On the Homology of Open-Closed String Theory
by Eric Harrelson

Introduction

In Zwiebach’s study of oriented open-closed string theory [18], he considered a certain moduli space of Riemann Surfaces with boundary having “closed” punctures in the interior and “open” punctures on the boundary coming with parameterizations by the unit disk and upper half disk. While he did not consider it as such, this moduli space forms a 2-colored PROP. The first purpose of this paper is to describe completely the homology of the biggest genus 0 structure inside this PROP. The operad inside of this PROP formed by spheres with no boundary is well known to be homotopy equivalent to the framed little disks operad. It is shown by Getzler that its homology defines a BV-algebra [7]. This extends the result by Cohen [4] showing that the homology of the non-framed little disks operad describes a Gerstenhaber algebra. In [17], Voronov invented the Swiss-cheese operad which is a (non framed) finite dimensional model of the operad inside this PROP formed by Riemann spheres with one or no boundary components. He computed its homology and calls the algebra that it defines a Swiss-cheese algebra. The algebra is defined on a pair of graded vector spaces \((V_C, V_O)\) and consists of a Gestenhaber structure on \(V_C\), an associative multiplication on \(V_O\), and an algebra action of \(V_C\) on \(V_O\). The framed version of Swiss, h.e. to the subspace of spheres with one or no boundary components, has the same result except that \(V_C\) is a BV-algebra.

The biggest genus 0 operad inside this PROP is formed by all spheres with boundary having exactly one puncture labeled as an output. It’s homology contains the structure of (framed) Swiss-cheese. However, there is a bigger genus 0 structure in this PROP containing this operad. The subspace of all genus 0 surfaces, with an arbitrary number of inputs and outputs, forms what’s called a dioperad, invented by Gan in [6]. A dioperad only considers compositions which attach one input to one output so as to create no genus. The first four sections of this paper together give a complete description of the homology of this dioperad.

Restricting the description of the dioperad to the generators with one output and the relations only involving them gives a description of the biggest genus 0 operad in the PROP. Sec 5 considers what extra structure is given by the cyclic structure of this operad and the “semi-modular” given by self sewing open punctures on the same boundary component.

The homology of the free loop space of an oriented manifold \(M, LM\), was shown to form a BV-algebra in Chas and Sullivan’s “String topology” [1]. In [14], Voronov invented the Cacti operad and announced that it is h.e. to the framed little disks operad. He showed how we can obtain \(H_*(LM)\) as an algebra over \(H_*(\text{Cacti})\) via a geometric action of \(\text{Cacti}\) on \(LM\). In [13], Sullivan considers the algebraic structure of open/closed string topology. In particular, he studies the homology of the space of paths in a manifold starting and ending in a fixed submanifold \(K, PM_K\). The purpose of the last section is to extend the \(\text{Cacti}\) result to this open/closed setting by defining an open/closed \(\text{Cacti}\), having the same homology as the biggest genus 0 operad inside the PROP, and showing how to obtain the pair \((H_*(LM), H_*(PM_K))\) as algebra over \(H_*(\text{open/closed Cacti})\).

In a soon to come section, I have extended this open-closed \(\text{Cacti}\) operad to an open-closed 2-colored graph PROP modeling the entire moduli space and acting in string topology. This extends the Sullivan chord diagrams used by R. Cohen and V. Godin in [5] which act in closed string topology. I have recently become aware that some similar work involving an open-closed Prop and string topology is being done by A. Ramirez for his thesis [12].

For good sources discussing the topics of operads and PROPs and their use in physics, see [11], [14], [15], and [16].

