi_r(x,t) = -\int \frac{d^3y}{4\pi|x-y|}\,\rho(y - q(t - |x-y|)), \tag{3.27}$$

$$\psi_0(x,t) = \frac{1}{4\pi t}\int_{S_t(x)} d^2y\,\pi(y,0) + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{S_t(x)} d^2y\,\psi(y,0)\right) \tag{3.28}$$

($S_t(x)$ is the sphere $|y-x|=t$).

Assume for simplicity that the initial fields vanish. The general case can easily be reduced to this situation by using the strong Huygens principle. We will comment on this reduction at the end of the proof.

In the case of zero initial data the solution reduces to the retarded potential:

$$\psi(x,t)=\psi_r(x,t), \qquad x\in\mathbb{R}^3, \quad t>0.$$

We construct the solution $Y_*(t)$ as a modification of $Y(t)$. First we modify the trajectory $q(t)$. The relaxation of acceleration (3.8) means that for any $\varepsilon>0$ there exists a $t_\varepsilon>0$ such that

$$|\ddot q(t)|\leqslant\varepsilon, \qquad t\geqslant t_\varepsilon.$$

Hence, for large times the trajectory tends locally to a straight line, that is, for any fixed $T>0$

$$q(t)=q(t_\varepsilon)+(t-t_\varepsilon)\dot q(t_\varepsilon)+r(t_\varepsilon,t), \quad \text{where} \quad \max_{t\in[t_\varepsilon,t_\varepsilon+T]}|r(t_\varepsilon,t)|\to 0, \quad t_\varepsilon\to\infty.$$

Let $\lambda_\varepsilon(t):=q(t_\varepsilon)+\dot q(t_\varepsilon)(t-t_\varepsilon)$ and define a modified trajectory by

$$q_*(t)=\begin{cases}\lambda_\varepsilon(t), & t\leqslant t_\varepsilon,\\ q(t), & t\geqslant t_\varepsilon.\end{cases} \tag{3.29}$$

Then

$$\ddot q_*(t)=\begin{cases}0, & t<t_\varepsilon,\\ \ddot q(t), & t>t_\varepsilon.\end{cases}$$

The next step is to define the modified field as a retarded potential of the form (3.27):

$$\psi_*(x,t)=-\int\frac{d^3y}{4\pi|x-y|}\,\rho(y-q_*(t-|x-y|)), \qquad x\in\mathbb{R}^3, \quad t\in\mathbb{R}. \tag{3.30}$$

Lemma 3.10. *The right-hand side of* (3.30) *depends on the trajectory* $q_*(\tau)$ *only on the bounded time interval* $\tau\in[t-T(x,t),t]$, *where*

$$T(x,t):=\frac{R_\rho+|x-q(t)|}{1-\overline v}\,. \tag{3.31}$$

Here $\overline v=\sup_{t\in\mathbb{R}}|\dot q(t)|<1$ *by* (2.32).

Proof. This lemma is obvious geometrically, and a formal proof of it is also easy.

The integrand in (3.30) vanishes for $|y-q_*(t-|x-y|)|\geqslant R_\rho$ by (2.35). Therefore, the integration is over the region $|y-q_*(t-|x-y|)|\leqslant R_\rho$, which implies that $|y-q_*(t)+q_*(t)-q_*(t-|x-y|)|\leqslant R_\rho$. Hence,

$$|y-q_*(t)|\leqslant R_\rho+\overline v|x-y|.$$

On the other hand, $|x-y|\leqslant|x-q_*(t)|+|y-q_*(t)|$, and thus

$$|y-q_*(t)|\geqslant-|x-q_*(t)|+|x-y|.$$

Therefore, $-|x-q_*(t)|+|x-y|\leqslant R_\rho+\overline v|x-y|$, which implies that $|x-y|\leqslant\dfrac{R_\rho+|x-q_*(t)|}{1-\overline v}$. $\square$

The potential (3.30) satisfies the wave equation

$$\ddot{\psi}_*(x,t) = \Delta\psi_*(x,t) - \rho(x - q_*(t)), \qquad x \in \mathbb{R}^3, \quad t \in \mathbb{R}.$$

We must also prove equations for the trajectory $q_*(t)$:

$$\dot{q}_*(t) = \frac{p_*(t)}{\sqrt{1+p_*^2(t)}}, \quad \dot{p}_*(t) = -\int \nabla\psi_*(x,t)\rho(x - q_*(t))\,dx, \qquad t > t_*, \quad (3.32)$$

with sufficiently large $t_* \geqslant t_\varepsilon$. Note that the integration here is over the ball $|x - q_*(t)| \leqslant R_\rho$. Lemma 3.10 now implies that $\psi_*(x,t)$ depends only on the trajectory $q_*(\tau)$ on the bounded time interval $\tau \in [t - \overline{T}, t]$, where

$$\overline{T} := \frac{2R_\rho}{1-\overline{v}}.$$

Let $t_* := t_\varepsilon + \overline{T}$. Then by Lemma 3.10

$$\psi_*(x,t) = \psi(x,t), \qquad t > t_*, \quad |x - q_*(t)| \leqslant R_\rho,$$

since $q_*(t) \equiv q(t)$ for $t > t_* - \overline{T} = t_\varepsilon$ by (3.29). Hence, the equations (3.32) hold not only for $q_*(t)$ but also for $q(t)$.

It remains to prove (3.26). The key observation is that outside the cone $K_\varepsilon := \{(x,t) \in \mathbb{R}^4 \colon |x - q(t_\varepsilon)| < t - t_\varepsilon\}$ the retarded potential (3.30) coincides with the soliton $\psi_{v,a}(x,t)$, where $v = \dot{q}(t_\varepsilon)$ and $a = q(t_\varepsilon)$ by our definition (3.29). In particular,

$$\psi(x,t_*) = \psi_{v,a}(x - a - vt_*), \qquad |x - q(t_\varepsilon)| > t_* - t_\varepsilon = \overline{T}.$$

In the ball $|x - q(t_*)| < \overline{T}$ the coincidence generally does not hold, but the difference between the left-hand and right-hand sides converges to zero as $\varepsilon \to 0$ uniformly for $|x - q(t_*)| < \overline{T}$, and the same uniform convergence holds for the gradient of the difference. This follows from the integral representation (3.30) by Lemma 3.10 since

$$\max_{t \in (t_* - T(x,t_*), t_*)} [|q_*(t) - \lambda_\varepsilon(t)| + |\dot{q}_*(t) - \dot{\lambda}_\varepsilon(t)|] \to 0, \qquad \varepsilon \to 0,$$

by the relaxation of acceleration (3.8). It is important that $T(x,t_*)$ is bounded for $|x - q(t_*)| < \overline{T}$ in view of (3.31). This proves Proposition 3.9 in the case of zero initial data.

The next step is to consider initial data with bounded support:

$$\psi(x,0) = \pi(x,0) = 0, \qquad |x| > R_0.$$

We now apply the strong Huygens principle: in this case the potential (3.28) vanishes in the future cone,

$$\psi_0(x,t) = 0, \qquad |x| < t - R_0.$$

Nevertheless, the estimate $|\dot{q}(t)| \leqslant \overline{v} < 1$ implies that the trajectory $(q(t),t)$ lies in this cone for all $t > t_0$. Hence, the solution for $t > t_0$ again reduces to the retarded potential, and the required conclusion follows.

Finally, arbitrary finite-energy initial data admit a splitting into two terms: the first vanishing for $|x| > R_0$ and the second vanishing for $|x| < R_0 - 1$. The energy of the second term is arbitrarily small for large R_0, and the energy of the corresponding potential (3.28) is conserved in time since it is a solution of the free wave equation. Consequently, its role is negligible for sufficiently large R_0. □

Now we can prove our main result.

Proof of Theorem 3.1. For any $\varepsilon > 0$ there is a $\delta > 0$ such that by Lemma 3.8, (3.26) implies that

$$\|\psi_*(q_*(t)+x,t)-\psi_v(x)\| + \|\pi_*(q_*(t)+x,t)-\pi_v(x)\| + |\dot q_*(t)-v| \leqslant \varepsilon$$

for $t > t_*$. Therefore, (3.24) and (3.25) imply that for any $R > 0$ and $t > t_* + \dfrac{R}{1-\overline{v}}$

$$\begin{aligned}&\|\psi(q(t)+x,t)-\psi_v(x)\|_R + \|\pi(q(t)+x,t)-\pi_v(x)\|_R + |\dot q(t)-v| \\ &\quad = \|\psi_*(q_*(t)+x,t)-\psi_v(x)\|_R + \|\pi_*(q_*(t)+x,t)-\pi_v(x)\|_R + |\dot q_*(t)-v| \leqslant \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain (3.10). □

3.2. Invariance of symplectic structure. The canonical equivalence of the Hamiltonian systems (3.1) and (3.15) can formally be seen from the Lagrangian point of view. We remain at the formal level. For a complete mathematical justification we would have to develop a certain theory of infinite-dimensional Hamiltonian systems, which is beyond the scope of this paper. By definition we have $\mathscr{H}^T(\Psi,\Pi,Q,P) = \mathscr{H}(\psi,\pi,q,p)$, where the arguments are related through the transformation T. With each Hamiltonian we associate a corresponding Lagrangian through the Legendre transformation

$$\begin{aligned}L(\psi,\dot\psi,q,\dot q) &= \langle\pi,\dot\psi\rangle + p\cdot\dot q - \mathscr{H}(\psi,\pi,q,p), \qquad \dot\psi = D_\pi\mathscr{H}, \quad \dot q = D_p\mathscr{H},\\ L^T(\Psi,\dot\Phi,Q,\dot Q) &= \langle\Pi,\dot\Psi\rangle + P\cdot\dot Q - \mathscr{H}^T(\Psi,\Pi,Q,P), \quad \dot\Psi = D_\Pi\mathscr{H}^T, \quad \dot Q = D_P\mathscr{H}^T.\end{aligned}$$

These Legendre transforms are well defined because the Hamiltonian functionals are convex in the momenta.

Lemma 3.11. *The following identity holds*:

$$L^T(\Psi,\dot\Psi,Q,\dot Q) = L(\psi,\dot\psi,q,\dot q).$$

Proof. Clearly, we have to check the invariance of the canonical 1-form,

$$\langle\Pi,\dot\Psi\rangle + P\cdot\dot Q = \langle\pi,\dot\psi\rangle + p\cdot\dot q. \tag{3.33}$$

For this purpose we substitute

$$\begin{aligned}\Pi(x) &= \pi(q+x), &\qquad \dot\Psi(x) &= \dot\psi(q+x) + \dot q\cdot\nabla\psi(q+x),\\ P &= p - \int\dot\psi\cdot\nabla\psi\,dx, &\qquad \dot Q &= \dot q.\end{aligned}$$

Then the left-hand side of (3.33) becomes

$$\langle\pi(q+x),\dot\psi(q+x)+\dot q\cdot\nabla\psi(q+x)\rangle + (p - \langle\pi(x),\nabla\psi(x)\rangle)\cdot\dot q = \langle\pi,\dot\psi\rangle + p\cdot\dot q. \qquad \square$$

This lemma implies that the corresponding action functionals become identical when transformed by T. Hence finally, the two Hamiltonian systems (3.1) and (3.15) are equivalent, since the dynamical trajectories are stationary points of the respective action functionals.

3.3. The translation-invariant Maxwell–Lorentz system. In [76] asymptotics of the form (3.8)–(3.10) were extended to the Maxwell–Lorentz translation-invariant system (2.61) without external fields. In this case the Hamiltonian coincides with (2.63), where $V(x) \equiv 0$. The extension of methods in [128] to this case required a new detailed analysis of the corresponding Hamiltonian structure, which is necessary for the canonical transformation. The key role in applying the strong Huygens principle is now played by new estimates of long-time decay for oscillations of energy and total momentum solutions for a perturbed Maxwell–Lorentz system (the estimates (4.24), (4.25) in [76]).

3.4. The case of weak interaction. Soliton asymptotic expressions of the type (3.8)–(3.10) for the system (2.22), (2.23) were proved in a stronger form for the case of weak coupling

$$\|\rho\|_{L^2(\mathbb{R}^3)} \ll 1. \tag{3.34}$$

Namely, in [78] initial fields were considered with decay $|x|^{-5/2-\varepsilon}$, where $\varepsilon > 0$ (condition (2.2) in [78]) provided that $\nabla V(q) = 0$ for $|q| >$ const. Under these assumptions, stronger decay holds:

$$|\ddot{q}(t)| \leqslant C(1+|t|)^{-1-\varepsilon}, \qquad t \in \mathbb{R}, \tag{3.35}$$

for 'outgoing' solutions that satisfy the condition

$$|q(t)| \to \infty, \qquad t \to \pm\infty. \tag{3.36}$$

With these assumptions the asymptotics (3.8)–(3.10) can be significantly strengthened: now

$$\dot{q}(t) \to v_\pm,$$
$$(\psi(x,t),\pi(x,t)) = (\psi_{v_\pm}(x-q(t)),\pi_{v_\pm}(x-q(t))) + W(t)\Phi_\pm + (r_\pm(x,t),s_\pm(x,t)),$$

where the 'dispersion waves' $W(t)\Phi_\pm$ are solutions of the free wave equation, and the remainder converges to zero in the *global energy norm*:

$$\|\nabla r_\pm(q(t),t)\| + \|r_\pm(q(t),t)\| + \|s_\pm(q(t),t)\| \to 0, \qquad t \to \pm\infty.$$

This progress in comparison with the local decay (3.10) is due to the fact that we could identify the dispersion wave $W(t)\Phi_\pm$ under the smallness condition (3.34). This was possible because of the more rapid decay (3.35), in contrast to (2.38).

All solitons propagate with velocities $v < 1$, and therefore they are spatially separated for large times from the dispersion waves $W(t)\Phi_\pm$, which propagate with unit velocity (see Fig. 2).

The proofs are based on the integral Duhamel representation and on the rapid dispersion decay of solutions of the free wave equation. A similar result was obtained in [75] for a system of the form (2.22), (2.23) with the Klein–Gordon equation

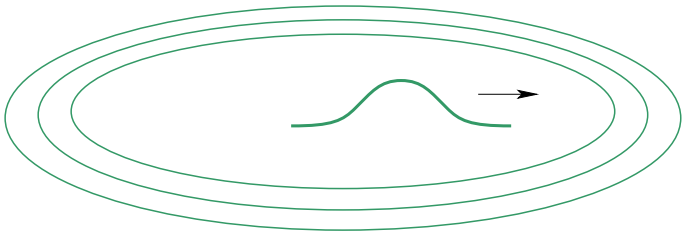

Figure 2. Soliton and dispersion waves.

and in [77] for the Maxwell–Lorentz system (2.61) with the same smallness condition (3.36) under the assumption that $E^{\mathrm{ext}}(x) = B^{\mathrm{ext}}(x) = 0$ for $|x| > \mathrm{const}$. In [79], this result was extended to the Maxwell–Lorentz system of the form (2.61) with a rotating charge.

Remark 3.12. The results in [78] and [79] imply Soffer's 'Grande Conjecture' ([190], p. 460) in a moving frame for translation-invariant systems under the smallness condition (3.34).

4. The adiabatic effective dynamics of solitons

The existence of solitons and the global attraction to them (1.6) are typical features of translation-invariant systems. However, if the deviation of a system from translational invariance is small in a certain sense, then the system can admit solutions which are always close to solitons with time-dependent parameters (the velocity, and so on). Moreover, in some cases it is possible to identify an 'effective dynamics' which describes the evolution of these parameters.

4.1. A 'wave-particle' system with a slowly varying external potential. The solitons (3.3) are solutions of the system (3.1) with zero external potential $V(x) \equiv 0$. However, even for the system (2.22), (2.23) with non-zero external potential, *soliton-like* solutions of the form

$$\psi(x,t) \approx \psi_{v(t)}(x - q(t)) \tag{4.1}$$

may exist if the potential is slowly changing:

$$|\nabla V(q)| \leqslant \varepsilon \ll 1. \tag{4.2}$$

In this case the total momentum (3.2) is generally not conserved, but its slow evolution and the evolution of the parameter $q(t)$ in (4.1) can be described in terms of some finite-dimensional Hamiltonian dynamics.

Namely, let $P = P_v$ be the total momentum of the soliton $S_{v,Q}$ in the notation (3.7). It is important that the map $\mathscr{P}\colon v \mapsto P_v$ is an isomorphism of the ball $|v| < 1$ on $\mathbb{R}^3$. Therefore, we can consider Q, P as global coordinates on the solitary manifold $\mathscr{S}$. We define the effective Hamiltonian functional

$$\mathscr{H}_{\mathrm{eff}}(Q,P_v) \equiv \mathscr{H}_0(S_{v,Q}), \qquad (Q,P_v) \in \mathbb{R}^3 \oplus \mathbb{R}^3, \tag{4.3}$$

where $\mathscr{H}_0$ is the *unperturbed* Hamiltonian (2.25) with $V = 0$. This functional can be represented as $\mathscr{H}_{\rm eff}(Q,\Pi) = E(\Pi) + V(Q)$, since the first integral in (2.25) does not depend on Q while the last integral vanishes on the solitons. Hence, the corresponding Hamiltonian equations have the form

$$\dot{Q}(t) = \nabla E(\Pi(t)), \qquad \dot{\Pi}(t) = -\nabla V(Q(t)). \tag{4.4}$$

The main result in [122] is the following theorem.

Theorem 4.1. *Let the condition* (4.2) *hold, and suppose that the initial state* $S_0 = (\psi_0, \pi_0, q_0, p_0) \in \mathscr{S}$ *is a soliton with total momentum* $P(0)$. *Then the corresponding solution* $\psi(x,t)$, $\pi(x,t)$, $q(t)$, $p(t)$ *of the system* (2.22), (2.23) *admits the 'adiabatic asymptotics'*

$$|q(t) - Q(t)| \leqslant C_0, \quad |P(t) - \Pi(t)| \leqslant C_1\varepsilon \quad \textit{for} \quad |t| \leqslant C\varepsilon^{-1}, \tag{4.5}$$

$$\sup_{t\in\mathbb{R}}\big[\|\nabla[\psi(q(t)+x,t) - \psi_{v(t)}(x)]\|_R + \|\pi(q(t)+x,t) - \pi_{v(t)}(x)\|_R\big] \leqslant C\varepsilon, \tag{4.6}$$

where $P(t)$ *denotes the total momentum* (3.2), $v(t) = \mathscr{P}^{-1}(\Pi(t))$, *and* $(Q(t),\Pi(t))$ *is the solution (trajectory) of the effective Hamiltonian equations* (4.4) *with initial conditions*

$$Q(0) = q(0), \quad \Pi(0) = P(0).$$

We note that such relevance of the effective dynamics (4.4) is due to the consistency of Hamiltonian structures:

1) The effective Hamiltonian (4.3) is the restriction of the Hamiltonian functional (2.25) with $V = 0$ to the solitary manifold $\mathscr{S}$.

2) As shown in [122], the canonical form of the Hamiltonian system (4.4) is also the restriction to $\mathscr{S}$ of the canonical form of the system (2.22), (2.23): formally,

$$P\,dQ = \left[p\,dq + \int \pi(x)\,d\psi(x)\,dx\right]\Big|_{\mathscr{S}}.$$

Therefore, the total momentum P is canonically conjugate to the variable Q on the solitary manifold $\mathscr{S}$. This fact justifies the definition (4.3) of the effective Hamiltonian as a function of the total momentum P_v, and not of the particle momentum p_v.

One of the important results in [122] is the following 'effective dispersion relation':

$$E(\Pi) \sim \frac{\Pi^2}{2(1+m_e)} + \text{const}, \qquad |\Pi| \ll 1. \tag{4.7}$$

It means that the non-relativistic mass of a slow soliton increases because of interaction with the field, by the amount

$$m_e = -\frac{1}{3}\langle\rho, \Delta^{-1}\rho\rangle. \tag{4.8}$$

This increment is proportional to the field energy of the soliton at rest:

$$\mathscr{H}(\Delta^{-1}\rho, 0, 0, 0) = -\frac{1}{2}\langle\rho, \Delta^{-1}\rho\rangle,$$

which agrees with the Einstein mass-energy equivalence principle (see below).

Remark 4.2. The relation (4.7) suggests that m_e is an increment of the effective mass. The true *dynamical justification* for such an interpretation is given by the asymptotics (4.5), (4.6), which demonstrate the relevance of the effective dynamics (4.4).

Generalizations. After [122], adiabatic effective asymptotics of the form (4.5), (4.6) were obtained in [56] and [55] for the nonlinear Hartree and Schrödinger equations with slowly varying external potentials, and in [39], [162], and [200] for the nonlinear Einstein–Dirac, Chern–Simon–Schrödinger, and Klein–Gordon–Maxwell equations with small external fields.

Recently, a similar adiabatic effective dynamics was established in [9] for an electron in the second-quantized Maxwell field in the presence of a slowly changing external potential.

4.2. Mass-energy equivalence. In [152], the asymptotic expressions (4.5), (4.6) were extended to solitons of the Maxwell–Lorentz equations (2.61) with small external fields. In this case the increment of non-relativistic mass also turns out to be proportional to the energy of the static soliton's own field.

Such equivalence of the self-energy of a particle with its mass was first discovered in 1902 by Abraham: he obtained by direct calculation that the electromagnetic self-energy E_{own} of an electron at rest adds

$$m_e = \frac{4}{3}\,\frac{E_{\text{own}}}{c^2}$$

to its non-relativistic mass (see [1], [2], and also [100], pp. 216, 217). It is easy to see that this self-energy is infinite for a point electron at the origin with charge density $\delta(x)$, because in this case the Coulomb electrostatic field is $|E(x)| = C/|x|^2$, so the integral in (2.63) diverges near $x = 0$. This means that the field mass for a point electron is infinite, which contradicts experiment. That is why Abraham introduced the model of electrodynamics with 'extended electron' (2.61), whose self-energy is finite.

At the same time, Abraham conjectured that the *entire mass* of an electron is due to its own electromagnetic energy; that is, $m = m_e$: "...*matter disappeared, only energy remains*...", as philosophically-minded contemporaries wrote (see [73], pp. 63, 87, 88).

This conjecture was justified in 1905 by Einstein, who discovered the famous universal relation $E = m_0c^2$ which follows from the special theory of relativity [49]. The additional factor 4/3 in the Abraham formula is because of the non-relativistic character of the system (2.61). According to the modern view, about 80% of an electron's mass is of electromagnetic origin [52].

