

Minimally modified gravity: a Hamiltonian construction

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Received May 21, 2019

Revised July 4, 2019

Accepted July 16, 2019

Published July 31, 2019

Abstract. Minimally modified gravity theories are modifications of general relativity with two local gravitational degrees of freedom in four dimensions. Their construction relies on the breaking of 4D diffeomorphism invariance keeping however the symmetry under 3D diffeomorphisms. Here, we construct these theories from the Hamiltonian point of view. We start with the phase space of general relativity in the ADM formalism. Then, we find the conditions that the Hamiltonian must satisfy for the theory to propagate (up to) two local gravitational degrees of freedom with the assumptions that the lapse and the shift are not dynamical, and that the theory remains invariant under 3D diffeomorphisms. This construction enables us to recover the well-known “cuscuton” class of scalar-tensor theories in the unitary gauge. We also exhibit a new class of interesting theories, that we dub $f(\mathcal{H})$ theories, where the usual Hamiltonian constraint \mathcal{H} of general relativity is replaced by $f(\mathcal{H})$ where f is an arbitrary function.

Keywords: dark energy theory, modified gravity

ArXiv ePrint: [1905.02000](https://arxiv.org/abs/1905.02000)

Contents

1	Introduction	1
2	Minimally modified Maxwell theory	4
2.1	Framework: symmetry breaking and degeneracy	4
2.2	Killing the extra degrees of freedom	5
2.3	Complete Hamiltonian description	7
2.4	Example: quadratic theories	8
2.4.1	General Hamiltonian analysis	9
2.4.2	Lagrangian	10
2.4.3	Modified gauge invariant Maxwell theories	11
3	Generalization to gravity	12
3.1	The modified phase space	12
3.2	Simple examples: \mathcal{H}_0 is the Hamiltonian constraint and \mathcal{V} is polynomial in π	14
3.3	A new class of theories: $f(\mathcal{H})$ theories	15
3.3.1	Hamilton equations	16
3.3.2	Cosmology	18
4	Conclusion	20

1 Introduction

A century after its discovery, the theory of general relativity continues to pass all validity tests. The latest is the fabulous detection of gravitational waves emitted by a neutron star merger [1] with the first measure of their propagation speed which is probably the same as the speed of light in vacuum, as predicted by Einstein. Despite all these successes, the reasons for believing that general relativity is not the ultimate theory of space-time and that it will have to be surpassed are numerous and so interesting that modifying gravity has become a very dynamical field of research per se in theoretical physics and cosmology these last years (see [2, 3] for example).

Going beyond relativity necessitates the relaxation of one of the fundamental hypothesis of the Lovelock theorem that makes Einstein theory unique: invariance under diffeomorphisms, locality, pure metric formulation in four space-time dimensions.

For example, four-dimensional massive gravity [4] gives up the invariance under diffeomorphisms, and scalar-tensor theories are based on the hypothesis that the metric is accompanied by a scalar degree of freedom for describing the physics of the spacetime at least on very large or very short scales. There exist many modified theories of gravity, and most of them (exactly like massive gravity or scalar-tensor theories) often share the property that one or more additional degree(s) of freedom propagate in the theory. When such theories are designed to account for dark energy for example, the extra degrees of freedom are responsible for the fifth force that accelerates the expansion of the universe. When constructed to solve the well-known problems of general relativity in the ultraviolet, these degrees of freedom play

the role of rendering the theory renormalizable and then eventually quantifying [5, 6]. Therefore, there may be a general opinion that gravity can not be changed without introducing at least a new degree of freedom in the scenario, in addition to the two massless degrees of freedom of the gravitational field.

In this spirit, scalar-tensor theories are sometimes considered as the “simplest” theories of modified gravity because they come with one extra degree of freedom only. In recent years, they have been at the heart of intense activity and scalar-tensor theories, whose actions involve up to second derivatives of the scalar field, have been systematically classified and extensively studied [7–20]. Adding higher order derivatives in a Lagrangian is potentially very dangerous because it could lead to the situation that not one but two scalars

in the theory, one of them being the Ostrogradski ghost. Degeneracy conditions [13] in a higher order scalar-tensor theory insure that at most three degrees of freedom propagate, but the theory must be studied in more detail to see whether these degrees of freedom are safe or not. Furthermore, it has been realized that degeneracy conditions in the unitary gauge (where the scalar field is fixed to be a function of time only) are sufficient to ensure that a unique scalar propagates in addition to the usual two tensor modes [21]. There exists the possibility that only two tensorial degrees of freedom propagate in a scalar-tensor theory which is, of course, different from gravity (the scalar mode is in fact shadowy in the sense of [21]). They form the class of “cuscuton” theories [22, 23]. These theories are particularly interesting and they can be considered as minimal modifications of general relativity.

A systematic construction of gravitational theories with only (up to) two degrees of freedom was initiated in [24, 25]. They were also considered in [26] where the coupling to matter has been discussed. The idea consists of renouncing to the invariance under four dimensional diffeomorphisms but keeping the three dimensional diff-invariance. This is equivalent to considering scalar-tensor theories in the unitary gauge. As generically Lorentz-breaking gravity have more than two degrees of freedom, one has to find the conditions for the theory to possess enough constraints that would kill the extra degrees of freedom, which would leave us with (at most) two gravitational degrees of freedom. More precisely, one starts with the ADM parametrization of the metric

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (1.1)$$

where N , N^i and h_{ij} are respectively the lapse function, the shift vector and the induced spatial metric. Then, one considers general actions of the form

$$S[N, N^i, h_{ij}] = \int d^3x dt \sqrt{h} \mathcal{L}(K_{ij}, R_{ij}, h^{ij}, N, \nabla_i), \quad (1.2)$$

where K_{ij} is the extrinsic curvature, R_{ij} the three-dimensional curvature and ∇_i the spatial covariant derivative. And finally, one performs a Hamiltonian analysis to find the necessary conditions for the theory to propagate (at most) two degrees of freedom. This program was completed in the case where the Lagrangian (1.2) was supposed to be linear in the lapse function [24]. In that way, one found a large class of modified theory of gravity with only two degrees of freedom that have been dubbed for obvious reasons “minimally modified gravity”.

In this paper, we construct minimally modified gravity theories from the Hamiltonian point of view with the idea that the Hamiltonian framework is more suited for studying and classifying Lorentz breaking theories than the Lagrangian framework. Indeed, we modify the phase space of general relativity (and not directly the Lagrangian) in such a way that the modified theory remains invariant under spatial diffeomorphisms only and still propagates

two tensorial degrees of freedom. More precisely, we start with a phase space which is parametrized by the usual ten pairs of conjugate variables (the metric variables in the ADM decomposition and their momenta), and we consider a “modified” Hamiltonian of the form

$$H = \int d^3x \sqrt{h} [\mathcal{H}(\pi^{ij}, R_{ij}, h^{ij}, N, \nabla_i) + N^i \mathcal{V}_i], \quad (1.3)$$

where \mathcal{V}_i is the usual vectorial constraint, and \mathcal{H} is a three dimensional diff-invariant function which is a priori different from the usual scalar constraint. Then, the problem consists of finding the conditions that \mathcal{H} must satisfy for the theory to propagate two (or less) degrees of freedom. We address this issue and find that \mathcal{H} must be an affine function of the lapse, of the form

$$\mathcal{H} = N \mathcal{H}_0(\pi^{ij}, R_{ij}, h^{ij}, \nabla_i) + \mathcal{V}(\pi^{ij}, R_{ij}, h^{ij}, \nabla_i), \quad (1.4)$$

with additional conditions on the functions \mathcal{H}_0 and \mathcal{V} . A necessary condition is that $\{\mathcal{H}_0(x), \mathcal{H}_0(y)\}$, viewed as an operator acting on the space of functions $\text{Fun}(M)$ on the space manifold by integration, has a non-trivial kernel, and a sufficient condition is that

$$\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} \approx 0, \quad (1.5)$$

where \approx means weakly vanishing (i.e. it vanishes up to constraints). In this construction, we recover the well-known class of “cuscuton” theories that can be extended to non-local theories. But we also find new classes of theories. In particular, we exhibit a remarkably simple class of theories with $\mathcal{H}_0 = f(\mathcal{H}_{\text{gr}})$ where f is an arbitrary function and \mathcal{H}_{gr} is the usual scalar constraint of general relativity. Such theories are invariant under a four-dimensional local symmetry (which contains the 3D diffeomorphisms) and possess very interesting properties that we discuss in the paper.