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Sec 1: Description of 2-colored dioperad

Consider the moduli space of genus 0 Riemann surfaces with boundary. RS with boundary means for us a complex surface based on the closed upper half plane. Add to this punctures which can be in the interior or on a boundary component. Each puncture is designated as an input or output. The inputs are labeled \( \{1, \ldots, n\} \) and the outputs labeled \( \{1_o, \ldots, m_o\} \). It is required that there be at least one output. A puncture in the interior of a surface, called a closed puncture, comes with an analytic parameterization given by a biholomorphic mapping of the standard disk \( \{ z \mid |z| \leq 1 \} \) into the surface sending 0 to the puncture. A puncture in the boundary, called an open puncture, comes with a biholomorphic mapping of the half disk \( \{ z \mid \text{Im}(z) \geq 0, |z| \leq 1 \} \) into the surface sending the real axis into the bndry and 0 to the puncture. These parameterizations may only overlap in their boundaries.

The space \( OC = \{ OC(n, m) \} \) forms a 2-colored dioperad via sewing closed punctures to closed punctures and open to open. This is done in the standard way, using \( w = 1/z \) for closed punctures and \( w = -1/z \) for open. When sewing closed to closed, the # of boundary components in the resulting surface is the sum of the boundary components. In the open to open case, it’s the sum minus one.

**Sewing closed to closed**
Sec 2: Description of $H_*(OC)$

Let’s start with a description of the path components of the moduli space and how composition (sewing) acts on them. For each point in $OC$ we have the following data:

1) a subset $C \subseteq \{1_i,...,n_i,1_o,...,m_o\}$ consisting of the labels of all closed punctures.

2) an unordered list of cyclically ordered subsets $(a_1,a_2,...,a_k),(b_1,b_2,...,b_k)$, etc... grouping together labels of open punctures lying on the same boundary component and giving them the cyclic order induced by the orientation of the boundary component (which is induced by the canonical orientation of the Riemann surface). An empty parentheses () is used for boundary components with no open punctures.

It’s clear that two points are in the same path component iff they have the same data. Call this data the type of the path component.

Example

Type: $(1_i,4_i,3_o),(1_o,2_o),(2_o,5_i,3_i),()$
It is also clear from the pictures for sewing how composition acts on the path components. Note that the path components of type \( \{1, 1_o\} \) and \((1, 1_o)\) are the identities for closed and open composition.

**Claim 2.1**

The following path components generate \( H_0(OC) \). (Listed with them are the degree 0 trees that we'll use to represent them and the corresponding degree 0 operations in an algebra \((V_C, V_o)\) over \( H_0(OC) \)).

1. **closed multiplication**

   Type \( \{1, 2_i, 1_o\} \)

   \[
   m_C : V_C \otimes V_C \rightarrow V_C
   \]

2. **open multiplication**

   Type \((1, 2_i, 1_o)\)

   \[
   m_O : V_O \otimes V_O \rightarrow V_O
   \]

3. **closed unit**

   Type \(\{1_o\}\)

   \[
   K \rightarrow V_C \quad 1 \mapsto e_C
   \]

4. **open unit**

   Type \((1_o)\)

   \[
   K \rightarrow V_O \quad 1 \mapsto e_O
   \]

5. **closed to open**

   Type \(\{1_i\}, (1_o)\)

   \[
   \phi_{C \rightarrow O} : V_C \rightarrow V_O
   \]
6. open to closed

Type \{1_o\},(1_i)

\[ \varphi_{O\rightarrow C}: V_O \rightarrow V_C \]

7. closed comultiplication

Type \{1_i,1_o,2_o\}

\[ \nabla_C: V_C \rightarrow V_C \otimes V_C \]

8. open comultiplication

Type \{1_o,1_i,2_o\}

\[ \nabla_O: V_O \rightarrow V_O \otimes V_O \]

proof:

In the proof of the main theorem of this section, it will be defined what it means for a tree to be in normal form. It is easy to see that all path components can be given by some tree in normal form.

