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Nonlinear stability of Minkowski spacetime in nonlocal gravity

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Abstract. We prove that the Minkowski spacetime is stable at nonlinear level and to all perturbative orders in the gravitational perturbation in a general class of nonlocal gravitational theories that are unitary and finite at quantum level.

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A class of nonlocal generalizations of the Einstein-Hilbert theory for gravity has been proposed and extensively studied in the last years [1–7], see [8] for review. A nonlocal gravitational model have been proposed for the first time by Krasnikov in 1988 and studied by Kuz'min in 1989 [9]. More recently, it has been proved that nonlocal gravity (NLG) is finite in odd dimension, while a slightly modification of the theory turns out to be finite in even dimension too [1–3, 8]. The unitarity issue in nonlocal field theory has been addressed in [10] where it has been proved that perturbative unitarity is preserved at any order in the loop expansion.

At classical level, all the solutions of Einstein's gravity in vacuum are solutions of NLG too [11]. Moreover, it has been shown that a Ricci flat spacetime is stable under linear perturbations if it is stable in Einstein's gravity [12, 13] (see also [14]). Therefore, based on the Wheeler-Regge result, it turns out that the Schwarzschild spacetime is stable in NLG at linear level.

In the context of Einstein's gravity, the global stability of the Minkowski spacetime has been established long time ago [15, 16], see also [17, 18]. In particular, any Strongly Asymptotically Flat (SAF) initial data set satisfying a Global Smallness Assumption (GSA) evolves in a smooth, geodesically complete and asymptotically flat solution of vacuum Einstein's equations; see the appendix A for a definition of the SAF condition and the GSA, and a discussion of the stability theorem of Minkowski spacetime in general relativity. Therefore, the SAF condition and the GSA define the class of small perturbations under which the Minkowski metric is stable. In this paper we show that this theorem is still valid in NLG.

We consider the following minimal nonlocal action for the gravitational field,

$$S_{\rm g} = -\frac{2}{\kappa_D^2}\int d^4x\sqrt{-g}\,[R + G_{\mu\nu}\gamma(\Box)R^{\mu\nu} + V(\mathcal{R})]\ . \tag{1}$$

where R, $R_{\mu\nu}$ and $G_{\mu\nu}$ are the Ricci scalar, Ricci curvature and the Einstein tensor respectively. Moreover, $V(\mathcal{R})$ is a generalized potential at least cubic in the Ricci and/or Einstein's tensor, while $\mathcal{R}$ stays for scalar, Ricci or Riemann curvatures, and derivatives thereof. Finally, the form factor $\gamma(\Box)$ depends on the non locality scale $\ell \equiv \sqrt{\sigma}$ and is defined by

$$\gamma(\Box) \equiv \frac{f(\sigma\Box) - 1}{\Box}\,, \tag{2}$$

where $f(z)$ is an entire analytic function without zeros for finite complex z, e.g. $f(z) = \exp H(z)$.

The equations of motion for the action (1) have been derived in [19] and read[1]

$$E_{\mu\nu} \equiv (1 + \Box\,\gamma(\Box))\,G_{\mu\nu} + (g_{\mu\nu}\nabla_\alpha\nabla_\beta - g_{\alpha\mu}\nabla_\beta\nabla_\nu)\,\gamma(\Box)\,G^{\alpha\beta} + Q_{2\mu\nu}\,({\rm Ric}) = 8\pi G_{\rm N}T_{\mu\nu}\,, \tag{3}$$

where $T_{\mu\nu} \equiv -(2/\sqrt{-g})\delta S_m/\delta g_{\mu\nu}$ is the matter stress-energy tensor. Moreover, $Q_2({\rm Ric})$ is a sum of local and nonlocal analytic terms at least quadratic in the Ricci tensor and/or the Ricci scalar [19], e.g.

$$\sigma\,((\sigma\Box)^n\,R_{\mu\alpha})\,((\sigma\Box)^m\,R^\alpha{}_\nu) \quad \text{or} \quad \sigma^2\,((\sigma\Box)^n\,R_{\mu\alpha})\,((\sigma\Box)^m\,R^\alpha{}_\nu)\,((\sigma\Box)^l\,R)\,, \tag{4}$$

for integer n, m, l (the label 2 stays exactly for at least quadratic in the Ricci tensor). However, regardless of the explicit form of $Q_2({\rm Ric})$, in what follows it will be only relevant that

[1] We signal an imprecision in eq. (A.22) in [19], which is valid only for the Minkowski metric, while (A.19-A.20) are valid in general.

$Q_2(\mathrm{Ric})$ is at least quadratic in Ric, which implies the following perturbative expansion: $Q_2\,(\mathrm{Ric}) = O(\epsilon^{2n})$ if $\mathrm{Ric} = O(\epsilon^n)$ for $\epsilon \ll 1$.