5. Global attraction to stationary orbits

Global attraction to stationary orbits (1.8) was first established in [99], [103], and [104] for the Klein–Gordon equation coupled to a nonlinear oscillator

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \delta(x)F(\psi(0,t)), \qquad x \in \mathbb{R}. \tag{5.1}$$

We consider complex solutions, identifying complex values $\psi \in \mathbb{C}$ with the real vectors $(\psi_1, \psi_2) \in \mathbb{R}^2$, where $\psi_1 = \operatorname{Re}\psi$ and $\psi_2 = \operatorname{Im}\psi$. Suppose that $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and

$$F(\psi) = -\nabla_{\overline{\psi}} U(\psi), \qquad \psi \in \mathbb{C}, \tag{5.2}$$

where U is a real function and $\nabla_{\overline{\psi}} := (\partial_1, \partial_2)$. In this case the equation (5.1) is formally equivalent to the Hamiltonian system (2.2) in the Hilbert phase space $\mathscr{E} := H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$. The Hamiltonian functional is

$$\mathscr{H}(\psi, \pi) = \frac{1}{2} \int \left[|\pi(x)|^2 + |\psi'(x)|^2 + m^2 |\psi(x)|^2 \right] dx + U(\psi(0)), \quad (\psi, \pi) \in \mathscr{E}. \tag{5.3}$$

Let us write (5.1) in the vector form as

$$\dot{Y}(t) = \mathscr{F}(Y(t)), \qquad t \in \mathbb{R}, \tag{5.4}$$

where $Y(t) = (\psi(t), \dot{\psi}(t))$. We assume that

$$\inf_{\psi \in \mathbb{C}} U(\psi) > -\infty. \tag{5.5}$$

In this case, a finite-energy solution $Y(t) \in C(\mathbb{R}, \mathscr{E})$ exists and is unique for any initial state $Y(0) \in \mathscr{E}$. The a priori bound

$$\sup_{t \in \mathbb{R}} [\|\dot{\psi}(t)\|_{L^2(\mathbb{R})} + \|\psi(t)\|_{H^1(\mathbb{R})}] < \infty \tag{5.6}$$

holds due to conservation of the energy (5.3). Note that the condition (2.10) is no longer necessary, since conservation of the energy (5.3) with $m > 0$ ensures boundedness of solutions.

Further, we assume the U(1)-invariance of the potential:

$$U(\psi) = u(|\psi|), \qquad \psi \in \mathbb{C}. \tag{5.7}$$

Then the differentiation in (5.2) gives us that

$$F(\psi) = a(|\psi|)\psi, \qquad \psi \in \mathbb{C}, \tag{5.8}$$

and therefore

$$F(e^{i\theta}\psi) = e^{i\theta} F(\psi), \qquad \theta \in \mathbb{R}. \tag{5.9}$$

By 'stationary orbits' we mean solutions of the form

$$\psi(x, t) = \psi_\omega(x) e^{-i\omega t} \tag{5.10}$$

with $\omega \in \mathbb{R}$ and $\psi_\omega \in H^1(\mathbb{R})$. Each stationary orbit corresponds to some solution of the equation

$$-\omega^2 \psi_\omega(x) = \psi_\omega''(x) - m^2 \psi_\omega(x) + \delta(x) F(\psi_\omega(0)), \qquad x \in \mathbb{R},$$

which is the *nonlinear eigenvalue problem.*

Solutions $\psi_\omega \in H^1(\mathbb{R})$ of this equation have the form

$$\psi_\omega(x) = C e^{-\kappa |x|}, \quad \text{where} \quad \kappa := \sqrt{m^2 - \omega^2} > 0,$$

and the constant C satisfies the nonlinear algebraic equation $2\kappa C = F(C)$. The solutions ψ_ω exist for ω in some set $\Omega \subset \mathbb{R}$ lying in the *spectral gap* $[-m, m]$. Denote the corresponding *solitary manifold* by $\mathscr{S}$:

$$\mathscr{S} = \{(e^{i\theta}\psi_\omega, -i\omega e^{i\theta}\psi_\omega) \in \mathscr{E} : \omega \in \Omega,\ \theta \in [0, 2\pi]\}. \tag{5.11}$$

Finally, suppose that the equation (5.4) is *strictly nonlinear*:

$$U(\psi) = u(|\psi|^2) = \sum_{j=0}^{N} u_j |\psi|^{2j}, \qquad u_N > 0, \quad N \geqslant 2. \tag{5.12}$$

For example, the well-known *Ginzburg–Landau* potential $U(\psi) = |\psi|^4/4 - |\psi|^2/2$ satisfies all the conditions (5.5), (5.7), and (5.12).

Definition 5.1. (i) $\mathscr{E}_F \subset H^1_{\mathrm{loc}}(\mathbb{R}^3) \oplus L^2_{\mathrm{loc}}(\mathbb{R}^3)$ is the space $\mathscr{E}$ endowed with the seminorms

$$\|Y\|_{\mathscr{E},R} := \|Y\|_{H^1(-R,R)} + \|Y\|_{L^2(-R,R)}, \qquad R = 1, 2, \dots . \tag{5.13}$$

(ii) Convergence in $\mathscr{E}_F$ is equivalent to convergence in every seminorm (5.13).

Convergence in $\mathscr{E}_F$ is equivalent to convergence in the metric of type (2.13),

$$\operatorname{dist}(Y_1, Y_2) = \sum_{R=1}^{\infty} 2^{-R} \frac{\|Y_1 - Y_2\|_{\mathscr{E},R}}{1 + \|Y_1 - Y_2\|_{\mathscr{E},R}}, \qquad Y_1, Y_2 \in \mathscr{E}. \tag{5.14}$$

Theorem 5.2. *Let the conditions* (5.2), (5.5), (5.7), *and* (5.12) *hold. Then any finite-energy solution* $Y(t) = (\psi(t), \dot\psi(t)) \in C(\mathbb{R}, \mathscr{E})$ *of* (5.4) *is attracted to the solitary manifold* (*see Fig.* 3):

$$Y(t) \xrightarrow{\mathscr{E}_F} \mathscr{S}, \qquad t \to \pm\infty, \tag{5.15}$$

where the attraction is in the sense of (2.15).

Generalizations. The attraction (5.15) was extended in [107] to the 1D Klein–Gordon equation with N nonlinear oscillators

$$\ddot\psi(x,t) = \psi''(x,t) - m^2\psi + \sum_{k=1}^{N} \delta(x - x_k) F_k(\psi(x_k, t)), \qquad x \in \mathbb{R}, \tag{5.16}$$

and in [30], [106], and [108] it was extended to the Klein–Gordon and Dirac equations in $\mathbb{R}^n$ with $n \geqslant 3$ and non-local interaction

$$\ddot\psi(x,t) = \Delta\psi(x,t) - m^2\psi + \sum_{k=1}^{N} \rho(x - x_k) F_k(\langle \psi(\,\cdot\,, t), \rho(\,\cdot\, - x_k)\rangle), \tag{5.17}$$

$$i\dot\psi(x,t) = (-i\alpha\cdot\nabla + \beta m)\psi + \rho(x) F(\langle \psi(\,\cdot\,, t), \rho\rangle), \tag{5.18}$$

under the Wiener condition (2.34), where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = \alpha_0$ are Dirac matrices.

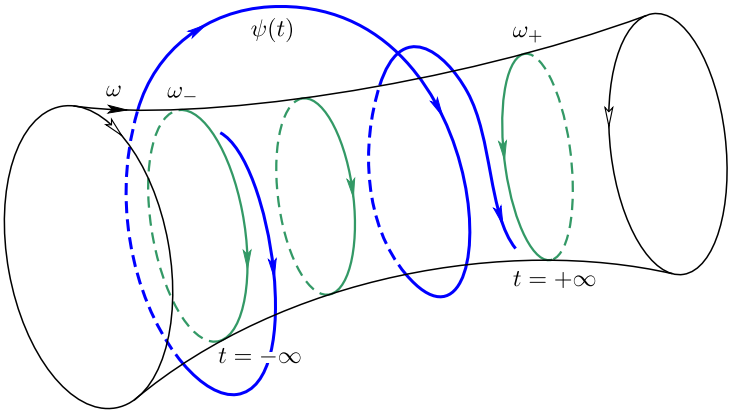

Figure 3. Convergence to stationary orbits.

Recently, the attraction (5.15) was extended in [143] to the 1D Dirac equation coupled to a nonlinear oscillator, and in [136], [137], [142] it was extended to the 3D wave and Klein-Gordon equations with concentrated nonlinearities.

In addition, the attraction (5.15) was extended in [31] to nonlinear space-time discrete Hamiltonian equations that are discrete approximations of equations of type (5.17), that is, they are the corresponding difference schemes. The proof relies on a new version in [92] of the Titchmarsh theorem for distributions on a circle.

Open questions.

I. Global attraction (1.8) to orbits with fixed frequencies $\omega_\pm$ has not yet been proved.

II. Global attraction to stationary orbits for nonlinear Schrödinger equations has also not been proved. In particular, such attraction is not known for the 1D Schrödinger equation coupled with a nonlinear oscillator

$$i\dot\psi(x,t) = -\psi''(x,t) + \delta(x)F(\psi(0,t)), \qquad x \in \mathbb{R}. \tag{5.19}$$

The main difficulty is the infinite 'spectral gap' $(-\infty, 0)$ (see Remark 5.15).

III. Global attraction to solitons (1.6) for the *relativistically invariant* nonlinear Klein–Gordon equations is an open problem. In particular, it is open for the one-dimensional equations

$$\ddot\psi(x,t) = \psi''(x,t) - m^2\psi(x,t) + F(\psi(x,t)). \tag{5.20}$$

The main difficulty is the presence of nonlinear interaction at every point $x \in \mathbb{R}$. The asymptotic stability of solitons (that is, *local attraction* to them) for such equations was first proved in [140] and [141] (see § 6.3 below).

5.1. Method of omega-limit trajectories. The proof of Theorem 5.2 is based on the general strategy of *omega-limit trajectories* first introduced in [99] and developed further in [103]–[108], [92], [136], [142], [143], [30], and [31].

Definition 5.3. An omega-limit trajectory for a function $Y(t) \in C(\mathbb{R}, \mathscr{E})$ is any limit function $Z(t)$ such that

$$Y(t + s_j) \xrightarrow{\mathscr{E}_F} Z(t), \qquad t \in \mathbb{R}, \tag{5.21}$$

as $s_j \to \infty$.

Definition 5.4. A function $Y(t) \in C(\mathbb{R}, \mathscr{E})$ is omega-compact if for any sequence $s_j \to \infty$ there exists a subsequence $s_{j'} \to \infty$ such that (5.21) holds.

These concepts are useful in view of the following lemma, which lies at the basis of our approach.

Lemma 5.5. *Suppose that any solution $Y(t) \in C(\mathbb{R}, \mathscr{E})$ of (5.4) is omega-compact, and any omega-limit trajectory is a stationary orbit*

$$Z(x,t) = (\psi_\omega(x)e^{-i\omega t}, -i\omega\psi_\omega(x)e^{-i\omega t}), \tag{5.22}$$

where $\omega \in \mathbb{R}$. Then the attraction (5.15) holds for each solution $Y(t) \in C(\mathbb{R}, \mathscr{E})$ of (5.4).

Proof. We need to show that

$$\lim_{t\to\infty} \operatorname{dist}(Y(t), \mathscr{I}) = 0.$$

Assume by contradiction that there exists a sequence $s_j \to \infty$ such that

$$\operatorname{dist}(Y(s_j), \mathscr{I}) \geqslant \delta > 0 \qquad \forall\, j \in \mathbb{N}. \tag{5.23}$$

According to the omega-compactness of the solution Y, the convergence (5.21) holds for some subsequence $s_{j'} \to \infty$ and some stationary orbit (5.22):

$$Y(t + s_j) \xrightarrow{\mathscr{E}_F} Z(t), \qquad t \in \mathbb{R}. \tag{5.24}$$

But this convergence with $t = 0$ contradicts (5.23) since $Z(0) \in \mathscr{I}$ by definition (5.11). □

For the proof of Theorem 5.2 it now suffices to check the conditions of Lemma 5.5:

(I) each solution $Y(t) \in C(\mathbb{R}, \mathscr{E})$ of (5.4) is omega-compact;

(II) any omega-limit trajectory is a stationary orbit (5.22).

We check these conditions by analyzing the Fourier transform of solutions with respect to time. The main steps of the proof are as follows.

(1) Spectral representation for solutions of the nonlinear equation (5.4):

$$\psi(t) = \frac{1}{2\pi} \int e^{-i\omega t} \tilde{\psi}(\omega)\, d\omega. \tag{5.25}$$

By the *spectrum* of a solution $\psi(t) := \psi(\,\cdot\,, t)$ we mean the support of its spectral density $\tilde{\psi}(\,\cdot\,)$, which is a tempered distribution of $\omega \in \mathbb{R}$ with values in $H^1(\mathbb{R})$.

(2) The *absolute continuity* of the spectral density $\tilde{\psi}(\omega)$ *on the continuous spectrum* $(-\infty, -m) \cup (m, \infty)$ of the free Klein–Gordon equation, which is an analogue of the Kato theorem on the absence of embedded eigenvalues.

(3) The *omega-limit compactness* of each solution.

(4) The reduction of the spectrum of *each omega-limit trajectory* to a subset of the *spectral gap* $[-m, m]$.

(5) The reduction of this spectrum to a *single point* using the *Titchmarsh convolution theorem*.

Below we follow this programme, referring at some points to the papers [99] and [104] for technically important properties of quasi-measures.

5.2. The spectral representation and the limiting absorption principle. It suffices to prove the attraction (5.15) only for positive times. For simplicity we consider only the solution $\psi(x,t)$ of (5.1) corresponding to zero initial data:

$$\psi(x,0)=0, \qquad \dot\psi(x,0)=0. \tag{5.26}$$

The general case of non-zero initial data can be reduced to this case by a trivial subtraction of the dispersion-wave solution of the free Klein–Gordon equation with these initial data ([99], [104]). We extend $\psi(x,t)$ and $f(t):=F(\psi(0,t))$ by zero for $t<0$:

$$\psi_+(x,t):=\begin{cases}\psi(x,t), & t>0,\\ 0, & t<0,\end{cases} \qquad f_+(t):=\begin{cases}f(t), & t>0,\\ 0, & t<0.\end{cases} \tag{5.27}$$

From (5.1) and (5.26) it follows that these functions satisfy the equation

$$\ddot\psi_+(x,t)=\psi''_+(x,t)-m^2\psi_+(x,t)+\delta(x)f_+(t), \qquad (x,t)\in\mathbb{R}^2, \tag{5.28}$$

in the distribution sense.

The Fourier–Laplace transform with respect to time. For tempered distributions $g(t)$ we let $\widetilde g(\omega)$ denote their Fourier transform, which is defined for $g\in C_0^\infty(\mathbb{R})$ by

$$\widetilde g(\omega)=\int_{\mathbb{R}} e^{i\omega t}g(t)\,dt, \qquad \omega\in\mathbb{R}.$$

The a priori estimates (5.6) imply that $\psi_+(x,t)$ and $f_+(t)$ are bounded functions of $t\in\mathbb{R}$ with values in the Sobolev space $H^1(\mathbb{R})$ and in $\mathbb{C}$, respectively. Therefore, their Fourier transforms are (by definition) *quasi-measures* with values in $H^1(\mathbb{R})$ and in $\mathbb{C}$, respectively [57]. Moreover, these Fourier transforms can be extended from the real axis to analytic functions in the upper complex half-plane $\mathbb{C}^+:=\{\omega\in\mathbb{C}\colon \operatorname{Im}\omega>0\}$ with values in $H^1(\mathbb{R})$ and in $\mathbb{C}$ respectively:

$$\tilde\psi_+(x,\omega)=\int_0^\infty e^{i\omega t}\psi(x,t)\,dt, \quad \widetilde f_+(\omega)=\int_0^\infty e^{i\omega t}f(t)\,dt, \qquad \omega\in\mathbb{C}^+.$$

Further, we have the following convergence of tempered distributions with values in $H^1(\mathbb{R})$ and $\mathbb{C}$, respectively:

$$e^{-\varepsilon t}\psi_+(x,t)\to\psi_+(x,t), \qquad e^{-\varepsilon t}f_+(t)\to f_+(t), \qquad \varepsilon\to 0+.$$

Hence, their Fourier transforms also converge in the same sense:

$$\tilde\psi_+(x,\omega+i\varepsilon)\to\tilde\psi_+(x,\omega), \quad \widetilde f_+(\omega+i\varepsilon)\to\widetilde f_+(\omega), \qquad \varepsilon\to 0+. \tag{5.29}$$

The analytic functions $\tilde\psi_+(x,\omega)$ and $\widetilde f(\omega)$ grow (in norm) no faster than $|\operatorname{Im}\omega|^{-1}$ as $\operatorname{Im}\omega \to 0+$ in view of (5.6). Hence, their boundary values at $\omega \in \mathbb{R}$ are tempered distributions of small singularity: they are the second-order derivatives of continuous functions, as in the case of $\widetilde f_+(\omega) = i/(\omega-\omega_0)$ with $\omega_0 \in \mathbb{R}$, which corresponds to $f_+(t) = \theta(t)e^{-i\omega_0 t}$.

The limiting absorption principle. By (5.26), in terms of the Fourier transform the equation (5.28) becomes the stationary Helmholtz equation

$$-\omega^2\tilde\psi_+(x,\omega) = \tilde\psi_+''(x,\omega) - m^2\tilde\psi_+(x,\omega) + \delta(x)\widetilde f_+(\omega), \qquad x \in \mathbb{R}. \tag{5.30}$$

This equation has two linearly independent solutions, but only one of them is analytic and bounded in $\operatorname{Im}\omega > 0$ with values in $H^1(\mathbb{R})$:

$$\tilde\psi_+(x,\omega) = -\widetilde f_+(\omega)\frac{e^{ik(\omega)|x|}}{2ik(\omega)}, \qquad \operatorname{Im}\omega > 0. \tag{5.31}$$

Here $k(\omega) := \sqrt{\omega^2 - m^2}$, where the branch has a positive imaginary part for $\operatorname{Im}\omega > 0$. For the other branch this function *grows exponentially* as $|x| \to \infty$. Such an argument in the selection of solutions of stationary Helmholtz equations is known as the 'limiting absorption principle' in diffraction theory ([112], [127]).

Spectral representation. We rewrite (5.31) in the form

$$\tilde\psi_+(x,\omega) = \widetilde\alpha(\omega)e^{ik(\omega)|x|}, \qquad \operatorname{Im}\omega > 0, \quad \text{where} \quad \alpha(t) := \psi_+(0,t). \tag{5.32}$$

It is a non-trivial fact that the identity (5.32) between analytic functions keeps its structure for their restrictions to the real axis:

$$\tilde\psi_+(x,\omega+i0) = \widetilde\alpha(\omega+i0)e^{ik(\omega+i0)|x|}, \qquad \omega \in \mathbb{R}, \tag{5.33}$$

where $\tilde\psi_+(\,\cdot\,,\omega+i0)$ and $\widetilde\alpha(\omega+i0)$ are the corresponding quasi-measures with values in $H^1(\mathbb{R})$ and $\mathbb{C}$, respectively. The problem is that the factor $M_x(\omega) := e^{ik(\omega+i0)|x|}$ is not smooth with respect to ω at the points $\omega = \pm m$. Correspondingly, the identity (5.33) must be justified, based on quasi-measure theory [104].

Finally, the inversion of the Fourier transform can be written as

$$\psi_+(x,t) = \frac{1}{2\pi}\langle\tilde\psi_+(x,\omega+i0), e^{-i\omega t}\rangle = \frac{1}{2\pi}\langle\widetilde\alpha(\omega+i0)e^{ik(\omega+i0)|x|}, e^{-i\omega t}\rangle, \qquad x,t \in \mathbb{R}, \tag{5.34}$$

where $\langle\,\cdot\,,\cdot\,\rangle$ is the bilinear duality between distributions and smooth bounded functions. The right-hand side exists by Theorem 5.6 (see below).

5.3. A nonlinear analogue of Kato's theorem. It turns out that the properties of the quasi-measure $\widetilde\alpha(\omega+i0)$ with $|\omega| < m$ and that with $|\omega| > m$ differ significantly. This is because the set $\{i\omega\colon |\omega| \geqslant m\}$ is the continuous spectrum of the generator

$$A = \begin{pmatrix} 0 & 1 \\ d^2/dx^2 - m^2 & 0 \end{pmatrix},$$

which is the generator of the linearization of (5.4). The following theorem plays a key role in the proof of Theorem 5.2. It is a nonlinear analogue of Kato's theorem

on the absence of embedded eigenvalues in the continuous spectrum (see Remark 5.9 below). Let $\Sigma := \{\omega \in \mathbb{R}\colon |\omega| > m\}$. Below we will also write $\widetilde{\alpha}(\omega)$ and $k(\omega)$ instead of $\widetilde{\alpha}(\omega + i0)$ and $k(\omega + i0)$ for $\omega \in \mathbb{R}$.

Theorem 5.6 (see [104], Proposition 3.2). *Let the conditions* (5.2), (5.5), *and* (5.7) *hold, and let* $\psi(t) \in C(\mathbb{R}, \mathscr{E})$ *be any finite-energy solution of* (5.1). *Then the corresponding tempered distribution* $\widetilde{\alpha}(\omega)$ *is absolutely continuous on* Σ. *Moreover,* $\alpha \in L^1(\Sigma)$ *and*

$$\int_\Sigma |\widetilde{\alpha}(\omega)|^2 |\omega k(\omega)|\, d\omega < \infty. \tag{5.35}$$

Proof. We first explain the main idea of the proof. By (5.34), the function $\psi_+(x,t)$ is formally a 'linear combination' of the functions $e^{ik|x|}$ with the *amplitudes* $\widehat{z}(\omega)$:

$$\psi_+(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{z}(\omega) e^{ik(\omega)|x|} e^{-i\omega t}\, d\omega, \qquad x \in \mathbb{R}.$$

For $\omega \in \Sigma$ the functions $e^{ik(\omega)|x|}$ have an infinite $L^2(\mathbb{R})$-norm, while $\psi_+(\,\cdot\,, t)$ has a finite $L^2(\mathbb{R})$-norm. This is possible only if the amplitude is absolutely continuous in Σ. This idea is suggested by the Fourier integral $f(x) = \int_{\mathbb{R}} e^{-ikx} g(k)\, dk$, which belongs to $L^2(\mathbb{R})$ if and only if $g \in L^2(\mathbb{R})$. For example, if one took $\widehat{z}(\omega) = \delta(\omega - \omega_0)$ with $\omega_0 \in \Sigma$, then $\psi_+(\,\cdot\,, t)$ would have infinite $L^2(\mathbb{R})$-norm.

