

# Bose-Einstein condensates as gravitational wave detectors

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**Abstract.** We investigate a Bose-Einstein condensate (BEC) as a gravitational wave detector, and study its sensitivity by optimizing the properties of the condensate and the measurement duration. We show that detecting kilohertz gravitational waves is limited by current experimental techniques in squeezing BEC phonons. Without focussing on a specific detector setup, our study shows that substantive future improvements in technology (e.g., increasing the squeezing of BEC states or their physical size) will be necessary for such a detector to be competitive in measuring gravitational waves of astrophysical and/or cosmological origin.

**Keywords:** gravitational wave detectors, quantum field theory on curved space, gravitational waves / experiments

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## 1 Introduction

With the direct detection of gravitational waves, an entirely new avenue of studying the universe has opened. It is incumbent upon the scientific community to consider both the theoretical aspects of this discovery and to propose new means of detecting gravitational waves.

At the present time, gravitational wave detection is primarily done with interferometers. A drawback is that they are sensitive to only a small range of frequencies, with LIGO being most sensitive around 100–300 Hz, allowing it to detect stellar-mass inspiralling black holes and neutron stars [1]. Gravitational wave detectors with sensitivities at lower frequencies have been proposed, which will allow the large-mass black hole binary inspirals to be studied [2]. At kilohertz frequencies, the creation of magnetars [3] and signals from more neutron star mergers [4, 5] can also be observed with improved sensitivity. Detection of kHz gravitational waves will have wide-ranging implications in further understanding the universe, such as constraining the equation of state of neutron stars [6].

Interferometers involving cold atoms have previously been suggested as a means to detect kHz frequency gravitational waves [7, 8]. Recently, a novel suggestion [9] making use of a Bose-Einstein condensate (BEC) as a gravitational wave detector has been proposed. A zero temperature quasi (1+1)-dimensional BEC with fluctuating boundaries was considered in the presence of a gravitational wave with plus polarization (in the BEC frame),  $h_+ = \epsilon \sin \Omega t$ , where  $\epsilon$  is the amplitude of the gravitational wave and  $\Omega$  is its frequency. By calculating the fidelity between phonon states, it was possible to determine the quantum Fisher information  $H_\epsilon$  of the phonon state of the BEC, which characterizes the amount of information contained in the amplitude of the gravitational wave [9, 10]:

$$M \langle (\Delta\epsilon)^2 \rangle \geq \frac{1}{H_\epsilon}, \quad (1.1)$$

where  $M$  is the number of independent measurements of the system and  $\langle (\Delta\epsilon)^2 \rangle$  is the mean-square-error in the amplitude of the gravitational wave. Assuming that the phonons are in squeezed two-mode states, it was claimed that, with a suitable number of measurements of the fidelity between phonons interacting with a gravitational wave of amplitude  $\epsilon$  and a

gravitational wave of amplitude  $\epsilon + d\epsilon$ , the strain sensitivity  $\sqrt{\langle \Delta \epsilon^2 \rangle}$  was able to exceed that of LIGO at frequencies in the kilohertz regime.

We offer here an alternate perspective by considering zero-temperature (3+1)-dimensional BECs, where  $h_+ = \epsilon e^{-t^2/\tau^2} \sin(\Omega t)$  is used to model the incoming gravitational wave; the quantity  $\tau$  captures the duration of a single measurement of the BEC. We treat the phonons as being in single-mode squeezed states and examine whether current techniques of squeezing phonons and producing BECs are sufficient in order to use the condensate as a gravitational wave detector.

In section 2, we introduce BECs in a curved spacetime and derive the Euler-Lagrange equation for the phonons being influenced by the gravitational waves. Then, in section 3, we apply techniques in quantum metrology to estimate the sensitivity for the detection of gravitational waves. In section 4, we consider whether or not current experimental techniques can be used to achieve the necessary sensitivity. We show that if the phonons are restricted to obey a linear dispersion relation, then the amount of phonon squeezing is the dominant limiting factor. We also address the damping present in the condensate at  $T = 0$  and comment on how to increase the sensitivity of the condensate to gravitational waves. Section 5 presents our conclusions.

## 2 Bose-Einstein condensates in a curved background

We will now derive the equation of motion for the phonons as well as its Bogoliubov coefficients. Related derivations can be found in [9, 11, 12].

The Lagrangian ( $\hbar = c = 1$ ) for a BEC in a curved background

$$\mathcal{L} = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^\dagger - m^2 |\phi|^2 - U(|\phi|^2) \quad (2.1)$$

where  $m$  is the mass of the atoms of the BEC,  $\phi$  is the field,  $U(|\phi|^2) = \lambda |\phi|^4 > 0$  describes the interaction of the BEC.<sup>1</sup> Let us write  $\phi = \hat{\phi} e^{i\chi}$  (with real  $\hat{\phi}$  and  $\chi$ ) and assume that the BEC is homogeneous. We want to determine the  $\hat{\phi}$  that extremizes  $\mathcal{L}$ . Differentiating with respect to  $\hat{\phi}$ , we find that the extremum occurs at

$$\hat{\phi}^2 = \frac{1}{2\lambda} [\partial^\mu \chi \partial_\mu \chi - m^2] . \quad (2.2)$$

Inserting (2.2) into (2.1) and writing  $\chi = \kappa t + \pi$  (where  $\pi \in \mathfrak{R}$  is the pseudo-Goldstone boson, describing the BEC acoustic perturbations or phonons), the action becomes

$$\begin{aligned} S &= \int \frac{d^4x}{4\lambda} \sqrt{-g} \{ (\kappa \delta_0^\nu + \partial^\nu \pi) (\kappa \delta_0^\mu + \partial^\mu \pi) g_{\mu\nu} - m^2 \}^2 , \\ &= \int \frac{d^4x}{4\lambda} \sqrt{-g} [\kappa^2 g_{00} + 2\kappa \dot{\pi} g_{00} + \kappa \partial^i \pi g_{0i} + |\dot{\pi}|^2 g_{00} + \partial^i \pi \partial^j \pi g_{ij} + 2\partial^i \pi \dot{\pi} g_{i0} - m^2]^2 . \end{aligned} \quad (2.3)$$

We are interested in low frequencies, using an effective field theory framework for  $\chi$ , where “heavy” fields are “integrated out”. As a result,  $\partial_\mu \hat{\phi}$  terms are higher order and are therefore

<sup>1</sup>As this describes the generalization of a BEC in curved spacetime, we note that it will only be accurate for  $na^3 = \frac{(\lambda c_s)^2}{(4\pi)^3 c^2} \ll 1$  as this is the regime of validity for the Gross-Pitaevskii equation, where  $n$  is the number density,  $a$  is the s-wave scattering wavelength, and  $c_s$  is the speed of sound defined below in equation (2.8) [13].

suppressed [14]. Let us work in the TT-gauge and take  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ , where

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(t) & h_\times(t) & 0 \\ 0 & h_\times(t) & -h_+(t) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.4)$$

describes a gravitational wave propagating in the  $z$ -direction and  $h_+$  and  $h_\times$  are the two polarizations of the gravitational wave.<sup>2</sup> Using  $\sqrt{-g} \sim 1 + \mathcal{O}(h_{\mu\nu}h^{\mu\nu})$  and a  $(+, -, -, -)$  signature, we can expand in terms of  $\pi$  to find

$$S = \int \frac{d^4x}{4\lambda} [\kappa^2 + 2\kappa\dot{\pi} + |\dot{\pi}|^2 + \partial^i \pi \partial^j \pi g_{ij} - m^2]^2, \quad (2.5)$$

$$\approx \int \frac{d^4x}{4\lambda} [|\dot{\pi}|^2(6\kappa^2 - 2m^2) + (2\kappa^2 - 2m^2)(\eta_{ij} + h_{ij})\partial^i \pi \partial^j \pi], \quad (2.6)$$

where the first-order terms can be written as a total derivative that integrates to zero on the boundary and we assume that the higher-order terms can be neglected. Therefore, the Euler-Lagrange equation is

$$\ddot{\pi} + c_s^2(\eta_{ij} + h_{ij})\partial^i \partial^j \pi + c_s^2(\partial^j h_{ij})\partial^i \pi = 0, \quad (2.7)$$

where

$$c_s^2 \equiv \frac{\kappa^2 - m^2}{3\kappa^2 - m^2}. \quad (2.8)$$

In appendix A, instead of making the low-energy (or adiabatic) approximation (2.2), we solve for the full linear perturbations (for  $\hat{\phi}$  and  $\pi$ ). This shows that the above derivation is valid as long as  $\omega = c_s k \ll \mu$ , where  $\mu = \kappa - m \approx mc_s^2$  is the chemical potential of the BEC.

