

Fundamental oscillation modes of self-interacting bosonic dark stars

C. Vásquez Flores,^a Alessandro Parisi,^b Chian-Shu Chen^b
and Germán Lugones^{c,1}

^aDepartamento de Física, CFM, Universidade Federal de Santa Catarina,
CP 476, CEP 88040-900, Florianópolis, SC, Brazil

^bDepartment of Physics, Tamkang University,
New Taipei 251, Taiwan

^cUniversidade Federal do ABC, Centro de Ciências Naturais e Humanas,
Avenida dos Estados, 5001, CEP 09210-170, Santo André, SP, Brazil

E-mail: cesarovfsky@gmail.com, alessandro26@live.it, chianshu@gmail.com,
german.lugones@ufabc.edu.br

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Abstract. We perform a detailed analysis of the fundamental f -mode frequencies and damping times of nonrotating stellar-mass boson stars in general relativity by solving the nonradial perturbation equations. Two parameters which govern the microscopic properties of the bosonic condensates, namely the self-coupling strength and the mass of the scalar particle, are explored. These two quantities characterize oscillations of boson stars. Specifically, we reexamine some empirical relations that describe the f -mode parameters in terms of the mass and radius of boson stars.

Keywords: dark matter theory, gravitational waves / sources, gravity, massive stars

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¹<https://orcid.org/0000-0002-2978-8079>.

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1 Introduction

Gravitational waves (GWs) from a binary black hole (BH) were detected at the Advanced LIGO interferometer in september 2015 [1] opening a new window to explore the Universe. More recently, the LIGO and Virgo Collaborations reported the first event, GW170817, where a gravitational-wave signal was observed from a merger of two neutron stars (NSs) [2]. Based on the GW170817 observation, several recent studies have imposed relevant constraints on NS equations of state (EoS) [3–10]. While BHs and NSs now represent the standard model of compact objects, it is worth exploring alternatives which differ in their GW signatures from the standard one. In this work, we compute the f -mode of an important class of hypothetical objects, composed of self-interacting scalar field configurations known as boson stars (BSs). The nature of these objects depends of the scalar self-interaction and its coupling to gravity. Examples of such nonstandard stars were widely discussed in literature, such as a geon, which is a self-gravitating star consisting of electromagnetic fields and was first considered by Wheeler [11]. The gravitational attraction by its own field energy confines the geon in a certain region. Later, Kaup solved the Einstein-Klein-Gordon (EKG) equations for a massive complex scalar field and found a new class of solutions for compact objects [12]. These BSs are stable with respect to spherically symmetric gravitational collapse. Ruffini and Bonazzola [13] demonstrated that BSs describe a family of self-gravitational scalar field configurations within general relativity.

Although the existence of these elementary bosons, their clustering, and their hypothetical role in the formation of galaxies and large scale structure of the Universe, is nowadays obscure [14], there are two theoretical arguments that support the possibility of self-gravitating objects made by bosonic particles in the Universe. First, the discovery of the Higgs boson [15, 16] confirmed the existence of scalar fields in nature. Second, the suggestion of a formation mechanism, dubbed as gravitational cooling [17], to produce BSs from a generic scalar field configuration. In the past few years BSs have been studied in many different contexts (see [18–20] for complete reviews). The stability of BSs against radial perturbations around the equilibrium state has also been studied by several authors [21–23]. In general, it

is found that BSs and NSs share a remarkable similarity on the stability properties. Furthermore, it has been argued that BSs could be composed by dark matter (DM) particles [24, 25] and they have been considered as viable models of black hole candidates [26].

Torres et al. [27, 28] explored the possibility that supermassive BSs (for a large range of boson masses and self-interactions) are capable of explaining the nature of the object in Sgr A^* without invoking the presence of a singularity. Olivares et al. [29] showed that the accretion flows onto BSs behave differently compared to that on Kerr black holes as they do not produce jets but stalled accretion tori that make them distinguishable from black holes. Recently, the Event Horizon Telescope (EHT) has mapped the central compact radio source of the elliptical galaxy M87 [30], finding that the observation is consistent with a spinning Kerr black hole as predicted by general relativity. It was also shown that some exotic compact objects are incompatible with this observation. For example, the shadow of a superspinar is very different from that of a black hole excluding any superspinar model for M87. Analogously, several types of wormholes can be excluded, while other compact-object candidates need to be analyzed with more care. BSs produce images with ring-like features similar to those observed by the EHT, but generically require masses that are substantially different from that explored for M87 [31]. The gravitational wave production from the merger of two BSs has been studied by Palenzuela et al. [32, 33] and, in particular, the emission of gravitational radiation by an oscillating BS has been studied under the Newtonian approximation by Ferrell and Gleiser [34]. It has been shown that the amount of gravitational energy corresponds to the transition energy from an excited state to the ground state of the oscillation modes. Quasinormal modes of BSs were also obtained within general relativity by Yoshida et al. [35], Balakrishna et al. [36], and more recently by Macedo et al. [37]. Kling and Rajaraman [38, 39] found a semianalytic solution describing dilute BSs in the Newtonian limit, and showed that the solution is stable to numerical errors. The tidal deformability of BSs has also been investigated in [40, 41] and this can be used to discriminate between BSs and NSs with the future aLIGO sensitivity.