The rigorous proof is based on estimates of Paley–Wiener type [94]. Namely, the Parseval identity and (5.6) imply that

$$\int_{\mathbb{R}} \|\tilde{\psi}_+(\,\cdot\,, \omega + i\varepsilon)\|^2_{H^1(\mathbb{R})}\, d\omega = 2\pi \int_0^\infty e^{-2\varepsilon t} \|\psi_+(\,\cdot\,, t)\|^2_{H^1(\mathbb{R})}\, dt \leqslant \frac{\text{const}}{\varepsilon}, \quad \varepsilon > 0. \tag{5.36}$$

On the other hand, we can estimate exactly the integral on the left-hand side of (5.36). Indeed, according to (5.34),

$$\tilde{\psi}_+(\,\cdot\,, \omega + i\varepsilon) = \widetilde{\alpha}(\omega + i\varepsilon) e^{ik(\omega + i\varepsilon)|x|}.$$

Consequently, (5.36) gives us that

$$\varepsilon \int_{\mathbb{R}} |\widetilde{\alpha}(\omega + i\varepsilon)|^2 \|e^{ik(\omega+i\varepsilon)|x|}\|^2_{H^1(\mathbb{R})}\, d\omega \leqslant \text{const}, \qquad \varepsilon > 0. \tag{5.37}$$

Here is a crucial observation about the asymptotics of the norm of $e^{ik(\omega+i\varepsilon)|x|}$ as $\varepsilon \to 0+$.

Lemma 5.7. (i) *For* $\omega \in \mathbb{R}$,

$$\lim_{\varepsilon \to 0+} \varepsilon \|e^{ik(\omega+i\varepsilon)|x|}\|^2_{H^1(\mathbb{R})} = n(\omega) := \begin{cases} \omega k(\omega), & |\omega| > m, \\ 0, & |\omega| < m, \end{cases} \tag{5.38}$$

where the norm in $H^1(\mathbb{R})$ *is chosen to be* $\|\psi\|_{H^1(\mathbb{R})} = (\|\psi'\|^2_{L^2(\mathbb{R})} + m^2 \|\psi\|^2_{L^2(\mathbb{R})})^{1/2}$.

(ii) *For any* $\delta > 0$ *there exists an* $\varepsilon_\delta > 0$ *such that for all* $|\omega| > m + \delta$ *and* $\varepsilon \in (0, \varepsilon_\delta)$,

$$\varepsilon \|e^{ik(\omega+i\varepsilon)|x|}\|^2_{H^1(\mathbb{R})} \geqslant \frac{n(\omega)}{2}. \tag{5.39}$$

Proof. Let us compute the $H^1(\mathbb{R})$-norm using the Fourier space representation. Setting $k_\varepsilon = k(\omega+i\varepsilon)$, so that $\operatorname{Im} k_\varepsilon > 0$, we get that $F_{x\to k}[e^{ik_\varepsilon|x|}] = 2ik_\varepsilon/(k_\varepsilon^2-k^2)$ for $k\in\mathbb{R}$. Hence, by the Cauchy theorem on residues

$$\|e^{ik_\varepsilon|x|}\|^2_{H^1(\mathbb{R})} = \frac{2|k_\varepsilon|^2}{\pi}\int_{\mathbb{R}}\frac{(k^2+m^2)\,dk}{|k_\varepsilon^2-k^2|^2} = -4\operatorname{Im}\left[\frac{(k_\varepsilon^2+m^2)\overline{k}_\varepsilon}{k_\varepsilon^2-\overline{k}_\varepsilon^2}\right].$$

Substituting here $k_\varepsilon^2 = (\omega+i\varepsilon)^2 - m^2$, we get that

$$\|e^{ik(\omega+i\varepsilon)|x|}\|^2_{H^1(\mathbb{R})} = \frac{1}{\varepsilon}\operatorname{Re}\left[\frac{(\omega+i\varepsilon)^2\overline{k(\omega+i\varepsilon)}}{\omega}\right], \qquad \varepsilon>0, \quad \omega\in\mathbb{R}, \quad \omega\neq 0.$$

The limits (5.38) now follow, since the function $k(\omega)$ is real for $|\omega|>m$ but purely imaginary for $|\omega|<m$. Therefore, the second statement of the lemma also follows, since $n(\omega)>0$ for $|\omega|>m$ and $n(\omega)\sim|\omega|^2$ for $|\omega|\to\infty$. □

Remark 5.8. Clearly, $n(\omega)\equiv 0$ for $|\omega|<m$ without any calculations, since in that case the function $e^{ik(\omega)|x|}$ decays exponentially in x, and hence the $H^1(\mathbb{R})$-norm of $e^{ik(\omega+i\varepsilon)|x|}$ remains finite when $\varepsilon\to 0+$.

Substituting (5.39) into (5.37), we get that

$$\int_{\Sigma_\delta}|\widetilde{\alpha}(\omega+i\varepsilon)|^2\omega k(\omega)\,d\omega \leqslant 2C, \qquad 0<\varepsilon<\varepsilon_\delta, \tag{5.40}$$

with the same C as in (5.37), and with the region $\Sigma_\delta := \{\omega\in\mathbb{R}\colon |\omega|>m+\delta\}$. We conclude that for each $\delta>0$ the set of functions

$$g_{\delta,\varepsilon}(\omega) = \widetilde{\alpha}(\omega+i\varepsilon)|\omega k(\omega)|^{1/2}, \qquad \varepsilon\in(0,\varepsilon_\delta),$$

is bounded in the Hilbert space $L^2(\Sigma_\delta)$, so that by the Banach theorem it is weakly compact. Hence, convergence of the distributions (5.29) implies weak convergence in $L^2(\Sigma_\delta)$:

$$g_\varepsilon \rightharpoonup g, \qquad \varepsilon\to 0+,$$

and the limit function $g(\omega)$ coincides with the distribution $\widehat{z}(\omega)|\omega k(\omega)|^{1/2}$ restricted to Σ_δ. It remains to note that the norms of g in $L^2(\Sigma_\delta)$ with all $\delta>0$ are bounded in view of (5.40), and this implies (5.35). Finally, $\widetilde{\alpha}(\omega)\in L^1(\overline{\Sigma})$ by (5.35) and the Cauchy–Schwarz inequality. Theorem 5.6 is proved. □

Remark 5.9. Theorem 5.6 is a nonlinear analogue of Kato's theorem on the absence of embedded eigenvalues in the continuous spectrum. Indeed, solutions of type $\psi_*(x)e^{-i\omega_* t}$ become $\psi_*(x)\left[\pi i\delta(\omega-\omega_*) + \text{p. v.}\,\dfrac{1}{i(\omega-\omega_*)}\right]$ in the Fourier–Laplace transform, and this is forbidden for $|\omega_*|>m$ by Theorem 5.6.

5.4. Splitting into dispersion and bound components. Theorem 5.6 presupposes a splitting of the solutions (5.34) into a 'dispersion' component and a 'bound' component:

$$\begin{aligned}\psi_+(x,t) &= \frac{1}{2\pi}\int_\Sigma(1-\zeta(\omega))\widetilde{\alpha}(\omega)e^{ik(\omega)|x|}e^{-i\omega t}\,d\omega + \frac{1}{2\pi}\langle\zeta(\omega)\widetilde{\alpha}(\omega)e^{ik(\omega)|x|}, e^{-i\omega t}\rangle\\ &= \psi_d(x,t)+\psi_b(x,t), \qquad t>0, \quad x\in\mathbb{R},\end{aligned} \tag{5.41}$$

where

$$\zeta(\omega) \in C_0^\infty(\mathbb{R}), \quad \text{and} \quad \zeta(\omega) = 1 \quad \text{for} \quad \omega \in [-m-1, m+1].$$

Note that $\psi_d(x,t)$ is a dispersion wave, because

$$\psi_d(x,t) := \frac{1}{2\pi}\int_\Sigma (1-\zeta(\omega))e^{-i\omega t}\widetilde{\alpha}(\omega)e^{ik(\omega)|x|}\,d\omega \to 0, \qquad t \to \infty,$$

according to the Riemann–Lebesgue theorem, since $\alpha \in L^1(\Sigma)$ by Theorem 5.6. Moreover, it is easy to prove that

$$(\psi_d(\,\cdot\,,t), \dot{\psi}_d(\,\cdot\,,t)) \to 0, \qquad t \to \infty, \tag{5.42}$$

in the seminorms (2.12). Therefore, it remains to prove the attraction (5.15) for $Y_b(t) := (\psi_b(\,\cdot\,,t), \dot{\psi}_b(\,\cdot\,,t))$ instead of $Y(t)$:

$$Y_b(t) \xrightarrow{\mathscr{E}_F} \mathscr{S}, \qquad t \to \infty. \tag{5.43}$$

5.5. Omega-compactness. Here we establish the omega-compactness of the trajectory $Y_b(t)$, which is necessary for the application of Lemma 5.5. First we note that the bound component $\psi_b(x,t)$ is a smooth function for $x \neq 0$, and

$$\partial_x^j\partial_t^l\psi_b(x,t) = \frac{1}{2\pi}\langle \zeta(\omega)(ik(\omega)\operatorname{sgn}x)^j\widetilde{\alpha}(\omega)e^{ik(\omega)|x|}, (-i\omega)^l e^{-i\omega t}\rangle, \quad t > 0, \quad x \neq 0, \tag{5.44}$$

for any $j, l = 0, 1, \dots$. These formulae must be justified, since the function $k(\omega)$ is not smooth at the points $\omega = \pm m$. The needed justification is done in [99], [104] by a suitable development of the theory of quasi-measures. These formulae imply the boundedness of each derivative.

Lemma 5.10 (see [104], Proposition 4.1). *For all* $j, l = 0, 1, 2, \dots$

$$\sup_{x\neq 0}\sup_{t\in\mathbb{R}} |\partial_x^j\partial_t^l\psi_b(x,t)| < \infty. \tag{5.45}$$

Proof. Note that the distribution $\widetilde{\alpha}(\omega)$ generally is not a finite measure, since we only know that $\alpha(t) := \psi_+(0,t)$ is a bounded function by (5.32) and (5.6). To prove the lemma, it suffices to check that

$$\zeta(\omega)(ik(\omega)\operatorname{sgn}x)^j e^{ik(\omega)|x|}(-i\omega)^l = \widetilde{g}_x(\omega),$$

where the function $g_x(\,\cdot\,)$ belongs to a bounded subset of $L^1(\mathbb{R})$ for $x \neq 0$ and $t \in \mathbb{R}$. This implies the lemma, since by the Parseval identity the right-hand side of (5.44) is the convolution

$$\langle \alpha(t-s), g_x(s)\rangle,$$

where $\alpha(t)$ is a bounded function. □

Remark 5.11. All the properties of quasi-measures used are justified in [99], [104] by similar arguments relying on the Parseval identity.

By the Ascoli–Arzelà theorem, for any sequence $s_j \to \infty$ there is a subsequence $s_{j'} \to \infty$ such that

$$\partial_x^j \partial_t^l \psi_b(x, s_{j'} + t) \to \partial_x^j \partial_t^l \beta(x,t), \qquad x \neq 0, \quad t \in \mathbb{R}, \tag{5.46}$$

for any $j, l = 0, 1, 2, \dots$, and this convergence is uniform for $|x| + |t| \leqslant R$. The estimates (5.45) imply that

$$\sup_{(x,t)\in\mathbb{R}^2} |\partial_x^j \partial_t^l \beta(x,t)| < \infty. \tag{5.47}$$

Corollary 5.12. *Each solution $Y(t) \in C(\mathbb{R}, \mathscr{E})$ of (5.4) is omega-compact.*

This follows from (5.41), (5.42), and (5.46).

5.6. Reduction of spectrum of omega-limit trajectories to a spectral gap. The convergence of the functions (5.46) implies the convergence of their Fourier transforms:

$$\widetilde{\psi}_b(x,\omega) e^{-i\omega s_{j'}} \to \widetilde{\beta}(x,\omega) \quad \forall\, x \in \mathbb{R}, \tag{5.48}$$

in the sense of tempered distributions of $\omega \in \mathbb{R}$.

Lemma 5.13. *For any $x \in \mathbb{R}$*

$$\widetilde{\beta}(x,\omega) = 0, \qquad |\omega| > m. \tag{5.49}$$

Proof. The convergence (5.48) and the representation (5.44) with $j = l = 0$ imply that

$$\zeta(\omega)\widetilde{\alpha}(\omega) e^{ik(\omega)|x|} e^{-i\omega s_{j'}} \to \widetilde{\beta}(x,\omega) \quad \forall\, x \in \mathbb{R}, \tag{5.50}$$

in the sense of tempered distributions of $\omega \in \mathbb{R}$. Moreover, this convergence takes place in the stronger *Ascoli–Arzelà topology* in the space of quasi-measures [104]. In addition, the function $e^{-ik(\omega)|x|}$ is a multiplier in the space of quasi-measures with this topology by Lemma B.3 of [104]). Therefore, (5.50) implies that

$$\zeta(\omega)\widetilde{\alpha}(\omega) e^{-i\omega s_{j'}} \to \widetilde{\gamma}(\omega) := \widetilde{\beta}(x,\omega) e^{-ik(\omega)|x|} \quad \forall\, x \in \mathbb{R}, \tag{5.51}$$

in the same topology of quasi-measures. Applying the same lemma again, we obtain

$$\beta(x,t) = \frac{1}{2\pi} \langle \widetilde{\gamma}(\omega) e^{ik(\omega)|x|}, e^{-i\omega t} \rangle, \qquad (x,t) \in \mathbb{R}^2. \tag{5.52}$$

Note that

$$\beta(0,t) = \gamma(t). \tag{5.53}$$

Finally, the key observation is that (5.51) and Theorem 5.6 imply that

$$\operatorname{supp} \widetilde{\gamma} \subset [-m, m] \tag{5.54}$$

by the Riemann–Lebesgue theorem. □

5.7. Reduction of spectrum of omega-limit trajectories to a single point.

5.7.1. *Equation for omega-limit trajectories and spectral inclusion.* The question arises of the available means for verifying the representation (5.22) for omega-limit trajectories. We have no formulae for solutions of equation (5.4), and so the only hope is to use the nonlinear equation itself. The key observation, albeit simple, is that $\beta(x,t)$ is a solution of this nonlinear equation *for all* $t \in \mathbb{R}$, despite the fact that $\psi_+(x,t)$ is a solution of the equation (5.4) only for $t > 0$, due to (5.27).

Lemma 5.14. *The function* $\beta(x,t)$ *satisfies the original equation* (5.4):

$$\ddot{\beta}(x,t) = \beta'(x,t) - m^2\beta(x,t) + \delta(x)F(\beta(0,t)), \qquad (x,t) \in \mathbb{R}^2. \tag{5.55}$$

Proof. This lemma follows by (5.42) and (5.46) in the limit as $s_{j'} \to \infty$ in the equation (5.4) for $\psi_+(x, s_{j'}+t) = \psi_d(x, s_{j'}+t) + \psi_b(x, s_{j'}+t)$ with $s_{j'}+t > 0$. □

Applying the Fourier transform to the equation (5.55), we now get the corresponding 'nonlinear stationary Helmholtz equation'

$$-\omega^2\widetilde{\beta}(x,\omega) = \widetilde{\beta}''(x,\omega) - m^2\widetilde{\beta}(x,\omega) + \delta(x)\widetilde{f}(\omega), \qquad (x,\omega) \in \mathbb{R}^2, \tag{5.56}$$

where we define $f(t) := F(\beta(0,t)) = F(\gamma(t))$ in accordance with (5.53). From (5.8) we get that

$$f(t) = a(|\gamma(t)|)\gamma(t) = A(t)\gamma(t), \qquad A(t) := a(|\gamma(t)|), \quad t \in \mathbb{R}.$$

Finally, in the Fourier transform we get the convolution $\widetilde{f} = \widetilde{A} * \widetilde{\gamma}$, which exists by (5.54). Correspondingly, (5.56) is now

$$-\omega^2\widetilde{\beta}(x,\omega) = \widetilde{\beta}''(x,\omega) - m^2\widetilde{\beta}(x,\omega) + \delta(x)[\widetilde{A} * \widetilde{\gamma}](\omega), \qquad (x,\omega) \in \mathbb{R}^2.$$

This identity implies the key *spectral inclusion*

$$\operatorname{supp} \widetilde{A} * \widetilde{\gamma} \subset \operatorname{supp} \widetilde{\gamma}, \tag{5.57}$$

because $\operatorname{supp} \widetilde{\beta}(x,\,\cdot\,) \subset \operatorname{supp} \widetilde{\gamma}$ and $\operatorname{supp} \widetilde{\beta}'(x,\,\cdot\,) \subset \operatorname{supp} \widetilde{\gamma}$ in view of the representation (5.52). From this inclusion, we will derive (5.22) below, using a fundamental result in harmonic analysis — the Titchmarsh convolution theorem.

5.7.2. *Titchmarsh convolution theorem.* In 1926 Titchmarsh proved a theorem on the distribution of zeros of entire functions (see [159], p. 119, and [203]), which implies, in particular, the following corollary (see [72], Theorem 4.3.3).

Theorem. *Let* $f(\omega)$ *and* $g(\omega)$ *be distributions of* $\omega \in \mathbb{R}$ *with bounded supports. Then*

$$[\operatorname{supp} f * g] = [\operatorname{supp} f] + [\operatorname{supp} g],$$

where $[X]$ *denotes the convex hull of a set* $X \subset \mathbb{R}$.

Note that in our situation, $\operatorname{supp}\widetilde{\gamma}$ is bounded in view of (5.54). Consequently, $\operatorname{supp}\widetilde{A}$ is also bounded, since $A(t) := a(|\gamma(t)|)$ is a polynomial in $|\gamma(t)|^2$ according to (5.12). Now the spectral inclusion (5.57) and the Titchmarsh theorem imply that

$$[\operatorname{supp}\widetilde{A}] + [\operatorname{supp}\widetilde{\gamma}] \subset [\operatorname{supp}\widetilde{\gamma}],$$

whence it immediately follows in turn that $[\operatorname{supp}\widetilde{A}] = \{0\}$. Besides, $A(t)$ is a bounded function in view of (5.47), because $\gamma(t) = \beta(0,t)$. Therefore, $\widetilde{A}(\omega) = C\delta(\omega)$, and hence

$$a(|\gamma(t)|) = C_1, \qquad t \in \mathbb{R}.$$

The strict nonlinearity condition (5.12) now implies that

$$|\gamma(t)| = C_2, \qquad t \in \mathbb{R}.$$

This immediately gives us that $\operatorname{supp}\widetilde{\gamma} = \{\omega_+\}$ by the same Titchmarsh theorem for the convolution $\widetilde{\gamma} * \overline{\widetilde{\gamma}} = C_3\delta(\omega)$. Therefore, $\widetilde{\gamma}(\omega) = C_4\delta(\omega - \omega_+)$, and now (5.22) follows from (5.52).

Remark 5.15. In the case of the Schrödinger equation (5.19), the Titchmarsh theorem does not work. The fact is that the continuous spectrum of the operator $-d^2/dx^2$ is the half-line $[0,\infty)$, so now the role of the 'spectral gap' is played by the unbounded interval $(-\infty, 0)$. Correspondingly, in this case the spectral inclusion (5.58) gives us only that $\operatorname{supp}\widetilde{\beta}(x,\,\cdot\,) \subset (-\infty, 0)$, while the Titchmarsh theorem applies only to distributions with bounded support.

5.8. Remarks on dispersion radiation and nonlinear energy transfer. Let us explain the informal arguments for attraction to stationary orbits behind the formal proof of Theorem 5.2. The main part of the proof involves the study of the spectrum of omega-limit trajectories

$$\beta(x,t) = \lim_{s_{j'}\to\infty} \psi(x, s_{j'} + t).$$

Theorem 5.6 implies the spectral inclusion (5.54), which leads to the inclusion

$$\operatorname{supp}\widetilde{\beta}(x,\,\cdot\,) \subset [-m, m], \qquad x \in \mathbb{R}. \tag{5.58}$$

The Titchmarsh theorem then lets us conclude that

$$\operatorname{supp}\widetilde{\beta}(x,\,\cdot\,) = \{\omega_+\}. \tag{5.59}$$

These two inclusions are suggested by the following two informal considerations.

A. *Dispersion radiation in the continuous spectrum.*

B. *Nonlinear spreading of the spectrum and energy transfer from lower to higher harmonics.*

A. *Dispersion radiation in the continuous spectrum.* The inclusion (5.58) is because of the dispersion mechanism, which can be illustrated by the example of energy radiation in a wave field of a harmonic source with a frequency lying in the continuous spectrum. Namely, let us consider a one-dimensional linear Klein–Gordon equation with a *harmonic source*

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + b(x)e^{-i\omega_0 t}, \qquad x \in \mathbb{R}, \tag{5.60}$$

where $b \in L^2(\mathbb{R})$ and the real frequency ω_0 is different from $\pm m$. Then the *limiting amplitude principle* holds [153], [171], [112]:

$$\psi(x,t) \sim a(x)e^{-i\omega_0 t}, \qquad t \to \infty. \tag{5.61}$$

For the equation (5.60), this follows directly from the Fourier–Laplace transform in time

$$\tilde{\psi}(\omega,t) = \int_0^\infty e^{i\omega t}\psi(x,t)\,dt, \qquad x \in \mathbb{R}, \quad \operatorname{Im}\omega > 0. \tag{5.62}$$

Indeed, applying this transform to equation (5.60), we get that

$$-\omega^2\tilde{\psi}(x,\omega) = \tilde{\psi}''(x,\omega) - m^2\tilde{\psi}(x,\omega) + \frac{b(x)}{i(\omega-\omega_0)}, \qquad x \in \mathbb{R}, \quad \operatorname{Im}\omega > 0,$$

where for simplicity we assume zero initial data. Hence,

$$\tilde{\psi}(\,\cdot\,,\omega) = \frac{R(\omega)b}{i(\omega-\omega_0)} = \frac{R(\omega_0+i0)b}{i(\omega-\omega_0)} + \frac{R(\omega)b - R(\omega_0+i0)b}{i(\omega-\omega_0)}, \quad \operatorname{Im}\omega > 0, \tag{5.63}$$

where

$$R(\omega) := (H-\omega^2)^{-1}$$

is the resolvent of the Schrödinger operator $H := -d^2/dx^2 + m^2$. This resolvent is a convolution operator with fundamental solution $-e^{ik(\omega)|x|}/(2ik(\omega))$, where $k(\omega) = \sqrt{\omega^2 - m^2} \in \overline{\mathbb{C}^+}$ for $\omega \in \overline{\mathbb{C}^+}$, as in (5.31). The last quotient in (5.63) is regular at $\omega = \omega_0$, and therefore its contribution is a dispersion wave which decays like (5.42) in local energy seminorms. Consequently, the long-time asymptotics of $\psi(x,t)$ is determined by the middle quotient in (5.63), and therefore (5.61) holds with the limiting amplitude $a(x) = R(\omega_0+i0)b$. The Fourier transform of this limiting amplitude is equal to

$$\widehat{a}(k) = -\frac{\widehat{b}(k)}{k^2 + m^2 - (\omega_0+i0)^2}, \qquad k \in \mathbb{R}.$$

This formula shows that the properties of the limiting amplitude differ significantly in the cases $|\omega_0| < m$ and $|\omega_0| \geqslant m$: $a(x) \in H^2(\mathbb{R})$ for $|\omega_0| < m$, but

$$a(x) \notin L^2(\mathbb{R}) \quad \text{for } |\omega_0| \geqslant m \tag{5.64}$$

if $|\widehat{b}(k)| \geqslant \varepsilon > 0$ in a neighbourhood of the 'sphere' $|k|^2 + m^2 = \omega_0^2$ (which consists of two points in the 1D case). *This means the following.*

I. *In the case* $|\omega_0| \geqslant m$ the energy of the solution $\psi(x,t)$ tends to infinity for large times according to (5.61) and (5.64). This means that energy is transmitted from the harmonic source to the wave field!