The rest of the paper is organized as follows. We start, in section 2 with the simpler case of a spin-1 field to illustrate our construction. Hence, we construct modified Maxwell theory in a four dimensional Minkowski space-time, where the dynamical variable is a one form A_μ . To mimic the construction of minimally modified theories of gravity, we relax some hypothesis which makes Maxwell theory unique: we break the U(1) gauge symmetry and also the global Lorentz invariance keeping, however, a symmetry under one rotational subgroup SO(3) (the one that leaves A_0 invariant). Then, we modify the Maxwell Hamiltonian and find the conditions for the new theory to propagate only (up to) two degrees of freedom. Finally, we give some concrete examples. In section 3, we turn to the more interesting case of minimally modified gravities. We write conditions that the modified Hamiltonian constraint (1.3) must satisfy to have (up to) two tensorial degrees of freedom. These conditions (1.4) and (1.5) appear to be very simple in the Hamiltonian framework, and they can be explicitly solved in some cases. As we mentioned previously, we recover the cuscuton theories, and we find an interesting and remarkably simple new class of theories, dubbed $f(\mathcal{H})$ theories, where the usual Hamiltonian constraint of general relativity \mathcal{H}_{gr} has been replaced by $f(\mathcal{H}_{\text{gr}})$ where f is an arbitrary function. We quickly study their cosmology to show interesting differences with general relativity. We conclude with a brief summary of our results and some perspectives.

2 Minimally modified Maxwell theory

Following the ideas that lead to the construction of minimally modified gravity theories, we build, in this section, a large class of modified Maxwell theories which propagates 2 (vectorial) degrees of freedom in the 4-dimensional Minkowski space-time. Maxwell theory provides us with a simpler but very interesting context to illustrate the construction of minimally modified gravity theories from a Hamiltonian point of view that we will present in section 3.

2.1 Framework: symmetry breaking and degeneracy

Maxwell theory is the unique free action for a U(1) connection A_μ evolving in a Minkowski space-time, which is invariant under the usual U(1) gauge symmetry, also invariant under the global Lorentz symmetry (i.e. the isometry group of the Minkowski metric SO(1,3)), and which in addition produces (at most) second order equations of motion. The U(1) invariance implies that the action is a functional of the curvature two-form only

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.1)$$

The global Lorentz symmetry implies that the curvature components must be contracted with the metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ (and its inverse) such that the Lagrangian density is a scalar for the Lorentz group. Finally, the freeness of the theory says that the action is at most quadratic in the connection. Hence the only possible theory is described by the action (in vacuum)

$$S[A_\mu] = -\frac{1}{4\mu_0} \int d^4x F_{\mu\nu} F^{\mu\nu}, \quad (2.2)$$

where μ_0 is the usual permeability, and indices are raised with $\eta^{\mu\nu}$. A simple analysis shows that this very well-known theory propagates only 2 degrees of freedom which are the 2 (transverse) photons. Generalizing the action to any space-time is straightforward.

In order to mimic the construction of minimally modified theories of gravity, we relax some of the conditions that make Maxwell theory unique. In minimally modified gravity theories, one breaks the full space-time diffeomorphism invariance and keep only symmetry under three dimensional diffeomorphisms. In the case of Maxwell theory, there is only the one-dimensional local symmetry group U(1) that we choose to break, and then there is no remaining local symmetry in the theory. However, to be close to the gravity case, we also decide to break the global Lorentz symmetry keeping only the invariance under the subgroup of rotations SO(3) that leaves A_0 invariant. In that sense, A_0 is similar to the lapse function in the context of Maxwell theory. As a consequence, we look for theories whose action is of the form

$$S[A_0, A_i] = \int d^4x \mathcal{L}(A_0, \dot{A}_0, A_i, \dot{A}_i, \partial_i), \quad (2.3)$$

where \mathcal{L} is the Lagrangian density. In other words, \mathcal{L} is constructed from A_0 , A_i , their first time derivatives and their space derivatives at any order.

As we are going to see in a few lines, this theory propagates generically more than 2 degrees of freedom. To find the conditions for the theory to propagate only 2 degrees of freedom, we perform a Hamiltonian analysis. Hence, we start by introducing the phase-space variables

$$\{A_\mu(x), P^\nu(y)\} = \delta_\mu^\nu \delta^3(x - y). \quad (2.4)$$

If there is no constraints, the theory propagates 4 degrees of freedom. The presence of a primary constraint is then an obvious necessary condition for the theory to propagate only 2 (vectorial) degrees of freedom. The theory admits a primary constraint if its action is degenerate, i.e. the 4 dimensional Hessian matrix defined by

$$\mathbb{H}^{\mu\nu} \equiv \frac{\partial^2 \mathcal{L}}{\partial \dot{A}_\mu \partial \dot{A}_\nu}, \quad (2.5)$$

for $\mu, \nu \in \{0, 1, 2, 3\}$ is not invertible. Furthermore, as we want vector modes to propagate, we add the condition that the submatrix \mathbb{H}^{ij} , for $i, j \in \{1, 2, 3\}$, is invertible. If this is the case, we can formally reformulate the Lagrangian density in (2.3) as a function

$$\mathcal{L}(A_0, \dot{A}_0, A_i, \dot{A}_i, \partial_i) = \mathcal{F}(A_0, A_i, \dot{A}_i - \alpha_i \dot{A}_0, \partial_i), \quad (2.6)$$

where α_i depends on the connection A_μ and their spatial derivatives in general. In general (even when there is no coupling to external current) time derivatives of A_0 cannot be absorbed into a redefinition of A_i . But, for simplicity, we assume that A_0 is not a dynamical variable as in the original Maxwell theory, and then it does not appear a priori with time derivatives in the action, which means that $\alpha_i = 0$. In that case, the theory possesses the simple primary constraint¹

$$\mathcal{P} \equiv P^0 \approx 0, \quad (2.7)$$

where we recall that \approx means weakly vanishing,

At this stage, there is no more primary constraint (which is a consequence of the fact that \mathbb{H}^{ij} is not degenerate), and then one can (in principle) uniquely express (at least locally, on any open set of the phase space) the velocities \dot{A}_i in terms of the momenta P^i . As a consequence, one can construct (formally) the canonical and the total Hamiltonians, respectively given by

$$H = \int d^3x \mathcal{H}(A_\mu, P^i, \partial_i), \quad H_{\text{tot}} = H + \int d^3x \lambda P^0, \quad (2.8)$$

where λ is a Lagrange multiplier which enforces the primary constraint.

As we have already emphasized above, the relation between the Lagrangian and the canonical Hamiltonian is, in general, implicit. It can be made explicit in simple cases only, for free (quadratic) Lagrangians for instance. Furthermore, it will be much more convenient to find the conditions for the theory to propagate (at most) 2 degrees of freedom in its Hamiltonian formulation than in its Lagrangian formulation. For all these reasons, we will construct modified Hamiltonian Maxwell theories, and in some cases, we will show how to recover the associated Lagrangian.

2.2 Killing the extra degrees of freedom

From now on, the starting point is the Hamiltonian (2.8) together with the primary constraint (2.7). The stability under time evolution of the primary constraint leads to a secondary constraint

$$\mathcal{S} \equiv \{\mathcal{P}, H\} = \frac{\partial \mathcal{H}}{\partial A_0} - \partial_i \left(\frac{\partial \mathcal{H}}{\partial (\partial_i A_0)} \right) + \partial_i \partial_j \left(\frac{\partial \mathcal{H}}{\partial (\partial_i \partial_j A_0)} \right) + \dots \approx 0, \quad (2.9)$$

¹The generalization to a non-zero α_i is immediate and the primary constraint is replaced by the combination $\mathcal{P} \equiv P^0 + \alpha_i P^i \approx 0$.

when \mathcal{H} depends explicitly on A_0 . In the particular case where \mathcal{H} does not depend on A_0 (and on its spatial derivatives), then the Lagrangian itself does not depend on A_0 and the theory propagates 3 degrees of freedom. For this reason, we assume from now on that \mathcal{H} depends on (the spatial derivatives of) A_0 . To be more precise, we exclude the case where \mathcal{H} depends on A_0 and its spatial derivatives only through a total spatial derivative.