In this work, we study the f -mode of BSs with different self-interaction strengths and different masses of the scalar boson. Recently these modes have been studied for many models of compact stars in [42, 43] and in the context of binary systems in [44, 45]. The scope of this work is to connect the microscopic properties of scalar bosons with the macroscopic observables of BSs. In particular we compute both real and imaginary parts of the oscillation frequencies which may be observed if pulsations are excited during the formation of BSs or during their evolution under the action of an external perturbation. The paper is organized as follows. In section 2 we present the main properties of quasi-normal modes of compact stars. Equilibrium configurations and the EoS of BSs are studied in section 3. In section 4 we present our results for the f -mode with various possible parametrizations of the EoS. We draw our conclusions and the possible connections to astronomical observations in section 5.

2 Quasi-normal modes of compact stars

The equations describing the nonradial pulsations of a compact star in a fully general relativistic context were first studied by Thorne and Campolattaro [46, 47]. They showed that Einstein's equations describing small, nonradial, quasi-periodic oscillations of general relativistic stellar models could be reduced to a system of ordinary differential equations for the perturbed functions. Here, we use the formulation of Lindblom and Detweiler (see appendix A) [48, 49], where the formalism is reduced to a system of four ordinary differential

equations describing the frequency and damping time of the star's oscillations as well as of the emitted gravitational waves.

We assume that the unperturbed spherically symmetric equilibrium state of a compact star is given by a solution of the Tolman-Oppenheimer-Volkoff (TOV) equations. For pulsations of spherical-harmonic indices ℓ and m and parity $\pi = (-1)^\ell$, the perturbed metric tensor inside the star in the Regge-Wheeler gauge [50] is given by

$$ds^2 = -e^\psi(1 + r^\ell H_0^{\ell m} Y_{\ell m} e^{i\omega t}) dt^2 + e^\lambda(1 - r^\ell H_2^{\ell m} Y_{\ell m} e^{i\omega t}) dr^2 - 2i\omega r^{\ell+1} H_1^{\ell m} Y_{\ell m} e^{i\omega t} dt dr + r^2(1 - r^\ell K^{\ell m} Y_{\ell m} e^{i\omega t})(d\theta^2 + \sin^2\theta d\varphi^2), \quad (2.1)$$

where ω is the frequency, $Y_{\ell m}$ denote the usual scalar spherical harmonics, the functions e^ψ and e^λ are the components of the metric of the unperturbed stellar model, while $H_i^{\ell m}(r)$ and $K^{\ell m}(r)$ characterize the metric perturbations. In this paper we do not consider perturbations with axial parity because they are not characterized by pulsations emitting gravitational waves [46].

The perturbation of the compact star fluid is described by the Lagrangian displacement vector ξ_a , having components

$$\begin{aligned} \xi_r(t, r, \theta, \varphi) &= e^{\lambda/2} r^{\ell-1} W^{\ell m}(r) Y_{\ell m}(\theta, \varphi) e^{i\omega t}, \\ \xi_\theta(t, r, \theta, \varphi) &= -r^\ell V^{\ell m}(r) \partial_\theta Y_{\ell m}(\theta, \varphi) e^{i\omega t}, \\ \xi_\varphi(t, r, \theta, \varphi) &= -r^\ell V^{\ell m}(r) \partial_\varphi Y_{\ell m}(\theta, \varphi) e^{i\omega t}. \end{aligned} \quad (2.2)$$

In this paper we use the formulation of Lindblom and Detweiler [48, 49], consisting of a system of four ordinary differential equations

$$\frac{d\mathbf{Y}(r)}{dr} = \mathbf{Q}(r, \ell, \omega) \mathbf{Y}(r) \quad (2.3)$$

for the functions $\mathbf{Y}(r) = (H_1^{\ell m}, K^{\ell m}, W^{\ell m}, X^{\ell m})$, where

$$X^{\ell m} = -e^{\psi/2} \Delta p^{\ell m} \quad (2.4)$$

and three algebraic relations which allow to compute the remaining functions $\{H_0^{\ell m}, H_2^{\ell m}, V^{\ell m}\}$ in terms of the others (see appendix A). We concentrate our attention on normal modes which belong to a particular even parity spherical harmonic $\pi = (-1)^\ell$ with the complex frequency

$$\omega = \sigma + \frac{i}{\tau}. \quad (2.5)$$

The normal modes of the coupled system are defined as those oscillations which lead to purely outgoing waves at spatial infinity. The real parts of their eigenfrequencies correspond to the oscillation rate and the imaginary parts describe the damping due to radiative energy loss.

A compact star at the end of its evolution is cold and isentropic, and can be described by a barotropic EoS $p = p(\varepsilon)$. In contrast, in a hot compact star the situation is more complicated because the pressure depends nontrivially on entropy s , i.e. $p = p(\varepsilon, s)$. Thermal effects on compact star oscillations have been studied for different modes in Burgio et al. [51] who showed that in general the frequencies and the damping times can change significantly with temperature. Thermal effects for self-gravitating BSs have been analysed in [52, 53]. These effects are studied within the effective field theory at finite temperature, where in general the

concept of entropy per particle can also be introduced for a star made up of bosons (see the book of Pitaevskii and Stringari [54] for a complete description of the bosonic properties).

In [53] the authors find that at high density the boson matter EoS does not depend sensitively on the temperature, so that the maximum mass of BSs is insensitive to temperature variations. However, at low densities, temperature effects on the EoS are large and lead to significant changes in the mass of BSs with large radii. Unfortunately, the thermal evolution of BSs is unknown and therefore in this paper we prefer to assume adiabatic oscillations for which the BS is described by a barotropic EoS $p = p(\varepsilon)$.