II. *On the contrary, for* $|\omega_0| < m$ the energy of the solution remains bounded, so there is no radiation.

It is this radiation in the case of $|\omega_0| \geqslant m$ that prohibits the presence of harmonics with such frequencies in omega-limit trajectories. Indeed, any omega-limit trajectory cannot radiate at all, because the total energy is finite and bounded from below, and hence the radiation cannot last forever. These physical arguments make the inclusion (5.58) plausible, although a rigorous proof of it, as was seen above, requires special arguments.

Recall that the set $i\Sigma := \{i\omega_0 \in \mathbb{R}\colon |\omega_0| \geqslant m\}$ coincides with the continuous spectrum of the generator of the free Klein–Gordon equation. Radiation in the continuous spectrum is well known in the theory of waveguides. Namely, waveguides can transmit only signals with a frequency $|\omega_0| > \mu$, where μ is a *threshold frequency*, which is an edge point of the continuous spectrum [160]. In our case, the waveguide occupies the 'entire space' $x \in \mathbb{R}$ and is described by the nonlinear Klein–Gordon equation (5.1) with the threshold frequency m.

B. *Nonlinear inflation of spectrum and energy transfer from lower to higher harmonics.* Let us show that the single-frequency spectrum (5.59) is due to inflation of the spectrum by nonlinear functions. For example, consider the potential $U(\psi) = |\psi|^4$. Correspondingly, $F(\psi) = -\nabla_{\overline{\psi}} U(\psi) = -4|\psi|^2\psi$. We consider the sum $\psi(t) = e^{i\omega_1 t} + e^{i\omega_2 t}$ of two harmonics, whose spectrum is shown in Fig. 4.

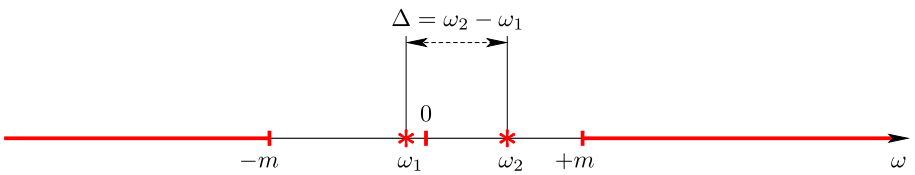

Figure 4. Two-point spectrum.

We substitute this sum into the nonlinearity:

$$F(\psi(t)) \sim \psi(t)\overline{\psi(t)}\psi(t) = e^{i\omega_2 t}e^{-i\omega_1 t}e^{i\omega_2 t} + \cdots = e^{i(\omega_2+\Delta)t} + \cdots, \quad \Delta := \omega_2 - \omega_1.$$

The spectrum of this expression contains harmonics with the new frequencies $\omega_1 - \Delta$ and $\omega_2 + \Delta$. As a result, all the frequencies $\omega_1 - \Delta, \omega_1 - 2\Delta, \dots$ and $\omega_2 + \Delta$, $\omega_2 + 2\Delta, \dots$ also will appear in the nonlinear dynamics described by (5.1) (see Fig. 5). Consequently, these frequencies will appear also in the nonlinear δ-function term which plays the role of a source.

As we already know, these frequencies lying in the continuous spectrum $|\omega| > m$ will surely cause energy radiation. This radiation will continue until the spectrum of the solution contains at least two different frequencies. It is this fact that prohibits the presence of two different frequencies in omega-limit trajectories because the total energy is finite, and thus the radiation cannot continue forever.

However, we underscore that the precise meaning of inflation of the spectrum by a nonlinearity is established by the Titchmarsh convolution theorem.

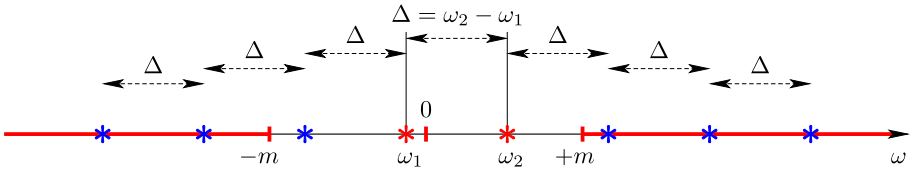

Figure 5. Nonlinear inflation of the spectrum.

Remark 5.16. The above arguments physically mean the following two-step *nonlinear radiation mechanism*:

(i) a nonlinearity inflates the spectrum, which means energy transfer from lower to higher harmonics;

(ii) the dispersion radiation transfers energy to infinity.

We have rigorously justified such a nonlinear radiation mechanism for the first time for the nonlinear U(1)-invariant Klein–Gordon and Dirac equations (5.4) and (5.16)–(5.18). Our numerical experiments demonstrate an analogous radiation mechanism for nonlinear *relativistic* wave equations (see Remark 7.1). However, a rigorous proof is still missing.

Remark 5.17. Let us comment on the term *generic equation* in our conjecture (1.4).

(i) The asymptotic expressions (2.36), (2.37) hold under the Wiener condition (2.34), which defines a certain 'open dense set' of functions ρ. This asymptotic expression may break down if the Wiener condition fails. For example, if $\rho(x) \equiv 0$, then the particle dynamics is generally independent of the fields, and hence the attraction to stationary states can fail.

(ii) Similarly, for an open set of U(1)-invariant equations corresponding to the polynomials (5.12) with $N \geqslant 2$ the asymptotic expression (5.15) is valid. However, this asymptotic expression may break down for 'exceptional' U(1)-invariant equations, in particular, for the linear equations corresponding to the polynomials (5.12) with $N = 1$. Such examples are constructed in [104].

(iii) The general situation is the following. Let a Lie group g be a (proper) subgroup of some larger Lie group G. Then G-invariant equations form an 'exceptional subset' among all g-invariant equations, and the corresponding asymptotics (1.4) may be completely different. For example, the trivial group $\{e\}$ is a subgroup in U(1) and in $\mathbb{R}^n$, and the asymptotic expressions (1.6) and (1.8) may differ significantly from (1.5).

6. Asymptotic stability of stationary orbits and solitons

Asymptotic stability of solitary manifolds means local attraction, that is, for states sufficiently close to the manifold. The main feature of this attraction is the instability of the dynamics *along the manifold*. This follows directly from the fact that solitons move with different speeds and therefore scatter apart over large times.

Analytically, this instability is caused by the presence of the eigenvalue $\lambda = 0$ in the spectrum of the generator of the linearized dynamics. Namely, the tangent

vectors to solitary manifolds are eigenvectors and associated vectors of the generator. They correspond to the zero eigenvalue. Therefore, the Lyapunov theory is not applicable to this case.

In a series of articles by Weinstein and Soffer and by Buslaev, Perelman, and Sulem in 1985–2003, an original strategy was developed for proving asymptotic stability of solitary manifolds. This strategy is based on (i) a special projection of a trajectory on the solitary manifold, (ii) modulation equations for the parameters of the projection, and (iii) long-time decay of the transversal component. This approach is a far-reaching development of the Lyapunov stability theory.

6.1. Asymptotic stability of stationary orbits. Orthogonal projection. This strategy arose in 1985–1992 in the pioneering work of Soffer and Weinstein ([191], [192], [208]; see also the review [190]) involving the nonlinear U(1)-invariant Schrödinger equations with real potential $V(x)$,

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t) + \lambda|\psi(x,t)|^p\psi(x,t), \qquad x \in \mathbb{R}^n, \tag{6.1}$$

where $\lambda \in \mathbb{R}$, $p = 3$ or 4, $n = 2$ or 3, and $\psi(x,t) \in \mathbb{C}$. The corresponding Hamiltonian functional is

$$\mathscr{H} = \int \left[\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}V(x)|\psi(x)|^2 + \frac{\lambda}{p}|\psi(x)|^p\right] dx.$$

For $\lambda = 0$, the equation (6.1) is linear. It is assumed that the discrete spectrum of the Schrödinger operator $H := -\Delta + V(x)$ with short-range potential is a single point $\omega_* < 0$, and the point zero is neither an eigenvalue nor a resonance for H. Let $\phi_*(x)$ denote the corresponding ground state:

$$(-\Delta + V(x))\phi_*(x) = \omega_*\phi_*(x). \tag{6.2}$$

Then the functions $C\phi_*(x)e^{-i\omega_* t}$ are periodic solutions for arbitrary complex constants C. The corresponding phase curves are circles filling the complex plane.

For nonlinear equations (6.1) with a small real $\lambda \neq 0$, it turns out that a remarkable *bifurcation* occurs: a small neighbourhood of the zero of the complex plane turns into an analytic invariant solitary manifold $\mathscr{S}$ that is still filled with invariant circles which are trajectories of *stationary orbits* of the form (5.10):

$$\psi(x,t) = \psi_\omega(x)e^{-i\omega t}, \tag{6.3}$$

with frequencies ω close to ω_*.

Remark 6.1. All these solutions $\psi_\omega(x)e^{-i\omega t}$ are called ground states in this case.

The main result in [191] and [192] (see also [179]) is the long-time attraction to one of these ground states for any solution with sufficiently small initial data,

$$\psi(x,t) = \psi_\pm(x)e^{-i\omega_\pm t} + r_\pm(x,t), \tag{6.4}$$

where the remainder term decays in weighted norms: for $\sigma > 2$

$$\|\langle x\rangle^{-\sigma} r_\pm(\,\cdot\,,t)\|_{L^2(\mathbb{R}^n)} \to 0, \qquad t \to \pm\infty,$$

where $\langle x\rangle := (1+|x|)^{1/2}$. The proof is based on linearization of the dynamics and decomposition of solutions into two components,

$$\psi(t) = e^{-i\Theta(t)}(\psi_{\omega(t)} + \phi(t)),$$

with the orthogonality condition

$$\langle \psi_{\omega(t)}, \phi(t)\rangle = 0 \tag{6.5}$$

(see [191], (3.2) and (3.4)). This orthogonality and the dynamics (6.1) imply the *modulation equations* for $\omega(t)$ and $\gamma(t)$, where $\gamma(t) := \Theta(t) - \int_0^t \omega(s)\,ds$ (see (3.2) and (3.9a), (3.9b) in [191]). The orthogonality (6.5) also implies that the component $\phi(t)$ lies in the space of the continuous spectrum of the Schrödinger operator $H(\omega_0) := -\Delta + V + \lambda|\psi_{\omega_0}|^p$, and this leads to the long-time decay of $\phi(t)$ (see [191], (4.2a) and (4.2b)). Finally, this decay implies the convergence $\omega(t) \to \omega_\pm$ and the asymptotics (6.4).

These results and methods have been further developed in the many publications for nonlinear Schrödinger, wave, and Klein–Gordon equations with potentials under various spectral assumptions on linearized dynamics ([24], [22], [120], [191], [192], [179], [193], [194], [190], [208]).

6.2. Asymptotic stability of solitons. Symplectic projection. A genuine breakthrough in the theory of asymptotic stability was achieved in 1990–2003 by Buslaev, Perelman, and Sulem [25]–[27], who first generalized asymptotics of the type (6.4) for translation-invariant 1D Schrödinger equations

$$i\dot\psi(x,t) = -\psi''(x,t) - F(\psi(x,t)), \qquad x \in \mathbb{R}, \tag{6.6}$$

which are also assumed to be U(1)-invariant. The latter means that the nonlinear function $F(\psi) = -\nabla_{\overline\psi} U(\psi)$ satisfies the identities (5.7)–(5.9). Also assumed is the condition

$$U(\psi) = \mathscr{O}(|\psi|^{10}), \qquad \psi \to 0, \tag{6.7}$$

which is apparently due to an inadequacy in the technique. Under some simple additional conditions on the potential U (see below), there exist *stationary orbits*, that is, finite-energy solutions of the form

$$\psi(x,t) = \psi_0(x)e^{i\omega_0 t} \tag{6.8}$$

with $\omega_0 > 0$. The amplitude $\psi_0(x)$ satisfies the corresponding stationary equation

$$-\omega_0\psi_0(x) = -\psi_0''(x) - F(\psi_0(x)), \qquad x \in \mathbb{R}, \tag{6.9}$$

which implies the 'conservation law'

$$\frac{|\psi_0'(x)|^2}{2} + U_e(\psi_0(x)) = E, \tag{6.10}$$

where the 'effective potential' $U_e(\psi) = U(\psi) + \omega_0\dfrac{|\psi|^2}{2}$ is equivalent to $\omega_0\dfrac{|\psi|^2}{2}$ as $\psi \to 0$ by (6.7). For the existence of a finite-energy solution (6.8), the graph of the

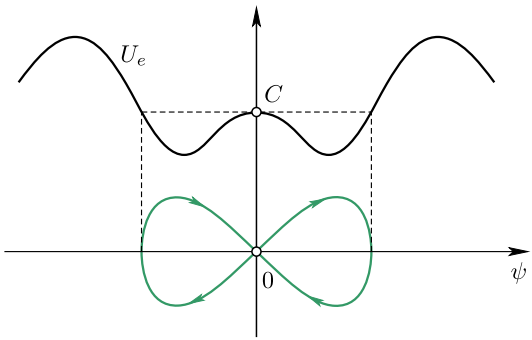

Figure 6. Reduced potential and soliton.

effective potential $U_e(\psi)$ must be similar to the graph of the potential in Fig. 6. The finite-energy solution is defined by (6.10) with the constant $E = U_e(0)$, since for other values of E the solutions of (6.10) do not converge to zero as $|x| \to \infty$. This equation with $E = U_e(0)$ implies that

$$\frac{|\psi_0'(x)|^2}{2} = U_e(0) - U_e(\psi_0(x)) \sim \frac{\omega_0}{2}\psi_0^2(x). \tag{6.11}$$

Hence, for finite-energy solutions

$$\psi_0(x) \sim e^{-\sqrt{\omega_0}\,|x|}, \qquad |x| \to \infty. \tag{6.12}$$

It is easy to verify that the following functions are also solutions (*moving solitons*):

$$\psi_{\omega,v,a,\theta}(x,t) = \psi_\omega(x - vt - a)e^{i(\omega t + kx + \theta)}, \qquad \omega = \omega_0 - \frac{v^2}{4}, \quad k = \frac{v}{2}. \tag{6.13}$$

The set of all such solitons with parameters ω, v, a, and θ forms a four-dimensional smooth submanifold $\mathscr{S}$ in the Hilbert phase space $\mathscr{X} := L^2(\mathbb{R})$. The moving solitons (6.13) are obtained from the standing soliton (6.8) by the Galilean transformation

$$G(a,v,\theta)\colon \psi(x,t) \mapsto \varphi(x,t) = \psi(x - vt - a, t)\exp\left(i\left(-\frac{v^2}{4}t + \frac{v}{2}x + \theta\right)\right). \tag{6.14}$$

It is easy to verify that the Schrödinger equation (6.6) is invariant with respect to this transformation group.

The linearization of the Schrödinger equation (6.6) on the stationary orbit (6.8) is obtained by substituting $\psi(x,t) = (\psi_0(x) + \chi(x))e^{-i\omega_0 t}$ and then retaining terms of first order in χ. This linearized equation contains χ and $\overline{\chi}$, and hence it is not linear over the field of complex numbers. This follows from the fact that the nonlinearity $F(\psi)$ is not complex-analytic because of the U(1)-invariance (5.7). The complexification of this linearized equation is

$$\dot{\Psi}(x,t) = C_0\Psi(x,t), \qquad C_0 = -jH_0, \tag{6.15}$$

where j is a real 2×2 matrix representing multiplication by i, $\Psi(x,t)\in\mathbb{C}^2$, and $H_0=-d^2/dx^2+\omega_0+V(x)$, with $V(x)$ a real matrix potential which decreases exponentially as $|x|\to\infty$ according to (6.12). Note that the operator $C_0=C_{\omega_0,0,0,0}$ corresponds to the linearization on the soliton (6.13) with parameters $\omega=\omega_0$ and $a=v=\theta=0$. Similar operators $C_{\omega,a,v,\theta}$ corresponding to linearization on the solitons (6.13) with various parameters ω, a, v, and θ are also connected by the linear Galilean transformation (6.14). Therefore, their spectral properties completely coincide. In particular, their continuous spectrum coincides with $(-i\infty,-i\omega_0]\cup[i\omega_0,i\infty)$.

The main results in [25]–[27] are asymptotics of the form (6.4) for solutions with initial data close to the solitary manifold $\mathscr{S}$:

$$\psi(x,t)=\psi_\pm(x-v_\pm t)e^{-i(\omega_\pm t+k_\pm x)}+W(t)\Phi_\pm+r_\pm(x,t),\qquad \pm t>0, \tag{6.16}$$

where $W(t)$ is the dynamical group of the free Schrödinger equation, the $\Phi_\pm$ are some scattering states of finite energy, and the $r_\pm$ are remainder terms which decay to zero in a global norm:

$$\|r_\pm(\,\cdot\,,t)\|_{L^2(\mathbb{R})}\to 0,\qquad t\to\pm\infty. \tag{6.17}$$

These asymptotics were obtained under the following assumptions about the spectrum of the generator B_0.

U1. The discrete spectrum of the operator C_0 consists of exactly three eigenvalues, 0 and $\pm i\lambda$, and

$$\lambda<\omega_0<2\lambda. \tag{6.18}$$

This condition means that the discrete mode can interact with the continuous spectrum already in the first order of perturbation theory.

U2. The edge points $\pm i\omega_0$ of the continuous spectrum are neither eigenvalues nor resonances of C_0.

U3. Furthermore, the condition (1.0.12) in [27] is assumed, which means a strong coupling of the discrete and continuous spectral components that ensures energy radiation, as in the case of the Wiener condition (2.34). This condition (1.0.12) ensures that the interaction of the discrete component with the continuous spectrum does not vanish in the first order of perturbation theory; it is a nonlinear version of the Fermi Golden Rule [181] introduced by Sigal in the context of nonlinear PDEs [189].

Examples of potentials satisfying all these conditions were constructed in [117].

In 2001, Cuccagna [33] extended results in [25]–[27] to nD translation-invariant Schrödinger equations with $n\geqslant 2$.

Method of symplectic projection in the Hilbert phase space. The novel approach in [25]–[27] is based on *symplectic projection* of solutions on the solitary manifold. This means that

$$Z:=\psi-S\quad\text{is symplectic-orthogonal to the tangent space}\quad \mathscr{T}:=T_S\mathscr{S}$$

for the projection $S := P\psi$. This projection is correctly defined in a small neighbourhood of $\mathscr{S}$ because $\mathscr{S}$ is a *symplectic manifold*, that is, the corresponding symplectic form is non-degenerate on the tangent spaces $T_S\mathscr{S}$. In particular, the approach in [25]–[27] does not require smallness of the initial data.

Thus, for each $t > 0$ the solution $\psi(t)$ decomposes as $\psi(t) = S(t) + Z(t)$, where $S(t) := P\psi(t)$, and the dynamics is linearized on the soliton $S(t)$. Similarly, for each $t \in \mathbb{R}$ the whole Hilbert phase space $\mathscr{X} := L^2(\mathbb{R})$ splits into a direct sum $\mathscr{X} = \mathscr{T}(t) \oplus \mathscr{Z}(t)$, where $\mathscr{Z}(t)$ is the *symplectic-orthogonal* complement of the tangent space $\mathscr{T}(t) := T_{S(t)}\mathscr{S}$. The corresponding equation for the *transversal component* $Z(t)$ is

$$\dot{Z}(t) = A(t)Z(t) + N(t),$$

where $A(t)Z(t)$ is the linear part, and $N(t) = \mathscr{O}(\|Z(t)\|^2)$ is the corresponding nonlinear part.

The main difficulties in studying this equation are: (i) it is *non-autonomous*, and (ii) the generators $A(t)$ *are not self-adjoint* (see the Appendix in [114]). It is important that the $A(t)$ are *Hamiltonian operators*, for which the existence of a spectral resolution is provided by the Krein–Langer theory of J-selfadjoint operators ([149], [157]). In [114] and [116] we developed a special version of this theory providing the corresponding eigenfunction expansion necessary for justification of the general approach in [25]–[27]. The main steps in this strategy are as follows.

• **Modulation equations.** The parameters of the soliton $S(t)$ satisfy the *modulation equations*: for example, for the speed $v(t)$ we have

$$\dot{v}(t) = M(\psi(t)),$$

where $M(\psi) = \mathscr{O}(\|Z\|^2)$ for small norms $\|Z\|$. This means that the change in the parameters is 'superslow' near the solitary manifold, like adiabatic invariants.