Even in that case, the theory could propagate up to 3 degrees of freedom (if there is no more constraint and if the two constraints are second class). To go further and to find the conditions on the Hamiltonian for the theory to propagate (at most) two degrees of freedom, we compute the Poisson bracket between the primary and the secondary constraints,

$$\Delta(x, y) \equiv \{ \mathcal{S}(x), \mathcal{P}(y) \}, \quad (2.10)$$

and one studies whether it (weakly) vanishes or not. Notice that we are using the shortened notations $F(x) = F(A_\mu(x), P^i(x), \partial_i)$ for any function F in the phase space.

First, we study the case where Δ is not weakly vanishing. There are no more constraints in the theory, and the pair $(\mathcal{P}, \mathcal{S})$ form a set of second class constraints. Hence, the theory propagates $[(2 \times 4) - 2]/2 = 3$ degrees of freedom, i.e. one more than Maxwell theory. The extra degree of freedom is the longitudinal mode which comes with the usual two polarizations of the graviton.

Now, we study the more interesting case where Δ is weakly vanishing. The number of degrees of freedom depends on whether the bracket $\Omega(x, y) \equiv \{ \mathcal{S}(x), \mathcal{H}(y) \}$ is vanishing or not. If Ω is weakly vanishing, the theory has no more constraints, the pair $(\mathcal{P}, \mathcal{S})$ forms a set of first class constraints, which means that there is a “hidden” local symmetry in the theory. Furthermore, the theory propagates $[(2 \times 4) - (2 \times 2)]/2 = 2$ degrees of freedom, as in the Maxwell theory. If Ω is not weakly vanishing, there is a tertiary constraint \mathcal{T} , but this may be not enough to insure that the theory propagates 2 degrees of freedom only. If one of the three constraints is first class (which is necessary the case if all the constraints are local), then the theory admits an extra symmetry and only 2 degrees of freedom. If this is not the case, one needs the presence of an extra quaternary constraint which would definitively imply that there is strictly less than 2 degrees of freedom.

As a consequence, in any cases, we see that a necessary condition for the theory to propagate 2 or less degrees of freedom is that

$$\{ \mathcal{P}(x), \{ \mathcal{P}(y), H \} \} \approx 0, \quad (2.11)$$

i.e. it vanishes up to terms proportional to \mathcal{S} . Let us make this condition more explicit, and show that it necessarily implies that $\mathcal{S} \equiv \{ \mathcal{P}, H_0 \}$ does not depend on A_0 . For that, let us assume the reverse is true, and then \mathcal{S} is supposed to depend at least on A_0 or on one of its spatial derivatives. Hence, the constraint $\mathcal{S}(A_0, \partial_i A_0, \dots) \approx 0$ can be viewed as a differential equation that we can solve for A_0 (with appropriate boundary conditions) in terms of the remaining phase space variables, at least formally. In that case, the secondary constraint can be (locally) replaced by the equivalent constraint

$$\tilde{\mathcal{S}} \equiv A_0 - \mathcal{A}_0(A_i, P^i, \partial_i) \approx 0, \quad (2.12)$$

where \mathcal{A}_0 is the explicit solution for A_0 . As a consequence, the new bracket between the constraints $\{ \tilde{\mathcal{S}}(x), \mathcal{P}(y) \} = \delta(x - y)$ is clearly non-vanishing, and then the theory propagates 3 degrees of freedom, which contradicts the initial assumption. As a consequence, the

condition (2.11) is (locally) equivalent to the condition that \mathcal{S} can be written as

$$\mathcal{S} = \nu(A_0) \mathcal{H}_0(A_i, P^i, \partial_i), \quad (2.13)$$

where \mathcal{H}_0 does not depend neither on A_0 nor on its derivatives, and ν is an arbitrary non-vanishing function of A_0 , say positive. Hence, the Hamiltonian density takes necessarily the form (up to a total spatial derivative)

$$\mathcal{H} = \mathcal{V} + N(A_0)\mathcal{H}_0, \quad (2.14)$$

where \mathcal{H}_0 and \mathcal{V} depends on A_i, P^i and their spatial derivatives only. The function N is an integral of ν , and then it is an increasing function of A_0 (as ν is supposed to be positive). Furthermore a simple canonical transformation allows us to fix (locally) $N(A_0) = A_0$ without loss of generality.

2.3 Complete Hamiltonian description

To summarize, we found that any Hamiltonian theory which satisfies the necessary condition (2.11) is defined (up to a canonical transformation) by a phase space parametrized by the 4 pairs of canonical variables (2.4) whose dynamics is governed by a Hamiltonian of the form

$$H = \int d^3x [\mathcal{V}(A_i, P^i, \partial_i) + A_0 \mathcal{H}_0(A_i, P^i, \partial_i)], \quad (2.15)$$

together with the primary constraint $\mathcal{P} \approx 0$ (2.7). Hence, the secondary constraint is now simply given by

$$\mathcal{S} \equiv \mathcal{H}_0(A_i, P^i, \partial_i) \approx 0. \quad (2.16)$$

The existence of this constraint implies immediately that the constraint $\mathcal{P} \approx 0$ is in fact first class, and it corresponds to the (on-shell) invariance of the theory under the arbitrary shift,

$$A_0 \mapsto A_0 + u, \quad (2.17)$$

of the non-dynamical variable A_0 , by an arbitrary function $u(x)$.

Requiring conservation under time evolution of the secondary constraint leads to the condition

$$\int d^3y (\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} A_0(y) + \{\mathcal{H}_0(x), \mathcal{V}(y)\}) \approx 0, \quad (2.18)$$

whose resolution depends on the properties of $\Delta(x, y) \equiv \{\mathcal{H}_0(x), \mathcal{H}_0(y)\}$ viewed as an operator acting on the space of functions $\text{Fun}(\mathbb{R}^3)$ by integration. When Δ is invertible, the condition (2.18) fixes completely the Lagrange multiplier A_0 in terms of the phase space variables. Furthermore, in that case, $\Delta(x, y)$ is necessary not scalar² (it involves derivatives of Dirac distributions) and the constraint on A_0 is in fact a partial differential equation which would need appropriate boundary conditions to be explicitly resolved. There is no quaternary constraint as the time evolution of $\mathcal{T} \approx 0$ fixes completely the Lagrange multiplier. Then the theory admits three secondary constraints with a non-scalar Dirac matrix and a

²A two-point distribution $F(x, y)$ is scalar if and only if $F(x, y) = F(x, 0)\delta(x - y)$ where δ is the Dirac distribution.

non-scalar Poisson bracket between \mathcal{P} and \mathcal{T} in particular. As a consequence, the theory is not well-posed.

The case where Δ is a non (weakly) vanishing operator with a non-trivial kernel is much more complicated to study. To understand this situation, it is convenient to decompose the space of functions on which Δ acts as the direct sum $\text{Fun}(\mathbb{R}^3) = \text{Im}(\Delta) \oplus \text{Ker}(\Delta)$ where $\text{Im}(\Delta)$ and $\text{Ker}(\Delta)$ are respectively the image and the kernel of Δ . Hence, the condition (2.18) not only fixes the component of A_0 in $\text{Im}(\Delta)$ but also can produce a new (quaternary) constraint obtained by projecting (2.18) into $\text{Ker}(\Delta)$. The new constraint may be non-scalar, the general Dirac analysis appears to be very subtil, and it should be done on a case-by-case basis. For that reason, we will exclusively consider the simpler case where Δ is weakly vanishing:

$$\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} \approx 0. \quad (2.19)$$

If this is the case, the conservation of the secondary constraint under time evolution leads either to a tertiary constraint

$$\mathcal{T}(x) \equiv \{\mathcal{H}_0(x), \int d^3y \mathcal{V}(y)\}, \quad (2.20)$$

or to no new constraint if \mathcal{T} is itself weakly vanishing. In any of these two cases, the theory propagates 2 degrees of freedom or less.

- Case where $\mathcal{T} \approx 0$ is automatically satisfied. The theory admits 2 first class constraints $\mathcal{P} \approx 0$ and $\mathcal{H}_0 \approx 0$. The constraint \mathcal{P} is associated to the (on-shell) symmetry described above (2.17), and the constraint \mathcal{S} generates a gauge symmetry acting on the phase space variables (A_i, P^i) , exactly as in Maxwell theory. As a result the theory propagates $[(2 \times 4) - (2 \times 2)]/2 = 2$ degrees of freedom.
- Case where $\mathcal{T} \approx 0$ is a new constraint, and \mathcal{T} does not commute with \mathcal{H}_0 . The Dirac analysis stops here with one first class constraint $\mathcal{P} \approx 0$ and two second class constraints $\mathcal{H}_0 \approx 0, \mathcal{T} \approx 0$, which lead to $[(2 \times 4) - (2 + 1 + 1)]/2 = 2$ degrees of freedom.
- Case where $\mathcal{T} \approx 0$ is a new constraint, and \mathcal{T} commutes with \mathcal{H}_0 . Either the Dirac analysis continues producing constraints, or \mathcal{T} and \mathcal{H}_0 are first class. In any case, the theory propagates 1 or 0 degree of freedom.