Next, we see that the operations satisfy the following relations:

1. Closed multiplication is associative and commutative with \( e_c \) as a unit.
2. Open multiplication is associative with \( e_o \) as a unit.
3. \( V_o \) is an algebra over \( V_c \) via \( \varphi_{c\rightarrow o} \). That is \( \varphi_{c\rightarrow o} \) is an algebra homomorphism into the (graded) center of \( V_o \) with \( \varphi_{c\rightarrow o}(e_c) = e_o \).
Type$\{1_i, 2_i, 1_o\}$

and

\[
\begin{array}{c}
1 \quad 2 \\
\end{array}
\]

\[
\begin{array}{c}
1 \quad 2 \\
\end{array}
\]
Type \{2_i\}, (1_i, 1_o)

4. \( \varphi_{o \to c}(\varphi_{c \to o}(a)b) = a\varphi_{o \to c}(b) \) (left side is open mult. and right side is closed)

Type \{1_i, 1_o\}, (2_i)
5. $\varphi_{o\rightarrow c}(ab) = (-1)^{|a|\cdot|b|}\varphi_{o\rightarrow c}(ba)$

\[\begin{array}{c}
1 & 2 \\
\downarrow & \downarrow \\
1 & 2
\end{array} \quad \begin{array}{c}
2 & 1 \\
\downarrow & \downarrow \\
1 & 1
\end{array} \]

Type $\{1_o\}, (1_i, 2_i)$

6. We have the dual relations except there is no counit (actually, in the course of the proof I found that this relation is not needed, it can be deduced from the others).

7.

\[\begin{array}{c}
1 & 2 & 1 & 2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 2 & 2
\end{array} \quad \begin{array}{c}
1 & 1 & 2 & 2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & 1 & 2
\end{array} \quad \begin{array}{c}
1 & 1 & 2 & 2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 2 & 2
\end{array} \]

Type $\{1_i, 2_i, 1_o, 2_o\}$

$a \cdot (\nabla_c (b)) = \nabla_c (a) \cdot b = \nabla_c (ab)$ where $a \cdot (b_1 \otimes b_2) = (ab_1) \otimes b_2$

and $(a_1 \otimes a_2) \cdot b = a_1 \otimes (a_2 b)$

8. The same relation holds for the open case as well.

In the closed case, there is only one path component with 2 closed inputs and 2 closed outputs. We can derive that any two green trees with two inputs and two outputs are equivalent using relations 7 and 1. In the open case, there are many path components with one boundary component, 2 open inputs, and 2 open outputs. Relations 2 and 8 are not enough to show that any two red trees with 2 inputs and 2 outputs going to the same path component are equivalent. Thus we add:
9. Type $(1_i, 2_o, 2_i, 1_o)$

$$b \cdot_R (\nabla_o (a)) = (\nabla_o (b) \cdot (1, 2)) \cdot_L a = \nabla_o (a) \cdot_L b \text{ where}$$

$$b \cdot_R (a_1 \otimes a_2) = (a_1 b) \otimes a_2, \quad (b_1 \otimes b_2) \cdot (1, 2) = (b_2 \otimes b_1), \quad \text{and}$$

$$(a_1 \otimes a_2) \cdot_L b = a_1 \otimes (b a_2)$$

10. Type $\{1_i, 1_o\}, (2_i, 2_o)$

$$\nabla_C (a) \cdot_{\varphi \rightarrow C} b = a \cdot (\varphi \rightarrow_C \cdot \nabla_o (b)) \text{ where} (a_1 \otimes a_2) \cdot \varphi = a_1 \otimes \varphi (a_2)$$

and $\varphi \cdot (b_1 \otimes b_2) = \varphi (b_1) \otimes b_2$
**Theorem 2.1**

These 10 relations completely describe $H_0(OC)$ or, equivalently, an algebra $(V_c, V_o)$ over it.

**Proof:**

Let $F(G)/S$ be the free 2-colored dioperad generated by the degree 0 trees representing the generators modulo the set of relations $S$, listed above. Then right now we have an onto dioperad morphism $F(G)/S \rightarrow H_0(OC)$. We need to see that it is in fact 1-1. I.e. we need to check that there is a 1-1 correspondence between path components of $OC(n)$ and equivalence classes of labeled trees in $F(G)/S$.