3 Equilibrium configurations of boson stars

A BS is a stellar object made of bosons, contrary to conventional stars which are formed of fermions [55]. They are similar in many respects to NSs, differing in that their pressure support derives from the Heisenberg uncertainty relation rather than the exclusion principle. The existence of BSs was first theoretically demonstrated by Ruffini and Bonazzola [13] for a non-interacting case. They analyzed only the zero-node solutions, corresponding to the lowest energy state. It has been shown that BSs are stable to small radial perturbations, provided that their central density does not exceed a critical value which also corresponds to the configuration with the maximum possible mass [56]. BSs are described by the EKG equations deriving from the action

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \nabla^\alpha \Phi \nabla_\alpha \Phi^* - V(|\Phi|^2) \right], \quad (3.1)$$

where R is the Ricci scalar, Φ is the scalar field, Φ^* its complex conjugate, and $V(|\Phi|^2)$ is the potential. Different BS models are classified according to their scalar potential and particle properties. Here we focus on the self-interaction potential

$$V(|\Phi|) = \frac{1}{2}m^2|\Phi|^2 + \frac{\lambda}{4}|\Phi|^4 \quad (3.2)$$

where m is the mass of the field and λ its self-interaction. The presence of a self-interaction in the scalar potential is known to have significant effects on the structure of BSs (see Colpi et al. [57]). The dimensionless ratio $\lambda\Phi^4/m^2\Phi^2$ characterizes the relative contribution of the potential energy due to self-interaction to the mass term. In the equilibrium state, the BS mass is characterized by

$$M \sim \frac{\sqrt{\Lambda} M_{\text{Pl}}^2}{m}, \quad (3.3)$$

where M_{Pl} is the Planck mass and the dimensionless quantity Λ is given by

$$\Lambda \equiv \frac{\lambda M_{\text{Pl}}^2}{4\pi m^2}. \quad (3.4)$$

This shows that the interaction can dominate the potential energy even for very small λ and that the mass of BSs can be comparable to the mass of typical fermion compact stars [57].

Since the action (3.1) is invariant under the U(1) global transformation $\Phi \rightarrow e^{i\theta}\Phi$, we obtain the continuity equation

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}J^\mu)_{,\mu} = 0, \quad (3.5)$$

where the comma denotes differentiation with respect to the following quantity and J_μ is the conserved four-vector current defined by

$$J_\mu = i(\Phi^* \nabla_\mu \Phi - \Phi \nabla_\mu \Phi^*). \quad (3.6)$$

The associated Noether charge,

$$Q = \int g^{0\mu} J_\mu \sqrt{-g} d^3x, \quad (3.7)$$

can be identified as the boson number. Note that the conservation of boson number here is due to the complex nature of the scalar; for a real scalar field there is no such conserved charge. By varying the action with respect to Φ^* and $g^{\mu\nu}$, we obtain the scalar field equation

$$\nabla^\mu \nabla_\mu \Phi = \frac{dV}{d|\Phi|^2} \Phi, \quad (3.8)$$

and the Einstein equations derived from the action (3.1) are given by

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi T_{\alpha\beta}^\Phi, \quad (3.9)$$

where $T_{\alpha\beta}^\Phi$ is stress-energy tensor of the scalar

$$T_{\alpha\beta}^\Phi = \nabla_\alpha \Phi^* \nabla_\beta \Phi + \nabla_\beta \Phi^* \nabla_\alpha \Phi - g_{\alpha\beta}(\nabla^\gamma \Phi^* \nabla_\gamma \Phi + V(|\Phi|^2)). \quad (3.10)$$

We seek for a time-independent, spherically symmetric solution of Einstein's field equations; thus, in Schwarzschild coordinates the metric has the form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2. \quad (3.11)$$

Assuming that the field has a time dependence $\Phi(r, t) = \Phi_0(r)e^{-i\tilde{\omega}t}$, the stress energy tensor becomes time independent, which implies that the space-time is stationary and the metric functions depend only on the radial coordinate r . Practically, $\Phi(r, t)$ is a complex field but we can choose our field definition such that the complex part vanishes at time $t = 0$.

The EKG equations reduce to a system of ordinary differential equations for the metric functions A , B , and for the scalar field Φ . The equilibrium configurations are found by numerically integrating the EKG along with suitable boundary conditions. For a given value Φ_c of the scalar field at the center of the star, the equilibrium equations are reduced to an eigenvalue problem for the frequency $\tilde{\omega}$. Colpi et al. [57] showed that, in the Thomas-Fermi limit, corresponding to $\Lambda \gg 1$, the scalar field becomes equivalent to a fluid with an EoS

$$p = \frac{c^4}{36K} \left[\left(1 + \frac{12K}{c^2} \rho \right)^{1/2} - 1 \right]^2 \quad (3.12)$$

with

$$K \equiv \frac{\lambda \hbar^3}{4m^4 c}, \quad (3.13)$$

where p and ρ represent the pressure and the density respectively. Chavanis and Harko [58] showed the accuracy of the hydrodynamical approach in this limit. Therefore under these conditions a BS can be treated as a perfect fluid [59]. In this limit the anisotropy parameter

$\delta \equiv (p_r - p_\perp)/p_r$ where p_r and p_\perp are the radial and tangential pressure approaches to zero. The parameter δ measures the deviation from local isotropy and was investigated by Gleiser [23] with the surprising conclusion that the value of δ at the surface of the star is only weakly dependent on its central density. This EoS was used in Maselli et al. [60] to study the I-Love-Q universal relations for BSs, showing that these relations exist for both fermion and boson dark stars, and could be extremely useful in the near future to combine multiple observations and perform redundancy tests of the stellar model. In the Newtonian limit, eq. (3.12) takes the form of a polytropic EoS with $n = 1$,

$$p = K\rho^2. \quad (3.14)$$

In the high density limit, it tends to the ultra-relativistic EoS

$$p = \frac{1}{3}\rho c^2, \quad (3.15)$$

as in the case of several typical fermion EoSs describing the core of neutron stars.