• **Tangent and transversal components.** The *transversal component* $Z(t)$ in the splitting $\psi(t) = S(t) + Z(t)$ belongs to the *transversal subspace* $\mathscr{Z}(t)$. The tangent space $\mathscr{T}(t)$ is the root space of the generator $A(t)$ and corresponds to the 'unstable' spectral point $\lambda = 0$. The key observation is that:

(i) the transversal subspace $\mathscr{Z}(t)$ is *invariant* with respect to the generator $A(t)$, since the subspace $\mathscr{T}(t)$ is invariant, and $A(t)$ is a Hamiltonian operator;

(ii) moreover, the transversal subspace $\mathscr{Z}(t)$ does not contain 'unstable' tangent vectors.

• **Continuous and discrete components.** The transversal component admits in turn a further splitting $Z(t) = z(t) + f(t)$, where $z(t)$ and $f(t)$ belong, respectively, to the discrete and the continuous spectral subspaces $\mathscr{Z}_{\mathrm{d}}(t)$ and $\mathscr{Z}_{\mathrm{c}}(t)$ of $A(t)$ in the space $\mathscr{Z}(t) = \mathscr{Z}_{\mathrm{d}}(t) + \mathscr{Z}_{\mathrm{c}}(t)$.

• **Poincaré normal forms and Fermi Golden Rule.** The component $z(t)$ satisfies a nonlinear equation, which reduces to the Poincaré normal form up to higher-order terms (see [27], (4.3.20)). (A similar reduction was done in [141], (5.18) for the relativistically invariant Ginzburg–Landau equation.) The normal form made it possible to obtain a certain 'conditional decay' for $z(t)$ using the Fermi Golden Rule (see [27], (1.0.12)).

• **Method of majorants.** A skillful combination of the conditional decay for $z(t)$ with the superslow evolution of the soliton parameters enables us to prove

decay for $f(t)$ and $z(t)$ by the method of majorants. Finally, this decay implies the asymptotics (6.16) and (6.17).

6.3. Generalizations and applications.

N-soliton solutions. The methods and results in [27] were developed in [166]–[170], [177], [178], [183], and [184] for N-soliton solutions of translation-invariant nonlinear Schrödinger equations.

Multiphoton radiation. In [35] Cuccagna and Mizumachi extended the methods and results in [27] to the case when the inequality (6.18) is changed to

$$N\lambda < \omega_0 < (N+1)\lambda$$

with some natural number $N > 1$, and the corresponding analogue of the condition U3 holds. This means that the interaction of discrete modes with the continuous spectrum occurs only in the Nth order of perturbation theory. The decay rate of the remainder term (6.17) worsens with growing N.

Linear equations coupled to nonlinear oscillators and particles. The methods and results in [27] were extended: (i) in [24] and [120] to the Schrödinger equation coupled to a nonlinear U(1)-invariant oscillator; (ii) in [80] and [82] to systems (3.1) and (2.61) with zero external fields; (iii) in [81], [109], and [119] to similar translation-invariant systems of the Klein–Gordon, Schrödinger and Dirac equations coupled to a particle. A survey of these results can be found in [74].

For example, the article [82] concerns solutions of the system (3.1) with initial data close to a solitary manifold (3.3) in the weighted norm

$$\|\psi\|_\sigma^2 = \int \langle x\rangle^{2\sigma} |\psi(x)|^2\, dx$$

with sufficiently large $\sigma > 0$. Namely, for an initial state close to the soliton (3.3) with some parameters v_0 and a_0 we have

$$\begin{aligned}\|\nabla\psi(x,0) - \nabla\psi_{v_0}(x-a_0)\|_\sigma + \|\psi(x,0) - \psi_{v_0}(x-a_0)\|_\sigma \\ + \|\pi(x,0) - \pi_{v_0}(x-a_0)\|_\sigma + |q(0)-a_0| + |\dot q(0) - v_0| \leqslant \varepsilon,\end{aligned}$$

where $\sigma > 5$, and $\varepsilon > 0$ is sufficiently small. Moreover, the Wiener condition (2.34) is assumed for $k \neq 0$. In addition, let

$$\partial^\alpha \widehat\rho(0) = 0, \qquad |\alpha| \leqslant 5,$$

which is equivalent to the equalities

$$\int x^\alpha \rho(x)\, dx = 0, \qquad |\alpha| \leqslant 5.$$

Under these conditions, the main results in [82] are the asymptotics

$$\ddot q(t) \to 0, \quad \dot q(t) \to v_\pm, \quad q(t) \sim v_\pm t + a_\pm, \quad t \to \pm\infty$$

(cf. (3.8) and (3.11)) and the attraction to the solitons (3.9), where the remainder now decays in *global weighted norms* in the comoving frame (cf. (3.10)):

$$\|\nabla r_\pm(q(t)+x,t)\|_{-\sigma} + \|r_\pm(q(t)+x,t)\|_{-\sigma} + \|s_\pm(q(t)+x,t)\|_{-\sigma} \to 0, \qquad t \to \pm\infty.$$

Relativistic equations. In [16], [23], [135], [140], and [141] the methods and results in [27] were extended for the first time to *relativistically invariant* nonlinear equations. Namely, in [16] and [135], [140], [141] asymptotics of the type (6.16) were obtained for the 1D relativistically invariant nonlinear wave equations (5.20) with potentials of Ginzburg–Landau type, and in [23] for the relativistically invariant nonlinear Dirac equations. In [117] we constructed examples of potentials providing all spectral properties of the linearized dynamics imposed in [135], [140], and [141].

In [114] and [116] we justified the eigenfunction expansions for non-selfadjoint Hamiltonian operators which were used in [135], [140], and [141]. For the justification we developed a special version of the Krein–Langer theory of J-selfadjoint operators [149], [157].

Vavilov–Cherenkov radiation. The article [54] concerns a system of type (3.1) with the Schrödinger equation instead of the wave equation (the system (1.9), (1.10) in [54]). This system is considered as a model of the Cherenkov radiation. The main result in [54] is long-time convergence to a soliton with sonic speed for initial solitons with a supersonic speed in the case of weak interaction ('Bogolyubov limit') and small initial field. Asymptotic stability of solitons for a similar system was established in [109].

6.4. Further generalizations. Results on asymptotic stability of solitons have been developed in different directions.

Systems with several bound states. The papers [10], [34], and [205]–[207] concern asymptotic stability of stationary orbits (6.3) for the nonlinear Schrödinger, Klein–Gordon, and wave equations in the case of several simple eigenvalues of the linearization. The typical assumptions are as follows:

(i) an endpoint of the continuous spectrum is neither an eigenvalue nor a resonance for the linearized equation;

(ii) the eigenvalues of the linearized equation satisfy several non-resonance conditions;

(iii) a new version of the Fermi Golden Rule.

One typical difficulty is the possible long stay of solutions near metastable tori which correspond to approximate resonances. Major efforts are being made to show that the role of metastable tori decreases like $t^{-1/2}$ as $t \to \infty$. A typical result is a long-time asymptotic expression of type 'ground state + dispersion wave' in the norm of $H^1(\mathbb{R}^3)$ for solutions close to the ground state.

General theory of relativity. The article [66] concerns so-called 'kink instability' of self-similar and spherically symmetric solutions of the equations of the general theory of relativity with a scalar field, as well as with a 'hard fluid' as sources. The authors have constructed examples of self-similar solutions that are unstable to the kink perturbations.

The article [36] examines linear stability of slowly rotating Kerr solutions for the Einstein equations in a vacuum. In [201] a pointwise damping of solutions of the wave equation is investigated for the case of stationary asymptotically flat space-time in the three-dimensional case.

In [7] the Maxwell equations are considered outside a slowly rotating Kerr black hole. The main results are:

(i) boundedness of a positive-definite energy on each hypersurface $t = \text{const}$;

(ii) convergence of each solution of a stationary Coulomb field.

In [41] pointwise decay was proved for linear waves on a Schwarzschild black hole background.

Concentration compactness method. In [89] the concentration compactness method was used for the first time to prove global well-posedness, scattering, and blow-up of solutions of the critical focusing nonlinear Schrödinger equation

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t) - |\psi(x,t)|^{4/(n-2)}\psi(x,t), \qquad x \in \mathbb{R}^n,$$

in the radial case. Later on, these methods were extended in [42], [44], [90], and [150] to general non-radial solutions and to nonlinear wave equations of the form

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) + |\psi(x,t)|^{4/(n-2)}\psi(x,t), \qquad x \in \mathbb{R}^n.$$

One of the main results is a splitting of the set of initial states close to the critical energy level, into three subsets with a certain long-term asymptotics: either a blow-up in a finite time, or an asymptotically free wave, or the sum of the ground state and an asymptotically free wave. All three alternatives are possible, and all nine combinations with $t \to \pm\infty$ are also possible. The lectures [174] give an excellent introduction to this area. The articles [43] and [91] concern supercritical nonlinear wave equations.

Recently these methods and results were extended to critical wave maps ([88], [87], [150], [151]). Namely, 'decay into solitons' was proved: every 1-equivariant finite-energy wave map of the exterior of a ball with Dirichlet boundary conditions into a three-dimensional sphere exists globally in time and dissipates into the unique stationary solution in its own topological class.

Weak convergence to equilibrium distributions in nonlinear Hamiltonian systems. The papers [145]–[148] concern weak convergence to an equilibrium distribution in the Liouville, Vlasov, and Schrödinger equations. In [146] the authors introduced the quantum Poincaré model.

6.5. Linear dispersion. The key role in all results on the long-time asymptotics for nonlinear Hamiltonian PDEs is played by dispersion decay of solutions of the corresponding linearized equations. A huge number of publications concern this decay, so we choose only those most important or most recent.

Dispersion decay in weighted Sobolev norms. Dispersion decay for wave equations was first proved in linear scattering theory [158].

A powerful systematic approach to dispersion decay for the Schrödinger equation with a potential was proposed by Agmon, Jensen, and Kato [5], [84]. This theory has been extended by many authors to the wave, Klein–Gordon, and Dirac equations and to the corresponding discrete equations (see [14], [15], [37], [38], [50], [60], [61], [47], [48] and [86], [110], [139], [111], [112], [127], [113], [115], [118], [121], [131]–[134], [138], [144], and references therein).

$L^1 - L^\infty$ decay. The estimate

$$\|P_c\psi(t)\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct^{-n/2}\|\psi(0)\|_{L^1(\mathbb{R}^n)}, \qquad t > 0, \tag{6.19}$$

for solutions of the linear Schrödinger equation

$$i\dot{\psi}(x,t) = H\psi(x,t) := (-\Delta + V(x))\psi(x,t), \qquad x \in \mathbb{R}^n, \tag{6.20}$$

with $n \geqslant 3$ was first proved by Journet, Soffer, and Sogge [86] under the condition that $\lambda = 0$ is neither an eigenvalue nor a resonance for H. The potential $V(x)$ is assumed to be sufficiently smooth and rapidly decaying as $|x| \to \infty$. Here P_c is the orthogonal projection on the space of the continuous spectrum of H. This result was later generalized by many authors (see below).

In [182] a decay of type (6.19) and Strichartz estimates were established for the 3D Schrödinger equations (6.20) with 'rough' and time-dependent potentials $V = V(x,t)$ (in the stationary case $V(x)$ belongs to both the Rollnik class and the Kato class). Similar estimates were obtained in [14] for the 3D Schrödinger and wave equations with (stationary) Kato-class potentials.

In [50] the 4D Schrödinger equations (6.20) are considered for the case when there is a resonance or an eigenvalue at zero energy. In particular, in the case of an eigenvalue at zero energy, there is a time-dependent operator F_t of rank 1 such that $\|F_t\|_{L^1\to L^\infty} \leqslant 1/\log t$ for $t > 2$, and

$$\|e^{itH}P_c - F_t\|_{L^1\to L^\infty} \leqslant Ct^{-1}, \qquad t > 2.$$

Similar dispersion estimates were proved also for solutions of the 4D wave equation with a potential.

In [60] and [61] the Schrödinger equation (6.20) is considered in $\mathbb{R}^n$ with $n \geqslant 5$ when there is an eigenvalue at the zero point of the spectrum. It is shown, in particular, that there is a time-dependent rank-1 operator F_t such that $\|F_t\|_{L^1\to L^\infty} \leqslant C|t|^{2-n/2}$ for $|t| > 1$ and

$$\|e^{itH}P_c - F_t\|_{L^1\to L^\infty} \leqslant C|t|^{1-n/2}, \qquad |t| > 1.$$

With a stronger decay of the potential, the evolution (dynamical group) admits an operator-valued decomposition

$$e^{itH}P_c(H) = |t|^{2-n/2}A_{-2} + |t|^{1-n/2}A_{-1} + |t|^{-n/2}A_0,$$

where A_{-2} and A_{-1} are finite-rank operators $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$, while A_0 maps weighted L^1-spaces to weighted L^∞-spaces. The main terms A_{-2} and A_{-1} are equal to zero under certain conditions of orthogonality of the potential V to an eigenfunction with zero energy. Under the same orthogonality conditions, the remainder term $|t|^{-n/2}A_0$ also maps $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, and therefore the group $e^{itH}P_c(H)$ has the same dispersion decay as the free evolution, despite its eigenvalue at zero.

$L^p - L^q$ decay was first established in [165] for solutions of the free Klein–Gordon equation $\ddot{\psi} = \Delta\psi - \psi$ with initial state $\psi(0) = 0$:

$$\|\psi(t)\|_{L^q} \leqslant Ct^{-d}\|\dot{\psi}(0)\|_{L^p}, \qquad t > 1, \tag{6.21}$$

where $1 \leqslant p \leqslant 2$, $1/p + 1/q = 1$, and $d \geqslant 0$ is a piecewise-linear function of $(1/p, 1/q)$. The proofs use the Riesz interpolation theorem.

In [13], the estimates (6.21) were extended to solutions of the perturbed Klein–Gordon equation

$$\ddot{\psi} = \Delta\psi - \psi + V(x)\psi$$

with $\psi(0) = 0$. The authors show that (6.21) holds for $0 \leqslant 1/p - 1/2 \leqslant 1/(n+1)$. The smallest value of p and the fastest decay rate d occur when $1/p = 1/2 + 1/(n+1)$ and $d = (n-1)/(n+1)$. The result is proved under the assumption that the potential V is smooth and small in a suitable sense. For example, the result is true when

$$|V(x)| \leqslant c(1 + |x|^2)^{-\sigma}$$

where $c > 0$ is sufficiently small. Here

$$\sigma > 2 \quad \text{for } n = 3; \qquad \sigma > \frac{n}{2} \quad \text{for odd } n \geqslant 5;$$

$$\sigma > \frac{2n^2 + 3n + 3}{4(n+1)} \quad \text{for even } n \geqslant 4.$$

The results also apply to the case when $\psi(0) \neq 0$.

The seminal article [86] concerns $L^p - L^q$ decay of solutions of the Schrödinger equation (6.20). It is assumed that $(1 + |x|^2)^\alpha V(x)$ is a multiplier in the Sobolev spaces H^η for some $\eta > 0$ and $\alpha > n + 4$, and that the Fourier transform of V belongs to $L^1(\mathbb{R}^n)$. Under these conditions the main result in [86] is the following theorem: if $\lambda = 0$ is neither an eigenvalue nor a resonance for H, then

$$\|P_c\psi(t)\|_{L^q} \leqslant Ct^{-n(1/p-1/2)}\|\psi(0)\|_{L^p}, \qquad t > 1, \tag{6.22}$$

where $1 \leqslant p \leqslant 2$ and $1/p + 1/q = 1$. The proofs are based on the $L^1 - L^\infty$ decay (6.19) and the Riesz interpolation theorem.

In [211] the estimates (6.22) were proved for all $1 \leqslant p \leqslant 2$ under suitable conditions on decay of $V(x)$ if $\lambda = 0$ is neither an eigenvalue nor a resonance for H, and for all $3/2 < p \leqslant 2$ otherwise.

The Strichartz estimates were extended: (i) in [38] to the Schrödinger equation with a magnetic potential in $\mathbb{R}^n$, $n \geqslant 3$; (ii) in [37] to wave equations with a magnetic potential in $\mathbb{R}^n$, $n \geqslant 3$; (iii) in [15] to the wave equation in $\mathbb{R}^3$ with Kato-class potential.

7. Numerical simulation of soliton asymptotics

Here we describe the results of joint work with Arkady Vinnichenko (1945–2009) on numerical simulation of (i) global attraction to solitons (1.6) and (1.7) and (ii) adiabatic effective dynamics of solitons (4.6) for relativistically invariant 1D nonlinear wave equations. Additional information can be found in [123].

7.1. Kinks of relativistically invariant Ginzburg–Landau equations. First let us describe numerical simulations of solutions of relativistically invariant 1D nonlinear wave equations with a polynomial nonlinearity:

$$\ddot{\psi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad \text{where} \quad F(\psi) := -\psi^3 + \psi. \tag{7.1}$$

Since $F(\psi) = 0$ for $\psi = 0, \pm 1$, there are three equilibrium states: $S(x) \equiv 0, +1, -1$. This equation is formally equivalent to a Hamiltonian system (2.2) with the Hamiltonian

$$\mathscr{H}(\psi, \pi) = \int \left[\frac{1}{2}|\pi(x)|^2 + \frac{1}{2}|\psi'(x)|^2 + U(\psi(x)) \right] dx, \tag{7.2}$$

where the potential is $U(\psi) = \psi^4/4 - \psi^2/2 + 1/4$. This Hamiltonian is finite for functions (ψ, π) in the space $\mathscr{E}_c$ defined in (2.3)–(2.5) with $C_\pm = \pm 1$, for which the convergence

$$\psi(x) \to \pm 1, \qquad |x| \to \pm\infty,$$

is sufficiently fast.

The potential $U(\psi)$ has minima at $\psi = \pm 1$ and a maximum at $\psi = 0$. Correspondingly, two finite-energy solutions $\psi = \pm 1$ are stable, and the solution $\psi = 0$ with infinite energy is unstable. Such potentials with two wells are called potentials of Ginzburg–Landau type.

In addition to the constant stationary solutions $S(x) \equiv 0, +1, -1$, there is also a non-constant solution $S(x) = \tanh(x/\sqrt{2}\,)$, called a 'kink'. Its shifts and reflections $\pm S(\pm x - a)$ are also stationary solutions, as well as their Lorentz transforms

$$\pm S(\gamma(\pm x - a - vt)), \qquad \gamma = \frac{1}{\sqrt{1 - v^2}}, \quad |v| < 1.$$

These are uniformly moving 'travelling waves' (that is, solitons). The kink is strongly compressed when the velocity v is close to ± 1. This compression is known as the 'Lorentz contraction'.

Numerical Simulation. Our numerical experiments show a decay of finite-energy solutions on a finite set of kinks and dispersion waves outside the kinks, which corresponds to the asymptotics (1.7). The result of one of the experiments is shown in Fig. 7: a finite-energy solution of the equation (7.1) decays to three kinks. Here the vertical line is the time axis, and the horizontal line is the space axis. The spatial scale redoubles at $t = 20$ and $t = 60$.

The red colour corresponds to the values $\psi > 1 + \varepsilon$, the blue colour to the values $\psi < -1 - \varepsilon$, and the yellow colour to the intermediate values $-1 + \varepsilon < \psi < 1 - \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Thus, the yellow stripes represent the kinks, while the blue and red zones outside the yellow stripes are filled with dispersion waves.

For $t = 0$ the solution begins with a rather chaotic behaviour, when there are no visible kinks. After 20 seconds, three separate kinks appear, which subsequently move almost uniformly.

The Lorentz contraction. The left kink moves to the left at a low speed $v_1 \approx 0.24$, the central kink is almost standing, because its velocity $v_2 \approx 0.02$ is very small, and the right kink moves very fast with speed $v_3 \approx 0.88$. The Lorentz spatial contraction $\sqrt{1 - v_k^2}$ is clearly visible in this picture: the central kink is the widest, the left is a bit narrower, and the right one is quite narrow.

The Einstein time delay. The Einstein time delay is also very pronounced. Namely, all three kinks oscillate (pulsate) because of the presence of a non-zero eigenvalue in

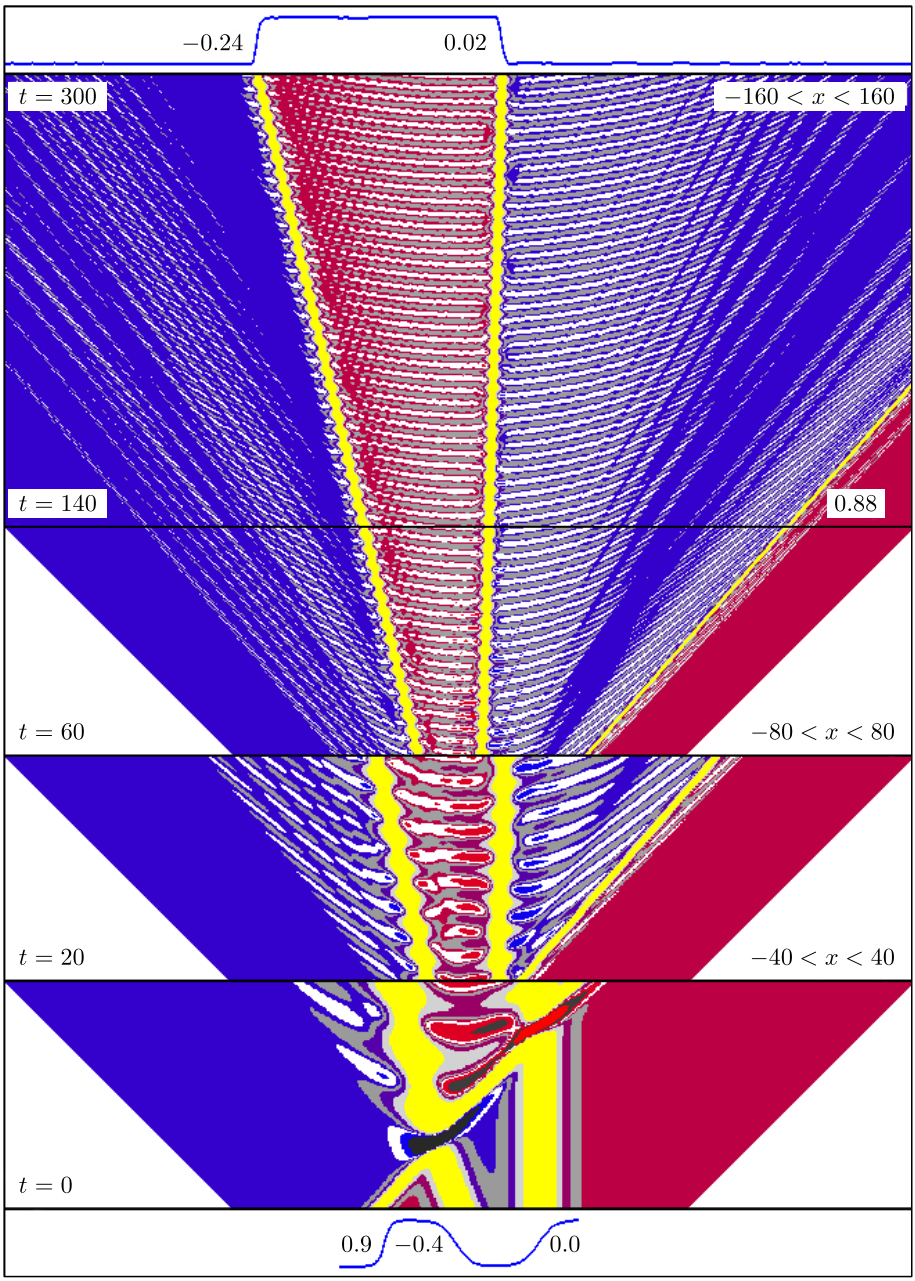

Figure 7. Decay to three kinks.

the equation linearized on the kink. Indeed, substituting $\psi(x,t) = S(x) + \varepsilon\varphi(x,t)$ into (7.1), we get in the first-order approximation the linearized equation

$$\ddot{\varphi}(x,t) = \varphi''(x,t) - 2\varphi(x,t) - V(x)\varphi(x,t), \tag{7.3}$$

where the potential

$$V(x) = 3S^2(x) - 3 = -\frac{3}{\cosh^2(x/\sqrt{2})}$$

decays exponentially for large $|x|$. It is very fortunate that for this potential the spectrum of the corresponding *Schrödinger operator*

$$H := -\frac{d^2}{dx^2} + 2 + V(x)$$

is well known [154]. Namely, the operator H is non-negative, and its continuous spectrum is the interval $[2, \infty)$. It turns out that H also has a two-point discrete spectrum: the points $\lambda = 0$ and $\lambda = 3/2$. It is this non-zero eigenvalue that is responsible for the pulsations that we observe for the central slow kink, with frequency $\omega_2 \approx \sqrt{3/2}$ and period $T_2 \approx 2\pi/\sqrt{3/2} \approx 5$. On the other hand, for the fast kinks the ripples are much slower, that is, the corresponding period is longer. This time delay agrees numerically with the Lorentz formulae, which confirms the relevance of these results of numerical simulation.