For simplicity, we will assume that  $h_\times = 0$ . We model the plus polarization as  $h_+ = \epsilon e^{-t^2/\tau^2} \sin \Omega t$  (ignoring its spatial dependence),<sup>3</sup> where  $\epsilon$  is the amplitude of the gravitational wave,  $\tau$  captures the duration of a single measurement of the gravitational wave, and  $\Omega$  is the frequency of the wave. Noting that  $(\partial_i h^{ij}) = 0$  in the TT gauge and using the ansatz  $\pi \propto e^{ik \cdot x} \psi(t)$ , we find

$$(3\kappa^2 - m^2)\ddot{\psi} + (\kappa^2 - m^2)(\eta_{ij} + h_{ij})k^i k^j \psi = 0 \quad (2.9)$$

up to a normalization of  $\psi$ . Thus, with  $k^1 = k_x$ ,  $k^2 = k_y$ , and  $k^3 = k_z$ ,

$$(3\kappa^2 - m^2)\ddot{\psi} + (\kappa^2 - m^2)|k|^2[1 + \tilde{\epsilon}e^{-t^2/\tau^2} \sin(\Omega t)]\psi = 0, \quad (2.10)$$

where  $\tilde{\epsilon} = \frac{(k_x^2 - k_y^2)}{|k|^2} \epsilon$ . We can then rewrite equation (2.10) as

$$\ddot{\psi} + [1 + \tilde{\epsilon}e^{-t^2/\tau^2} \sin(\Omega t)]c_s^2 k^2 \psi = 0. \quad (2.11)$$

<sup>2</sup>We assume that the trapped particles, representing the boundary of the BEC box, move on geodesics. Therefore, starting at rest, they will not see the gravitational waves to linear order.

<sup>3</sup>This is justified when the speed of sound is much smaller than that of gravitational waves,  $c_s \ll 1$ .

Let us now look at the validity of equation (2.11), which was determined by neglecting higher-order terms in equation (2.5). The third-order terms in the Lagrangian are

$$\mathcal{L}_3 = \frac{\kappa}{\lambda} \left[ |\dot{\pi}|^3 + \dot{\pi} g_{ij} \partial^i \pi \partial^j \pi \right]. \quad (2.12)$$

For  $\kappa \approx m$ , we have  $c_s^2 \approx \frac{\kappa^2 - m^2}{2m^2} \ll 1$ , so  $4m^2 |\dot{\pi}|^2 \approx 2m^2 c_s^2 |\nabla \pi|^2$  on average. Now, comparing equations (2.6) and (2.12), we see that the linear theory is only valid when

$$c_s^{-1} |\dot{\pi}| \sim |\nabla \pi| \ll m c_s. \quad (2.13)$$

Solving (2.11) perturbatively by writing  $\psi = \psi^{(0)} + \tilde{\epsilon} \psi^{(1)}$  yields

$$\ddot{\psi}^{(0)} + \omega^2 \psi^{(0)} = 0, \quad (2.14)$$

$$\ddot{\psi}^{(1)} + \omega^2 \psi^{(1)} = -\omega^2 e^{-t^2/\tau^2} \sin(\Omega t) \psi^{(0)}, \quad (2.15)$$

which has the solutions

$$\psi^{(0)}(t) = C_1^{(0)} e^{i\omega t} + C_2^{(0)} e^{-i\omega t}, \quad (2.16)$$

$$\psi^{(1)}(t) = C_1^{(1)} e^{i\omega t} + C_2^{(1)} e^{-i\omega t} - \int_{-\infty}^{\infty} dt_1 \omega^2 e^{-t_1^2/\tau^2} \sin(\Omega t_1) G(t, t_1) \psi^{(0)}(t_1), \quad (2.17)$$

where

$$G(t, t_1) = \frac{\sin[\omega(t - t_1)]}{\omega} \Theta(t - t_1), \quad (2.18)$$

is the Green's function of a harmonic oscillator, while  $\Theta$  represents the Heaviside function. Combining our ansatz of  $\pi \propto e^{ik \cdot x} \psi(t)$  with equations (2.16) and (2.17), we find

$$\pi(x, t) \propto e^{ikx} \left[ e^{-i\omega t} + \frac{\sqrt{\pi} \tilde{\epsilon} \omega \tau}{4} e^{-\frac{1}{4}(\Omega + 2\omega)^2 \tau^2} \left( e^{2\omega \Omega \tau^2} - 1 \right) e^{i\omega t} \right]. \quad (2.19)$$

As we are working in curved spacetime with a single mode, we can write  $\pi(x, t) \propto e^{ikx} [\alpha e^{-i\omega t} + \beta e^{i\omega t}]$ , where  $\alpha$  and  $\beta$  are Bogoliubov coefficients. This immediately yields

$$\alpha = 1, \quad (2.20)$$

$$\beta = \frac{\tilde{\epsilon} \sqrt{\pi} \omega}{4} \tau e^{-(\Omega + 2\omega)^2 \tau^2 / 4} \left( e^{2\omega \Omega \tau^2} - 1 \right), \quad (2.21)$$

from equation (2.19).

Let us briefly comment on the large and small  $\tau$  limits. We note that  $\beta \rightarrow 0$  as  $\tau \rightarrow 0$ . Intuitively, this makes sense because a vanishing measurement duration implies that no information about the gravitational wave would be obtained. We further discuss the information that can be acquired in section 3. For  $\tau \rightarrow \infty$  we recover (as expected [9])  $\beta \rightarrow 0$ . We note that for non-geodesic boundaries [9], non-zero Bogoliubov coefficients  $\alpha_{nm}$  and  $\beta_{nm}$  result for modes  $n \neq m$ . However in our case coefficients with  $n \neq m$  are zero since we assume non-interacting modes.

As we are considering an odd function for the gravitational waveform, we find  $\alpha = 1$ . If, for example, we had instead considered an even function  $h_+ = \epsilon e^{-t^2/\tau^2} \cos(\Omega t)$ , then  $\alpha$  would include a non-zero imaginary  $\mathcal{O}(\epsilon)$  term. Such effects are necessary to consider if a BEC gravitational wave detector were constructed; we shall neglect this additional effect henceforth in order to keep the discussion as simple as possible.

### 3 Detection of gravitational waves via quantum metrology

Quantum metrology is the study of making precision measurements by exploiting quantum mechanical properties, rather than solely relying on classical measurements of a system. This can be used, for example, to overcome shot noise in a detector [15–17]. Another advantage is that quantum metrology can be used to estimate a parameter  $\theta$  that is not an operator observable of a system. This is done by determining how an infinitesimal change of the parameter affects the statistical distance between two quantum states, thereby defining their distinguishability (fidelity) and quantum Fisher information.

An estimate in the error in the measurement of  $\theta$  is obtained from the quantum Fisher information [10],

$$H(\theta) = \frac{8 \left( 1 - \sqrt{F(\rho_\theta, \rho_{\theta+d\theta})} \right)}{d\theta^2}, \quad (3.1)$$

where  $F(\rho_\theta, \rho_{\theta+d\theta})$  is the fidelity between the states  $\rho_\theta$  and  $\rho_{\theta+d\theta}$ . The fidelity between two states  $\rho', \rho''$  is defined as [18, 19]

$$F(\rho', \rho'') = \left[ \text{Tr} \sqrt{\rho' \sqrt{\rho''} \rho'} \right]^2, \quad (3.2)$$

which describes the overlap between  $\rho'$  and  $\rho''$ . When both  $\rho'$  and  $\rho''$  correspond to Gaussian states, it is often easier to use covariance matrices. In this case, the covariance matrix for a Gaussian state is  $\sigma_{mn} = \frac{1}{2} \langle X_m X_n + X_n X_m \rangle - \langle X_m \rangle \langle X_n \rangle$ , where  $X_{2n-1} = \frac{1}{\sqrt{2}}(a_n + a_n^\dagger)$ ,  $X_{2n} = \frac{1}{\sqrt{2}i}(a_n - a_n^\dagger)$ , and  $a_n, a_n^\dagger$  are the creation and annihilation operators. Note that this normalization convention is different from what was used in [9, 20]. Suppose that  $M$  independent measurements are done to determine  $\theta$ . Then,

$$\langle (\Delta\theta)^2 \rangle \geq \frac{1}{MH(\theta)} \quad (3.3)$$

is the minimum error in measuring  $\theta$  [9, 10].