4 Results

The scalar potential is symmetric under a Z_2 discrete transformation $\Phi \rightarrow -\Phi$. Therefore, the lightest Z_2 -odd component is stable and is a good dark matter candidate. Additionally, observations of the central regions of galaxies, along with the missing of galaxy satellites in the Local Group and the so-called “too big to fail” problem has led some to question the non-interacting DM paradigm. The inclusion of self-interactions in the DM sector could resolve these issues without creating tension with other astrophysical constraints. These requirements can be satisfied at the same time if the cross section per unit mass for DM verifies [61, 62]:

$$0.1 \frac{\text{cm}^2}{\text{g}} \leq \frac{\bar{\sigma}}{m} \leq 10 \frac{\text{cm}^2}{\text{g}}. \quad (4.1)$$

Here $\bar{\sigma}$ is the scattering cross-section among four scalars relating to λ by

$$\bar{\sigma} = \frac{\lambda^2}{64\pi m^2}. \quad (4.2)$$

This interaction plays an important role in establishing an equilibrium BS configuration because, together with the quantum repulsive force generated by the Heisenberg uncertainty principle, it counterbalances the attractive pull of gravity. In order to make the correspondence between Bose-Einstein condensate (BEC) with short-range interactions described by the Gross-Pitaevskii equation [54] and scalar fields with a $\frac{\lambda}{4}|\Phi|^4$ interaction described by the Klein-Gordon, we set [58]:

$$\frac{\lambda}{8\pi} \equiv \frac{a}{\lambda_c} = \frac{amc}{\hbar}, \quad (4.3)$$

where $\lambda_c = \hbar/mc$ is the Compton wavelength of the bosons. In our calculations, four benchmark values of the scattering length a ($a = 5$ fm, $a = 10$ fm, $a = 15$ fm and $a = 20$ fm) and five benchmark values of the mass m ($1m_n$, $1.25m_n$, $1.5m_n$, $1.75m_n$ and $2m_n$, where m_n is the neutron mass) are considered. In the main text we present the result for $a = 5$ fm and we show the remaining three cases in appendix B. For these parameter values we calculate the mass-radius relationship, the compactness, the f -mode frequency defined as $f = \text{Re}(\omega)/2\pi$,

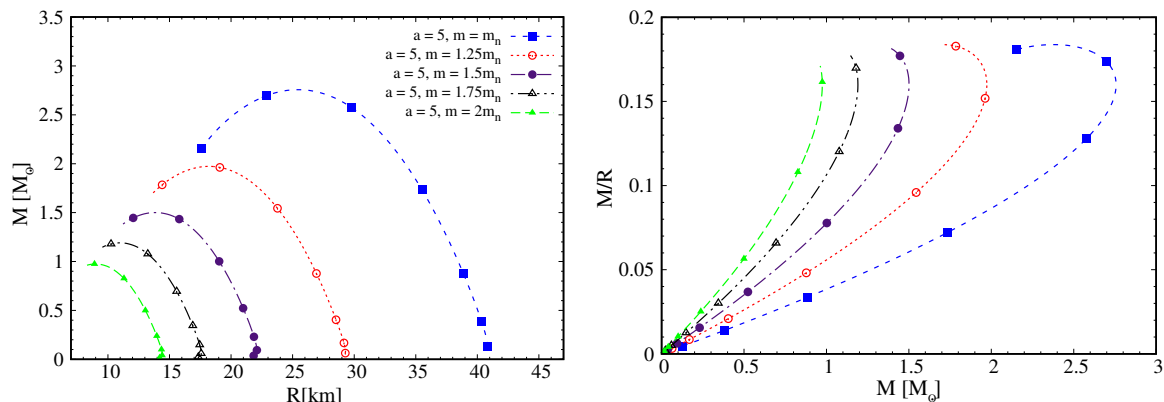


Figure 1. Mass radius relation and compactness for relativistic BSs with the EoS given in eq. (3.12). We assume $a = 5$ fm and consider different values of the mass m . Changing these parameters it is possible to span a large range of values of mass and radius. From top to bottom: $m = m_n$, $m = 1.25m_n$, $m = 1.5m_n$, $m = 1.75m_n$, and $m = 2m_n$.