Since we are interested in the stability of Minkowski spacetime in vacuum, hereafter we set $T_{\mu\nu} = 0$. We start our analysis considering small perturbations of the Minkowski metric, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, h_{\mu\nu}, \qquad \text{with} \qquad |\epsilon\, h_{\mu\nu}| \ll 1 \tag{5}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $\epsilon \ll 1$ is a small dimensionless parameter. We then proceed to determine the vacuum solutions of (3) perturbatively. We will show that equations (3) are verified iff the Einstein's tensor $G_{\mu\nu}$ vanish at any perturbative order in ϵ. Therefore, we conclude that the evolution of small perturbations of Minkowski spacetime is the same in NLG and in general relativity. This implies that the stability of Minkowski spacetime against small perturbations is the same in NLG and Einsteins gravity.

In order to obtain the perturbative expansion of the equations of motion (3), we start expanding in power series in the small parameter ϵ the tensors and tensor operators that appeared in (3), see [20] for a review. We first expand $h_{\mu\nu}$ and then the Einstein's tensor $G_{\mu\nu}$, namely

$$h_{\mu\nu} = \sum_{n=0}^{\infty} \epsilon^n h_{\mu\nu}^{(n)} \quad \text{and} \quad G_{\mu\nu}(g_{\mu\nu}) = \sum_{n=1}^{\infty} \epsilon^n G_{\mu\nu}^{(n)}\,. \tag{6}$$

Notice that the leading contribution to the Einstein tensor is of order ϵ because $G_{\mu\nu}^{(0)} \equiv G_{\mu\nu}(\eta) = 0$. Similarly, we can expand the covariant derivative as

$$\nabla_\alpha = \sum_{n=0}^{\infty} \epsilon^n \nabla_\alpha^{(n)} = \partial_\alpha + \sum_{n=1}^{\infty} \epsilon^n \nabla_\alpha^{(n)}\,, \tag{7}$$

where we have shown explicitly only the first term $\nabla_\alpha^{(0)} = \partial_\alpha$, since the other terms will not be needed. Now let us define and expand the following differential operator in (3)

$$\begin{aligned} A_{\mu\nu\alpha\beta} &\equiv \left(g_{\mu\nu}\nabla_\alpha\nabla_\beta - g_{\alpha\mu}\nabla_\beta\nabla_\nu\right)\gamma\,(\Box) = \sum_{n=0}^{\infty} \epsilon^n A_{\mu\nu\alpha\beta}^{(n)} \\ &= \left(\eta_{\mu\nu}\partial_\alpha\partial_\beta - \eta_{\alpha\mu}\partial_\beta\partial_\nu\right)\gamma\left(\Box^{(0)}\right) + \sum_{n=1}^{\infty} \epsilon^n A_{\mu\nu\alpha\beta}^{(n)}\,, \end{aligned} \tag{8}$$

and $\Box^{(0)} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma$. We also define and expand the following differential operator again present in (3), namely

$$f(\sigma\Box) \equiv (1 + \Box\,\gamma\,(\Box)) = \sum_{n=0}^{\infty} \epsilon^n f^{(n)} = 1 + \Box^{(0)}\,\gamma\left(\Box^0\right) + \sum_{n=1}^{\infty} \epsilon^n f^{(n)} = f(\sigma\Box^{(0)}) + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}\,, \tag{9}$$

From the definitions and expansions (8) and (9) it follows that

$$\begin{aligned} A_{\mu\nu\alpha\beta}G^{\alpha\beta} &= \sum_{n=1}^{\infty} \epsilon^n \sum_{m=0}^{n-1} A_{\mu\nu\alpha\beta}^{(m)} G^{(n-m)\alpha\beta} \\ &= \epsilon\, A_{\mu\nu\alpha\beta}^{(0)} G^{(1)\alpha\beta} + \epsilon^2 \left(A_{\mu\nu\alpha\beta}^{(0)} G^{(2)\alpha\beta} + A_{\mu\nu\alpha\beta}^{(1)} G^{(1)\alpha\beta}\right) + \ldots\,, \end{aligned} \tag{10}$$

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and

$$f(\Box)G^{\alpha\beta}=\sum_{n=1}^{\infty}\epsilon^n\sum_{m=0}^{n-1}f^{(m)}G^{(n-m)\alpha\beta}=\epsilon\, f^{(0)}G^{(1)\alpha\beta}+\epsilon^2\left(f^{(0)}G^{(2)\alpha\beta}+f^{(1)}G^{(1)\alpha\beta}\right)+\dots\,. \tag{11}$$