Dispersion waves. An analysis of dispersion waves provides additional confirmation. Namely, the space outside the kinks in Fig. 7 is filled with dispersion waves whose values are very close to ± 1, with an accuracy of 0.01. These waves satisfy with high accuracy the linear Klein–Gordon equation obtained by linearization of the Ginzburg–Landau equation (7.1) on the stationary solutions $\psi_\pm \equiv \pm 1$:

$$\ddot{\varphi}(x,t) = \varphi''(x,t) + 2\varphi(x,t).$$

The corresponding dispersion relation $\omega^2 = k^2 + 2$ determines the group velocities of high-frequency wave packets:

$$\omega'(k) = \frac{k}{\sqrt{k^2+2}} = \pm\frac{\sqrt{\omega^2 - 2}}{\omega}\,. \tag{7.4}$$

These wave packets are clearly visible in Fig. 7 as straight lines whose propagation speeds converge to ± 1. This convergence is explained by the high-frequency limit $\omega'(k) \to \pm 1$ as $\omega \to \pm\infty$. For example, for dispersion waves emitted by the central kink the frequencies $\omega = \pm n\omega_2 \to \pm\infty$ are generated by the polynomial nonlinearity in (7.1) in accordance with Fig. 5.

Remark 7.1. These observations of dispersion waves agree with the radiation mechanism in Remark 5.16.

The nonlinearity in (7.1) is chosen exactly because of the well-known spectrum of the linearized equation (7.3). In numerical experiments [123] more general nonlinearities of Ginzburg–Landau type have also been considered. The results were qualitatively the same: for 'any' initial data, the solution decays for large times to a sum of kinks and dispersion waves. Numerically, this is clearly visible, but rigorous justification remains an open problem.

7.2. Numerical observation of soliton asymptotics. Besides the kinks the numerical experiments [123] also revealed soliton-like asymptotics of type (1.7) and adiabatic effective dynamics of the form (4.6) for complex solutions of the 1D relativistically invariant nonlinear wave equations (5.20). Polynomial potentials of the form

$$U(\psi) = a|\psi|^{2m} - b|\psi|^{2n} \tag{7.5}$$

were considered with $a, b > 0$ and $m > n = 2, 3, \dots$. Correspondingly,

$$F(\psi) = 2am|\psi|^{2m-2}\psi - 2bn|\psi|^{2n-2}\psi. \tag{7.6}$$

The parameters a, b, m, and n were taken as follows.

N	a	m	b	n
1	1	3	0.61	2
2	10	4	2.1	2
3	10	6	8.75	5

Various 'smooth' initial functions $\psi(x,0)$, $\dot\psi(x,0)$ with supports on the interval $[-20, 20]$ were considered. The second-order finite-difference scheme with $\Delta x \sim 0.01$ and $\Delta t \sim 0.001$ was employed. In all cases the asymptotics of type (1.7) were observed with the numbers 0, 1, 3, and 5 of solitons for $t > 100$.

7.3. Adiabatic effective dynamics of relativistic solitons. In the numerical experiments [123] the adiabatic effective dynamics of the form (4.6) was also observed for soliton-like solutions of type (4.1) of the 1D equations (5.20) with a slowly varying external potential (4.2):

$$\ddot\psi(x,t) = \psi''(x,t) - \psi(x,t) + F(\psi(x,t)) - V(x)\psi(x,t), \qquad x \in \mathbb{R}. \tag{7.7}$$

This equation is formally equivalent to the Hamiltonian system (2.2) with the Hamiltonian

$$\mathscr{H}_V(\psi,\pi) = \int \left[\frac{1}{2}|\pi(x)|^2 + \frac{1}{2}|\psi'(x)|^2 + U(\psi(x)) + \frac{1}{2}V(x)|\psi(x)|^2\right] dx. \tag{7.8}$$

The soliton-like solutions are of the form (cf. (4.1))

$$\psi(x,t) \approx e^{i\Theta(t)}\phi_{\omega(t)}\big(\gamma_{v(t)}(x - q(t))\big). \tag{7.9}$$

The numerical experiments [123] qualitatively confirm the adiabatic effective Hamiltonian dynamics for the parameters Θ, ω, q, and v, but it has not yet been rigorously justified. Figure 8 represents solutions of equation (7.7) with the potential (7.5), where $a = 10$, $m = 6$ and $b = 8.75$, $n = 5$. The potential is $V(x) = -0.2\cos(0.31x)$ and the initial conditions are

$$\psi(x,0) = \phi_{\omega_0}(\gamma_{v_0}(x - q_0)), \qquad \dot\psi(x,0) = 0, \tag{7.10}$$

where $v_0 = 0$, $\omega_0 = 0.6$, and $q_0 = 5.0$. We note that the initial state does not belong to the solitary manifold. The effective width (half-amplitude) of the solitons is in the range $[4.4, 5.6]$. It is quite small when compared with the spatial period of

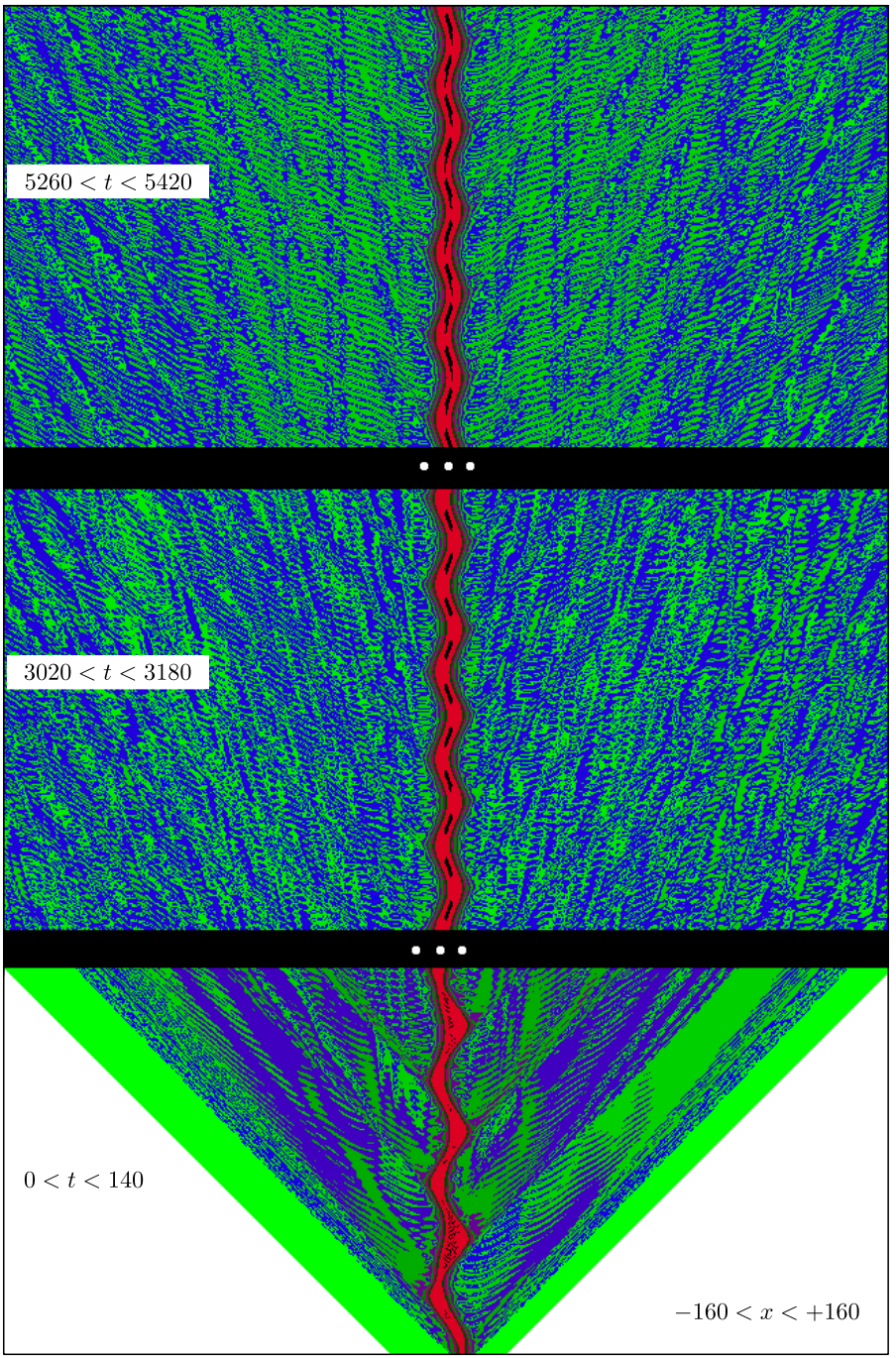

Figure 8. Adiabatic effective dynamics of relativistic solitons.

the potential $2\pi/0.31 \sim 20$. The results of the numerical simulations are shown in Fig. 8.

• The blue and green colours represent a dispersion wave with values $|\psi(x,t)| < 0.01$, while the red colour represents a soliton with values $|\psi(x,t)| \in [0.4, 0.8]$.

• The soliton trajectory ('red snake') corresponds to oscillations of a classical particle in the potential $V(x)$.

• For $0 < t < 140$ the solution is rather distant from the solitary manifold, and the radiation is rather intense.

• For $3020 < t < 3180$ the solution approaches the solitary manifold, and the radiation weakens. The oscillation amplitude of the soliton is almost unchanged over a long time, confirming the Hamiltonian type of the effective dynamics.

• However, for $5260 < t < 5420$ the amplitude of the soliton oscillation is halved. This suggests that on a large time scale the deviation from Hamiltonian effective dynamics becomes essential. Consequently, the effective dynamics gives a good approximation only on an adiabatic time scale of type $t \sim \varepsilon^{-1}$.

• The deviation of the effective dynamics from being Hamiltonian is due to radiation, which plays the role of dissipation.

• The radiation is realized as dispersion waves which bring energy to infinity. The dispersion waves combine into uniformly moving wave packets with a discrete set of group velocities, as in Fig. 7. The magnitude of the solution is of order ~ 1 on the trajectory of the soliton, while the values of the dispersion waves are less than 0.01 for $t > 200$, so that their energy density does not exceed 0.0001. The amplitude of the dispersion waves decays at large times.

• In the limit as $t \to \pm\infty$ the soliton should converge to a static position corresponding to a local minimum of the potential $V(x)$. However, the numerical observation of this 'ultimate stage' is hopeless, since the rate of the convergence strongly decays with decrease of the radiation.

8. Appendix: attractors and quantum mechanics

The foregoing results on attractors for nonlinear Hamiltonian equations were suggested by fundamental postulates of quantum theory, primarily the Bohr postulate on transitions between quantum stationary orbits. Namely, in 1913 Bohr proposed the following two postulates, which give the 'Columbus' solution of the problem of stability and radiation of atoms and molecules [20].

B1. Atoms and molecules stay in some stationary orbits $|E_n\rangle$ with energies E_n, and sometimes make transitions between the orbits,

$$|E_n\rangle \mapsto |E_{n'}\rangle.$$

B2. Such a transition is accompanied by radiation of an electromagnetic wave of frequency

$$\omega_{nn'} = \omega_{n'} - \omega_n, \quad \text{where} \quad \omega_k = \frac{E_k}{\hbar}.$$

Both these postulates should have become theorems in the quantum theory of Schrödinger and Heisenberg later discovered. However, this did not happen, and both postulates are still actively used in quantum theory. This lack of theoretical clarity hinders the progress in the theory (for example, in superconductivity and in

nuclear reactions), and in numerical simulation of many engineering processes (of laser radiation and quantum amplifiers, for instance) since a computer can solve dynamical equations but cannot take the postulates into account.

8.1. On dynamical interpretation of quantum jumps. The simplest dynamic interpretation of the postulate B1 is the attraction to stationary orbits (1.8) for any finite-energy quantum trajectory $\psi(t)$. This means that the stationary orbits form a global attractor of the corresponding quantum dynamics. However, this attraction contradicts the linear Schrödinger equation because of the superposition principle. Thus, in the linear theory Bohr transitions B1 do not exist.

It is natural to suggest that the attraction (1.8) holds for a nonlinear modification of the linear Schrödinger theory. On the other hand, it turns out that even the original Schrödinger theory is nonlinear, because it involves interaction with the Maxwell field. The corresponding nonlinear Maxwell–Schrödinger system is essentially already contained in Schrödinger's first papers [186]:

$$\begin{cases} i\dot\psi(x,t) = \dfrac{1}{2}[-i\nabla + \mathbf{A}(x,t) + \mathbf{A}^{\mathrm{ext}}(x,t)]^2\psi + [A_0(x,t) + A_0^{\mathrm{ext}}(x)]\psi, \\ \Box A_\alpha(x,t) = 4\pi J_\alpha(x,t), \quad \alpha = 0,1,2,3, \end{cases} \qquad x \in \mathbb{R}^3, \tag{8.1}$$

where the system of units is chosen so that $\hbar = e = m = c = 1$. The Maxwell equations are written here in the four-dimensional form, where $A = (A_0, \mathbf{A}) = (A_0, A_1, A_2, A_3)$ denotes four-dimensional potential of the Maxwell field with the Lorentz gauge $\dot A_0 + \nabla \cdot \mathbf{A} = 0$. Further, $A^{\mathrm{ext}} = (A_0^{\mathrm{ext}}, \mathbf{A}^{\mathrm{ext}})$ denotes the external four-dimensional potential, and $J = (\rho, j_1, j_2, j_3)$ is the four-dimensional current. To make these equations a closed system, we must also express the density of charges and currents via the wave function:

$$J_0(x,t) = |\psi(x,t)|^2; \quad J_k(x,t) = \psi(x,t) \cdot [(-i\nabla_k + A_k(x,t) + A_k^{\mathrm{ext}}(x,t))\psi(x,t)], \tag{8.2}$$

where $k = 1,2,3$, and '$\cdot$' denotes the scalar product of two-dimensional real vectors corresponding to complex numbers. In particular, these expressions satisfy the continuity equation $\dot\rho + \operatorname{div} j = 0$ for any solution of the Schrödinger equation with arbitrary real potentials (see [100], § 3.4).

The system (8.1) is nonlinear in the functions (ψ, A) although the Schrödinger equation is formally linear in the wave function ψ and the Maxwell equations are linear in the potential A. Now the question arises as to what the 'stationary orbits' should be for the nonlinear hyperbolic system (8.1). It is natural to suggest that these are solutions of the form

$$(\psi(x)e^{-i\omega t}, A(x)) \tag{8.3}$$

in the case of static external potentials $A^{\mathrm{ext}}(x,t) = A^{\mathrm{ext}}(x)$.

Indeed, in this case the functions (8.3) give stationary distributions of the charges and currents (8.2). Moreover, these functions are the trajectories of one-parameter subgroups of the symmetry group U(1) of the system (8.1). Namely, for any solution $(\psi(x,t), A(x,t))$ and any $\theta \in \mathbb{R}$ the functions

$$U_\theta(\psi(x,t), A(x,t)) := (\psi(x,t)e^{i\theta}, A(x,t)) \tag{8.4}$$

are also solutions. The same remarks apply to the Maxwell–Dirac system introduced by Dirac in 1927:

$$\begin{cases} \sum_{\alpha=0}^{3} \gamma^\alpha [i\nabla_\alpha - A_\alpha(x,t) - A^{\rm ext}_\alpha(x,t)]\psi(x,t) = m\psi(x,t), \\ \Box A_\alpha(x,t) = J_\alpha(x,t) := \overline{\psi(x,t)}\gamma^0\gamma_\alpha\psi(x,t), \quad \alpha = 0,1,2,3, \end{cases} \qquad x \in \mathbb{R}^3, \tag{8.5}$$

where $\nabla_0 := \partial_t$. Thus, Bohr transitions B1 for the systems (8.1) and (8.5) with a static external potential $A^{\rm ext}(x,t) = A^{\rm ext}(x)$ can be interpreted as the long-time asymptotics

$$(\psi(x,t), A(x,t)) \sim (\psi_\pm(x)e^{-i\omega_\pm t}, A_\pm(x,t)), \qquad t \to \pm\infty, \tag{8.6}$$

for any finite-energy solution, where the asymptotics hold in local energy norms. The maps U_θ form a group isomorphic to U(1), and the functions (8.3) are the trajectories of its one-parametric subgroups. Hence, the asymptotics (8.6) correspond to our general conjecture (1.4) with the symmetry group $G = {\rm U}(1)$.

Furthermore, in the case of zero external potentials these systems are translation-invariant. Correspondingly, for their solutions one should expect soliton asymptotics of type (1.7) in global energy norms as $t \to \pm\infty$:

$$\psi(x,t) \sim \sum_k \psi^k_\pm(x - v^k_\pm t)\exp(i\Phi^k_\pm(x,t)) + \varphi_\pm(x,t), \tag{8.7}$$

$$A(x,t) \sim \sum_k A^k_\pm(x - v^k_\pm t) + A_\pm(x,t). \tag{8.8}$$

Here $\Phi^k_\pm(x,t)$ are suitable phase functions, and each soliton

$$(\psi^k_\pm(x - v^k_\pm t)\exp(i\Phi^k_\pm(x,t)), A^k_\pm(x - v^k_\pm t))$$

is a solution of the corresponding nonlinear system, while $\varphi_\pm(x,t)$ and $A_\pm(x,t)$ are some dispersion waves which are solutions of the free Schrödinger and free Maxwell equations, respectively. The existence of the solitons for the Maxwell–Schrödinger and Maxwell–Dirac systems was established in [29] and [51], respectively.

For the Maxwell–Schrödinger and Maxwell–Dirac equations (8.1) and (8.5) the asymptotics (8.6) and (8.7) have not yet been proved. One might expect that these asymptotics should follow by a suitable modification of the arguments in § 5. Indeed, let the time spectrum of an omega-limit trajectory $\psi(x,t)$ contain at least two different frequencies $\omega_1 \neq \omega_2$: for example, $\psi(x,t) = \psi_1(x)e^{-i\omega_1 t} + \psi_2(x)e^{-i\omega_2 t}$. Then the currents $J_\alpha(x,t)$ in the systems (8.1) and (8.5) contain terms with harmonics $e^{in\Delta t}$ with $n \in \mathbb{Z}$, where $\Delta := \omega_1 - \omega_2 \neq 0$. Thus, the nonlinearity inflates the spectrum as in the U(1)-invariant equations considered in § 5.

In turn, these harmonics $e^{in\Delta t}$ with $n \neq 0$ on the right-hand side of the Maxwell equations induce radiation of electromagnetic waves with frequencies $n\Delta$ according to the limiting amplitude principle (5.61), since the continuous spectrum of the Maxwell generator is $\mathbb{R}\setminus 0$. Finally, this radiation brings energy to infinity, which is impossible for omega-limit trajectories, a contradiction proving the validity of the single-frequency asymptotics (8.6).

The methods in § 5 give a rigorous justification of similar arguments for the U(1)-invariant equations (5.4) and (5.16)–(5.18). However, a rigorous justification for the systems (8.1) and (8.5) is still an open problem.

8.2. Bohr postulates via perturbation theory. The remarkable success of the Schrödinger theory was the explanation of the Bohr postulates in the case of *static external potentials* with the help of *perturbation theory* applied to the *coupled Maxwell–Scrödinger equations* (8.1). Namely, as a first approximation, the time-dependent fields $\mathbf{A}(x,t)$ and $A^0(x,t)$ in the Schrödinger equation of the system (8.1) can be neglected:

$$i\hbar\dot\psi(x,t) = H\psi(x,t) := \frac{1}{2m}\left[-i\hbar\nabla - \frac{e}{c}\mathbf{A}_{\rm ext}(x)\right]^2\psi(x,t) + eA^0_{\rm ext}(x)\psi(x,t). \quad (8.9)$$

For 'sufficiently good' external potentials and initial conditions, any finite-energy solution can be expanded in eigenfunctions:

$$\psi(x,t) = \sum_n C_n\psi_n(x)e^{-i\omega_n t} + \psi_{\rm c}(x,t), \qquad \psi_{\rm c}(x,t) = \int C(\omega)e^{-i\omega t}\,d\omega, \quad (8.10)$$

where the integration is over the continuous spectrum of the Schrödinger operator H, and the integral decays as $t\to\infty$ in each bounded domain $|x|\leqslant R$ (see, for example, [112], Theorem 21.1). The substitution of this expansion into the expression for currents (8.2) gives the series

$$J(x,t) = \sum_{n,n'} J_{nn'}(x)e^{-i\omega_{nn'}t} + (\text{complex conjugate}) + J_{\rm c}(x,t), \quad (8.11)$$

where $J_{\rm c}(x,t)$ has a continuous frequency spectrum. Therefore, the currents on the right-hand side of the Maxwell equation in (8.1) contain, besides the continuous spectrum, only the discrete frequencies $\omega_{nn'}$. Hence, the discrete spectrum of the corresponding Maxwell field also contains only these frequencies $\omega_{nn'}$. This proves the Bohr rule B2 *in the first order of perturbation theory*, since this calculation ignores the inverse effect of radiation on the atom.