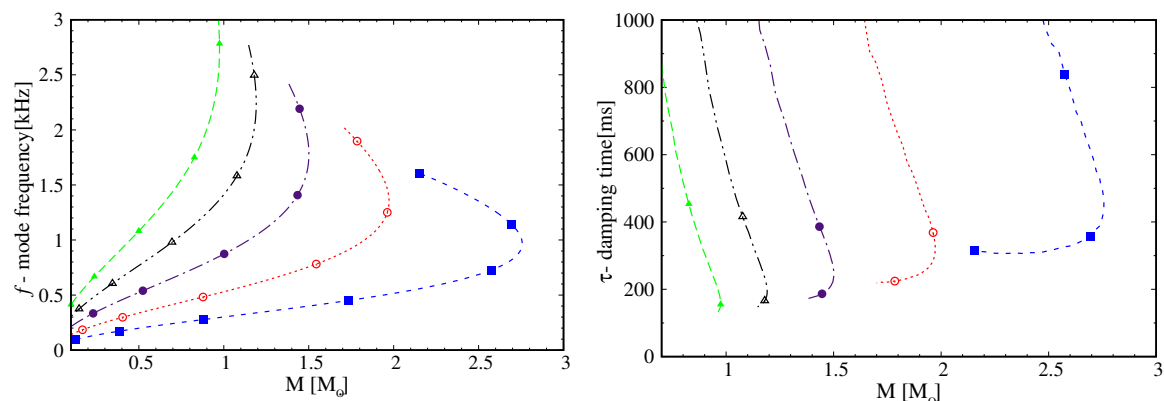


Figure 2. Frequency and damping time of the fundamental mode as a function of the stellar mass for the EoS given in eq. (3.12) using $a = 5$ fm and different values of the mass m .

and the damping time τ of BSs; see figures 1–2. Compactness C is defined as $C \equiv M/R$, so that $C = 0.5$ for a Schwarzschild BH and $C_{\text{max}} \approx 0.18$ for a BS.

It is important to note that we obtain our results within the formulation of Lindblom and Detweiler (see appendix A) [48, 49], valid in the case of a perfect fluid star. In the case of a generic BS, Yoshida et al. [35] obtained the perturbation equations which become a system of six first-order differential equations. This is in clear contrast with the case of perfect fluid stars where the basic equations can be reduced to a system of four first-order differential equations. The reason for the difference would be that both real and imaginary parts of the perturbed scalar field are dynamical degrees of freedom for the material source. It would be very useful to compare both approaches for stellar configurations with the same mass. Unfortunately, in [35] the self-interaction Λ was set to zero for simplicity; thus, a meaningful comparison is not possible. In Balakrishna et al. [36], the authors study the quasinormal modes of BSs for low values of Λ , i.e. outside the Thomas-Fermi regime. However, the damping times are not given because they are more interested in studying the decay of excited states. In Macedo et al. [37], three of the most popular models for BSs are discussed. For each case, two

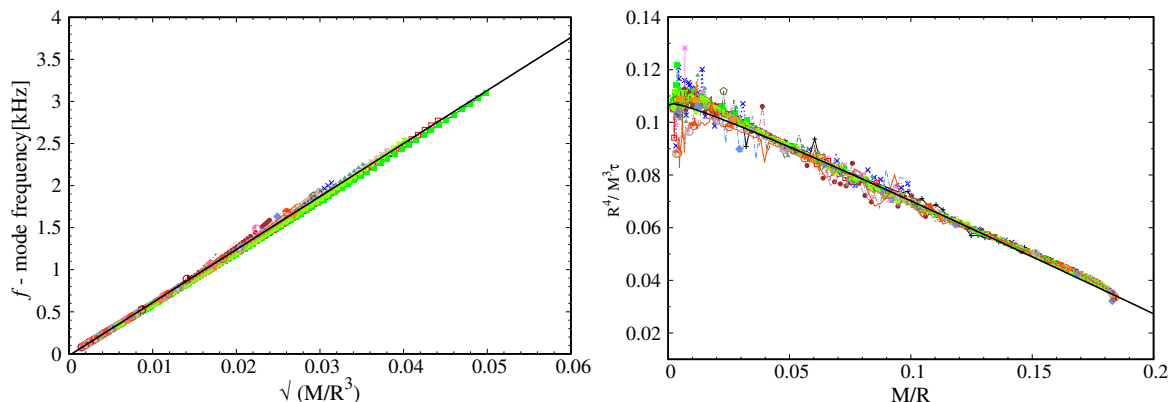


Figure 3. The frequency of the fundamental mode is plotted as functions of the square root of the average stellar density, while the normalized damping time of the f -modes as functions of the stellar compactness M/R , for the EoS considered in this paper with different parameter values.

specific stellar configurations are selected, one corresponding to the maximum total mass of the model and the other to the maximum compactness M/R . In the case of massive BSs with self-interaction, they fix the value of the dimensionless quantity $\tilde{\lambda} = \lambda M_{\text{Pl}}^2 / (8\pi m^2)$ to $\tilde{\lambda} = 100$. Notice that $\tilde{\lambda}$ in ref. [37] is closely related to the dimensionless quantity $\Lambda \equiv \lambda M_{\text{Pl}}^2 / (4\pi m^2)$ defined in eq. (3.4); i.e. $\Lambda = 2\tilde{\lambda}$. Using eq. (4.3), the quantity Λ can be written in terms of the scattering length a and the boson mass m as:

$$\Lambda = \frac{\lambda M_{\text{Pl}}^2}{4\pi m^2} = \frac{8\pi \frac{amc}{\hbar} M_{\text{Pl}}^2}{4\pi m^2} = \frac{2acM_{\text{Pl}}^2}{m\hbar} \approx 16 \times 10^{38} \left(\frac{a}{1 \text{ fm}} \right) \left(\frac{m_n}{m} \right). \quad (4.4)$$

Since in ref. [37] the value $\Lambda = 2\tilde{\lambda} = 200$ is chosen, it is clear that the value of Λ used here is several orders of magnitude larger. Therefore, in all these cases it is not possible to make a comparison with our results.