We will now use the quantum Fisher information to estimate the minimum error in the amplitude of the gravitational wave. We will restrict ourselves to the case in which  $\langle X_i \rangle = 0$ . First, consider two Gaussian states described by the covariance matrices  $\sigma_A, \sigma_B$ . Let us define [21]

$$\Delta = \det[\sigma_A + \sigma_B], \quad (3.4a)$$

$$\Lambda = 2^{2n} \det \left[ \sigma_A + \frac{i}{2} J \right] \det \left[ \sigma_B + \frac{i}{2} J \right], \quad (3.4b)$$

$$J = \bigoplus_{k=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.4c)$$

where  $n$  is the number of modes. Now, the fidelity between the two covariance matrices of two single-mode states is given by [21]

$$F(\sigma_A, \sigma_B) = \frac{1}{\sqrt{\Delta + \Lambda} - \sqrt{\Lambda}}. \quad (3.5)$$

Consider preparing the phonons in the BEC in a squeezed Gaussian single-mode state, described by an initial covariance matrix  $\sigma(0)$  (at zero-temperature) [22]:

$$\sigma(0) = \frac{1}{2} \begin{pmatrix} \cosh(2r) + \cos(\phi) \sinh(2r) & -\sin(\phi) \sinh(2r) \\ -\sin(\phi) \sinh(2r) & \cosh(2r) - \cos(\phi) \sinh(2r) \end{pmatrix}, \quad (3.6)$$

where  $r$  is the squeezing parameter and  $\phi$  is the squeezing angle. When a gravitational wave passes by the BEC, it affects the phonons by transforming its covariance matrix to  $\sigma_\ell(\tilde{\epsilon})$  where [20]

$$\sigma_\ell(\tilde{\epsilon}) = \mathcal{M}_{\ell\ell}(\tilde{\epsilon})\sigma(0)\mathcal{M}_{\ell\ell}(\tilde{\epsilon}) + \sum_{j \neq \ell} \mathcal{M}_{\ell j}(\tilde{\epsilon})\mathcal{M}_{\ell j}^T(\tilde{\epsilon}), \quad (3.7)$$

with  $\ell$  the mode number of the phonon and

$$\mathcal{M}_{mn}(\tilde{\epsilon}) = \begin{pmatrix} \Re[\alpha_{mn} - \beta_{mn}] & \Im[\alpha_{mn} + \beta_{mn}] \\ -\Im[\alpha_{mn} - \beta_{mn}] & \Re[\alpha_{mn} + \beta_{mn}] \end{pmatrix}. \quad (3.8)$$

Our Bogloliubov coefficients in equations (2.20) and (2.21) do not couple different modes, so we note that  $\alpha_{mn} = \alpha\delta_{mn}$  and  $\beta_{mn} = \beta\delta_{mn}$ .

As shown in [20], equation (3.1) can be written as

$$H(\epsilon) = 2 \left( \sigma_{11}^{(0)} \sigma_{22}^{(2)} + \sigma_{11}^{(2)} \sigma_{22}^{(0)} - 2\sigma_{12}^{(0)} \sigma_{12}^{(2)} \right) + \frac{1}{2} \left( \sigma_{11}^{(1)} \sigma_{22}^{(1)} - 2\sigma_{12}^{(1)} \sigma_{12}^{(1)} \right). \quad (3.9)$$

where  $\sigma_{ij}^{(n)}$  is the  $ij$  matrix element in an expansion of the covariance matrix,

$$\sigma_{ij}(\epsilon) = \sigma_{ij}^{(0)} + \sigma_{ij}^{(1)}\epsilon + \sigma_{ij}^{(2)}\epsilon^2 + \mathcal{O}(\epsilon^3) \quad (3.10)$$

in powers of  $\epsilon$ , and we have assumed that the amount of displacement of the squeezed state is zero. Using equations (2.20), (2.21), and (3.6)–(3.9), we calculate

$$H(\epsilon) = \frac{1}{64} \pi \tau^2 \omega^2 e^{-\frac{1}{2}\tau^2(2\omega+\Omega)^2} \left( e^{2\tau^2\omega\Omega} - 1 \right)^2 R \quad (3.11)$$

where

$$R \equiv \sinh^2(2r)(6\sin^2\phi - 2) + \cosh(4r) + 1. \quad (3.12)$$

Therefore, we find

$$\frac{1}{M \langle (\Delta\tilde{\epsilon})^2 \rangle} \leq \frac{\pi\omega^2\tau^2}{64} R \left( e^{-\frac{(\Omega-2\omega)^2\tau^2}{4}} - e^{-\frac{(\Omega+2\omega)^2\tau^2}{4}} \right)^2. \quad (3.13)$$

Equation (3.13) describes a BEC possessing a single mode, but we can further exploit all modes of the BEC to improve the sensitivity. From  $\tilde{\epsilon} = \frac{k_x^2 - k_y^2}{k^2} \epsilon$ , we have  $\langle (\Delta\tilde{\epsilon}_{\vec{k}})^2 \rangle = \left( \frac{k_x^2 - k_y^2}{k^2} \right)^2 \langle (\Delta\epsilon_{\vec{k}})^2 \rangle$ , where  $\langle (\Delta\epsilon_{\vec{k}})^2 \rangle$  is the error in  $\epsilon$  for mode  $\vec{k}$ . Averaging over the solid angle,  $\frac{1}{4\pi} \int d\theta d\varphi \langle (\Delta\tilde{\epsilon}_{\vec{k}})^2 \rangle \sin\theta = \frac{4}{15} \langle (\Delta\epsilon_{\vec{k}})^2 \rangle$ . Substituting this average into  $\langle (\Delta\tilde{\epsilon}_{\vec{k}})^2 \rangle$ , we find

$$\frac{1}{\langle (\Delta\epsilon)^2 \rangle_{\text{tot}}} = \sum_{\vec{k}} \left\langle \left( \frac{k_x^2 - k_y^2}{k^2} \right)^2 \right\rangle \frac{1}{\langle (\Delta\tilde{\epsilon}_{\vec{k}})^2 \rangle} = \frac{4}{15} \sum_{\vec{k}} \frac{1}{\langle (\Delta\tilde{\epsilon}_{\vec{k}})^2 \rangle}, \quad (3.14)$$

where  $\langle(\Delta\epsilon)^2\rangle_{\text{tot}}$  is the total error in the measurement of the amplitude of the gravitational wave. For a large number of single-mode states and assuming that the modes are non-interacting, we can convert equation (3.14) into an integral. With  $\omega = c_s k$ , assuming the BEC has a volume of  $L^3$ ,  $k = \sqrt{\left(\frac{n_x\pi}{L}\right)^2 + \left(\frac{n_y\pi}{L}\right)^2 + \left(\frac{n_z\pi}{L}\right)^2}$ , and using spherical coordinates (with the both the altitudinal and azimuthal integrals between 0 and  $\frac{\pi}{2}$ ), we have

$$\begin{aligned} \frac{1}{\langle(\Delta\epsilon)^2\rangle_{\text{tot}}} &\lesssim \frac{\pi^4 M c_s^2 R \tau^2}{480 L^2} \int_0^\infty n^4 e^{-\frac{(2\pi c_s n + L\Omega)^2 \tau^2}{2L^2}} \left( e^{\frac{2\pi c_s n \Omega \tau^2}{L}} - 1 \right)^2 dn, \\ &= \frac{M L^3 e^{-\frac{1}{2}\tau^2 \Omega^2} R \left( e^{\frac{\tau^2 \Omega^2}{2}} (\tau^4 \Omega^4 + 6\tau^2 \Omega^2 + 3) - 3 \right)}{7680 \sqrt{2\pi} c_s^3 \tau^3}, \end{aligned} \quad (3.15)$$

neglecting  $\mathcal{O}(\epsilon^2)$  terms and only considering the cases in which the integrand is peaked at  $n \gg 1$  (corresponding to gravitational wave frequency much bigger than the lowest acoustic harmonic of the BEC,  $\Omega \gg \frac{2\pi c_s}{L}$ ).

