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On the structure of the critical group of a circulant graph with non-constant jumps

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1. The notion of the critical group of a graph, which is also known as the sandpile group, the Jacobian group, the Picard group, or the dollar group, was independently introduced by several authors [1]–[6]. The order of the critical group coincides with the number of skeleton trees of the graph.

Consider a connected finite graph G, allowed to have multiple edges but without loops. Denote the vertex and edge sets of G by $V(G)$ and $E(G)$, respectively. Given $u,v\in V(G)$, we let a_{uv} be the number of edges between u and v.

The matrix $A=A(G)=\{a_{uv}\}_{u,v\in V(G)}$ is called *the incidence matrix* of the graph G. Let $D=D(G)$ be the diagonal matrix with $d_{vv}=d(v)$, where $d(v)$ is the degree of the vertex $v\in V(G)$. The matrix $L(G)=D(G)-A(G)$ is called *the Laplacian matrix* of G.

Consider an integer $m\times n$ matrix M as a homomorphism $\mathbb{Z}^m\to\mathbb{Z}^n$. Then M has the image $\operatorname{im}M=M^t\mathbb{Z}^m$ and the cokernel $\operatorname{coker}M=\mathbb{Z}^n/\operatorname{im}M$. The *critical group* or the *Jacobian* $\operatorname{Jac}(G)$ of the graph G is defined to be the torsion group of $\operatorname{coker}(L(G))$.

From the standpoint of spectral theory and statistical physics it is most interesting to study Jacobians $\operatorname{Jac}(G_n)$ for infinite families of graphs G_n with infinitely increasing number of vertices. The size of the Laplacian matrix $L(G_n)$ also tends to infinity, and calculating its cokernel by means of direct analytic or computer methods is rather difficult. In this note we suggest a new approach to the solution of this problem for the infinite family of graphs indicated in the title. We show that the critical group of the graphs of this family is determined by the cokernel of a matrix of fixed size whose entries depend on the parameter n in a specific way.

Let $s_1,\dots,s_K$ be integers such that $1\leqslant s_j\leqslant N/2$, $j=1,\dots,K$. A *circulant graph* $C_N(s_1,\dots,s_K)$ on the N vertices $0,1,\dots,N-1$ is the graph in which each vertex i, $0\leqslant i\leqslant N-1$, is incident to the vertices $i\pm s_1,\dots,i\pm s_K \pmod N$. If $s_j=N/2$, then the vertices i and $i\pm s_j$ are joined by a pair of edges. The number N of vertices and the integers $s_1,\dots,s_K$ (*the jumps of the graph*) can exhibit non-trivial dependence on the parameter n. This is called a circulant graph with non-constant jumps.

The aim of this note is to describe the structure of the critical group for the infinite family of circulant graphs of the form $G_n=C_{\beta n}(s_1,\dots,s_k,\alpha_1 n,\dots,\alpha_\ell n)$ having βn vertices and $k+\ell$ jumps, the first k of which are constant, whereas the other ℓ depend linearly on n. The particular case $\beta=1$, $\ell=0$ (a circulant graph with constant jumps) was studied earlier in [7].

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2. Our main result is the following theorem.

Theorem 1. *Let* $G_n = C_{\beta n}(s_1, \dots, s_k, \alpha_1 n, \dots, \alpha_\ell n)$ *be a connected circulant graph, where the* s_j *and* a_j *with* $1 \leqslant s_1 < \dots < s_k \leqslant [\beta n/2]$ *and* $1 \leqslant \alpha_1 < \dots < \alpha_\ell \leqslant [\beta/2]$ *are integers. Let* $L(z) = \sum_{j=1}^{k}(z^{s_j} + z^{-s_j} - 2)$ *and* $l(z) = \sum_{j=1}^{\ell}(z^{\alpha_j} + z^{-\alpha_j} - 2)$. *Following* [8], *denote by* $\mathscr{A}$ *the companion matrix of the Laurent polynomial* $R_\beta(z) = \prod_{p=1}^{\beta}(L(z) + l(e^{2\pi i\, p/\beta}))$. *Then the critical group* $\operatorname{Jac}(G_n)$ *is isomorphic to the torsion group of the cokernel of the* $2\beta s_k \times 4\beta s_k$ *integer matrix* $(L(\mathscr{A}) + l(\mathscr{A}^n), \mathscr{A}^{\beta\, n} - I_{2\beta s_k})$. *Moreover, the rank of the Abelian group* $\operatorname{Jac}(G)$ *does not exceed* $2\beta s_k - 1$. *This estimate is sharp.*