Finally, we suggest two universal relations for the fundamental oscillation mode of BSs. These empirical fits were originally proposed for compact stars by Andersson and Kokkotas [63] and recently studied by several authors for different compact star EoS [42, 43, 64]. It is known that the f -mode frequency scales naturally with the square root of the average stellar density, $\sqrt{M/R^3}$. In fact, from our results we obtain:

$$f = b_1 + b_2 \sqrt{\frac{M}{R^3}}, \quad (4.5)$$

with $b_1 = -0.0195 \pm 0.0008$, $b_2 = 62.997 \pm 0.058$. Also, the damping time τ is usually fitted with a simple formula involving the stellar mass and radius. In the case of BSs we find:

$$\left(\frac{M^3 \tau}{R^4} \right)^{-1} = c_1 + c_2 \sqrt{\frac{M}{R}} + c_3 \frac{M}{R} \quad (4.6)$$

with $c_1 = 0.106 \pm 0.0005$, $c_2 = 0.035 \pm 0.005$, and $c_3 = -0.474 \pm 0.010$. It is interesting to notice that the coefficients of this expansion are quite independent on the choice of the BS model. We show in table 1 all the coefficients for the various models analyzed in this paper. The universal relations are shown in figure 3.

$a[\text{fm}]$	m/m_n	$b_1[\text{kHz}]$	$b_2[\text{km/kHz}]$	c_1	c_2	c_3
5	1.00	-0.030	64.93	0.107	0.031	-0.468
5	1.25	-0.028	63.59	0.112	0.003	-0.430
5	1.50	-0.029	62.79	0.110	0.018	-0.448
5	1.75	-0.028	62.18	0.110	0.018	-0.451
5	2.00	-0.028	61.72	0.111	0.011	-0.444
10	1.00	-0.030	66.41	0.104	0.038	-0.473
10	1.25	-0.031	64.98	0.104	0.059	-0.519
10	1.50	-0.030	63.95	0.110	0.009	-0.429
10	1.75	-0.029	63.14	0.108	0.031	-0.470
10	2.00	-0.029	62.56	0.110	0.013	-0.438
15	1.00	-0.029	67.26	0.090	0.201	-0.792
15	1.25	-0.030	65.77	0.106	0.037	-0.484
15	1.50	-0.031	64.72	0.107	0.024	-0.452
15	1.75	-0.030	63.82	0.110	0.015	-0.442
15	2.00	-0.029	63.14	0.110	0.014	-0.463
20	1.00	-0.033	68.06	0.057	0.356	-0.981
20	1.25	-0.018	63.17	0.079	0.190	-0.703
20	1.50	-0.016	61.92	0.086	0.154	-0.646
20	1.75	-0.016	61.30	0.087	0.157	-0.658
20	2.00	-0.030	62.29	0.089	0.147	-0.647

Table 1. Set of parameters obtained for our model.

5 Conclusions

In this paper we have studied the fundamental oscillation mode of BSs. We restricted our attention to the case of massive boson stars described by a relativistic EoS given by equation (3.12), and we choose the values of the scattering length a and the boson mass m in order to satisfy observational constraints on the self-interaction of dark matter. We have also limited our choice of the EoS parameters to values that result in stellar-mass BSs. In principle, a different choice of the parameters could lead to models with a million or billion times the mass of the sun. However, we consider a range of masses for the boson stars of $1 - 6M_\odot$, in order to extend the typical mass limit of a neutron star of $1.4 - 2.5M_\odot$. For these parameter values we calculated the mass, the radius, the f -mode frequency, the damping time and the compactness of BSs. Our results were obtained by solving the linear perturbation equations that describe the nonradial oscillations of relativistic compact stars.

Our results contribute to a series of empirical fits proposed in the literature that describe the general behavior of the f -mode frequency and damping time τ as functions of the stars' average density and compactness [63, 64]. We find that the relations of universality are valid for BSs not only for objects with mass comparable to that of normal NSs but also for BSs with masses between 3–6 solar masses, i.e. compatible with those of some BH candidates or other large-mass exotic objects such as self-bound strange quark matter stars [65]. Our main

conclusions are that BSs could radiate in the optimal range for present gravitational wave detectors and that the properties of the f -mode could be used to distinguish BSs from other families of compact objects because they are described by significantly different universal fits.

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A The Lindblom-Detweiler equations

The polar non-radial perturbations of a non-rotating star can be described through a set of first-order differential equations derived by Lindblom and Detweiler [48, 49] for the quantities $H_1^{\ell m}(r)$, $K^{\ell m}(r)$, $W^{\ell m}(r)$, $X^{\ell m}(r)$:

$$\begin{aligned}
H_1^{\ell m} &= -\frac{1}{r} \left[\ell+1 + \frac{2Me^\lambda}{r} + 4\pi r^2 e^\lambda (p-\varepsilon) \right] H_1^{\ell m} + \frac{e^\lambda}{r} [H_0^{\ell m} + K^{\ell m} - 16\pi(p+\varepsilon)V^{\ell m}], \\
K^{\ell m} &= \frac{1}{r} H_0^{\ell m} + \frac{\ell(\ell+1)}{2r} H_1^{\ell m} - \left[\frac{\ell+1}{r} - \frac{\psi'}{2} \right] K^{\ell m} - 8\pi(p+\varepsilon) \frac{e^{\lambda/2}}{r} W^{\ell m}, \\
W^{\ell m} &= -\frac{\ell+1}{r} W^{\ell m} + r e^{\lambda/2} \left[\frac{e^{-\psi/2}}{(p+\varepsilon)c_s^2} X^{\ell m} - \frac{\ell(\ell+1)}{r^2} V^{\ell m} + \frac{1}{2} H_0^{\ell m} + K^{\ell m} \right], \\
X^{\ell m} &= -\frac{\ell}{r} X^{\ell m} + \frac{(p+\varepsilon)e^{\psi/2}}{2} \left[\left(\frac{1}{r} - \frac{\psi'}{2} \right) H_0^{\ell m} + \left(r\omega^2 e^{-\psi} + \frac{\ell(\ell+1)}{2r} \right) H_1^{\ell m} + \left(\frac{3}{2}\psi' - \frac{1}{r} \right) K^{\ell m} \right. \\
&\quad \left. - \frac{\ell(\ell+1)}{r^2} \psi' V^{\ell m} - \frac{2}{r} \left(4\pi(p+\varepsilon)e^{\lambda/2} + \omega^2 e^{\lambda/2-\psi} - \frac{r^2}{2} \left(\frac{e^{-\lambda/2}}{r^2} \psi' \right)' \right) W^{\ell m} \right]. \quad (\text{A.1})
\end{aligned}$$

The remaining perturbation functions, $H_0^{\ell m}(r)$, $V^{\ell m}(r)$, $H_2^{\ell m}(r)$, are given by the algebraic relations:

$$\begin{aligned}
0 &= \left[3M + \frac{1}{2}(\ell-1)(\ell+2)r + 4\pi r^3 p \right] H_0^{\ell m} - 8\pi r^3 e^{-\psi/2} X^{\ell m} \\
&\quad + \left[\frac{1}{2}\ell(\ell+1)(M + 4\pi r^3 p) - \omega^2 r^3 e^{-(\lambda+\psi)} \right] H_1^{\ell m} \\
&\quad - \left[\frac{1}{2}(\ell-1)(\ell+2)r - \omega^2 r^3 e^{-\psi} - \frac{e^\lambda}{r}(M + 4\pi r^3 p)(3M - r + 4\pi r^3 p) \right] K^{\ell m}, \\
X^{\ell m} &= \omega^3(\varepsilon + p)e^{-\psi/2} V^{\ell m} - \frac{p'}{r} e^{(\psi-\lambda)/2} W^{\ell m} + \frac{1}{2}(\varepsilon + p)e^{\psi/2} H_0^{\ell m}, \\
H_0^{\ell m} &= H_2^{\ell m}. \quad (\text{A.2})
\end{aligned}$$

Equations (A.1) and (A.2) are solved numerically inside the star, assuming that the perturbation functions are nonsingular near the stellar center. An asymptotic expansion in power

series about $r = 0$ shows that:

$$X^{\ell m}(0) = (\varepsilon_0 + p_0)e^{\psi_0/2} \left\{ \left[\frac{4\pi}{3}(\varepsilon_0 + 3p_0) - \frac{\omega^2}{\ell}e^{-\psi_0} \right] W^{\ell m}(0) + \frac{1}{2}K^{\ell m}(0) \right\}, \quad (\text{A.3})$$

$$H_1^{\ell m}(0) = \frac{1}{\ell(\ell+1)}[2\ell K^{\ell m}(0) + 16\pi(\varepsilon_0 + p_0)W^{\ell m}(0)], \quad (\text{A.4})$$

where the constants ε_0 , p_0 , and ψ_0 appearing in these expressions are simply the first terms in the power-series expansions for the density, pressure, and gravitational potential. At the stellar surface, $r = R$, one assumes continuity of the perturbation functions and the vanishing of the Lagrangian pressure perturbation, i.e.,

$$X^{\ell m}(R) = 0. \quad (\text{A.5})$$

In the exterior, the metric perturbations are described by the Zerilli functions:

$$Z^{\ell m} = \frac{r^{\ell+2}}{nr + 3M}(K^{\ell m} - e^\psi H_1^{\ell m}), \quad (\text{A.6})$$

where $n = (\ell - 1)(\ell + 2)/2$, which is solution of the Zerilli equation

$$\frac{d^2 Z^{\ell m}}{dr_\star^2} + [\omega^2 - V_Z(r)]Z^{\ell m} = 0, \quad (\text{A.7})$$

with $r_\star \equiv r + 2M \ln(r/2M - 1)$ and

$$V_Z \equiv e^{-\lambda} \frac{2n^2(n+1)r^3 + 6n^2Mr^2 + 18nM^2r + 18M^3}{r^3(nr + 3M)^2}. \quad (\text{A.8})$$

The transformation between $H_1^{\ell m}$, $K^{\ell m}$, and the Zerilli function is nonsingular [66]. Chandrasekhar has proven that the reflection and transmission coefficients obtained from the Zerilli equation are identical to those derived from the Regge-Wheeler equation [50].

The solutions of eq. (A.7) representing outgoing and ingoing waves have the asymptotic behavior

$$Z_{\text{out}} \sim e^{r_\star/\tau} \quad \text{and} \quad Z_{\text{in}} \sim e^{-r_\star/\tau}. \quad (\text{A.9})$$

In order to describe the free oscillations of the star we must impose the outgoing wave boundary condition

$$Z^{\ell m}(r) \rightarrow e^{-i\omega r_\star} \quad (r \rightarrow \infty). \quad (\text{A.10})$$

A solution of eqs. (A.1) and (A.7) satisfying the boundary conditions (A.3), (A.4), (A.5), and (A.10) only exists for a discrete set of complex values of the frequency ω , which are the quasinormal modes of the star.