Let us consider the operator $Q_2(\mathrm{Ric})$ and express it as

$$\begin{aligned}Q_2(\mathrm{Ric})\,=&\sum_{n_1,n_2=0}^{\infty}c_{n_1,n_2}\,\sigma\,(D^{n_1}\mathrm{Ric})\,(D^{n_2}\mathrm{Ric})\\&+\sum_{n_1,n_2,n_3=0}^{\infty}c_{n_1,n_2,n_3}\,\sigma^2\,(D^{n_1}\mathrm{Ric})\,(D^{n_2}\mathrm{Ric})\,(D^{n_3}\mathrm{Ric})+\dots\,,\end{aligned} \tag{12}$$

where Ric is the Ricci scalar or the Ricci curvature, D is a short notation for some operator, e. g. $D=\sigma\Box$ or $D=g$, c_{n_1,n_2,n_3} are dimensionless parameters, and the dots indicate a sum of terms of order higher than third in Ric. Note that we have omitted indices in (12). We can expand D and Ric in powers of ϵ as

$$D=\sum_{n=0}^{\infty}\epsilon^n D^{(n)},\quad \mathrm{Ric}=\sum_{n=1}^{\infty}\epsilon^n\mathrm{Ric}^{(n)}\,, \tag{13}$$

where we have used the fact that $\mathrm{Ric}^{(0)}(\eta)=0$. From eq. (12) it follows that if $\mathrm{Ric}^{(k)}=0$ $\forall k<n$, then one has $\mathrm{Ric}=O(\epsilon^n)$. Moreover, by means of (13) one has $D=O(1)$, so that

$$\mathrm{Ric}^{(k)}=0\quad\forall\,k<n\quad\Longrightarrow\quad Q_2(\mathrm{Ric})\sim\epsilon^{2n}\,. \tag{14}$$

Finally, the Bianchi identity can be expressed perturbatively by the means of (6) and (7) as

$$0=\nabla_\alpha G^{\alpha\beta}=\sum_{n=1}^{\infty}\epsilon^n\sum_{m=0}^{n-1}\nabla_\alpha^{(m)}G^{(n-m)\alpha\beta}=\epsilon\,\partial_\alpha G^{(1)\alpha\beta}+\epsilon^2\left(\partial_\alpha G^{(2)\alpha\beta}+\nabla_\alpha^{(1)}G^{(1)\alpha\beta}\right)+O(\epsilon^3)\,. \tag{15}$$

Now we are ready to solve (3) perturbatively. At the lowest order ϵ^1, eq. (15) gives $\partial_\alpha G^{(1)\alpha\beta}=0$, which also implies:

$$A^{(0)}_{\mu\nu\alpha\beta}G^{(1)\alpha\beta}\propto(\eta_{\mu\nu}\partial_\alpha\partial_\beta-\eta_{\alpha\mu}\partial_\beta\partial_\nu)\,\gamma(\Box^{(0)})G^{(1)\alpha\beta}=0\,, \tag{16}$$

where we have used the fact that the operator $\gamma(\Box^{(0)})$ commute with the ordinary derivates ∂_α. Therefore, the operator in (10) is null at order ϵ^1. Moreover, since $G^{(0)}_{\mu\nu}=0$ implies $\mathrm{Ric}^{(0)}=0$ and $\mathrm{Ric}=O(\epsilon)$, from eq. (14) we see that the term $Q_2(\mathrm{Ric})$ is at least of order ϵ^2, hence it does not contribute to the equations at the order ϵ^1. In conclusion, in vacuum and at first perturbative order ϵ^1, eq. (3) reads

$$f(\sigma\Box^{(0)})G^{(1)}_{\mu\nu}=\left(1+\Box^{(0)}\,\gamma\left(\Box^{(0)}\right)\right)G^{(1)}_{\mu\nu}=0\,, \tag{17}$$

and, since $f(\Box^{(0)})$ is invertible, this equations admits the unique solution $G^{(1)}_{\mu\nu}=0$.