As a conclusion, any deformation of Maxwell theory which breaks the $U(1)$ symmetry, which is invariant under the global $SO(3)$ group that leaves A_0 invariant and which propagates at most 2 degrees of freedom has necessarily a Hamiltonian of the form (2.15). Furthermore, the condition (2.19) is sufficient to insure that the theory propagates at most 2 degrees of freedom, but it has not been rigorously proven that it is also necessary because the theory admits a quaternary (eventually non-local) constraint when Δ is non-vanishing with a non-trivial kernel.

2.4 Example: quadratic theories

Let us illustrate the previous analysis with a simple example. We consider a Hamiltonian which is, at most, quadratic in the phase space variables (A_0, A_i, P^i) .

2.4.1 General Hamiltonian analysis

Furthermore, we assume that the Hamiltonian can be written in terms of the fields and their first order (spatial) derivatives only. In that case \mathcal{H}_0 is linear in (A_i, P^i) whereas \mathcal{V} is quadratic in (A_i, P^i) . Hence, these two functions can be written as

$$\begin{aligned} \mathcal{V} = & \alpha_1 A^2 + \alpha_2 P^2 + \alpha_3 (AP) + \alpha_4 (\partial A)^2 + \alpha_5 (\partial P)^2 + \alpha_6 (\partial A)(\partial P) \\ & - \alpha_7 \partial_j A_i \partial^j A^i - \alpha_8 \partial_j P_i \partial^j P^i - \alpha_9 \partial_j A_i \partial^j P^i, \end{aligned} \quad (2.21)$$

$$\mathcal{H}_0 = \beta_1 \partial A + \beta_2 \partial P, \quad (2.22)$$

where α_I and β_I are constant, and we used the shortened notations

$$\partial X \equiv \partial_i X^i, \quad XY \equiv X_i Y^i, \quad X^2 \equiv X_i X^i, \quad (2.23)$$

for X being A or P . Notice that indices are lowered and raised with the flat metric δ_{ij} and its inverse δ^{ij} . As \mathcal{H}_0 trivially satisfies the condition (2.19), the theory propagates at most 2 degrees of freedom for any values of the coefficients α_I and β_I .

Let us study these theories in details. First, using canonical transformations, we can simplify the shape of the Hamiltonian. Indeed, canonical transformations (with no explicit time dependency) which preserves quadratic and first order Hamiltonians are of the form

$$A \mapsto xA + yP, \quad P \mapsto zA + wP, \quad xw - yz = 1. \quad (2.24)$$

Hence, (when $\beta_2 \neq 0$) one can find a canonical transformation such that

$$\begin{aligned} \mathcal{V} = & \alpha_1 A^2 + \alpha_2 P^2 + \alpha_4 (\partial A)^2 + \alpha_5 (\partial P)^2 + \alpha_6 (\partial A)(\partial P) \\ & - \alpha_7 \partial_j A_i \partial^j A^i - \alpha_8 \partial_j P_i \partial^j P^i - \alpha_9 \partial_j A_i \partial^j P^i, \end{aligned} \quad (2.25)$$

$$\mathcal{H}_0 = -\partial P, \quad (2.26)$$

which corresponds to taking $\alpha_3 = 0$, $\beta_1 = 0$ and $\beta_2 = -1$ in the general expression (2.22). As a consequence, the expression of the constraint $\mathcal{H}_0 \approx 0$ has exactly the same form as in Maxwell theory, and then, one can fix $\alpha_5 = 0$ without loss of generality (by a redefinition of the Lagrange multiplier A_0). Notice that, even though the constraint $\mathcal{H}_0 \approx 0$ is the same as in Maxwell theory, it is not necessarily first class. This can be easily seen if one re-expresses the total Hamiltonian as follows

$$\begin{aligned} H = \int d^3x \left[& -A_0 \partial P + \alpha_2 P^2 - \frac{1}{2} \alpha_4 F_{ij} F^{ij} + \alpha_6 F_{ij} \partial^j P^i + (\alpha_9 - \alpha_6) P_i \Delta A^i \right] \\ & + [\alpha_1 A^2 + (\alpha_7 - \alpha_4) A_i \Delta A^i], \end{aligned} \quad (2.27)$$

where $F_{\mu\nu}$ is the curvature of the connection (2.1). The first line in (2.27) is invariant under the U(1) gauge symmetry $\delta_\varepsilon A_i = \partial_i \varepsilon$. The second line is clearly not, which makes the constraint second class, and, from its expression, we see that the conditions for the theory to be U(1) gauge invariant are immediately given by $\alpha_1 = 0$ and $\alpha_7 = \alpha_4$.

For the moment, let us complete the Hamiltonian analysis. Using the notations of section 2.2, the secondary constraint is $\mathcal{S} \equiv \partial P$. To compute the remaining constraints, it is convenient to first write the equations of motion:

$$\dot{A}_i = \{A_i, H_0\} \approx \partial_i A_0 + 2\alpha_2 P_i - \alpha_6 \partial_i (\partial A) + 2\alpha_8 \Delta P_i + \alpha_9 \Delta A_i, \quad (2.28)$$

$$\dot{P}_i = \{P_i, H_0\} \approx -2\alpha_1 A_i + 2\alpha_4 \partial_i (\partial A) - 2\alpha_7 \Delta A_i - \alpha_9 \Delta P_i. \quad (2.29)$$

The tertiary constraint is obtained from the requirement that $\mathcal{S} \approx 0$ has to be weakly conserved under time evolution, which means that

$$\dot{\mathcal{S}} \approx 0 \iff \partial_i \dot{P}^i \approx 0 \iff \mathcal{T} \equiv [\alpha_1 + (\alpha_7 - \alpha_4)\Delta](\partial A) \approx 0. \quad (2.30)$$

Using suitable boundary conditions, one can replace the constraints \mathcal{T} by the condition

$$\partial A \approx 0, \quad (2.31)$$

except if $\alpha_1 = 0$ and $\alpha_4 = \alpha_7$, in which case the constraint \mathcal{S} is first class, as we have already seen previously. Clearly, the constraints \mathcal{S} and \mathcal{T} do not commute, and then the conservation of \mathcal{T} under time evolution does not lead to any new constraints. As a conclusion, the Dirac analysis of the theory closes with one first class constraint $\mathcal{P} \approx 0$ and the two second class constraints $\mathcal{S} \approx 0$ and $\mathcal{T} \approx 0$. This leads to 2 degrees of freedom, as expected.

2.4.2 Lagrangian

Let us focus on the case with $\alpha_1 \neq 0$ or $\alpha_4 \neq \alpha_7$, in which the theory has one first-class constraint and two second-class constraints. The first class constraint allows us to choose a gauge where $A_0 = 0$. In this gauge, the equations of motion (2.28) and (2.29) simplify into

$$\dot{A}_i = 2(\alpha_2 + \alpha_8\Delta)P_i + \alpha_9\Delta A_i, \quad (2.32)$$

$$-\dot{P}_i = 2(\alpha_1 + \alpha_7\Delta)A_i + \alpha_9\Delta P_i, \quad (2.33)$$

with the constraints that $\partial P = 0 = \partial A$, which means that both vectors are transverse. From this, we immediately see that the theory admits 2 degrees of freedom only which are governed, after decoupling the previous system, by the equation

$$-\ddot{A}_i + \alpha_9(\Delta\dot{A}_i - \dot{A}_i + \alpha_9\Delta A) - 4(\alpha_2 + \alpha_8\Delta)(\alpha_1 + \alpha_7\Delta)A_i = 0. \quad (2.34)$$

Notice that this equation is second order in time but higher order (up to fourth order) in space. This ensures that the theory is healthy and does not propagate Ostrogradski ghosts. Notice that the presence of higher space derivatives in the equations of motion could mean the existence of generalized instantaneous mode (or shadowy modes) which would appear in a ‘‘covariantization’’ of the theory similar to what happens in scalar-tensor theories [21].