Now, for a total observation time of  $t_{\text{obs}}$ , we can approximately run  $M \sim t_{\text{obs}}/\tau$  separate measurements of the BEC state. Therefore,

$$\langle(\Delta\epsilon)^2\rangle_{\text{tot}} \geq \frac{7680 \sqrt{2\pi} c_s^3 \tau^4 e^{\frac{\tau^2 \Omega^2}{2}}}{L^3 t_{\text{obs}} R \left( e^{\frac{\tau^2 \Omega^2}{2}} (\tau^4 \Omega^4 + 6\tau^2 \Omega^2 + 3) - 3 \right)}. \quad (3.16)$$

Note that  $R$  is maximized at  $\phi = \pi/2$ , such that  $R_{\text{max}} = 3 \cosh(4r) - 1$ . We point out that squeezing at specific angles is physical and has previously been done [23, 24].

For  $\Omega\tau \ll 1$ ,

$$\langle(\Delta\epsilon)^2\rangle_{\text{tot}} \geq \frac{1024 \sqrt{2\pi} c_s^3 \tau^2}{L^3 t_{\text{obs}} \Omega^2 R} + \mathcal{O}(\Omega^4 \tau^4), \quad (3.17)$$

indicating that shorter (individual) measurement times,  $\tau$ , for a fixed total observation time  $t_{\text{obs}}$ , will maximize the sensitivity of the BEC to an incoming gravitational wave. However,  $\tau$  cannot be made arbitrarily short; we shall briefly discuss this in section 4.

Let us now investigate the maximum squeezing of the system. From equation (2.6), for  $4|\dot{\pi}|^2 \approx 3c_s^2 |\nabla\pi|^2$ , which is true on average, the Hamiltonian is

$$H = \frac{7m^2}{4\lambda} \int d^3x |\nabla\pi|^2 \sim \frac{7m^2}{4\lambda} L^3 |\nabla\pi|^2, \quad (3.18)$$

where  $V$  is the volume of the condensate. In the ground state of the system, we would be able to write the Hamiltonian as

$$H = L^3 \int \frac{d^3k}{(2\pi)^2} \frac{1}{2} \omega \sim \frac{L^3}{32\pi^2} k^4 c_s, \quad (3.19)$$

where  $k$  is the maximum effective wavenumber obeying the linearity condition (2.13):  $\langle|\nabla\pi|^2\rangle \ll m^2 c_s^2$ . Comparing equations (3.18) and (3.19), we find  $|\nabla\pi|^2 \sim \frac{\lambda k^4 c_s}{56m^2 \pi^2}$ . If we squeeze the ground state, then we require  $\frac{\lambda k^4 c_s}{56m^2 \pi^2} e^{2r} \ll m^2 c_s^2$ . Therefore, after re-inserting  $\hbar$  and  $c$ , we have

$$e^{2r} \ll \frac{56\pi^2}{\lambda} \left( \frac{mc}{\hbar k} \right)^4 \left( \frac{c_s}{c} \right). \quad (3.20)$$



To calculate  $\lambda$ , we need only to consider the background, such that  $\pi = 0$ . With

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}, \quad (3.21)$$

and

$$S = \int d^4x \sqrt{-g} \mathcal{L}, \quad (3.22)$$

where  $\mathcal{L} = \frac{1}{4\lambda} [g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - m^2]^2$ , we find

$$T_{\mu\nu} = g_{\mu\nu} \mathcal{L} + 4 \left[ g^{\mu'\nu'} \partial_{\mu'} \chi \partial_{\nu'} \chi - m^2 \right] \partial_\mu \chi \partial_\nu \chi. \quad (3.23)$$

Let  $\rho = T_{00}$  be the energy density. Then, with  $\dot{\pi} = 0$ ,  $\nabla \pi \sim \dot{\pi}$ , and  $2m^2 c_s^2 \approx \kappa^2 - m^2$ , we see (again re-inserting  $\hbar$  and  $c$ ),

$$\rho \approx \frac{1}{\lambda} \frac{m^4 c_s^2 c^3}{\hbar^3} \quad (3.24)$$

which is a relation that is already established in the BEC community.

Equation (3.20) becomes

$$e^{2r} \ll \frac{56\pi^2 c_s^3 \rho}{\omega^4 \hbar}, \quad (3.25)$$

where we have also used  $\omega = c_s k$ . For a BEC with number density  $7 \times 10^{20} \text{ m}^{-3}$  containing atoms of mass  $10^{-25} \text{ kg}$  and in the case of phonons of frequency  $\frac{\omega}{2\pi} = 10^4 \text{ Hz}$  and speed  $c_s = 1.2 \text{ cm/s}$ , we have  $r \lesssim 27$ .

## 4 Sensitivity for specific experimental parameters

### 4.1 Non-interacting modes

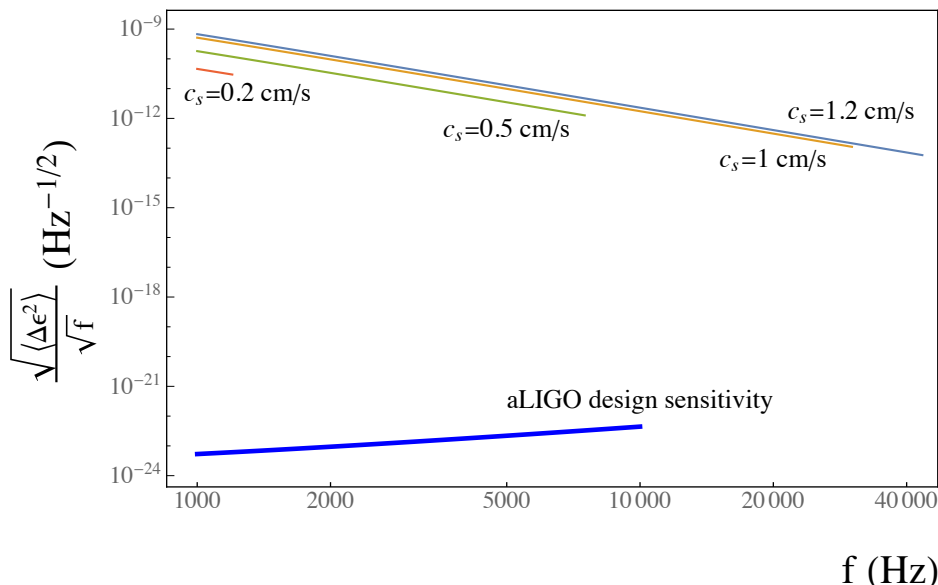
Let us now analyze the experimental feasibility of using a BEC to detect gravitational waves. We will consider condensates with non-interacting modes, which corresponds to the model described in sections 2 and 3. In reality, this is an oversimplification, and we discuss how our model will be affected by damping effects in section 4.2.

Numerical simulations involving optomechanics and trichromatic lasers have been able to squeeze phonons by at least 7.2 dB [25], corresponding to a squeezing parameter of  $r = 0.83$ .<sup>4</sup> Phonons have been squeezed using second-order Raman scattering [27, 28], though this was in the presence of a crystal lattice. Even though we are interested in a BEC in a curved spacetime, it may still be possible to exploit this feature.

As shown in [29], the presence of an optical lattice potential implies that the flat space BEC Hamiltonian can be written as a Bose-Hubbard Hamiltonian. In this case, a modification of the methods of squeezing phonons in crystal lattices, such as second order Raman scattering [27] or pump-probe detection scheme [30] might potentially be used. The current

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<sup>4</sup>Note that position squeezing  $s$  is reported in decibels and can be converted to the squeezing parameter  $r$  with  $s = -10 \log_{10} (e^{-2r})$  [26].