The invertibility of $f(\sigma\Box^{(0)})$ is a simple consequence of the fact that $f(z)$ is an entire analytic function without zeros for finite complex z. In facts, under such hypothesis we can

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expand $f(\sigma\Box^{(0)})$ in power series of σ, so that the Kernel of such operator will be given by the solutions of the following equation

$$f(\sigma\Box^{(0)})\phi = \sum_{n=0}^{\infty} c_n \left(\Box^{(0)}\right)^n \sigma^n \, \phi = 0 \tag{18}$$

with $c_0 \neq 0$. Since (18) is analytic in σ, it must vanish at any order, i.e., it must be $(\Box^{(0)})^n \phi = 0$, $\forall n \geq 0$, which implies that $\phi = 0$. Therefore, the operator $f(\sigma\Box^{(0)})$ is invertible because it is linear and its kernel is the zero function. An equivalent proof can be given using Fourier transforms, writing

$$f(\sigma\Box^{(0)})\,\phi(x) = \int \frac{d^4k}{(2\pi)^4}\, f(-\sigma k^2)\, \tilde{\phi}(k)\, e^{i\,k\,x}\,, \tag{19}$$

where $\tilde{\phi}(k)$ is the Fourier transform of $\phi(x)$. Since $f(z)$ has no zeros for finite z, one has $f(-\sigma k^2) \neq 0$, $\forall k^2 < \infty$, and the equation $f(\sigma\Box^{(0)})\,\phi(x) = 0$ has the only solution $\tilde{\phi}(k) = 0$, that is $\phi(x) = 0$.

Let us here emphasize the crucial role of the invertibility of the function $f(z)$ in our result. In order to better understand this point, let us consider a function $f(z)$ with a root of order q in $z = m^2$, namely

$$f(\sigma\Box^{(0)})G^{(1)}_{\mu\nu} = \left[\sum_{n=0}^{\infty} c_n \left(\sigma\Box^{(0)}\right)^n\right] \left(\Box^{(0)} - m^2\right)^q G^{(1)}_{\mu\nu} = 0\,, \tag{20}$$

which has non null solutions $G^{(1)}_{\mu\nu} \neq 0$, duo to the solutions of the equation $\left(\Box^{(0)} - m^2\right)^q G^{(1)}_{\mu\nu} = 0$. Therefore, if the function $f(\sigma\Box^{(0)})$ would be non invertible, we would have $G^{(1)}_{\mu\nu} \neq 0$, and our proof could not be implemented.

The outcome of our analysis of (3) at first perturbative order in ϵ is that $G^{(1)}_{\mu\nu}$ must vanish. Before generalizing this result to any order, let us repeat our analysis at second order ϵ^2. Using the Bianchi identity together with $G^{(1)}_{\mu\nu} = 0$ we see that it must be $\partial_\alpha G^{(2)\alpha\beta} = 0$, which in turn gives

$$A^{(0)}_{\mu\nu\alpha\beta}G^{(2)\alpha\beta} \propto (\eta_{\mu\nu}\partial_\alpha\partial_\beta - \eta_{\alpha\mu}\partial_\beta\partial_\nu)\,\gamma(\Box^{(0)})G^{(2)\alpha\beta} = 0. \tag{21}$$

The above equation, together with $G^{(1)}_{\mu\nu} = 0$, implies that $A_{\mu\nu\alpha\beta}G^{\alpha\beta}$ vanishes at second order in epsilon (see eq. (10)). Moreover, $G^{(1)}_{\mu\nu} = 0$ implies that $\mathrm{Ric}^{(1)} = 0$, indeed from eq. (14) it also comes that $Q_2(\mathrm{Ric}) \sim \epsilon^4$. Thus, at order ϵ^2 eq. (3) reads

$$f(\sigma\Box^{(0)})G^{(2)}_{\mu\nu} = \left(1 + \Box^{(0)}\,\gamma\left(\Box^{(0)}\right)\right) G^{(2)}_{\mu\nu} = 0\,, \tag{22}$$

which, due to the invertibility of $f(\sigma\Box^{(0)})$, implies that $G^{(2)}_{\mu\nu} = 0$.

Now it is easy to infer that eq. (3) implies that $G^{(n)}_{\mu\nu} = 0$, $\forall n \geq 0$. This result can be proved by induction showing that, if $G^{(m)}_{\mu\nu} = 0$, $\forall m \leq n$, then, eq. (3) implies $G^{(n+1)}_{\mu\nu} = 0$. Indeed, if $G^{(n)}_{\mu\nu} = 0$ the Bianchi identity (15) gives $\partial_\alpha G^{(n+1)\alpha\beta} = 0$, which implies, by the means of (10), that $A_{\mu\nu\alpha\beta}G^{\alpha\beta} \sim \epsilon^{n+2}$. Moreover, since $G^{(m)}_{\mu\nu} = 0$, $\forall m \leq n$ implies $\mathrm{Ric}^{(m)} = 0$, $\forall m \leq n$, eq. (14) tells us that $Q_2(\mathrm{Ric}) \sim \epsilon^{2(n+1)}$. Finally, using (11) we conclude that at the order ϵ^{n+1}, equation (3) turns into $f(\sigma\Box^{(0)})G^{(n+1)}_{\mu\nu} = 0$, which implies $G^{(n+1)}_{\mu\nu} = 0$.