Moreover, these arguments also clarify the asymptotics (8.6). Namely, the currents (8.11) on the right-hand side of the Maxwell equation in (8.1) produce the radiation when non-zero frequencies $\omega_{nn'}$ are present. However, this radiation cannot last forever since the total energy is finite. Hence, only the zero frequency $\omega_{nn'}=0$ should remain in the long-time limit, which means exactly the single-frequency asymptotics (8.6) and the limiting stationary Maxwell field.

8.3. Conclusion. The discussion above suggests that the Bohr postulates cannot be explained by the linear Schrödinger equation alone, but admit a hypothetical explanation in the framework of the coupled Maxwell–Schrödinger equation.

This fact was the cause of heated discussions among Einstein, Bohr, and other physicists [21]. In [68] and [69], Heisenberg began developing a nonlinear theory of elementary particles.

Bibliography

[1] M. Abraham, "Prinzipien der Dynamik des Elektrons", *Phys. Z.* **4** (1902), 57–63.

[2] M. Abraham, *Theorie der Elektrizität*, vol. 2: *Elektromagnetische Theorie der Strahlung*, B. G. Teubner, Leipzig 1905, 404 pp.

[3] R. K. Adair and E. C. Fowler, *Strange particles*, Intersci. Publ. John Wiley & Sons, New York–London 1963, viii+151 pp.

[4] R. Adami, D. Noja, and C. Ortoleva, "Orbital and asymptotic stability for standing waves of a nonlinear Schrödinger equation with concentrated nonlinearity in dimension three", *J. Math. Phys.* **54**:1 (2013), 013501, 33 pp.

[5] S. Agmon, "Spectral properties of Schrödinger operators and scattering theory", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **2**:2 (1975), 151–218.

[6] S. Albeverio and R. Figari, "Quantum fields and point interactions", *Rend. Mat. Appl.* (7) **39**:2 (2018), 161–180.

[7] L. Andersson and P. Blue, "Uniform energy bound and asymptotics for the Maxwell field on a slowly rotating Kerr black hole exterior", *J. Hyperbolic Differ. Equ.* **12**:4 (2015), 689–743.

[8] А. В. Бабин, М. И. Вишик, *Аттракторы эволюционных уравнений*, Наука, М. 1989, 296 с.; English transl., A. V. Babin and M. I. Vishik, *Attractors of evolution equations*, Stud. Math. Appl., vol. 25, North-Holland Publishing Co., Amsterdam 1992, x+532 pp.

[9] V. Bach, T. Chen, J. Faupin, J. Fröhlich, and I. M. Sigal, "Effective dynamics of an electron coupled to an external potential in non-relativistic QED", *Ann. Henri Poincaré* **14**:6 (2013), 1573–1597.

[10] D. Bambusi and S. Cuccagna, "On dispersion of small energy solutions to the nonlinear Klein Gordon equation with a potential", *Amer. J. Math.* **133**:5 (2011), 1421–1468.

[11] D. Bambusi and L. Galgani, "Some rigorous results on the Pauli–Fierz model of classical electrodynamics", *Ann. Inst. H. Poincaré Phys. Théor.* **58**:2 (1993), 155–171.

[12] V. E. Barnes et al., "Observation of a hyperon with strangeness minus three", *Phys. Rev. Lett.* **12**:8 (1964), 204–206.

[13] M. Beals and W. Strauss, "L^p estimates for the wave equation with a potential", *Comm. Partial Differential Equations* **18**:7-8 (1993), 1365–1397.

[14] M. Beceanu and M. Goldberg, "Schrödinger dispersive estimates for a scaling-critical class of potentials", *Comm. Math. Phys.* **314**:2 (2012), 471–481.

[15] M. Beceanu and M. Goldberg, "Strichartz estimates and maximal operators for the wave equation in $\mathbb{R}^3$", *J. Funct. Anal.* **266**:3 (2014), 1476–1510.

[16] A. Bensoussan, C. Iliine, and A. Komech, "Breathers for a relativistic nonlinear wave equation", *Arch. Ration. Mech. Anal.* **165**:4 (2002), 317–345.

[17] H. Berestycki and P.-L. Lions, "Nonlinear scalar field equations. I. Existence of a ground state", *Arch. Ration. Mech. Anal.* **82**:4 (1983), 313–345.

[18] H. Berestycki and P.-L. Lions, "Nonlinear scalar field equations. II. Existence of infinitely many solutions", *Arch. Ration. Mech. Anal.* **82**:4 (1983), 347–375.

[19] Ф. А. Березин, Л. Д. Фаддеев, "Замечание об уравнении Шредингера с сингулярным потенциалом", *Докл. АН СССР* **137**:5 (1961), 1011–1014; English transl., F. A. Berezin and L. D. Faddeev, "A remark on Schrödinger's equation with a singular potential", *Soviet Math. Dokl.* **2** (1961), 372–375.

[20] N. Bohr, "On the constitution of atoms and molecules. I", *Philos. Mag.* (6) **26**:151 (1913), 1–25; "II", no. 153, 476–502; "III", no. 155, 857–875.

[21] N. Bohr, "Discussions with Einstein on epistemological problems in atomic physics", *Albert Einstein: philosopher-scientist*, The Library of Living Philosophers, vol. 7, The Library of Living Philosophers, Inc., Evanston, IL 1949, pp. 201–241.

[22] N. Boussaid, "Stable directions for small nonlinear Dirac standing waves", *Comm. Math. Phys.* **268**:3 (2006), 757–817.

[23] N. Boussaid and S. Cuccagna, "On stability of standing waves of nonlinear Dirac equations", *Comm. Partial Differential Equations* **37**:6 (2012), 1001–1056.

[24] V. S. Buslaev, A. I. Komech, E. A. Kopylova, and D. Stuart, "On asymptotic stability of solitary waves in Schrödinger equation coupled to nonlinear oscillator", *Comm. Partial Differential Equations* **33**:4 (2008), 669–705.

[25] В. С. Буслаев, Г. С. Перельман, "Рассеяние для нелинейного уравнения Шрёдингера: состояния, близкие к солитону", *Алгебра и анализ* **4**:6 (1992), 63–102; English transl., V. S. Buslaev and G. S. Perel'man, "Scattering for the nonlinear Schrödinger equation: states close to a soliton", *St. Petersburg Math. J.* **4**:6 (1993), 1111–1142.

[26] V. S. Buslaev and G. S. Perelman, "On the stability of solitary waves for nonlinear Schrödinger equations", *Nonlinear evolution equations*, Amer. Math. Soc. Transl. Ser. 2, vol. 164, Adv. Math. Sci., 22, Amer. Math. Soc., Providence, RI 1995, pp. 75–98.

[27] V. S. Buslaev and C. Sulem, "On asymptotic stability of solitary waves for nonlinear Schrödinger equations", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20**:3 (2003), 419–475.

[28] V. V. Chepyzhov and M. I. Vishik, *Attractors for equations of mathematical physics*, Amer. Math. Soc. Colloq. Publ., vol. 49, Amer. Math. Soc., Providence, RI 2002, xii+363 pp.

[29] G. M. Coclite and V. Georgiev, "Solitary waves for Maxwell–Schrödinger equations", *Electron. J. Differential Equations* **2004** (2004), 94, 31 pp. (electronic).

[30] A. Comech, "On global attraction to solitary waves. Klein–Gordon equation with mean field interaction at several points", *J. Differential Equations* **252**:10 (2012), 5390–5413.

[31] A. Comech, "Weak attractor of the Klein–Gordon field in discrete space-time interacting with a nonlinear oscillator", *Discrete Contin. Dyn. Syst.* **33**:7 (2013), 2711–2755.

[32] F. H. J. Cornish, "Classical radiation theory and point charges", *Proc. Phys. Soc.* **86**:3 (1965), 427–442.

[33] S. Cuccagna, "Stabilization of solutions to nonlinear Schrödinger equations", *Comm. Pure Appl. Math.* **54**:9 (2001), 1110–1145.

[34] S. Cuccagna, "The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states", *Comm. Math. Phys.* **305**:2 (2011), 279–331.

[35] S. Cuccagna and T. Mizumachi, "On asymptotic stability in energy space of ground states for nonlinear Schrödinger equations", *Comm. Math. Phys.* **284**:1 (2008), 51–77.

[36] M. Dafermos and I. Rodnianski, "A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds", *Invent. Math.* **185**:3 (2011), 467–559.

[37] P. D'Ancona, "Kato smoothing and Strichartz estimates for wave equations with magnetic potentials", *Comm. Math. Phys.* **335**:1 (2015), 1–16.

[38] P. D'Ancona, L. Fanelli, L. Vega, and N. Visciglia, "Endpoint Strichartz estimates for the magnetic Schrödinger equation", *J. Funct. Anal.* **258**:10 (2010), 3227–3240.

[39] S. Demoulini and D. Stuart, "Adiabatic limit and the slow motion of vortices in a Chern–Simons–Schrödinger system", *Comm. Math. Phys.* **290**:2 (2009), 597–632.

[40] P. A. M. Dirac, "Classical theory of radiating electrons", *Proc. Roy. Soc. London Ser. A* **167**:929 (1938), 148–169.

[41] R. Donninger, W. Schlag, and A. Soffer, "On pointwise decay of linear waves on a Schwarzschild black hole background", *Comm. Math. Phys.* **309**:1 (2012), 51–86.

[42] T. Duyckaerts, C. Kenig, and F. Merle, "Profiles of bounded radial solutions of the focusing, energy-critical wave equation", *Geom. Funct. Anal.* **22**:3 (2012), 639–698.

[43] T. Duyckaerts, C. Kenig, and F. Merle, "Scattering for radial, bounded solutions of focusing supercritical wave equations", *Int. Math. Res. Not. IMRN* **2014**:1 (2014), 224–258.

[44] T. Duyckaerts, C. Kenig, and F. Merle, "Concentration-compactness and universal profiles for the non-radial energy critical wave equation", *Nonlinear Anal.* **138** (2016), 44–82.

[45] А. В. Дымов, "Диссипативные эффекты в одной линейной лагранжевой системе с бесконечным числом степеней свободы", *Изв. РАН. Сер. матем.* **76**:6 (2012), 45–80; English transl., A. V. Dymov, "Dissipative effects in a linear Lagrangian system with infinitely many degrees of freedom", *Izv. Math.* **76**:6 (2012), 1116–1149.

[46] W. Eckhaus and A. van Harten, *The inverse scattering transformation and the theory of solitons. An introduction*, North-Holland Math. Stud., vol. 50, North-Holland Publishing Co., Amsterdam–New York 1981, xi+222 pp.

[47] И. Е. Егорова, Е. А. Копылова, В. А. Марченко, Г. Тешль, "Об уточнении дисперсионных оценок для одномерных уравнений Шрёдингера и Клейна–Гордона", *УМН* **71**:3(429) (2016), 3–26; English transl., I. E. Egorova, E. A. Kopylova, V. A. Marchenko, and G. Teschl, "Dispersion estimates for one-dimensional Schrödinger and Klein–Gordon equations revisited", *Russian Math. Surveys* **71**:3 (2016), 391–415.

[48] I. Egorova, E. A. Kopylova, and G. Teschl, "Dispersion estimates for one-dimensional discrete Schrödinger and wave equations", *J. Spectr. Theory* **5**:4 (2015), 663–696.

[49] A. Einstein, "Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?", *Ann. der Phys.* (4) **18** (1905), 639–641.

[50] M. B. Erdoğan, M. Goldberg, and W. R. Green, "Dispersive estimates for four dimensional Schrödinger and wave equations with obstructions at zero energy", *Comm. Partial Differential Equations* **39**:10 (2014), 1936–1964.

[51] M. J. Esteban, V. Georgiev, and E. Séré, "Stationary solutions of the Maxwell–Dirac and the Klein–Gordon–Dirac equations", *Calc. Var. Partial Differential Equations* **4**:3 (1996), 265–281.

[52] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics*, vol. 2: *Mainly electromagnetism and matter*, Addison-Wesley Publishing Co., Inc., Reading, MA–London 1964, xii+569 pp.

[53] C. Foias, O. Manley, R. Rosa, and R. Temam, *Navier–Stokes equations and turbulence*, Encyclopedia Math. Appl., vol. 83, Cambridge Univ. Press, Cambridge 2001, xiv+347 pp.

[54] J. Fröhlich and Z. Gang, "Emission of Cherenkov radiation as a mechanism for Hamiltonian friction", *Adv. Math.* **264** (2014), 183–235.

[55] J. Fröhlich, S. Gustafson, B. L. G. Jonsson, and I. M. Sigal, "Solitary wave dynamics in an external potential", *Comm. Math. Phys.* **250**:3 (2004), 613–642.

[56] J. Fröhlich, T.-P. Tsai, and H.-T. Yau, "On the point-particle (Newtonian) limit of the non-linear Hartree equation", *Comm. Math. Phys.* **225**:2 (2002), 223–274.

[57] G. I. Gaudry, "Quasimeasures and operators commuting with convolution", *Pacific J. Math.* **18**:3 (1966), 461–476.

[58] M. Gell-Mann, "Symmetries of baryons and mesons", *Phys. Rev.* (2) **125**:3 (1962), 1067–1084.

[59] H.-P. Gittel, J. Kijowski, and E. Zeidler, "The relativistic dynamics of the combined particle-field system in renormalized classical electrodynamics", *Comm. Math. Phys.* **198**:3 (1998), 711–736.

[60] M. Goldberg and W. R. Green, "Dispersive estimates for higher dimensional Schrödinger operators with threshold eigenvalues. I. The odd dimensional case", *J. Funct. Anal.* **269**:3 (2015), 633–682.

[61] M. Goldberg and W. R. Green, "Dispersive estimates for higher dimensional Schrödinger operators with threshold eigenvalues. II. The even dimensional case", *J. Spectr. Theory* **7**:1 (2017), 33–86.

[62] M. Grillakis, J. Shatah, and W. Strauss, "Stability theory of solitary waves in the presence of symmetry. I", *J. Funct. Anal.* **74**:1 (1987), 160–197.

[63] M. Grillakis, J. Shatah, and W. Strauss, "Stability theory of solitary waves in the presence of symmetry. II", *J. Funct. Anal.* **94**:2 (1990), 308–348.

[64] J. K. Hale, *Asymptotic behavior of dissipative systems*, Math. Surveys Monogr., vol. 25, Amer. Math. Soc., Providence, RI 1988, x+198 pp.

[65] F. Halzen and A. D. Martin, *Quarks and leptons*: *an introductory course in modern particle physics*, John Wiley & Sons, Inc., New York 1984, xvi+396 pp.

[66] T. Harada and H. Maeda, "Stability criterion for self-similar solutions with a scalar field and those with a stiff fluid in general relativity", *Classical Quantum Gravity* **21**:2 (2004), 371–389.

[67] A. Haraux, *Systèmes dynamiques dissipatifs et applications*, Rech. Math. Appl., vol. 17, Masson, Paris 1991, xii+132 pp.

[68] W. Heisenberg, "Der derzeitige Stand der nichtlinearen Spinortheorie der Elementarteilchen", *Acta Phys. Austriaca* **14** (1961), 328–339.

[69] W. Heisenberg, *Introduction to the unified field theory of elementary particles*, Intersci. Publ. John Wiley & Sons, London–New York–Sydney 1966, ix+177 pp.

[70] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin–New York 1981, iv+348 pp.

[71] E. Hopf, "Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen", *Math. Nachr.* **4** (1951), 213–231.

[72] L. Hörmander, *The analysis of linear partial differential operators*, т. I: *Distribution theory and Fourier analysis*, 2nd ed., Grundlehren Math. Wiss., **256**, Springer-Verlag, Berlin 1990, xii+440 с.

[73] L. Houllevigue, *L'évolution des sciences*, Librairie A. Collin, Paris 1908, xi+287 pp.

[74] В. М. Имайкин, "Солитонные асимптотики для систем типа 'поле-частица' ", *УМН* **68**:2(410) (2013), 33–90; English transl., V. M. Imaykin, "Soliton asymptotics for systems of 'field-particle' type", *Russian Math. Surveys* **68**:2 (2013), 227–281.

[75] V. Imaykin, A. Komech, and P. A. Markowich, "Scattering of solitons of the Klein–Gordon equation coupled to a classical particle", *J. Math. Phys.* **44**:3 (2003), 1202–1217.

[76] V. Imaykin, A. Komech, and N. Mauser, "Soliton-type asymptotics for the coupled Maxwell–Lorentz equations", *Ann. Henri Poincaré* **5**:6 (2004), 1117–1135.

[77] V. Imaykin, A. Komech, and H. Spohn, "Soliton-type asymptotic and scattering for a charge coupled to the Maxwell field", *Russ. J. Math. Phys.* **9**:4 (2002), 428–436.

[78] V. Imaykin, A. Komech, and H. Spohn, "Scattering theory for a particle coupled to a scalar field", *Discrete Contin. Dyn. Syst.* **10**:1-2 (2004), 387–396.

[79] V. Imaykin, A. Komech, and H. Spohn, "Rotating charge coupled to the Maxwell field: scattering theory and adiabatic limit", *Monatsh. Math.* **142**:1-2 (2004), 143–156.

[80] V. Imaykin, A. Komech, and H. Spohn, "Scattering asymptotics for a charged particle coupled to the Maxwell field", *J. Math. Phys.* **52**:4 (2011), 042701, 33 pp.

[81] V. Imaykin, A. Komech, and B. Vainberg, "On scattering of solitons for the Klein–Gordon equation coupled to a particle", *Comm. Math. Phys.* **268**:2 (2006), 321–367.

[82] V. Imaykin, A. Komech, and B. Vainberg, "Scattering of solitons for coupled wave-particle equations", *J. Math. Anal. Appl.* **389**:2 (2012), 713–740.

[83] J. D. Jackson, *Classical electrodynamics*, 3rd ed., John Wiley & Sons, Inc., New York 1999, xxi+808 c.

[84] A. Jensen and T. Kato, "Spectral properties of Schrödinger operators and time-decay of the wave functions", *Duke Math. J.* **46**:3 (1979), 583–611.

[85] K. Jörgens, "Das Anfangswertproblem im Großen für eine Klasse nichtlinearer Wellengleichungen", *Math. Z.* **77** (1961), 295–308.

[86] J.-L. Journé, A. Soffer, and C. D. Sogge, "Decay estimates for Schrödinger operators", *Comm. Pure Appl. Math.* **44**:5 (1991), 573–604.

[87] C. Kenig, A. Lawrie, B. Liu, and W. Schlag, "Stable soliton resolution for exterior wave maps in all equivariance classes", *Adv. Math.* **285** (2015), 235–300.

[88] C. E. Kenig, A. Lawrie, and W. Schlag, "Relaxation of wave maps exterior to a ball to harmonic maps for all data", *Geom. Funct. Anal.* **24**:2 (2014), 610–647.

[89] C. E. Kenig and F. Merle, "Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case", *Invent. Math.* **166**:3 (2006), 645–675.

[90] C. E. Kenig and F. Merle, "Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation", *Acta Math.* **201**:2 (2008), 147–212.

[91] C. E. Kenig and F. Merle, "Nondispersive radial solutions to energy supercritical non-linear wave equations, with applications", *Amer. J. Math.* **133**:4 (2011), 1029–1065.

[92] А. А. Комеч, А. И. Комеч, "Вариант теоремы Титчмарша о свертке для распределений на окружности", *Функц. анализ и его прил.* **47**:1 (2013), 26–32; English transl., A. A. Komech and A. I. Komech, "On the Titchmarsh convolution theorem for distributions on the circle", *Funct. Anal. Appl.* **47**:1 (2013), 21–26.

[93] А. И. Комеч, "О стабилизации взаимодействия струны с нелинейным осциллятором", *Вестн. Моск. ун-та. Сер.* 1. *Матем., мех.*, 1991, № 6, 35–41; English transl., A. I. Komech, "On the stabilization of interaction of a string with a nonlinear oscillator", *Moscow Univ. Math. Bull.* **46**:6 (1991), 34–39.

[94] А. И. Комеч, "Линейные уравнения в частных производных с постоянными коэффициентами", *Дифференциальные уравнения с частными производными* – 2, Итоги науки и техн. Сер. Соврем. пробл. матем. Фундам. направления, **31**, ВИНИТИ, М. 1988, с. 127–261; English transl., A. I. Komech, "Linear partial differential equations with constant coefficients", *Partial differential equations* II, Encyclopaedia Math. Sci., vol. 31, Springer, Berlin 1994, pp. 121–255.

[95] A. I. Komech, "On stabilization of string-nonlinear oscillator interaction", *J. Math. Anal. Appl.* **196**:1 (1995), 384–409.

[96] A. I. Komech, "On the stabilization of string-oscillator interaction", *Russ. J. Math. Phys.* **3**:2 (1995), 227–247.

[97] A. Komech, "On transitions to stationary states in one-dimensional nonlinear wave equations", *Arch. Ration. Mech. Anal.* **149**:3 (1999), 213–228.

[98] А. И. Комеч, "Аттракторы нелинейных гамильтоновых одномерных волновых уравнений", *УМН* **55**:1(331) (2000), 45–98; English transl, A. I. Komech, "Attractors of non-linear Hamiltonian one-dimensional wave equations", *Russian Math. Surveys* **55**:1 (2000), 43–92.

[99] A. I. Komech, "On attractor of a singular nonlinear U(1)-invariant Klein–Gordon equation", *Progress in analysis*, vols. I, II (Berlin, 2001), World Sci. Publ., River Edge, NJ 2003, pp. 599–611.

[100] A. Komech, *Quantum mechanics*: *genesis and achievements*, Springer, Dordrecht 2013, xviii+285 pp.