**Notation**

Since open/closed (co) multiplication is associative, we’ll just use a green or red tree like the following to denote (co) multiplication with $n$ inputs (outputs):

![Diagram of a tree in normal form](image)

Now, say a tree is in normal form if it is in one of the following forms:

**Form 1**

The tree is completely red:

![Diagram of a completely red tree](image)

Explanation: All possible labelings are allowed. The first and last stems can be either output or input stems.
The following example shows what the image of a tree of this form is in $H_{\delta}(OC)$:

Form 2
There is at least one closed output:

Explanation: All, none, or some of the green input stems can have a tree in form 1 connected to it via $\varphi_{O \rightarrow C}$.

The following example shows what path component a tree of this form corresponds to:
Form 3
There is no closed output, but the tree is not equivalent to an entirely red tree:

The red tree at the bottom should be a tree in form 1 with its left most stem an input stem.

Note: Requiring that the tree not be equivalent to a red tree is the same thing as requiring that the form 2 tree not be $e_C$. This is so that there is no overlap between the forms. We could not require this and just have 2 forms, the first being included in the third, but separating these cases is helpful for clearness in arguing.
Here’s an example showing the corresponding path component:

Claim 2.2
There is a bijection between the equivalence classes of trees in normal form and the path components of the moduli space.

Proof:
As noted earlier in Claim 2.1, it is clear that every path component corresponds to some tree in normal form.

Two trees in form 1 give the same path component iff they give the same circular permutation of the labels. To show that two red trees giving the same circular permutation are equivalent, it suffices to show the following equivalence:

Note that this covers the cases where the output that we are “rotating” to the root is the first or last stem. To see this, just plug $e_o$ into the first or second input.
We have (number on top of equal sign says which relation is being used):

Now consider trees of form 2. Given that two trees of the following form are equivalent as long as they have the same # of inputs and outputs, we see that it suffices to check that two trees of the form below giving the same circular permutation of the OPEN labels are equivalent:

But this follows directly from relation 5.

Finally, consider form 3 trees. What we first need to see is that for each output of a form 3 tree, the tree is equivalent to another form 3 tree which has the output as the main root output of the bottom red tree. If the output is already on the bottom red tree, then use the form 1 result to make this output the root output, then use relation 3b to move the form 2 stem back onto the left most stem of the bottom red tree.

Otherwise, the output belongs to one of the top red trees. Using relation 5, we can assume this output stem is the right most stem of this red tree. Then the following equivalence suffices:
Now take two trees of form 3 which correspond to the same path component. By what was just shown, we can assume that their main root outputs are labeled the same. Then it must be that their bottom red trees are exactly the same since otherwise the trees would go to different path components. It must also be that their trees of form 2 correspond to the same path component so that they are equivalent by the last case. Thus the two form 3 trees must be equivalent.

This concludes the proof of Claim 2.2.

So now, to complete the Thm, all that needs to be shown is that any tree in $F(G)/S$ is equivalent to some tree in normal form. To see this, first note that every generator is equivalent to a tree in normal form:
Next we verify, case by case, that composing a tree in normal form with a generator gives a tree equivalent to a tree in normal form:

**Form 1**

1. Plugging $e_O$ into an input of a form 1 tree results in another tree of form 1.
2. Plugging $e_{C\rightarrow O}$ into an input is equivalent to a tree of form 3 by relation 3b.
3. Consider plugging $e_{O\rightarrow C}$ into an output. By Claim 2.2 we can assume that the output is the main root output. Thus we get a tree of form 2.
4. Composing with $m_O$, open multiplication, clearly gives another tree of form 1 after we assume again that when composing with an output of the tree, this output is the main root output.
5. Finally, consider composing with $V_O$. If we assume again that when we compose with an output of the tree it is the main root output, then using one of its normal forms below gives a tree of form 1:

![Diagram](image1)

**Form 2**

1. Plugging in units give trees of form 2.
2. If we plug the output of $m_c$ into an input we get another tree of form 2. If we plug an input into an output, then we can assume the output is the rightmost output stem. Then relation 7 suffices.
3. Same argument for $V_c$.
4. Plugging $m_c$ into an input gives another tree of form 2. Now say we plug an input into an output $o_1$ of the tree. Then using the form 1 result, we first take an equivalent tree which has $o_1$ as the base output of the red tree:

![Diagram](image2)
Then plug the input into $o_1$. Then “rotate back” so that the green stem is again attached to the main red output stem, resulting in a tree of type 2.