Summarizing, we have proved that eq. (3) implies that, at any perturbative order in the gravitational perturbation, it must be $G_{\mu\nu}^{(n)} = 0$. This, by means of eq. (6), implies that the Einstein's tensor must vanish, namely

$$G_{\mu\nu}(g_{\mu\nu}) = G_{\mu\nu}(\eta_{\mu\nu} + \epsilon h_{\mu\nu}) = \sum_{n=1}^{\infty} \epsilon^n G_{\mu\nu}^{(n)} = 0 \,. \tag{23}$$

Therefore, the stability analysis for the Minkowski spacetime in NLG is exactly the same then in Einstein's gravity at all perturbative orders. Notice that the inverse implication is straightforward, since $G_{\mu\nu} = 0$ implies that (3) is automatically satisfied.

Equation (23) makes evident that the stability of the Minkowski spacetime is the same in NLG and in Einstein's gravity. Indeed, the dynamics of small perturbations around Minkowski is the same (at any perturbative order) in the two theories, which immediately allows us to infer about the evolution of small perturbations in NLG. Therefore, any Strongly Asymptotically Flat initial data set satisfying a Global Smallness Assumption leads to a unique smooth, geodesically complete, and asymptotically flat solution of eq. (3) in the vacuum, which is in facts a solution of the Einstein's equations in the vacuum. This is a further confirmation of the result found in [25] where it was shown that all the n-points tree-level scattering amplitudes of NLG coincide with those in Einstein's gravity.

Let us discuss the validity of the perturbative approach used in the derivation of eq. (23). This is based on the assumption that, the metric and all the tensors constructed with it can be expanded in the quantity $|\epsilon\, h_{\mu\nu}| \ll 1$. Therefore we expand in powers of just one small parameter ϵ, that represents the amplitude of the perturbations of the Minkowski metric, as expressed by eq. (5). This is a standard assumption — for instance it is the basis of all the post-Newtonian and post-Minkowskian studies of the gravitational emission by compact objects, see [21] — and it is well justified by the choice of the class of small initial perturbations of Minkowski spacetime considered so far.

In facts, the GSA assumption implies that, at the initial time, the metric (5) must satisfy the condition (6) for a sufficiently small μ, that implies that ϵ must be sufficiently small. Thus, we can take ϵ small enough in order to guarantee that all the series expansions in ϵ are convergent at the initial time. As a consequence, equation (23) is valid, and we infer that the evolution of $h_{\mu\nu}$ in NLG is the same as in general relativity. In [17, 18] it has been shown that, in the harmonic gauge and for asymptotically flat initial data satisfying the global smallness assumption, the solutions of vacuum Einstein equations converges asymptotically in time to Minkowski spacetime; and more precisely it has been shown that $|h_{\mu\nu}| \lesssim t^{-1}\ln(t)$ converges asymptotically to zero. Using this result, we conclude that all the series expansions in $|\epsilon\, h_{\mu\nu}|$ are convergent at any time, since they are convergent at the initial time. This proves the validity of our perturbative scheme.

We emphasize that the condition of strong asymptotic flatness of the initial data is crucial for the time convergence to zero of the perturbation $h_{\mu\nu}$ proved in [17, 18]. In facts, in [22] it has been shown that, in the harmonic gauge, the collision of two plane gravitational waves produces a secular divergence of $h_{\mu\nu}$ and the break down of the series expansion (5). However, the occurrence of this secularity is due to the fact that plane waves do not satisfy the condition of asymptotic spatial flatness because they are not localized in the space but infinitely extended. We also stress that the initial asymptotic flatness condition implies that the initial perturbation $h_{\mu\nu}(t_0, \vec{x})$ is confined into a finite volume $\mathcal{V}$ of the space, say $\vec{x} \in \mathcal{V}$. Thus, far from that region $\mathcal{V}$, i.e. for $|\vec{x}| \to \infty$, one has $h_{\mu\nu}(t_0, \vec{x}) \to 0$. On the other

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hand, the asymptotic behavior $h_{\mu\nu} \sim t^{-1}\ln(t)$ for large times means that the perturbation $h_{\mu\nu}$, initially confined in $\mathcal{V}$, will be dinamically spread in all the space, to end up with the Minkowski metric.