It is also instructive to compute the Lagrangian and study some of its properties. As the Hamiltonian is quadratic, the associated Lagrangian is easily obtained from the Legendre transformation

$$L[A_\mu] = \int d^4x \left(P\dot{A} - \mathcal{V} + A_0\partial P \right), \quad (2.35)$$

where the momenta P^i are expressed in terms of the velocities \dot{A}_i solving the equation of motion (2.28). Formally the momenta variables are given by

$$\begin{aligned} P_i &= \frac{1}{2}(\alpha_2 + \alpha_8\Delta)^{-1}[\dot{A}_i - \partial_i A_0 + \alpha_6\partial_i(\partial A) - \alpha_9\Delta A_i] \\ &= \frac{1}{2}(\alpha_2 + \alpha_8\Delta)^{-1} [F_{0i} + \alpha_6\partial^j F_{ij} + (\alpha_6 - \alpha_9)\Delta A_i], \end{aligned} \quad (2.36)$$

which, to be defined, needs suitable spatial boundary conditions.

First, we immediately remark that a non-vanishing α_8 coefficient in the Hamiltonian makes the Lagrangian (spatially) non-local. In general, any terms which involve spatial derivatives of the momenta in the Hamiltonian will produce non-local terms in the Lagrangian, even though we started from a local Hamiltonian. For simplicity, we restrict the analysis to local Lagrangians, which, in this case, implies $\alpha_8 = 0$. Notice that P_i is not U(1) gauge invariant when $\alpha_6 \neq \alpha_9$. The calculation of the Lagrangian is now immediate and shows that it contains higher spatial derivatives but not higher time derivatives as expected. This is obviously consistent with the equation of motion for the vector field A_i (2.34).

2.4.3 Modified gauge invariant Maxwell theories

To finish with this example, let us consider quadratic theories which are gauge invariant, i.e. $\alpha_1 = \alpha_7 - \alpha_4 = 0$. For simplicity, we assume $\alpha_9 - \alpha_6 = 0$ as well as $\alpha_8 = 0$. In that case, $\mathcal{H}_0 \approx 0$ is first class, and the full connection transforms as expected according to $A_\mu \mapsto A_\mu + \partial_\mu \theta$ where θ is an arbitrary function, under the symmetry. The infinitesimal transformation law of A_i comes from the Poisson action of \mathcal{H}_0 ,

$$\delta_\varepsilon A_i = \{A_i, \int d^3x \varepsilon(x) \mathcal{H}_0\}. \quad (2.37)$$

The transformation law for A_0 under gauge transformations can be seen from the gauge invariance of the full (covariant) Lagrangian. In that context, the canonical Hamiltonian is simply given by

$$H = \int d^3x \left[-A_0 \partial P + \alpha_2 P^2 - \frac{1}{2} \alpha_4 F_{ij} F^{ij} + \alpha_6 F_{ij} \partial^j P^i \right], \quad (2.38)$$

and, after some calculations, one finds that the action is given by

$$S[A_0, A_i] = \frac{1}{2} \int d^4x \left[-\frac{1}{2\alpha_2} F_{0i} F^{0i} + \alpha_4 F_{ij} F^{ij} - \frac{\alpha_6^2}{2\alpha_2} (\partial_j F^{ij})^2 \right]. \quad (2.39)$$

As expected, the action is not Lorentz invariant, it contains spatial higher derivatives terms and its equations of motion are given by

$$\partial_i F^{0i} = 0, \quad \partial_0 F^{0i} + 4\alpha_2 \alpha_4 \partial_j F^{ij} + \frac{\alpha_6^2}{2} \Delta \partial_j F^{ij} = 0, \quad (2.40)$$

which we can compare to standard Maxwell equations $\partial_\mu F^{\mu\nu} = 0$. Using the usual definitions of the electromagnetism field $E^i \equiv F^{0i}$ and $B^i \equiv \varepsilon^{ijk} F_{jk}$, we obtain the following modified Maxwell equations in the vacuum,

$$\text{div} \vec{E} = 0, \quad (1 + \mu \Delta) \text{rot} \vec{B} = \lambda \frac{\partial \vec{E}}{\partial t}, \quad (2.41)$$

where $\mu = \alpha_6^2 / (8\alpha_2 \alpha_4)$ and $\lambda = -1 / (4\alpha_2 \alpha_4)$. The equations $\text{div} \vec{B} = 0$ and $\text{rot} \vec{E} + \partial \vec{B} / \partial t = 0$ which are equivalent to the existence of the gauge field A_μ are obviously unchanged. Hence, the propagation equations become

$$\Delta \vec{V} - \lambda \frac{\partial^2 \vec{V}}{\partial t^2} + \mu \Delta^2 \vec{V} = \vec{0}, \quad (2.42)$$

where \vec{V} can be either \vec{E} or \vec{B} . It is obvious that λ parametrizes the deviation to the speed of light and μ parametrizes higher derivatives deviations. As the theory is still linear and λ is constant, we can fix it to $\lambda = 1$ by a rescaling of the time variable, provided that λ is neither vanishing nor divergent.

To close the analysis of this example, let us make a couple of remarks. First, It is easy to generalize our analysis to cases where higher derivatives have an order higher than two, including in H terms with higher than 2 spatial derivatives of the fields A_μ . Introducing higher derivatives of the momenta variables would produce non local actions.

Second, as we briefly discussed below (2.5), one could have started with a dynamical A_0 variables in the Hamiltonian framework. For that, one would have replaced the primary constraint by a more general constraint $\mathcal{P}(P^0, P^i) \approx 0$ which would mix all the components of the momenta. The analysis would be similar to what we have done. Another way to make A_0 dynamical would be to consider “disformal-like” transformations on the connection which preserve the quadratic form:

$$A_0 \mapsto A_0 + x\partial_i A^i, \quad A_i \mapsto A_i + y\partial_i A_0, \quad (2.43)$$

where x and y are constant.

3 Generalization to gravity

In this section, we adapt the previous construction to gravity, and we construct a large class of minimally modified gravity theories from the Hamiltonian point of view. We first find (sufficient) conditions on the Hamiltonian for the theory to propagate at most two tensorial degrees of freedom. Then, we illustrate our construction with examples. In particular, we will exhibit a new interesting class of minimally modified gravities, dubbed $f(\mathcal{H})$ theories.

We start with the ADM parametrization of the metric in terms of the lapse function N , the shift vector N^i and the induced spatial h_{ij} , as it was recalled in the introduction (1.1).

3.1 The modified phase space

The phase space is parametrized by the usual ten pairs of canonical variables

$$\{h_{ij}(x), \pi^{kl}(y)\} = \delta_{ij}^{kl} \delta(x - y), \quad (3.1)$$

$$\{N^i(x), \pi_j(y)\} = \delta_j^i \delta(x - y), \quad (3.2)$$

$$\{N(x), \pi_N(y)\} = \delta(x - y). \quad (3.3)$$

We want to construct a Hamiltonian in this phase space which satisfies the properties of minimally modified gravity, i.e.

- It is invariant under space-like diffeomorphisms;
- It propagates only 2 tensorial degrees of freedom (or less);
- The lapse and the shift are non-dynamical.

Notice that the last requirement is not necessary, and one can relax the condition that the lapse function is not dynamical at the price to add a degeneracy condition as it is done in the context of DHOST theories [13]. Another way to make the lapse function dynamical

would be to perform a disformal transformation on the metric variables. For simplicity, we will consider only the case where N is not dynamical.

The invariance under space-like diffeomorphisms implies immediately that the canonical Hamiltonian takes the form

$$H = \int d^3x \sqrt{h} [\mathcal{H}(h_{ij}, \pi^{ij}, N, \nabla_i) + N^i \mathcal{V}_i], \quad (3.4)$$

where $\mathcal{V}_i \equiv -2\nabla^j(\pi_{ij}/\sqrt{h})$ is the usual vectorial constraint of gravity, and \mathcal{H} is a priori an arbitrary scalar. At this stage, with no restriction on the function \mathcal{H} , it is straightforward to see that the theory generically propagates 3 degrees of freedom.