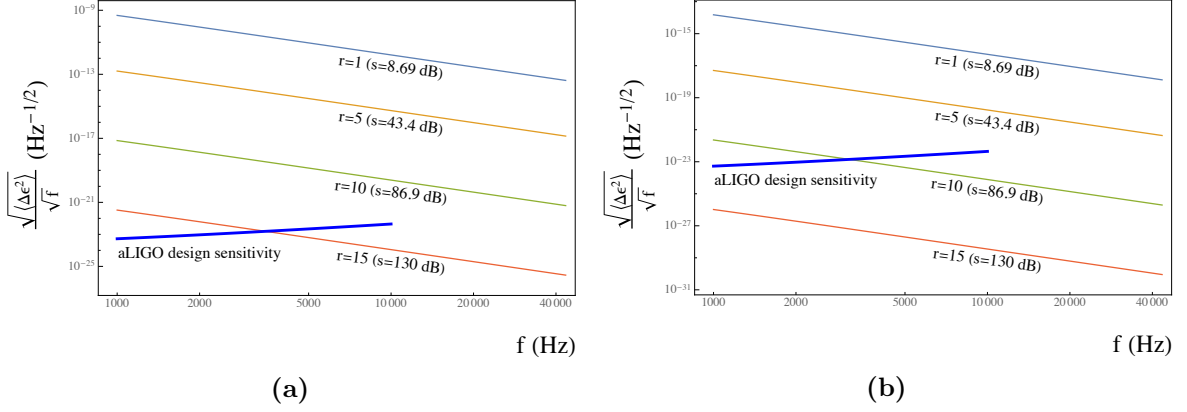
**Figure 1.** Theoretical sensitivities of a BEC gravitational wave detector (top curve) using current experimental capabilities, where  $f$  is the frequency of the gravitational wave. With  $L = 10^{-3}$  m,  $t_{\text{obs}} = 10^6$  s,  $\tau = 10^{-3}$  s,  $\phi = \pi/2$ , and  $r = 0.83$ , a BEC is unable to detect kHz gravitational waves. The thick blue line corresponds to the maximum design sensitivity of aLIGO (extrapolated to  $10^4$  Hz) [37, 38]. Note that phonons of frequency  $\frac{f}{2}$  are necessary to detect gravitational waves of frequency  $f$ .

limitation on squeezing phonons arguably represents one of the greatest challenges for using a BEC as a gravitational wave detector. With  $r = 0.83$  and  $\phi = \pi/2$ , we see that  $R \approx 41$ . If phonons were squeezed 20 dB, corresponding to  $r = 2.3$ , then  $R \approx 1.5 \times 10^4$ . An increase in the amount of squeezing would exponentially increase the sensitivity to the gravitational wave [31], though we note that this will be extremely difficult to achieve. For simplicity, we have assumed that the BEC is cubic in shape with no external potential, but realistically, we note that only certain trap geometries and trapping potentials lend themselves to large (number) squeezing [24].

Let us now consider the ratio  $\frac{c_s^3}{L^3 t_{\text{obs}}}$ . Experiments have been done to create condensates with lengths on the order of tens of microns to mm [32–34]. However, these lengths are only in a single direction, with the other length (in the case of quasi-two-dimensional BECs) much smaller. As shown in [35, 36], speeds of sound in BECs were analyzed as a function of the density, with a speed of approximately 1.2 cm/s being obtained at a number density of  $7 \times 10^{20} \text{ m}^{-3}$ .

For a gravitational wave of period  $T$ , sensitivity is optimized for  $T \lesssim \tau \lesssim t_d$ , where  $t_d$  is the decoherence time of the phonons, which we discuss in more detail in section 4.2. For gravitational waves in the kHz frequency range, the minimum time required is  $\tau \gtrsim 10^{-3}$  s. One proposal in [9] is to use quantum dots to make measurements on the BEC in which they suggested using 1500 dots to make  $10^6$  measurements per second.

In figure 1, we assume that a BEC can be constructed with the best experimental/numerical parameters that have been achieved: modes are squeezed by 7.2 dB ( $r = 0.83$ ) [25], the BEC has a length of  $L = 10^{-3}$  m [32–34], the speed of sound of  $c_s = 1.2 \times 10^{-2}$  m/s [35, 36], the quadrature angle is  $\phi = \pi/2$ , and there is a total ob-



**Figure 2.** Theoretical sensitivity to kHz gravitational waves for a futuristic BEC with  $c_s = 1.2 \times 10^{-2}$  m/s,  $t_{\text{obs}} = 10^6$  s,  $\phi = \pi/2$ , and  $\tau = 10^{-3}$  s. In (a),  $L = 10^{-3}$  m, while  $L = 1$  m in (b). The thick blue line is the design sensitivity of aLIGO (extrapolated to  $10^4$  Hz) [37, 38].

servational time of  $10^6$  s. It is seen that our maximum sensitivity for a gravitational wave in the kHz regime is approximately  $6 \times 10^{-14}$  Hz $^{-1/2}$ . From equation (3.16), though smaller speeds of sound will increase the sensitivity, the available frequency range will also decrease because the chemical potential  $\mu = mc_s^2$  becomes smaller. To detect a gravitational wave of 1 kHz using atoms of  $m = 10^{-25}$  kg, the maximum sensitivity of  $4 \times 10^{-11}$  Hz $^{-1/2}$  occurs when  $\mu = 500$  Hz ( $c_s \approx 1.8$  mm/s).

In figure 2a, we illustrate how increased squeezing can affect the BEC's sensitivity to gravitational waves. We assume that a BEC in the future can be constructed with similar properties as those in figure 1, but with  $r$  in excess of 0.83. It is necessary to squeeze phonons above  $r = 15$  in order to rival LIGO-level sensitivities. In an alternate scenario shown in figure 2b, we suppose that metre-long BECs can be constructed. For this case, we can exceed a LIGO-level sensitivity at  $r \approx 10$ . We acknowledge that several difficulties exist in constructing large-scale BECs, such as how to cool to sufficiently low temperatures. This is a major experimental challenge for the future that we will not further consider here. Indeed, attaining values  $r \geq 1$  is currently unfeasible, and would require advances in squeezing techniques.

The average number of particles in a squeezed vacuum state is  $\langle n \rangle = \sinh^2(r)$ , with its energy expectation value being  $\sinh^2(r)\hbar\omega$  [39]. It should be mentioned that, even though figures 2a and 2b demonstrate that the desired sensitivity can be achieved for  $r \approx 10$  and  $r \approx 15$ , depending on the size of the condensate, we note that this results in  $\langle n \rangle \approx 10^8$  and  $\langle n \rangle \approx 2 \times 10^{12}$ , respectively. In fact, for  $r \approx 27$ , which is the maximum squeezing possible before our model breaks down, we have  $\langle n \rangle \approx 7 \times 10^{22}$ . Using current techniques, these expectation values exceed the number of particles that can be confined in rubidium BECs, which is on the order of  $2 \times 10^7$  [40], though  $10^9$  particles were confined in a hydrogen BEC [34]. If such values of squeezing were achieved without also increasing the number of particles present in a condensate, additional effects may manifest themselves that might act to decrease the sensitivity to gravitational waves. However, in the case of figure 2b, an increase in the length of the condensate will probably also correspond to an increase in the number of particles present in the BEC, thereby negating this problem.

## 4.2 Decoherence (damping) from interacting modes

Let us now look at the maximum value of  $\tau$ . As derived in [41], in the case that  $r_0 > \max[\mu_0/\mu_\infty, \mu_\infty/\mu_0]$  the decoherence time of a BEC is

$$t_d = \frac{1}{\gamma_B} \ln \left( \frac{\frac{\mu_0}{\mu_\infty} + \frac{\mu_\infty}{\mu_0} - 2 \cosh(2r_0)}{\frac{\mu_0}{\mu_\infty} - \cosh(2r_0)} \right), \quad (4.1)$$

where  $\mu_0$  is the initial purity,  $\mu_\infty$  is the purity as  $t \rightarrow \infty$ ,  $r_0$  is the initial squeezing (which will decay over time), and  $\gamma_B$  quantifies the damping rate. For low temperatures, Beliaev damping is dominant and at zero-temperature is given by [42]

$$\gamma_B \approx \frac{3}{640\pi} \frac{\hbar \omega_{\vec{k}}^5}{m n c_s^5}, \quad (4.2)$$