Let us stress again the importance that $Q_2(\mathrm{Ric})$ is at least quadratic in Ric in our derivation of the stability. To clarify this point, let us consider the following EOM:

$$f(\sigma\Box)\, G_{\mu\nu} + (g_{\mu\nu}\nabla_\alpha\nabla_\beta - g_{\alpha\mu}\nabla_\beta\nabla_\nu)\,\gamma(\Box)\, G^{\alpha\beta} = \sigma R_{\mu\tau\rho\sigma} R_\nu{}^{\tau\rho\sigma} + \dots\,, \tag{24}$$

which is (3) with the replacement $Q_{2\mu\nu} = -\sigma R_{\mu\tau\rho\sigma}R_\nu^{\tau\rho\sigma} + \dots$, so that $Q_{2\mu\nu}$ is now quadratic in the Riemann curvature. If we exploit the perturbative expansion of (24) at the first order in ϵ we find

$$G^{(1)}_{\mu\nu} = \sigma R^{(1)}_{\mu\tau\rho\sigma} R^{(0)\tau\rho\sigma}_\nu + \sigma R^{(0)}_{\mu\tau\rho\sigma} R^{(1)\tau\rho\sigma}_\nu + \dots \tag{25}$$

which implies $G^{(1)}_{\mu\nu} = 0$ and $R^{(1)}_{\mu\nu} = 0$, but does not imply $R^{(1)}_{\alpha\beta\mu\nu} = 0$. At the order ϵ^2 we get

$$G^{(2)}_{\mu\nu} = \sigma R^{(2)}_{\mu\tau\rho\sigma} R^{(0)\tau\rho\sigma}_\nu + \sigma R^{(0)}_{\mu\tau\rho\sigma} R^{(2)\tau\rho\sigma}_\nu + \sigma R^{(1)}_{\mu\tau\rho\sigma} R^{(1)\tau\rho\sigma}_\nu + \dots \tag{26}$$

which does not imply $G^{(2)}_{\mu\nu} = 0$ because of the term $R^{(1)}_{\mu\tau\rho\sigma} R^{(1)\tau\rho\sigma}_\nu \neq 0$. Therefore, if $Q_{2\mu\nu}$ in eq. (3) is not assumed at least quadratic in the Ricci curvature, with the exclusion of terms quadratic in the Riemann tensor, one does not get eq. (23), and can not conclude that small perturbations of the Minkowski metric are stable.

Also note that the outcome of this paper is in agreement with the unitarity of the theory (1) at quantum level [10]. Once again, unitarity is guaranteed by the invertibility of the operator $f(\sigma\Box)$ in eq. (18), which is crucial for the derivation of eq. (23). Therefore, the stability of Minkowski spacetime at classical level is strongly related to the unitarity of the quantum theory.

Finally, we comment on the existence and occurrence of up to six extra degrees of freedom of the NLG (1), as it has been derived in [23, 24]. This theory can be reformulated in terms of auxiliary fields, considering the following action

$$\begin{aligned} S[g,\phi,\chi] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g} \\ \times\left[R + 2\,G_{\mu\nu}\,\gamma(\Box)\,\phi^{\mu\nu} - \phi_{\mu\nu}\,\gamma(\Box)\,\phi^{\mu\nu} + R\,\gamma(\Box)\,\varphi + \frac{1}{2}\varphi\,\gamma(\Box)\,\varphi + V(\mathcal{R})\right], \end{aligned} \tag{27}$$

containing the extra fields $\phi_{\mu\nu}$ and φ. The EOM for the scalar φ and the tensor $\phi_{\mu\nu}$ are

$$\frac{\delta S}{\delta\varphi} = 0 \quad\Longrightarrow\quad \varphi = G = -R\,, \qquad \frac{\delta S}{\delta\phi^{\mu\nu}} = 0 \quad\Longrightarrow\quad \phi_{\mu\nu} = G_{\mu\nu}\,. \tag{28}$$

Notice that the auxiliary fields coincide with the Einstein's tensor and the Ricci scalar respectively. Furthermore, one has $\nabla^\mu\phi_{\mu\nu} = 0$ and $\varphi = \phi^\mu_\mu$, so that one is left with up to six extra degrees of freedom. Finally, eliminating the auxiliary fields from (27) we end up with (1), thus the two actions are equivalent.

It happens that the initial data satisfying the SAF condition and the GSA around the Minkowski metric set to zero the extra modes $\phi_{\mu\nu}$ because the evolution of such initial data obeys the Einstein's equations in vacuum (according to the equation (23)), which set $\phi_{\mu\nu} = G_{\mu\nu} = 0$. On the other hand, the extra degrees of freedom could arise when considering more general initial data. However, such initial data will not be small perturbations of the

Minkowski spacetime, because they imply that the corresponding spacetime has a curvature $R = O(\ell^{-2})$ at any time (including of course at the initial time), as we will show below. Since $\ell = \sqrt{\sigma}$ is the scale of nonlocality, which can be as small as the Planck length, R is huge, and a spacetime with such a large curvature is not a globally small perturbation of the Minkowski spacetime. This explains why we do not see the modes $\phi_{\mu\nu}$ in our stability analysis. We stress that the existence of classical exact solutions with $\phi_{\mu\nu} = G_{\mu\nu} \neq 0$ in the vacuum, makes evident that NLG is not identical to general relativity.