Following what has been done for Maxwell theory in the previous section, we can immediately show that a necessary condition (up to a redefinition of the lapse function by a canonical transformation) for the theory to propagate (up to) 2 degrees of freedom is that

$$\mathcal{H} = \mathcal{V} + N \mathcal{H}_0, \quad (3.5)$$

where \mathcal{H}_0 and \mathcal{V} are three-dimensional scalar which depend on h_{ij} , π^{ij} and their covariant spatial derivatives only. The fact that there are scalars insures that they commute with the vectorial constraint. Hence, the conservation under time evolution of the constraints $\pi_N \approx 0$ and $\pi_i \approx 0$ creates respectively the constraints

$$\mathcal{H}_0 \approx 0, \quad \mathcal{V}_i \approx 0. \quad (3.6)$$

By construction, the vectorial constraints $\mathcal{V}_i \approx 0$, together with $\pi_i \approx 0$, are necessarily first class.

Then, requiring that the theory has enough constraints to kill the extra degrees of freedom implies, as in the vector case, leads to the condition that $\{\mathcal{H}_0(x), \mathcal{H}_0(y)\}$ has necessarily a non-trivial kernel (see (2.18) and the paragraph below). A sufficient condition is that

$$\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} \approx 0, \quad (3.7)$$

and we restrict our analysis to that case only where the conservation under time evolution of $\mathcal{H}_0 \approx 0$ implies the condition

$$\{\mathcal{H}_0(x), \mathcal{V}(y)\} \approx 0. \quad (3.8)$$

If this condition is trivially (weakly) satisfied, then there is no tertiary constraints in the theory. The constraints $\mathcal{H}_0 \approx 0$ and $\pi_N \approx 0$ are also first class, and the theory propagates $[10 \times 2 - (3 \times 2 + 3 \times 2 + 1 \times 2 + 1 \times 2)]/2 = 2$ degrees of freedom, as in Einstein theory. Furthermore, in that case, the theory admits an extra symmetry in addition to three-dimensional diffeomorphisms.

If, on the contrary, the condition (3.8) is not trivially satisfied, then the theory admits a new constraint which is

$$\mathcal{T}(x) \equiv \{\mathcal{H}_0(x), H\} \approx 0. \quad (3.9)$$

The existence of this new constraint is sufficient to conclude that the theory propagates at most 2 degrees of freedom. Indeed, as the constraint $\pi_N \approx 0$ is necessarily first class (because the theory is invariant by any redefinition of the lapse), the theory admits 7 first

class constraints in addition to the two constraints $\mathcal{H}_0 \approx 0$ and $\mathcal{T} \approx 0$, which implies immediately that the theory propagates 2 or less degrees of freedom. It has exactly 2 degrees of freedom if \mathcal{H}_0 and \mathcal{T} are second class, and no degrees of freedom if they are first class.

As a conclusion, the following Hamiltonian satisfies the three conditions recalled at the beginning of this section and thus defines a class of minimally modified theories of gravity:

$$H = \int d^3x \sqrt{h} \left[\mathcal{V}(h_{ij}, \pi^{ij}, \nabla_i) + N \mathcal{H}_0(h_{ij}, \pi^{ij}, \nabla_i) - 2N^i \nabla^j \left(\frac{\pi_{ij}}{\sqrt{h}} \right) \right], \quad (3.10)$$

$$\text{with } \{ \mathcal{H}_0(x), \mathcal{H}_0(y) \} \approx 0. \quad (3.11)$$

In that case, the function \mathcal{V} is totally free. Notice that, as N and N^i are not dynamical, the Hamiltonian comes with the primary constraints $\pi_i \approx 0$ and $\pi_N \approx 0$ which are first class. They are associated to the invariance of the theory under arbitrary redefinitions of the lapse and shift.

3.2 Simple examples: \mathcal{H}_0 is the Hamiltonian constraint and \mathcal{V} is polynomial in π

To illustrate the previous general construction, let us consider the simple example defined by

$$\mathcal{H}_0 = \frac{1}{|h|} \left(\pi_{ij} \pi^{ij} - \frac{1}{2} \pi^2 \right) - R, \quad \mathcal{V} = \lambda \pi - \mu \sqrt{|h|}, \quad (3.12)$$

where λ and μ are constant, and R is the three-dimensional curvature. We notice that, as \mathcal{H}_0 is the Hamiltonian constraint of gravity, it trivially satisfies the condition (3.7). In fact, if we fix \mathcal{H}_0 to this expression, one could have chosen any arbitrary function for \mathcal{V} but for simplicity we make the choice above.

With this example, one can easily compute the explicit action which is given by

$$S = \int d^4x N \sqrt{h} \left[K_{ij} K^{ij} - K^2 + R + \lambda \left(\frac{K}{N} - \frac{3\lambda}{4N^2} \right) + \frac{\mu}{N} \right], \quad (3.13)$$

where K_{ij} is the usual extrinsic curvature

$$K_{ij} = \frac{1}{2N} \left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i \right), \quad (3.14)$$

and $K \equiv K^i_i$ is its trace.

Let us remark that the change of variable

$$K_{ij} \equiv \bar{K}_{ij} + \frac{\lambda}{2N} h_{ij}, \quad (3.15)$$

allows to see that the previous action (3.13) takes exactly the same form as the general relativity action

$$S = \int d^4x N \sqrt{h} \left(\bar{K}_{ij} \bar{K}^{ij} - \bar{K}^2 + R + \frac{\mu}{N} \right), \quad (3.16)$$

up to the μ -term. However, as \bar{K}_{ij} cannot be interpreted as the extrinsic curvature of a metric, the theory is not equivalent to general relativity. To illustrate the difference between the modified theory and general relativity, let us now make the following time dependent change of variable on the metric components

$$\hat{h}_{ij} \equiv e^{-\lambda t} h_{ij}, \quad \hat{N}_i \equiv e^{-\lambda t} N_i, \quad \hat{N} \equiv e^{-\lambda t/2} N. \quad (3.17)$$

Hence, the action takes the form

$$S = \int d^4x \hat{N} \sqrt{\hat{h}} \left(\hat{K}_{ij} \hat{K}^{ij} - \hat{K}^2 + e^{-\lambda t} \hat{R} + e^{3\lambda t} \frac{\mu}{\hat{N}} \right) \quad (3.18)$$

which makes obvious that the theory propagates only 2 tensorial degrees of freedom because the modification affects only terms with spatial derivatives in the action.

To finish with this example, let us remark that the action (3.13) can easily be made covariance introducing, as usual, a scalar field ϕ whose gradient is orthogonal to the space-like hyper-surfaces. Using the results of [20], one obtains

$$S[g_{\mu\nu}, \phi] = \int d^4x \sqrt{|g|} \left[\mathcal{R} - \frac{\lambda}{2} \ln(X^2) \square \phi + \frac{3\lambda^2}{2} X + 2\mu \sqrt{-X} \right]. \quad (3.19)$$

From this action, it is clearly not obvious that only two gravitational degrees of freedom are propagating. But the theory belongs to the class of “cuscuton” theories [22, 23].

A more interesting example would be to assume that \mathcal{V} is a scalar quadratic in π_{ij} , in which case, it can be written as

$$\mathcal{V} = \frac{1}{|h|} \left(\lambda_1 \pi^{ij} \pi_{ij} - \frac{\lambda_2}{2} \pi^2 \right), \quad (3.20)$$

where λ_1 and λ_2 are constant.

Using the results of the Hamiltonian analysis of DHOST theories [14], we see that such a Hamiltonian can be obtained from a DHOST theory in the unitary gauge with a k-essence term, a generalized cubic galileon term and a quadratic DHOST term with

$$\frac{a_1}{N^2} + 1 = \frac{1}{N + \lambda_1}, \quad \frac{a_2}{N^2} - 1 = \frac{N + \lambda_2}{(N + \lambda_1)(2\lambda_1 - 3\lambda_2 - N)}, \quad (3.21)$$

in the unitary gauge. We notice that the theory belongs to (the safe) class I only if $a_1 + a_2 = 0$, which implies that $\lambda_1 = \lambda_2$. Otherwise, perturbations about any cosmological background develop gradient instabilities. Furthermore, all these theories belong by definition to the class of extended cuscuton [22, 23].

3.3 A new class of theories: $f(\mathcal{H})$ theories

In this section, we introduce a new interesting class of minimally modified theories of gravity. To explain the construction of this class, we first recall that a Hamiltonian of the form (3.4) corresponds to a theory with (up to) two tensorial modes only if the “modified” Hamiltonian constraint \mathcal{H}_0 (weakly) commutes with itself (3.11):

$$\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} \approx 0. \quad (3.22)$$

The function \mathcal{V} is a priori free, but to have a modified theory which is very close to general relativity, we make the choice $\mathcal{V} = 0$.