where  $\omega_{\vec{k}}$  is the frequency of the single phonon mode  $\vec{k}$ ,  $m$  is the mass of the atoms making up the BEC, and  $n$  is the number density. For simplicity, consider the case in which  $r_0$  is large enough such that  $\ln \left( \frac{\frac{\mu_0}{\mu_\infty} + \frac{\mu_\infty}{\mu_0} - 2 \cosh(2r_0)}{\frac{\mu_0}{\mu_\infty} - \cosh(2r_0)} \right) \sim \mathcal{O}(1)$ . From equation (3.15), note that phonons with a frequency  $\omega_{\vec{k}} \approx \frac{\Omega}{2}$  are most important for the gravitational wave detection. In this case, sensitivity to gravitational waves is maximized when our measurement duration is in the range  $\frac{2\pi}{\Omega} \leq \tau \lesssim \frac{1}{\gamma_B}$ . Taking our BEC to have  $c_s = 1.2 \times 10^{-2}$  m/s,  $n = 7 \times 10^{20}$  m<sup>-3</sup>,  $m = 10^{-25}$  kg, our sensitivity to gravitational waves is optimal when  $\frac{2\pi}{\Omega} \leq \tau \lesssim (3.6 \times 10^2 \text{ sec}) \left( \frac{f}{2\pi \cdot 10^3 \text{ Hz}} \right)^{-5}$ . For a 10 kHz gravitational wave, we find  $t_d \approx 3.62$  s, which is greater than the period of the gravitational wave.

We can analyze decoherence effects more rigorously by noting that the squeezing parameter evolves in time as [41]

$$\cosh[2r(t)] = \mu(t) \left( e^{-\gamma_B t} \frac{\cosh(2r_0)}{\mu_0} + \frac{1 - e^{-\gamma_B t}}{\mu_\infty} \right), \quad (4.3)$$

where

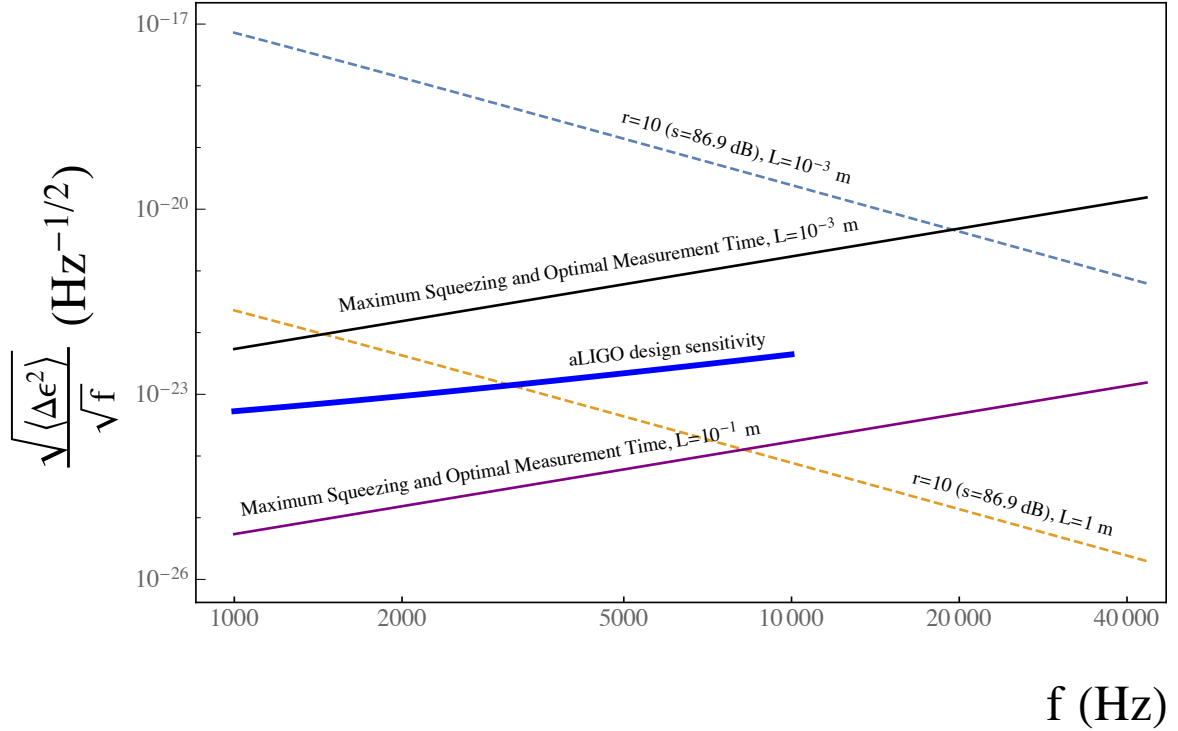
$$\mu(t) = \mu_0 \left( e^{-2\gamma_B t} + \frac{\mu_0^2}{\mu_\infty^2} (1 - e^{-\gamma_B t})^2 + 2 \frac{\mu_0}{\mu_\infty} e^{-\gamma_B t} (1 - e^{-\gamma_B t}) \cosh(2r_0) \right)^{-1/2} \quad (4.4)$$

is the purity. We will now determine the measurement time of  $\tau = t$  in order to maximize the sensitivity to gravitational waves. Taking  $e^r, e^{r_0} \gg 1$  and  $\mu_0 = \mu_\infty = 1$ , we note that equation (4.3) behaves as

$$e^{2r} \sim \frac{e^{-\gamma_B \tau} e^{2r_0} + 1 - e^{-\gamma_B \tau}}{\sqrt{e^{-2\gamma_B \tau} + (1 - e^{-\gamma_B \tau})^2 + 2e^{-\gamma_B \tau} (1 - e^{-\gamma_B \tau}) e^{2r_0}}}. \quad (4.5)$$

We see that the squeezing decays from  $e^{2r_0} \rightarrow e^{r_0}$  on a time-scale of

$$\tau \sim -\frac{1}{\gamma_B} \ln \left[ \frac{-\sqrt{e^{2r_0} - e^{4r_0} - e^{6r_0} + e^{8r_0}} + 2e^{2r_0} - e^{4r_0} - 1}{4e^{2r_0} - 3e^{4r_0} - 1} \right] \sim \frac{2}{5\gamma_B} \quad (4.6)$$



**Figure 3.** The theoretical sensitivity of a BEC to gravitational waves, with  $\tau$  optimized and  $r_0$  maximized within the validity of the model (solid black and purple lines), including decoherence due to interacting modes. Over this frequency range, the measurement duration corresponds to  $\tau \sim \frac{1}{\omega}$ . We have set  $t_{\text{obs}} = 10^6$  s,  $m = 10^{-25}$  kg,  $n = 7 \times 10^{20} \text{ m}^{-3}$ ,  $\mu_0 = \mu_\infty = 1$ ,  $c_s = 1.2 \times 10^{-2} \text{ m/s}$ , and  $\phi = \pi/2$ . To facilitate comparison, we have also considered  $r = 10$  for two different condensate lengths, assuming damping to be negligible (dashed blue and orange lines), where we have  $\tau = 10^{-3}$  s in both cases. The overall trend of the design sensitivity of aLIGO is indicated by the thick blue line (extrapolated to  $10^4$  Hz) [37, 38].

Consider phonons at a frequency of  $\frac{\Omega}{2}$ . For  $\tau \approx \frac{2}{5\gamma_B}$  and large  $e^r, e^{r_0}$ , we can combine equations (4.5) and (3.13) to find

$$\frac{1}{\langle (\Delta \tilde{\epsilon})^2 \rangle} \sim \tau^4 e^{4r_0} e^{-2\gamma_B \tau}, \quad (4.7)$$

which is maximized at  $\tau = \frac{2}{\gamma_B}$ .

Let us now consider more formally how decoherence could affect the sensitivity to gravitational waves. We can incorporate decoherence into (3.13) with equation (4.3) by letting  $r \rightarrow r(t)$  and following the same steps that were used to arrive at equation (3.16). We also note from [43, 44] that the purity divides the covariance matrix, equation (3.6), so  $\mu(t)$  also multiplies equation (3.13). By integrating over all the modes, we can then determine an equation analogous to (3.16), such that  $\langle \Delta \epsilon^2 \rangle_{\text{tot}}$  now includes effects arising from decoherence. We note that this is only an approximation; decoherence should, strictly speaking, be introduced prior to solving equation (2.11).