To explain these claims with more details, let us consider the classical exact solutions of the theory, and let us recast the exact equations of motion in the vacuum (3) as

$$W_{\mu\nu\alpha\beta}\phi^{\alpha\beta} = -Q_{2\mu\nu}(\phi) \,, \tag{29}$$

where Ric has been expressed in terms of ϕ using the relation $R_{\mu\nu} = G_{\alpha\beta} - g_{\alpha\beta}G/2 = \phi_{\alpha\beta} - g_{\alpha\beta}\phi/2$, and the operator $W_{\mu\nu\alpha\beta}$ is defined as

$$W_{\mu\nu\alpha\beta} \equiv (1 + \Box\,\gamma(\Box))\,\delta_{\mu\alpha}\,\delta_{\nu\beta} + (g_{\mu\nu}\nabla_\alpha\nabla_\beta - g_{\alpha\mu}\nabla_\beta\nabla_\nu)\,\gamma(\Box) \,. \tag{30}$$

From the expression (12) of the operator $Q_2(\phi)$, one easily recognizes that equation (29) has the form

$$W_{\mu\nu\alpha\beta}\phi^{\alpha\beta} = \ell^2\, O(\phi^2) \,, \tag{31}$$

where, once more, ℓ is the length scale of nonlocality.

Now two situations can occur: in the first case, which is the one considered in this paper, one has $\ell^2\,|\phi| \ll 1$, so that the r.h.s. of (31) is negligible and one finds that $\phi_{\mu\nu}$ must be zero, as we concluded. In other words, the only possibility compatible with the assumption that the curvature does not exceed certain value, say $|R| = |\phi| \ll 1/\ell^2$, at the initial time, is that curvature tensor must be zero at any time. Therefore, in that case, the equations of motion (29) coincide with the Einstein's equations in vacuum, i.e., $\phi_{\mu\nu} = 0$.

In the second case, in which $\ell^2\,|\phi| \gtrsim 1$, the r.h.s. of (31) cannot be neglected, and one can have solutions of the equations of motion with $\phi_{\mu\nu} \neq 0$, so that the extra degrees of freedom can show up. However, in this case $|R| = |\phi| \gtrsim 1/\ell^2$, so that the curvature of the spacetime cannot be arbitrarily small, but it is fixed by the scale of nonlocality. Since ℓ is very small, one understand that a spacetime with such a large curvature is not a globally small perturbation of the Minkowski spacetime. Particularly, such a spacetime cannot fulfil the SAF condition and the GSA around the Minkowski metric at the initial time.

Therefore, choosing the class of small initial perturbations of the Minkowski spacetime satisfying the SAF condition and the GSA, one a automatically sets to zero the extra degrees of freedom of the theory. Such extra degrees of freedom can be excited only on exact background solutions of nonlocal gravity, that are not solutions of general relativity. In facts, if a spacetime is solution of Einstein's equations, it must contain only the graviton. Since, any initial data satisfying the SAF condition and the GSA evolves according to the vacuum Einstein's equations, such class of initial data can excite only the graviton. For the same reason, background metrics containing the extra degrees of freedom cannot be seeded by small (in the SAF and GSA sense) perturbations of the Minkowski spacetime.

Our result can be resumed as follows: the NLG (1) is stable under the same class of initial conditions under which GR is stable too, namely those satisfying the SAF condition and the GSA around Minkowski spacetime. Furthermore, such initial conditions do not excite the extra degrees of freedom contained in the theory, and the graviton is the only propagating mode in the evolution of the initial data. The extra modes $\phi_{\mu\nu}$ can be nonzero in the vacuum, but their occurrence implies that the curvature of the spacetime must be very large $R = O(\ell^{-2})$ at any time, including the initial time. This is why initial states containing the $\phi_{\mu\nu}$ modes are not small perturbations of the Minkowski spacetime.

We stress that, since one is left with the two polarizations of general relativity, the evolution of gravitational waves on a Minkowski background in NLG is the same as in Einstein's gravity. Thus, NLG is in agreement with the recent observations of gravitational radiation from binary systems achieved in gravitational interferometers [26, 27].

The results presented in this paper can be extended to more general classes of metrics, including Ricci flat and (A)dS spacetimes as will be proved elsewhere [28]. So that, such spacetimes, e.g., (A)dS, will be stable in NLG provided they are stable in GR; for a review of (A)dS stability in general relativity see [29–33].