Let us give a sketch of the proof of Theorem 1. First of all, note that $R_\beta(z)$ is the resultant of the Laurent polynomials $L(z) + l(x)$ and $x^\beta - 1$ in the variable x. Therefore, $R_\beta(z)$ is a monic polynomial with integer coefficients. Following the proof of the polynomial remainder theorem for resultants (for example, see [9], Theorem 3.2), we find Laurent polynomials $p(x, z)$ and $q(x, z)$ in x and z with integer coefficients such that $R_\beta(z) = p(z^n, z)(L(z) + l(z^n)) + q(z^n, z)(z^{\beta n} - 1)$.

Note that the Laplacian $\mathbb{L}$ of the graph G is specified by the matrix $-L(T_{\beta n}) - l(T_{\beta n}^n)$, where $T_{\beta n} = \operatorname{circ}(0, 1, 0, \dots, 0)$ is the circulant $\beta\, n \times \beta\, n$ shift matrix. Let $\mathbb{A}$ be the Abelian group freely generated by the infinite set of elements x_i, $i \in \mathbb{Z}$. Consider the endomorphism T acting on $\mathbb{A}$ by the rule $T\colon x_i \to x_{i+1}$. With the notation in [8], we represent the cokernel $\mathbb{L}$ as an Abelian group with the following presentation:

$$
\begin{aligned}
\operatorname{coker} \mathbb{L} &= \langle x_i, i \in \mathbb{Z} \mid (L(T) + l(T^n))x_j = 0,\ (T^{\beta\, n} - 1)x_j = 0,\ j \in \mathbb{Z} \rangle \\
&= \langle x_i,\ i \in \mathbb{Z} \mid (L(T) + l(T^n))x_j = 0,\ (T^{\beta\, n} - 1)x_j = 0, \\
&\qquad p(T^n, T)(L(T) + l(T^n))x_j + q(T^n, T)(T^{\beta\, n} - 1)x_j = 0,\ j \in \mathbb{Z} \rangle \\
&= \langle x_i,\ i \in \mathbb{Z} \mid (L(T) + l(T^n))x_j = 0,\ (T^{\beta\, n} - 1)x_j = 0,\ R_\beta(T)x_j = 0 \rangle.
\end{aligned}
$$

Put $A = L(T) + l(T^n)$, $B = T^{\beta\, n} - 1$, and $C = R_\beta(T)$. Since the polynomial $R_\beta(z)$ is monic, the $\mathbb{Z}$-module $\operatorname{coker} C = \mathbb{A}/\operatorname{im} C$ is freely generated by the elements $u_1, \dots, u_s$, where $s = 2\beta s_k$ is the degree of $R_\beta(z)$ and $u_i = [x_i]$ is the image of x_i under the canonical map $\mathbb{A} \to \mathbb{A}/\operatorname{im} C$. In the basis $u_1, \dots, u_s$ the action of $T\big|_{\operatorname{coker} C}$ is determined by the companion matrix $\mathscr{A}$ of the polynomial $R_\beta(z)$. Hence the actions of the endomorphisms $A\big|_{\operatorname{coker} C}$ and $B\big|_{\operatorname{coker} C}$ on $\operatorname{coker} C$ are given by the $s \times s$ matrices $L(\mathscr{A}) + l(\mathscr{A}^n)$ and $\mathscr{A}^{\beta\, n} - I_{2\beta s_k}$, respectively. Use of Lemma 1 in [8] completes the proof.

As a corollary of the theorem proved above we obtain the following result.

Theorem 2. *The critical group of the Möbius ladder* $C_{2n}(1, n)$ *with double rungs is isomorphic to* $\mathbb{Z}_{(n,a(n))} \oplus \mathbb{Z}_{a(n)} \oplus \mathbb{Z}_{4[n,a(n)]/(2,n)}$, *where* $(l, m) = \gcd(l, m)$, $[l, m] = \operatorname{lcm}(l, m)$, *and* $a(n)$ *is the sequence* A079496 *in the* On-Line Encyclopedia of Integer Sequences. *Moreover,* $a(n) = T_m(3)$ *if* $n = 2m$, *and* $a(n) = 2U_{m-1}(3) + T_m(3)$ *if* $n = 2m + 1$, *where* $T_m(x)$ *and* $U_{m-1}(x)$ *are Chebyshev polynomials of the first and second kind, respectively.*

The critical group of the Möbius ladder with single rungs was described in [10] and [7].

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