[101] A. Komech, "Attractors of Hamilton nonlinear PDEs", *Discrete Contin. Dyn. Syst.* **36**:11 (2016), 6201–6256.

[102] A. Komech, "Quantum jumps and attractors of Maxwell–Schrödinger equations", *Nonlinearity* (to appear); 2019, 14 pp., arXiv: 1907.04297.

[103] A. I. Komech and A. A. Komech, "On the global attraction to solitary waves for the Klein–Gordon equation coupled to a nonlinear oscillator", *C. R. Math. Acad. Sci. Paris* **343**:2 (2006), 111–114.

[104] A. Komech and A. Komech, "Global attractor for a nonlinear oscillator coupled to the Klein–Gordon field", *Arch. Ration. Mech. Anal.* **185**:1 (2007), 105–142.

[105] A. I. Komech and A. A. Komech, "Global attraction to solitary waves in models based on the Klein–Gordon equation", *SIGMA* **4** (2008), 010, 23 pp.

[106] A. Komech and A. Komech, "Global attraction to solitary waves for Klein–Gordon equation with mean field interaction", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:3 (2009), 855–868.

[107] A. Komech and A. Komech, "On global attraction to solitary waves for the Klein–Gordon field coupled to several nonlinear oscillators", *J. Math. Pures Appl.* (9) **93**:1 (2010), 91–111.

[108] A. Komech and A. Komech, "Global attraction to solitary waves for a nonlinear Dirac equation with mean field interaction", *SIAM J. Math. Anal.* **42**:6 (2010), 2944–2964.

[109] A. Komech and E. Kopylova, "Scattering of solitons for the Schrödinger equation coupled to a particle", *Russ. J. Math. Phys.* **13**:2 (2006), 158–187.

[110] A. I. Komech and E. A. Kopylova, "Weighted energy decay for 1D Klein–Gordon equation", *Comm. Partial Differential Equations* **35**:2 (2010), 353–374.

[111] A. I. Komech and E. A. Kopylova, "Weighted energy decay for 3D Klein–Gordon equation", *J. Differential Equations* **248**:3 (2010), 501–520.

[112] A. Komech and E. Kopylova, *Dispersion decay and scattering theory*, John Wiley & Sons, Inc., Hoboken, NJ 2012, xxvi+175 pp.

[113] A. I. Komech and E. A. Kopylova, "Dispersion decay for the magnetic Schrödinger equation", *J. Funct. Anal.* **264**:3 (2013), 735–751.

[114] A. Komech and E. Kopylova, "On eigenfunction expansion of solutions to the Hamilton equations", *J. Stat. Phys.* **154**:1-2 (2014), 503–521.

[115] A. I. Komech and E. A. Kopylova, "Weighted energy decay for magnetic Klein–Gordon equation", *Appl. Anal.* **94**:2 (2015), 218–232.

[116] A. Komech and E. Kopylova, "On the eigenfunction expansion for Hamilton operators", *J. Spectr. Theory* **5**:2 (2015), 331–361.

[117] A. I. Komech, E. A. Kopylova, and S. A. Kopylov, "On nonlinear wave equations with parabolic potentials", *J. Spectr. Theory* **3**:4 (2013), 485–503.

[118] A. I. Komech, E. A. Kopylova, and M. Kunze, "Dispersive estimates for 1D discrete Schrödinger and Klein–Gordon equations", *Appl. Anal.* **85**:12 (2006), 1487–1508.

[119] A. I. Komech, E. A. Kopylova, and H. Spohn, "Scattering of solitons for Dirac equation coupled to a particle", *J. Math. Anal. Appl.* **383**:2 (2011), 265–290.

[120] A. Komech, E. Kopylova, and D. Stuart, "On asymptotic stability of solitons in a nonlinear Schrödinger equation", *Commun. Pure Appl. Anal.* **11**:3 (2012), 1063–1079.

[121] A. I. Komech, E. A. Kopylova, and B. R. Vainberg, "On dispersive properties of discrete 2D Schrödinger and Klein–Gordon equations", *J. Funct. Anal.* **254**:8 (2008), 2227–2254.

[122] A. Komech, M. Kunze, and H. Spohn, "Effective dynamics for a mechanical particle coupled to a wave field", *Comm. Math. Phys.* **203**:1 (1999), 1–19.

[123] A. I. Komech, N. J. Mauser, and A. P. Vinnichenko, "Attraction to solitons in relativistic nonlinear wave equations", *Russ. J. Math. Phys.* **11**:3 (2004), 289–307.

[124] A. I. Komech and A. E. Merzon, "Scattering in the nonlinear Lamb system", *Phys. Lett. A* **373**:11 (2009), 1005–1010.

[125] A. I. Komech and A. E. Merzon, "On asymptotic completeness for scattering in the nonlinear Lamb system", *J. Math. Phys.* **50**:2 (2009), 023514, 10 pp.

[126] A. I. Komech and A. E. Merzon, "On asymptotic completeness of scattering in the nonlinear Lamb system. II", *J. Math. Phys.* **54**:1 (2013), 012702, 9 pp.

[127] A. Komech and A. Merzon, *Stationary diffraction by wedges. Method of automorphic functions on complex characteristics*, Lecture Notes in Math., vol. 2249, Springer, Cham 2019, xi+165 pp.

[128] A. Komech and H. Spohn, "Soliton-like asymptotics for a classical particle interacting with a scalar wave field", *Nonlinear Anal.* **33**:1 (1998), 13–24.

[129] A. Komech and H. Spohn, "Long-time asymptotics for the coupled Maxwell–Lorentz equations", *Comm. Partial Differential Equations* **25**:3-4 (2000), 559–584.

[130] A. Komech, H. Spohn, and M. Kunze, "Long-time asymptotics for a classical particle interacting with a scalar wave field", *Comm. Partial Differential Equations* **22**:1-2 (1997), 307–335.

[131] Е. А. Копылова, "Дисперсионные оценки для дискретных уравнений Шредингера и Клейна–Гордона", *Алгебра и анализ* **21**:5 (2009), 87–113; English transl., E. A. Kopylova, "Dispersion estimates for discrete Schrödinger and Klein–Gordon equations", *St. Petersburg Math. J.* **21**:5 (2010), 743–760.

[132] E. A. Kopylova, "On asymptotic stability of solitary waves in the discrete Schrödinger equation coupled to a nonlinear oscillator", *Nonlinear Anal.* **71**:7-8 (2009), 3031–3046.

[133] E. A. Kopylova, "On asymptotic stability of solitary waves in discrete Klein–Gordon equation coupled to a nonlinear oscillator", *Appl. Anal.* **89**:9 (2010), 1467–1492.

[134] Е. А. Копылова, "Дисперсионные оценки для уравнений Шрёдингера и Клейна–Гордона", *УМН* **65**:1(391) (2010), 97–144; English transl., E. A. Kopylova, "Dispersive estimates for the Schrödinger and Klein–Gordon equations", *Russian Math. Surveys* **65**:1 (2010), 95–142.

[135] Е. А. Копылова, "Асимптотическая устойчивость солитонов для нелинейных гиперболических уравнений", *УМН* **68**:2(410) (2013), 91–144; English transl., E. A. Kopylova, "Asymptotic stability of solitons for nonlinear hyperbolic equations", *Russian Math. Surveys* **68**:2 (2013), 283–334.

[136] E. Kopylova, "On global attraction to solitary waves for the Klein–Gordon equation with concentrated nonlinearity", *Nonlinearity* **30**:11 (2017), 4191–4207.

[137] E. Kopylova, "On global attraction to stationary states for wave equation with concentrated nonlinearity", *J. Dynam. Differential Equations* **30**:1 (2018), 107–116.

[138] E. Kopylova, "On dispersion decay for 3D Klein–Gordon equation", *Discrete Contin. Dyn. Syst.* **38**:11 (2018), 5765–5780.

[139] E. A. Kopylova and A. I. Komech, "Long time decay for 2D Klein–Gordon equation", *J. Funct. Anal.* **259**:2 (2010), 477–502.

[140] E. A. Kopylova and A. I. Komech, "On asymptotic stability of moving kink for relativistic Ginzburg–Landau equation", *Comm. Math. Phys.* **302**:1 (2011), 225–252.

[141] E. Kopylova and A. I. Komech, "On asymptotic stability of kink for relativistic Ginzburg–Landau equations", *Arch. Ration. Mech. Anal.* **202**:1 (2011), 213–245.

[142] E. Kopylova and A. Komech, "On global attractor of 3D Klein–Gordon equation with several concentrated nonlinearities", *Dyn. Partial Differ. Equ.* **16**:2 (2019), 105–124.

[143] E. Kopylova and A. Komech, "Global attractor for 1D Dirac field coupled to nonlinear oscillator", *Comm. Math. Phys.*, Publ. online 2019, 1–31.

[144] E. Kopylova and G. Teschl, "Dispersion estimates for one-dimensional discrete Dirac equations", *J. Math. Anal. Appl.* **434**:1 (2016), 191–208.

[145] V. V. Kozlov, "Kinetics of collisionless continuous medium", *Regul. Chaotic Dyn.* **6**:3 (2001), 235–251.

[146] В. В. Козлов, О. Г. Смолянов, "Функция Вигнера и диффузия в бесстолкновительной среде, состоящей из квантовых частиц", *Теория вероятн. и ее примен.* **51**:1 (2006), 109–125; English transl., V. V. Kozlov and O. G. Smolyanov, "Wigner function and diffusion in a collision-free medium of quantum particles", *Theory Probab. Appl.* **51**:1 (2007), 168–181.

[147] В. В. Козлов, Д. В. Трещёв, "Слабая сходимость решений уравнения Лиувилля для нелинейных гамильтоновых систем", *ТМФ* **134**:3 (2003), 388–400; English transl., V. V. Kozlov and D. V. Treshchev, "Weak convergence of solutions of the Liouville equation for nonlinear Hamiltonian systems", *Theor. Math. Phys.* **134**:3 (2003), 339–350.

[148] В. В. Козлов, Д. В. Трещёв, "Эволюция мер в фазовом пространстве нелинейных гамильтоновых систем", *ТМФ* **136**:3 (2003), 496–506; English transl., V. V. Kozlov and D. V. Treshchev, "Evolution of measures in the phase space of nonlinear Hamiltonian systems", *Theor. Math. Phys.* **136**:3 (2003), 1325–1335.

[149] М. Г. Крейн, Г. К. Лангер, "О спектральной функции самосопряженного оператора в пространстве с индефинитной метрикой", *Докл. АН СССР* **152**:1 (1963), 39–42; English transl., M. G. Kreĭn and G. K. Langer, "The spectral function of a selfadjoint operator in a space with indefinite metric", *Soviet Math. Dokl.* **4** (1963), 1236–1239.

[150] J. Krieger, K. Nakanishi, and W. Schlag, "Center-stable manifold of the ground state in the energy space for the critical wave equation", *Math. Ann.* **361**:1-2 (2015), 1–50.

[151] J. Krieger and W. Schlag, *Concentration compactness for critical wave maps*, EMS Monogr. Math., Eur. Math. Soc., Zürich 2012, vi+484 pp.

[152] M. Kunze and H. Spohn, "Adiabatic limit for the Maxwell–Lorentz equations", *Ann. Henri Poincaré* **1**:4 (2000), 625–653.

[153] О. А. Ладыженская, "О принципе предельной амплитуды", *УМН* **12**:3(75) (1957), 161–164. [O. A. Ladyzhenskaya, "On the principle of limit amplitude", *Uspekhi Mat. Nauk* **12**:3(75) (1957), 161–164.]

[154] G. L. Lamb, Jr., *Elements of soliton theory*, Pure Appl. Math., John Wiley & Sons, Inc., New York 1980, xiii+289 pp.

[155] H. Lamb, "On a peculiarity of the wave-system due to the free vibrations of a nucleus in an extended medium", *Proc. London Math. Soc.* **32** (1900), 208–211.

[156] L. Landau, "On the problem of turbulence", *Докл. АН СССР* **44** (1944), 311–314.

[157] H. Langer, "Spectral functions of definitizable operators in Krein spaces", *Functional analysis* (Dubrovnik, 1981), Lecture Notes in Math., vol. 948, Springer, Berlin–New York 1982, pp. 1–46.

[158] P. D. Lax, C. S. Morawetz, and R. S. Phillips, "Exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle", *Comm. Pure Appl. Math.* **16**:4 (1963), 477–486.

[159] B. Ya. Levin, *Lectures on entire functions*, Transl. Math. Monogr., vol. 150, Amer. Math. Soc., Providence, RI 1996, xvi+248 pp.

[160] L. Lewin, *Advanced theory of waveguides*, Iliffe and Sons, Ltd., London 1951, 192 pp.

[161] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris; Gauthier-Villars, Paris 1969, xx+554 pp.

[162] E. Long and D. Stuart, "Effective dynamics for solitons in the nonlinear Klein–Gordon–Maxwell system and the Lorentz force law", *Rev. Math. Phys.* **21**:4 (2009), 459–510.

[163] Л. А. Люстерник, Л. Г. Шнирельман, *Топологические методы в вариационных задачах*, МГУ, М. 1930, 68 с.; French transl., L. Lusternik and L. Schnirelmann, *Méthodes topologiques dans les problèmes variationnels*, Actualités Sci. Indust., vol. 188, Hermann, Paris 1934, 51 pp.

[164] Л. Люстерник, Л. Шнирельман, "Топологические методы в вариационных задачах и их приложения к дифференциальной геометрии поверхностей", *УМН* **2**:1(17) (1947), 166–217. [L. Lusternik and L. Schnirelmann, "Topological methods in variational problems and their application to the differential geometry of surfaces", *Uspekhi Mat. Nauk* **2**:1(17) (1947), 166–217.]

[165] B. Marshall, W. Strauss, and S. Wainger, "$L^p - L^q$ estimates for the Klein–Gordon equation", *J. Math. Pures Appl.* (9) **59**:4 (1980), 417–440.

[166] Y. Martel, "Asymptotic N-soliton-like solutions of the subcritical and critical generalized Korteweg–de Vries equations", *Amer. J. Math.* **127**:5 (2005), 1103–1140.

[167] Y. Martel and F. Merle, "Asymptotic stability of solitons of the subcritical gKdV equations revisited", *Nonlinearity* **18**:1 (2005), 55–80.

[168] Y. Martel, F. Merle, and T.-P. Tsai, "Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations", *Comm. Math. Phys.* **231**:2 (2002), 347–373.

[169] M. Merkli and I. M. Sigal, "A time-dependent theory of quantum resonances", *Comm. Math. Phys.* **201**:3 (1999), 549–576.

[170] J. R. Miller and M. I. Weinstein, "Asymptotic stability of solitary waves for the regularized long-wave equation", *Comm. Pure Appl. Math.* **49**:4 (1996), 399–441.

[171] C. S. Morawetz, "The limiting amplitude principle", *Comm. Pure Appl. Math.* **15**:3 (1962), 349–361.

[172] C. S. Morawetz, "Time decay for the nonlinear Klein–Gordon equations", *Proc. Roy. Soc. London Ser. A* **306** (1968), 291–296.

[173] C. S. Morawetz and W. A. Strauss, "Decay and scattering of solutions of a nonlinear relativistic wave equation", *Comm. Pure Appl. Math.* **25**:1 (1972), 1–31.

[174] K. Nakanishi and W. Schlag, *Invariant manifolds and dispersive Hamiltonian evolution equations*, Zur. Lect. Adv. Math., Eur. Math. Soc., Zürich 2011, vi+253 pp.

[175] Y. Ne'eman, "Unified interactions in the unitary gauge theory", *Nuclear Phys.* **30** (1962), 347–349.

[176] D. Noja and A. Posilicano, "Wave equations with concentrated nonlinearities", *J. Phys. A* **38**:22 (2005), 5011–5022.

[177] R. L. Pego and M. I. Weinstein, "Asymptotic stability of solitary waves", *Comm. Math. Phys.* **164**:2 (1994), 305–349.

[178] G. Perelman, "Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equations", *Comm. Partial Differential Equations* **29**:7-8 (2004), 1051–1095.

[179] C.-A. Pillet and C. E. Wayne, "Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations", *J. Differential Equations* **141**:2 (1997), 310–326.

[180] M. Reed and B. Simon, *Methods of modern mathematical physics*, vol. III: *Scattering theory*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London 1979, xv+463 pp.

[181] M. Reed and B. Simon, *Methods of modern mathematical physics*, vol. IV: *Analysis of operators*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London 1978, xv+396 pp.

[182] I. Rodnianski and W. Schlag, "Time decay for solutions of Schrödinger equations with rough and time-dependent potentials", *Invent. Math.* **155**:3 (2004), 451–513.

[183] I. Rodnianski, W. Schlag, and A. Soffer, *Asymptotic stability of N-soliton states of NLS*, 2003, 70 pp., arXiv: math/0309114.

[184] I. Rodnianski, W. Schlag, and A. Soffer, "Dispersive analysis of charge transfer models", *Comm. Pure Appl. Math.* **58**:2 (2005), 149–216.

[185] W. Rudin, *Functional analysis*, McGraw-Hill Series in Higher Mathematics, New York–Düsseldorf–Johannesburg 1973, xiii+397 pp.

[186] E. Schrödinger, "Quantisierung als Eigenwertproblem. I", *Ann. Phys.* **79(384)**:4 (1926), 361–376; "II" **79(384)**:6 (1926), 489–527; "III" **80(385)**:13 (1926), 437–490; "IV" **81(386)**:18 (1926), 109–139.

[187] I. Segal, "Quantization and dispersion for nonlinear relativistic equations", *Mathematical theory of elementary particles* (Dedham, MA 1965), M.I.T. Press, Cambridge, MA 1966, pp. 79–108.

[188] I. Segal, "Dispersion for non-linear relativistic equations. II", *Ann. Sci. École Norm. Sup.* (4) **1**:4 (1968), 459–497.

[189] I. M. Sigal, "Non-linear wave and Schrödinger equations. I. Instability of periodic and quasiperiodic solutions", *Comm. Math. Phys.* **153**:2 (1993), 297–320.

[190] A. Soffer, "Soliton dynamics and scattering", *International congress of mathematicians*, vol. III, Eur. Math. Soc., Zürich 2006, pp. 459–471.

[191] A. Soffer and M. I. Weinstein, "Multichannel nonlinear scattering for nonintegrable equations", *Comm. Math. Phys.* **133**:1 (1990), 119–146.

[192] A. Soffer and M. I. Weinstein, "Multichannel nonlinear scattering for nonintegrable equations. II. The case of anisotropic potentials and data", *J. Differential Equations* **98**:2 (1992), 376–390.

[193] A. Soffer and M. I. Weinstein, "Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations", *Invent. Math.* **136**:1 (1999), 9–74.

[194] A. Soffer and M. I. Weinstein, "Selection of the ground state for nonlinear Schrödinger equations", *Rev. Math. Phys.* **16**:8 (2004), 977–1071.

[195] H. Spohn, *Dynamics of charged particles and their radiation field*, Cambridge Univ. Press, Cambridge 2004, xvi+360 pp.

[196] W. A. Strauss, "Decay and asymptotics for $\Box u = F(u)$", *J. Funct. Anal.* **2**:4 (1968), 409–457.

[197] W. A. Strauss, "Existence of solitary waves in higher dimensions", *Comm. Math. Phys.* **55**:2 (1977), 149–162.

[198] W. A. Strauss, "Nonlinear scattering theory at low energy", *J. Funct. Anal.* **41**:1 (1981), 110–133.

[199] W. A. Strauss, "Nonlinear scattering theory at low energy: sequel", *J. Funct. Anal.* **43**:3 (1981), 281–293.

[200] D. Stuart, "Existence and Newtonian limit of nonlinear bound states in the Einstein–Dirac system", *J. Math. Phys.* **51**:3 (2010), 032501, 13 pp.

[201] D. Tataru, "Local decay of waves on asymptotically flat stationary space-times", *Amer. J. Math.* **135**:2 (2013), 361–401.

[202] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, 2nd ed., Appl. Math. Sci., vol. 68, Springer-Verlag, New York 1997, xxii+648 pp.

[203] E. C. Titchmarsh, "The zeros of certain integral functions", *Proc. London Math. Soc.* (2) **25**:1 (1926), 283–302.

[204] D. Treschev, "Oscillator and thermostat", *Discrete Contin. Dyn. Syst.* **28**:4 (2010), 1693–1712.

[205] T.-P. Tsai, "Asymptotic dynamics of nonlinear Schrödinger equations with many bound states", *J. Differential Equations* **192**:1 (2003), 225–282.

[206] T.-P. Tsai and H.-T. Yau, "Classification of asymptotic profiles for nonlinear Schrödinger equations with small initial data", *Adv. Theor. Math. Phys.* **6**:1 (2002), 107–139.

[207] T.-P. Tsai and H.-T. Yau, "Asymptotic dynamics of nonlinear Schrödinger equations: resonance-dominated and dispersion-dominated solutions", *Comm. Pure Appl. Math.* **55**:2 (2002), 153–216.

[208] M. I. Weinstein, "Modulational stability of ground states of nonlinear Schrödinger equations", *SIAM J. Math. Anal.* **16**:3 (1985), 472–491.

[209] D. R. Yafaev, "On a zero-range interaction of a quantum particle with the vacuum", *J. Phys. A* **25**:4 (1992), 963–978.

[210] D. R. Yafaev, "A point interaction for the discrete Schrödinger operator and generalized Chebyshev polynomials", *J. Math. Phys.* **58**:6 (2017), 063511, 24 pp.

[211] K. Yajima, "Dispersive estimates for Schrödinger equations with threshold resonance and eigenvalue", *Comm. Math. Phys.* **259**:2 (2005), 475–509.

[212] Я. Б. Зельдович, "Рассеяние сингулярным потенциалом в теории возмущений и в импульсном представлении", *ЖЭТФ* **38**:3 (1960), 819–824; English transl.,

Ya. B. Zel'dovich, "Scattering by a singular potential in perturbation theory and in the momentum representation", *Soviet Physics. JETP* **11**:3 (1960), 594–597.

Alexander I. Komech
Institute for Information Transmission Problems
of the Russian Academy of Sciences
(Kharkevich Institute)
E-mail: alexander.komech@univie.ac.at

Elena A. Kopylova
Institute for Information Transmission Problems
of the Russian Academy of Sciences
(Kharkevich Institute)
E-mail: elena.kopylova@univie.ac.at

Received 13/JUL/19
Translated by THE AUTHORS