In figure 3, we consider the maximum value of the squeezing parameter from equation (3.25) and the optimal measurement duration to maximize the sensitivity to gravitational waves in the kHz range. As we are considering all modes, in this regime  $\tau \approx \frac{1}{\omega}$  (verified numerically). We see that, because of decoherence, the sensitivity decreases for higher frequencies.

It will be necessary to constantly regenerate the BEC [40, 45] in order to repeatedly perform measurements over the  $t_{\text{obs}} = 10^6$  s. With such a BEC machine, it will be possible to generate multiple BECs simultaneously. In equation (3.16), we assumed that the number of measurements was  $M \sim t_{\text{obs}}/\tau$ . For  $N$  BECs, the sensitivity to gravitational waves will improve by  $1/\sqrt{N}$ . To obtain a sensitivity  $\frac{\sqrt{\langle \Delta \epsilon^2 \rangle}}{\sqrt{f}} \approx 10^{-23} \text{ Hz}^{-1/2}$  to a gravitational wave of frequency  $f = 10^4 \text{ Hz}$  using the parameters in figure 1, we would therefore require  $\mathcal{O}(10^{22})$  BECs, which is impractical. This number is commensurate with other work [46], though an alternative detection scheme has been argued to be possible [47]. Improved techniques of increasing the squeezing of the phonons and increasing the volume of the condensate will be necessary for a BEC kHz gravitational wave detector to be achievable.

## 5 Conclusion

We have investigated the feasibility of using a BEC as a gravitational wave detector by modelling the wave as  $h_+ = e^{-t^2/\tau^2} \sin(\Omega t)$ , where the exponential prefactor is included to model the measurement duration. We have derived an analytic expression for the mean-square error in the amplitude of the gravitational wave, which depends on the squeezing of the phonons in the BEC, volume of the BEC, speed of sound, and frequency of the gravitational wave. Turning to a consideration of currently available techniques to improve sensitivity within the linear dispersion regime, we find that a BEC constructed using the best possible parameters to maximize the sensitivity will be unable to detect gravitational waves in the kHz range, in contrast to the work of [9]. A full analysis would require making use of the non-linear dispersion relation in appendix A.

Though a BEC as a gravitational wave detector is currently not feasible for observing kHz gravitational waves, it could be a promising method for observing waves at this frequency once it is understood how to increase phonon squeezing as well as the volumes of BECs. In the meantime, it will be productive to analyze how properties of the BEC can be optimized to improve sensitivity to gravitational waves by investigating different trap geometries and understanding the effects of vortices and inhomogeneities on the sensitivity. Furthermore, different trapping and optical potentials should be examined as this could have ramifications for the amount of squeezing that can be done and the speed of the phonons. The introduction of additional effects into the BEC may result in a way of using a BEC to detect gravitational waves without resorting to such large values of the squeezing parameter.

While here we focused on BECs at zero temperature, finite temperature effects can further affect the metrology and decoherence (through Landau damping), and will be studied in future work. Furthermore, we have considered a homogeneous BEC. Recent work [48] has emphasized the importance of using inhomogeneous BEC condensates, since they scale with the number of condensate atoms instead of the number of phonons as in the homogeneous case. It would be of interest to extend our work to the inhomogeneous case to see how to better optimize detection. It is only once these questions are answered that it will be possible to rival the sensitivities of LIGO (and its successors) for kHz gravitational waves.

## A Calculation of the BEC's dispersion relation

Let us now determine the general dispersion relation for the BEC. Inserting

$$\phi = \sqrt{\frac{\kappa^2 - m^2}{2\lambda}} \exp[i(\kappa t + \pi) + \sigma], \quad (\text{A.1})$$

(with real  $\sigma$  and  $\pi$ ) into equation (2.1), we find the Lagrangian to be

$$\mathcal{L} = \frac{(\kappa^2 - m^2)}{2\lambda} e^{2\sigma} [\partial_\mu \sigma \partial^\mu \sigma - (\nabla \pi)^2 + (\dot{\pi} + \kappa)^2] - \frac{m^2(\kappa^2 - m^2)e^{2\sigma}}{2\lambda} - \frac{(\kappa^2 - m^2)^2 e^{4\sigma}}{4\lambda}, \quad (\text{A.2})$$

where

$$2\sigma = \ln \left[ \frac{(\kappa + \dot{\pi})^2 - (\nabla \pi)^2 - m^2}{\kappa^2 - m^2} \right]. \quad (\text{A.3})$$

Writing  $\sigma$  and  $\pi$  in terms of their inverse Fourier transforms,

$$\sigma = \int \hat{\sigma}(\omega, \vec{k}) e^{i(k \cdot x - \omega t)} d\omega d^3k, \quad (\text{A.4})$$

$$\pi = \int \hat{\pi}(\omega, \vec{k}) e^{i(k \cdot x - \omega t)} d\omega d^3k, \quad (\text{A.5})$$

we find

$$\frac{2\lambda}{\kappa^2 - m^2} \mathcal{L}_2 = (\hat{\pi}^* \ \hat{\sigma}^*) \begin{pmatrix} \omega^2 - k^2 + 2(m^2 - \kappa^2) & -2i\kappa\omega \\ 2i\kappa\omega & \omega^2 - k^2 \end{pmatrix} \begin{pmatrix} \hat{\pi} \\ \hat{\sigma} \end{pmatrix} \quad (\text{A.6})$$

for the quadratic term in the Lagrangian.

Setting the determinant of this matrix equal to zero and solving for  $\omega$  gives the dispersion relation,

$$\omega^2 = k^2 - m^2 + 3\kappa^2 \pm \sqrt{m^4 + 4k^2\kappa^2 - 6m^2\kappa^2 + 9\kappa^4}. \quad (\text{A.7})$$

From equation (2.8), we can write

$$\kappa = \frac{\sqrt{1 - c_s^2}}{\sqrt{1 - 3c_s^2}} m. \quad (\text{A.8})$$

Requiring  $k \ll m$  and  $c_s \ll 1$ , equation (A.7) (for the minus sign, associated with the low-frequency Goldstone mode) can be expanded in  $k$  and  $c_s$  to find

$$\omega^2 = c_s^2 k^2 + \mathcal{O}\left(\frac{k^4}{m^2}\right). \quad (\text{A.9})$$

Therefore  $\omega \approx c_s k$  (indicating that  $c_s$  does represent the speed of sound), as long as  $\omega \ll \mu \equiv mc_s^2$ , i.e. frequency is much smaller than the chemical potential  $\mu$ .