B Results for various possible parameters

In figures 4–7, we present additional results for the mass versus radius relationship, the compactness, the oscillation frequency, and the oscillation damping time of BSs for three other benchmark values of a , and five values of the boson mass m .

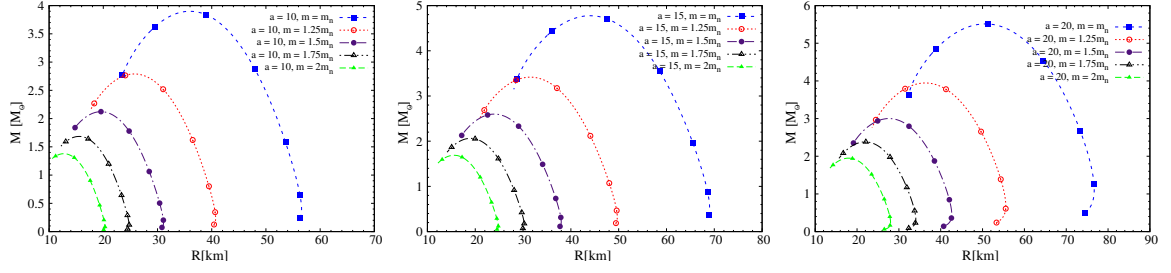


Figure 4. Mass versus radius relation for relativistic BSs for the EoS given in eq. (3.12) using $a = 10$ fm (left panel), $a = 15$ fm (central panel), $a = 20$ fm (right panel) and different values of the boson mass m (from top to bottom, $m/m_n = 1.0, 1.25, 1.5, 1.75, 2.0$). Changing these parameters it is possible to span a wide range of values of mass and radius.

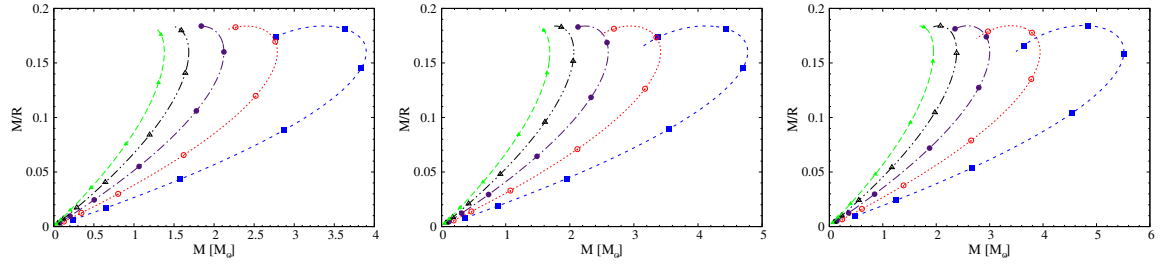


Figure 5. Compactness of relativistic BSs as a function of the stellar mass. The parameters and the labels are the same as in figure 4.

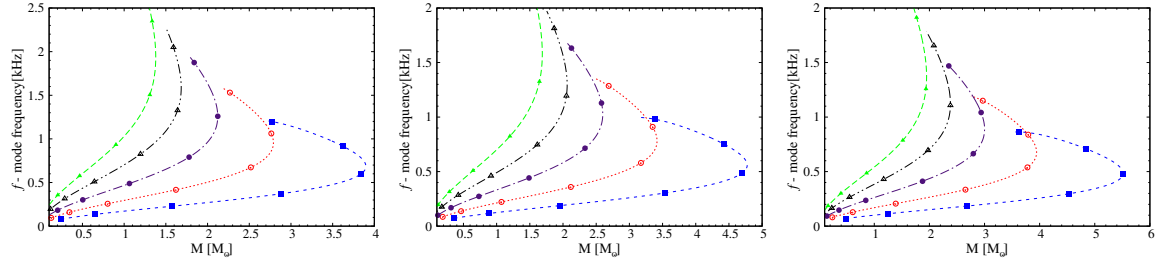


Figure 6. Frequency of the fundamental mode as a function of the stellar mass. The parameters and the labels are the same as in figure 4.

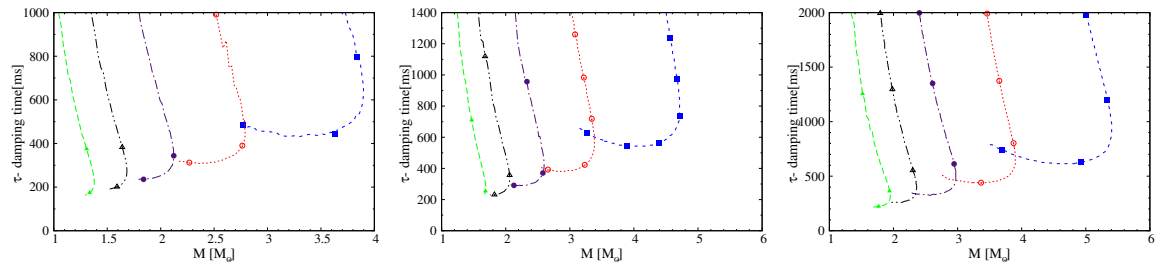


Figure 7. Damping time of the fundamental mode as a function of the stellar mass. The parameters and the labels are the same as in figure 4.

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