A Stability of the Minkowski spacetime in general relativity

In this appendix we recall the notions of Strongly Asymptotically Flat (SAF) condition and Global Smallness Assumption (GSA) of initial data sets, and the stability theorem for the Minkowski spacetime in GR, as given in [15, 16].

Let us first consider a foliation of the spacetime Σ_τ depending on a the time parameter τ, such that $D\tau$ is always time-like. Indeed, the spacetime is diffeomorphic to a product manifold $\mathbb{R} \times \Sigma$, where Σ is a three dimensional manifold. Such a spacetime can be parameterized by the points of a slice Σ_{τ_0} by following the integral curves of $D\tau$, so that the metric in such reference frame takes the form

$$ds^2 = \phi^2(\tau, \xi)\, d\tau^2 + \chi_{ij}(\tau, \xi)\, d\xi^i\, d\xi^j\,. \tag{1}$$

where χ_{ij} is a 3×3 Riemannian metric. Moreover, we define the following quantity, corresponding to the extrinsic curvature of the leaves Σ_τ as

$$\kappa_{ij} = (2\phi)^{-1}\, \partial_\tau \chi_{ij}\,. \tag{2}$$

An initial data set for the Einstein's equations is given by a set $(\Sigma_{\tau_0}, \chi, \kappa)$ corresponding to initial conditions at some initial time τ_0. We define the class of initial data satisfying the SAF condition as follows.

Definition (SAF): we say that the initial data set $(\Sigma_{\tau_0}, \chi, \kappa)$ satisfies the SAF condition if there is a coordinate system $\left(\xi^1, \xi^2, \xi^3\right)$ on Σ_{τ_0} such that, asymptotically for $|\xi|^2 = \sum_i \left(\xi^i\right)^2 \to \infty$ one has

$$\begin{aligned} \chi_{ij} &= \left(1 + \frac{M}{\xi}\right)\delta_{ij} + o_4\left(\xi^{-3/2}\right)\,, \\ \kappa_{ij} &= o_3\left(\xi^{-5/2}\right)\,, \end{aligned} \tag{3}$$

where a function is said to be $o_m(\xi^{-n})$ if $\partial^l f\,(\xi) = o\left(r^{-n-l}\right)$ for $|\xi|^2 \to \infty$.

In order to express the GSA, we define the following quantity:

$$\begin{aligned} Q\,(\xi_0) &= \sup_{\Sigma_{\tau_0}} \left\{\left(d_0^2+1\right)^3 |R^{(3)}_{ij}|^2\right\} + \int_{\Sigma_{\tau_0}} \sum_{l=0}^{3} \left(d_0^2+1\right)^{l+1} |\nabla^{(3)\,l}\kappa|^2 \\ &\quad + \int_{\Sigma_{\tau_0}} \sum_{l=0}^{3} \left(d_0^2+1\right)^{l+3} |\nabla^l B_{ij}|^2\,, \end{aligned} \tag{4}$$

where $d_0(\xi) = d(\xi_0, \xi)$ is the Riemannian geodesic distance between $\xi_0 \in \Sigma_{\tau_0}$ and $\xi \in \Sigma_{\tau_0}$. Moreover, the curvature $R^{(3)}_{ij}$ and the covariant derivatives are constructed with the 3-metric χ_{ij}, and B_{ij} is the symmetric and traceless 2-tensor

$$B_{ij} = \epsilon_j^{ab} \nabla_a^{(3)} \left(R^{(3)}_{ib} - \frac{1}{4} \chi_{ib} R^{(3)} \right) . \tag{5}$$

Now we can give the following definition (GSA): the metric χ_{ij} satisfies the GSA condition if there is a sufficiently small positive parameter μ such that

$$\inf_{\xi_0 \in \Sigma_{\tau_0}} Q(\xi_0) < \mu . \tag{6}$$

The stability of the Minkowski metric in general relativity GR is established by the following [15, 16].

Theorem: any SAF initial data set which satisfies the GSA leads to a unique, globally hyperbolic, maximal, smooth, and geodesically complete solution of the Einstein Vacuum Equations foliated by a normal, maximal time foliation. Moreover, this development is globally asymptotically flat.

This theorem in facts means that it exists a μ_0 such that, if the initial data satisfies the SAF condition and the GSA with $\mu < \mu_0$, such initial data have a regular evolution (see for instance the discussion in [17, 18]). Therefore, the stability of Minkowski spacetime is inferred proving the existence of such μ_0, without determining its size. Finally, note that the GSA (6) for the metric (5) ($g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$) implies that ϵ must be sufficiently small. Therefore, the stability theorem means that it exists a positive $\epsilon_0 < 1$ such that, if $\epsilon < \epsilon_0$, the perturbations have a regular evolution.

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