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## References

- [1] LIGO SCIENTIFIC and VIRGO collaborations, *Observation of Gravitational Waves from a Binary Black Hole Merger*, *Phys. Rev. Lett.* **116** (2016) 061102 [[arXiv:1602.03837](#)] [[INSPIRE](#)].
- [2] C.J. Moore, R.H. Cole and C.P.L. Berry, *Gravitational-wave sensitivity curves*, *Class. Quant. Grav.* **32** (2015) 015014 [[arXiv:1408.0740](#)] [[INSPIRE](#)].
- [3] L. Stella, S. Dall’Osso, G. Israel and A. Vecchio, *Gravitational radiation from newborn magnetars*, *Astrophys. J.* **634** (2005) L165 [[astro-ph/0511068](#)] [[INSPIRE](#)].
- [4] N. Andersson, *Gravitational waves from instabilities in relativistic stars*, *Class. Quant. Grav.* **20** (2003) R105 [[astro-ph/0211057](#)] [[INSPIRE](#)].
- [5] N. Andersson et al., *Gravitational waves from neutron stars: Promises and challenges*, *Gen. Rel. Grav.* **43** (2011) 409 [[arXiv:0912.0384](#)] [[INSPIRE](#)].
- [6] N. Andersson and K.D. Kokkotas, *Towards gravitational wave asteroseismology*, *Mon. Not. Roy. Astron. Soc.* **299** (1998) 1059 [[gr-qc/9711088](#)] [[INSPIRE](#)].
- [7] S. Dimopoulos, P.W. Graham, J.M. Hogan, M.A. Kasevich and S. Rajendran, *Gravitational Wave Detection with Atom Interferometry*, *Phys. Lett. B* **678** (2009) 37 [[arXiv:0712.1250](#)] [[INSPIRE](#)].
- [8] J.M. Hogan et al., *An Atomic Gravitational Wave Interferometric Sensor in Low Earth Orbit (AGIS-LEO)*, *Gen. Rel. Grav.* **43** (2011) 1953 [[arXiv:1009.2702](#)] [[INSPIRE](#)].
- [9] C. Sabín, D.E. Bruschi, M. Ahmadi and I. Fuentes, *Phonon creation by gravitational waves*, *New J. Phys.* **16** (2014) 085003 [[arXiv:1402.7009](#)] [[INSPIRE](#)].
- [10] S.L. Braunstein and C.M. Caves, *Statistical distance and the geometry of quantum states*, *Phys. Rev. Lett.* **72** (1994) 3439 [[INSPIRE](#)].
- [11] A. Nicolis, R. Penco and R.A. Rosen, *Relativistic Fluids, Superfluids, Solids and Supersolids from a Coset Construction*, *Phys. Rev. D* **89** (2014) 045002 [[arXiv:1307.0517](#)] [[INSPIRE](#)].
- [12] S. Fagnocchi, S. Finazzi, S. Liberati, M. Kormos and A. Trombettoni, *Relativistic Bose-Einstein condensates: a new system for analogue models of gravity*, *New J. Phys.* **12** (2010) 095012.
- [13] T.D. Lee, K. Huang and C.N. Yang, *Eigenvalues and Eigenfunctions of a Bose System of Hard Spheres and Its Low-Temperature Properties*, *Phys. Rev.* **106** (1957) 1135 [[INSPIRE](#)].
- [14] D.T. Son, *Low-energy quantum effective action for relativistic superfluids*, [hep-ph/0204199](#) [[INSPIRE](#)].
- [15] V. Giovannetti, S. Lloyd and L. Maccone, *Quantum metrology*, *Phys. Rev. Lett.* **96** (2006) 010401.
- [16] R. Demkowicz-Dobrzański, J. Kołodyński and M. Guţă, *The elusive Heisenberg limit in quantum-enhanced metrology*, *Nature Commun.* **3** (2012) 1063.
- [17] V. Giovannetti, S. Lloyd and L. Maccone, *Advances in quantum metrology*, *Nature Photon.* **5** (2011) 222.
- [18] A. Uhlmann, *The “transition probability” in the state space of a \*-algebra*, *Rept. Math. Phys.* **9** (1976) 273.
- [19] R. Jozsa, *Fidelity for mixed quantum states*, *J. Mod. Opt.* **41** (1994) 2315.
- [20] M. Ahmadi, D.E. Bruschi and I. Fuentes, *Quantum metrology for relativistic quantum fields*, *Phys. Rev. D* **89** (2014) 065028 [[arXiv:1312.5707](#)] [[INSPIRE](#)].
- [21] P. Marian and T.A. Marian, *Uhlmann fidelity between two-mode Gaussian states*, *Phys. Rev. A* **86** (2012) 022340.
- [22] A. Ferraro, S. Olivares and M.G.A. Paris, *Gaussian states in continuous variable quantum information*, Napoli series on physics and astrophysics, Bibliopolis (2005) [[quant-ph/0503237](#)].



- [23] S. Chelkowski et al., *Experimental characterization of frequency-dependent squeezed light*, *Phys. Rev. A* **71** (2005) 013806 [INSPIRE].
- [24] M.T. Johnsson, G.R. Dennis and J.J. Hope, *Squeezing in Bose-Einstein condensates with large numbers of atoms*, *New J. Phys.* **15** (2013) 123024.
- [25] W. Gu, G. Li, S. Wu and Y. Yang, *Generation of non-classical states of mirror motion in the single-photon strong-coupling regime*, *Opt. Express* **22** (2014) 18254.
- [26] A.I. Lvovsky, *Squeezed light*, in *Photonics, Volume 1: Fundamentals of Photonics and Physics*, chapter 5, Wiley (2015) [arXiv:1401.4118] [INSPIRE].
- [27] X. Hu and F. Nori, *Phonon squeezed states generated by second-order Raman scattering*, *Phys. Rev. Lett.* **79** (1997) 4605.
- [28] G.A. Garrett, A.G. Rojo, A.K. Sood, J.F. Whitaker and R. Merlin, *Vacuum squeezing of solids: Macroscopic quantum states driven by light pulses*, *Science* **275** (1997) 1638.
- [29] D. Jaksch, C. Bruder, J.I. Cirac, C.W. Gardiner and P. Zoller, *Cold Bosonic Atoms in Optical Lattices*, *Phys. Rev. Lett.* **81** (1998) 3108 [cond-mat/9805329] [INSPIRE].
- [30] F. Benatti, M. Esposito, D. Fausti, R. Floreanini, K. Titimbo and K. Zimmermann, *Generation and detection of squeezed phonons in lattice dynamics by ultrafast optical excitations*, *New J. Phys.* **19** (2017) 023032.
- [31] O. Hosten, N.J. Engelsen, R. Krishnakumar and M.A. Kasevich, *Measurement noise 100 times lower than the quantum-projection limit using entangled atoms*, *Nature* **529** (2016) 505.
- [32] M. Vengalattore, J.M. Higbie, S.R. Leslie, J. Guzman, L.E. Sadler and D.M. Stamper-Kurn, *High-resolution magnetometry with a spinor Bose-Einstein condensate*, *Phys. Rev. Lett.* **98** (2007) 200801.
- [33] I. Barr, *Investigating the dynamics of a Bose Einstein condensate on an atom chip*, Ph.D. Thesis, Imperial College London (2015).
- [34] T. Greytak et al., *Bose-Einstein condensation in atomic hydrogen*, *Physica B* **280** (2000) 20.
- [35] M.R. Andrews et al., *Propagation of Sound in a Bose-Einstein Condensate*, *Phys. Rev. Lett.* **79** (1997) 553 [INSPIRE].
- [36] M.R. Andrews et al., *Erratum: Propagation of sound in a Bose-Einstein condensate [Phys. Rev. Lett. 79, 553 (1997)]*, *Phys. Rev. Lett.* **80** (1998) 2967.
- [37] LIGO SCIENTIFIC collaboration, *Advanced LIGO: The next generation of gravitational wave detectors*, *Class. Quant. Grav.* **27** (2010) 084006 [INSPIRE].
- [38] M. Pitkin, S. Reid, S. Rowan and J. Hough, *Gravitational Wave Detection by Interferometry (Ground and Space)*, *Living Rev. Rel.* **14** (2011) 5 [arXiv:1102.3355] [INSPIRE].
- [39] S. Barnett and P. Radmore, *Methods in Theoretical Quantum Optics*, Oxford Series in Optical and Imaging Sciences, Clarendon Press, (2002).
- [40] E.W. Streed et al., *Large atom number Bose-Einstein condensate machines*, *Rev. Sci. Instrum.* **77** (2006) 023106.
- [41] A. Serafini, M.G.A. Paris, F. Illuminati and S.D. Siena, *Quantifying decoherence in continuous variable systems*, *J. Opt.* **B 7** (2005) R19.
- [42] S. Giorgini, *Damping in dilute Bose gases: A mean-field approach*, *Phys. Rev. A* **57** (1998) 2949.
- [43] S. Olivares, *Quantum optics in the phase space*, *Eur. Phys. J. ST* **203** (2012) 3.
- [44] R. Howl, C. Sabín, L. Hackermüller and I. Fuentes, *Quantum decoherence of phonons in Bose-Einstein condensates*, *J. Phys. B* **51** (2018) 015303.
- [45] T.G. Tiecke, S.D. Gensemer, A. Ludewig and J.T.M. Walraven, *High-flux two-dimensional magneto-optical-trap source for cold lithium atoms*, *Phys. Rev. A* **80** (2009) 013409.

- [46] R. Schützhold, *Reply to comment on “Interaction of a BEC with a gravitational wave”*, [arXiv:1901.08999](#) [INSPIRE].
- [47] R. Howl, D. Rätzel and I. Fuentes, *Comment on “Interaction of a Bose-Einstein condensate with a gravitational wave”*, [arXiv:1811.10306](#) [INSPIRE].
- [48] R. Schützhold, *Interaction of a Bose-Einstein condensate with a gravitational wave*, *Phys. Rev. D* **98** (2018) 105019 [[arXiv:1807.07046](#)] [INSPIRE].