In order for the theory to propagate gravitational wave, it is necessary that \mathcal{H}_0 contains both K_{ij} terms and three dimensional curvature terms (like the Ricci scalar R) as in the expression of the Hamiltonian constraint of general relativity (3.12). The presence of such terms makes difficult the problem of finding an expression of \mathcal{H}_0 which is different from the

usual Hamiltonian constraint. However, there is a simple modification that we can think about which is

$$\mathcal{H}_0 = f(\mathcal{H}_{\text{gr}}) \quad \text{with} \quad \mathcal{H}_{\text{gr}} \equiv \frac{2\pi_{ij}\pi^{ij} - \pi^2}{2|h|} - R, \quad (3.23)$$

where f is an arbitrary function. As \mathcal{H}_{gr} is dimensionful, the function f needs at least a mass scale to be defined, which could be the Planck mass and something else, like the cosmological constant.

In that case, the modified Hamiltonian constraint satisfies the Poisson algebra

$$\{\mathcal{H}_0(N_1), \mathcal{H}_0(N_2)\} = [f'(\mathcal{H}_{\text{gr}})]^2 (N_1 \nabla_i N_2 - N_2 \nabla_i N_1) \mathcal{V}^i, \quad (3.24)$$

which is, in general, non-linear. Obviously, the Poisson bracket weakly vanishes. Hence, we have found a new class of minimally modified theories of gravity that we dub $f(\mathcal{H})$ theories with reference to $f(R)$ theories. Contrary to $f(R)$ theories, $f(\mathcal{H})$ theories do not propagate scalar modes, and the main reason is that the associated equations of motion remain second order.

From a Legendre transformation, one can easily compute the corresponding action. Indeed, the equation of motion for h_{ij} enables us to relate the momenta π_{ij} to the extrinsic curvature K_{ij} as follows

$$K_{ij} = \frac{f'(\mathcal{H}_{\text{gr}})}{\sqrt{|h|}} \left(\pi_{ij} - \frac{1}{2} \pi h_{ij} \right), \quad (3.25)$$

from which we can implicitly obtain π_{ij} in terms of K_{ij} because, in general, this equation is non-linear in π_{ij} . Nonetheless, one can compute the action which, after a simple calculation, is given by

$$S[h_{ij}, N, N^i] = \int d^4x \sqrt{|g|} \left[\frac{2}{f'(C)} (K_{ij} K^{ij} - K^2) - f(C) \right], \quad (3.26)$$

where C is formally obtained by solving the equation

$$C = \frac{K_{ij} K^{ij} - K^2}{[f'(C)]^2} - R. \quad (3.27)$$

In the case where $f(x) = x$, one immediately recovers the action of general relativity. However, any other choice for f leads to a different theory which admits a four dimensional symmetry algebra (the constraints satisfy a deformed diffeomorphisms algebra) and propagates only two tensorial modes. For instance, the choice $f(x) = x(1 - x/(2\Lambda))$ could be interesting for dark energy because the solutions of the deformed Hamiltonian constraint contain both a sector with no cosmological constant and a sector with a cosmological constant Λ . In fact, in any situation where $f(x) = 0$ has a non-vanishing solution x_0 , there is in the theory an effective cosmological constant given by $x_0 = 2\Lambda$. For this reason, this new class of theories is very interesting and certainly deserves a deeper study.

3.3.1 Hamilton equations

We can easily compute the Hamilton equations of motion for any function $\mathcal{O}(h_{ij}, \pi^{ij})$ in the phase space using the definition of the time derivative

$$\dot{\mathcal{O}}(x) = \{\mathcal{O}(x), H\}. \quad (3.28)$$

The explicit form of the time derivative is easily obtained from the Hamilton equations of general relativity due to the fact that

$$\begin{aligned} \{\mathcal{O}(x), H\} &= \int d^3y \sqrt{h(y)} [f'(\mathcal{H}_{\text{gr}}(y)) N(y) \{\mathcal{O}(x), \mathcal{H}_{\text{gr}}(y)\} + N^i(y) \{\mathcal{O}(x), \mathcal{V}_i(y)\}] \\ &\quad + \int d^3y f(\mathcal{H}_{\text{gr}}(y)) N(y) \{\mathcal{O}(x), \sqrt{h(y)}\}. \end{aligned} \quad (3.29)$$

In the vacuum, the second line vanishes due to the constraint, but this is not the case in the presence of matter.

Applying this formula to the spatial metric h_{ij} and its momenta π^{ij} and using well-known results of Hamiltonian general relativity (see [27] for instance) leads immediately to the expressions

$$\dot{h}_{ij} = D_i N_j + D_j N_i + \frac{N f'}{\sqrt{h}} (2\pi_{ij} - \pi h_{ij}), \quad (3.30)$$

$$\begin{aligned} \dot{\pi}^{ij} &= -\sqrt{h} N \left[f' R^{ij} + \frac{1}{2} f h^{ij} \right] + \sqrt{h} (D^i D^j - h^{ij} D^2) (N f') - D_k \left[2N^{(i} \pi^{j)k} - N^k \pi^{ij} \right] \\ &\quad - \frac{N f'}{\sqrt{h}} \left[2\pi_k^i \pi^{kj} - \pi \pi^{ij} - \left(\pi_{kl} \pi^{kl} - \frac{1}{2} \pi^2 \right) h^{ij} \right], \end{aligned} \quad (3.31)$$

where f and f' are evaluated at \mathcal{H}_{gr} . Combining these two equations would allow us in principle to obtain the modified Einstein equations. To do so, one has to express π_{ij} in terms of the extrinsic curvature K_{ij} using the first equation

$$K_{ij} = \frac{f'(\mathcal{H}_{\text{gr}})}{\sqrt{h}} \left(\pi_{ij} - \frac{1}{2} \pi h_{ij} \right), \quad (3.32)$$

and then one substitutes the obtained expression in the second equation of motion for π_{ij} . When $f(x) = x$, we recover immediately the Hamilton equations of general relativity using the Hamiltonian constraint $\mathcal{H}_{\text{gr}} = 0$.

In the presence of matter, these equations have to be supplemented with source terms. However, describing explicitly how matter is coupled to the (modified) gravitational field is subtle and has been analyzed in great details in [25, 28]. A “naive” minimal coupling³ of the matter fields, for instance, would break the gauge invariance generated by the first class constraint \mathcal{H}_0 which, as a consequence, would become second class. Therefore, in general, one extra mode (besides those of the matter field) appears in the phase space. A consistent way to introduce the matter field is, before inclusion of the matter fields, to split the first class

³If the matter is minimally coupled (with no derivative couplings) and is described by an action S_M associated to an energy-momentum tensor $T^{\mu\nu}$, then the equation for h_{ij} (3.30) is unchanged, the deformed Hamiltonian constraint becomes

$$f(\mathcal{H}_{\text{gr}}) + 16\pi G_N N^2 T^{00} \approx 0, \quad (3.33)$$

and the equation for the momenta π_{ij} contains a source term

$$\dot{\pi}^{ij} = \dot{\pi}_0^{ij} + \frac{\delta S_M}{\delta h_{ij}} = \dot{\pi}_0^{ij} + 8\pi G_N N \sqrt{h} \left(T^{ij} - N^i N^j T^{00} \right), \quad (3.34)$$

where $\dot{\pi}_0^{ij}$ is the expression of $\dot{\pi}^{ij}$ in vacuum given by (3.31). However, as we said, in general such a coupling leads to new propagating degrees of freedom in addition to the tensors and the matter.

constraint into a pair of second class constraints by introducing a “gauge fixing condition”. Since these constraints remain second class after introducing the matter field, the number of gravitational degrees of freedom remains four in the phase space, i.e., two in the real space. This strategy has been successfully applied in [25, 28] by adding to the Hamiltonian a gauge fixing term \mathcal{H}_{gf} which is, by definition, not commuting with the Hamiltonian constraint. In our case, one need to introduce a gauge fixing term which does not commute with the \mathcal{H}_0 or equivalently \mathcal{H}_{gr} . Following [25], one could think about adding to the total Hamiltonian a term which imposes, using a Lagrange multiplier, a new constraint either of the form $\partial_i \mathcal{S} \approx 0$ or of the form $\mathcal{S} \approx 0$ where \mathcal{S} is a three-dimensional scalar, such that, together with \mathcal{H}_0 , they form a pair of second class constraints while the invariance under space-like diffeomorphisms is preserved. The coupling to matter (particularly the choice of \mathcal{H}_{gf}) needs to be studied in great details and goes beyond the scope of the present work. For this reason, we leave this analysis for future investigations.

3.3.2 Cosmology

To illustrate the difference between $f(\mathcal{H})$ theories and general relativity, we consider simple examples. First, let us study the cosmology of these theories in the presence of a perfect fluid (of density ρ and pressure p) which corresponds to taking a time dependent lapse function $N(t)$, a vanishing shift vector $N^i = 0$, homogeneous and isotropic spatial metric and momenta as follows

$$h_{ij} = a^2(t)\delta_{ij}, \quad \pi^{ij} = b(t)\delta^{ij}. \quad (3.35)$$

Here we assume that the spatial slices are flat. To make the dynamics in the cosmological sector more interesting, we consider the coupling to matter in the form a perfect fluid, as we have said previously. In that case, contrary to the generic situation, we do not really need an explicit form of \mathcal{H}_{gf} . Indeed, if the gauge-fixing condition is of the form $\partial_i \mathcal{S} \approx 0$, then it is trivially satisfied by FLRW space-time with the (space-independent) time reparametrization symmetry unbroken (namely, the lapse function is arbitrary). On the other hand, if the gauge condition of the form $\mathcal{S} \approx 0$, it may imply a specific choice of the lapse function if \mathcal{S} involves a fixed function of time, for instance. In this case, the (space-independent) time reparametrization symmetry is broken. In any cases, the gauge fixing term does not explicitly show up in the equations of motion, and we can consider a minimal coupling to matter (as described in the footnote 3) where the lapse is either free or fixed to a specific value. Hence, the deformed Hamiltonian constraint simplifies drastically and becomes

$$f(\mathcal{H}_{\text{gr}}) + 16\pi G_N \rho = 0 \quad \text{with} \quad \mathcal{H}_{\text{gr}} = -\frac{3}{2} \left(\frac{b}{a}\right)^2. \quad (3.36)$$

Furthermore, the Hamilton equations of motion reduces to

$$\dot{a} = -\frac{N f'(\mathcal{H}_{\text{gr}})}{2} b, \quad \dot{b} = -\frac{N}{2} a \left[f(\mathcal{H}_{\text{gr}}) + f'(\mathcal{H}_{\text{gr}}) \left(\frac{b}{a}\right)^2 - 16\pi G_N p \right]. \quad (3.37)$$

Notice that FLRW cosmology could also be analyzed starting from the Lagrangian (3.26) where C has been defined by the relation (3.27). The result is, as expected, the same as in the Hamiltonian formalism.

In general, the Friedmann equations are strongly modified compared to the classical ones. To write them, it is useful to introduce

$$F(\rho) = f^{-1}(-16\pi G_N \rho) \quad \Longrightarrow \quad f'(\mathcal{H}_{\text{gr}}) = -\frac{16\pi G_N}{F'(\rho)}, \quad (3.38)$$

in order to reformulate the previous three equations (with $N = 1$) equivalently as follows

$$\left(\frac{\dot{b}}{b}\right)^2 = -\frac{2}{3}F(\rho), \quad \dot{b} = \frac{F'(\rho)}{8\pi G_N} \dot{a}, \quad \dot{b} = 8\pi G_N a \left[\rho + p - \frac{2F(\rho)}{3F'(\rho)} \right], \quad (3.39)$$

which lead to the following modified Friedmann equations

$$H^2 = -\frac{2}{3}(8\pi G_N)^2 \frac{F(\rho)}{[F'(\rho)]^2}, \quad (3.40)$$

$$F'(\rho) \frac{\ddot{a}}{a} - 3H^2 F''(\rho)(\rho + p) = (8\pi G_N)^2 \left[\rho + p - \frac{2F(\rho)}{3F'(\rho)} \right], \quad (3.41)$$

whereas the conservation equation for the fluid remains unchanged

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (3.42)$$

When $f(x) = x$, one immediately recover the usual Friedmann equations. Furthermore, in vacuum (when $\rho = 0 = p$), these equations admit a self-accelerating solution if $F(0) < 0$. This is for instance the case for

$$f(x) = x \left(1 - \frac{x}{2\Lambda}\right) \quad \Longrightarrow \quad F(\rho) = \Lambda \left[-1 \pm \sqrt{1 - 32\pi G_N \rho / \Lambda}\right] \quad (3.43)$$

where Λ is a non-negative constant. The function $F(\rho)$ has two branches, and the minus branch, which is such that $F(0) = -2\Lambda < 0$, admits a self-accelerating solution in vacuum with cosmological constant Λ . This result has a simple interpretation. Indeed, in vacuum, the modified Hamiltonian constraint reduces to $f(\mathcal{H}_{\text{gr}}) = 0$ whose solutions fall into two branches: $\mathcal{H}_{\text{gr}} = 0$ which corresponds to general relativity with no cosmological constant and $\mathcal{H}_{\text{gr}} = 2\Lambda$ which corresponds to general relativity with a cosmological constant. In general, any deformation of general relativity associated to $f(x)$ admits a self accelerating solution if $f(x) = 0$ admits a non-negative solution x_0 .

Notice that in the absence of matter, the FLRW background reduces to a de Sitter spacetime and that the analysis of scalar perturbations about the de Sitter background (without matter nor gauge fixing term) confirms that no scalar modes are propagating in these theories.

To close this section, let us mention that the analysis of linear perturbations about a cosmological background can be performed. It shows that the no-ghost condition reduces to $f'(x) > 0$ where x is the value of \mathcal{H}_{gr} on the background. In addition, we also need to require that $f' \approx 1$ at low energy so that the speed of gravitational waves at low energy is unity.

4 Conclusion

In this paper, we constructed theories of minimally modified gravity (MMG) from the Hamiltonian point of view. To illustrate the construction, we started in section 2 with a complete study of minimally modified Maxwell theories which propagate 2 (vectorial) degrees of freedom in the 4-dimensional Minkowski space-time. The minimally modified Maxwell theory provides us with a simpler but very interesting context to present the main ingredients that enter in the construction of minimally modified gravity theories from the Hamiltonian point of view. Then, we considered the more interesting case of gravity. We started with the phase space of general relativity parametrized with 10 pairs of canonically conjugate variables (the metric components and their momenta) and whose dynamics is governed by the Hamiltonian and vectorial constraints. We modified the theory in such a way that, first, the lapse function and the shift vector remain non-dynamical (i.e. with vanishing conjugate momenta), second, the theory is still invariant under 3D diffeomorphisms, and third the theory propagates only two tensorial degrees of freedom. We found that these three requirements lead to a Hamiltonian of the form (3.10) with the condition (3.11).

We showed that these MMG theories encompass the so-called cuscuton theories (in the unitary gauge) which are (higher derivative) scalar-tensor theories with only two tensorial modes. In these theories, the scalar degree of freedom is in fact a shadow mode [21] and thus does not propagate. Notice that our construction naturally extends the cuscuton models to non-local theories which involve infinite spatial derivatives. We also found a particularly interesting and simple novel class of MMG whose Hamiltonian differs from the Hamiltonian of general relativity by the fact that the Hamiltonian constraint \mathcal{H}_{gr} has been replaced by $f(\mathcal{H}_{\text{gr}})$ where f is an arbitrary function. We dubbed them $f(\mathcal{H})$ -theories.

The class of $f(\mathcal{H})$ -theories opens numerous new windows in cosmology and in astrophysics. We have quickly studied cosmological solutions for a generic choice of function $f(x)$, but it would be interesting to make a systematic analysis of cosmological perturbations and of the constraints that observations put on these theories if they account for dark energy. For that, it is important to first understand in details how to consistently couple matter in these theories following the analysis of [25, 28]. This would also allow us to study, for instance, the structure of stars in these theories and to see how Newton laws are modified in this framework. From a more formal point of view, we are curious to understand the relations and the differences with the very well-studied $f(R)$ or $f(R, T)$ theories. We hope to investigate all these questions in the future.

Note added. After submission of the present paper, we were notified that ref. [26] also studied $f(\mathcal{H})$ -theories from a different point of view.

Acknowledgments

We want to thank David Langlois for very interesting and useful discussions. The work of S.M. was supported by Japan Society for the Promotion of Science (JSPS) Grants-in-Aid for Scientific Research (KAKENHI) No. 17H02890, No. 17H06359, and by World Premier International Research Center Initiative (WPI), MEXT, Japan. He is grateful to Institut Denis Poisson for hospitality during his stay.

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