5. The argument for $\nabla_c$ is the same after we use its appropriate normal form representation as in the last case.

6. Plugging $\varphi_{o\rightarrow c}$ into an input gives another tree of type 2. The following relation shows that plugging into an output gives a tree equiv to a tree of form 2:

7. If we plug $\varphi_{c\rightarrow o}$ into the output of a form 2 tree with one closed output then we get a form 3 tree. If there is more than one output, then we can assume the output being composed with is the right most output. Then the following suffices:
For plugging into an input, use the following equivalence:

![Diagram](image)

**Form 3**

1. Plugging in units gives trees of form 3 (unless the form 2 tree plugged into the left stem is the identity. Then plugging in $e_C$ gives a form 1 tree.)

2. If we plug $\varphi_{C \rightarrow O}$ into an open input which is not on the bottom red tree, then it is plugged into the form 2 tree and this case has been covered. If it is attached to an open input on the bottom red tree, then use this equivalence:

![Diagram](image)

3. Plugging $\varphi_{O \rightarrow C}$ into the input of a form 3 tree gives a form 3 tree. If we plug into an output, then we can assume it is the main root output of the bottom red tree. Then using relation 4 we immediately see it is equivalent to a tree of form 2.

4. Attaching $m_O$ and $\nabla_O$ works as before, again, we can assume the output being plugged into is the main root output.

5. Plugging $m_C$ into an input gives another form 3 tree (and there are no closed outputs to compose with).

6. For plugging $\nabla_C$ into an input, we should get a tree equivalent to form 2. To see that this is so, the following suffices:
This completes the form 3 case.

So inductively we now have that any tree formed by composing generators is equivalent to a tree in normal form. And since the action by the symmetric groups is invariant on the set of trees in normal form, we see that all trees are equivalent to a tree in normal form.

This concludes the proof of Thm 2.1.

Sec 3: Description of \( H_0(C) \)

Before looking at the full \( H_0(OC) \) let’s first see what we need to add to our green degree 0 generators and relations in order to give a complete description of \( H_0(C) \), where \( C \) is the moduli space of Riemann spheres with closed inputs/outputs and no boundary.

We add to the list of generators the degree one BV operator given by rotating the input parameterization 360 degrees:

9. **BV operator**

\[ \Delta : V_C \rightarrow V_C \]

Then we have the following new relations:

11. The usual BV relations:
12. The following relation holds since rotating the input on a sphere with an input at the north pole and an output at the south pole is conformally equivalent to rotating the output:

\[
(\nabla_C(a) \cdot \Delta) \cdot b = a \cdot (\Delta \cdot \nabla_C(b))
\]

**Theorem 3.1**

The green generators and green relations completely describe \( H_*(C) \).

**Proof:**

Again we consider the dioperad morphism from the free dioperad generated by the green trees representing our green generators modulo the green relations to \( H_*(C) \). By the description of the homology of the operad formed by spheres with closed inputs and one closed output, we know that there is a vector space isomorphism between \( H_*(C) \) and the span of equivalence classes of trees of the following form:

Here, it is required that the main root output be labeled 1.

Now, say a tree is in normal form if it is of the above form except that the main root output is not required to be labeled 1.
Claim 3.1
Restricting the morphism to the span of equivalence classes of trees in normal form gives a vector space isomorphism onto $H_* (C)$.

Proof:
By the above fact, it suffices to show that any tree in normal form is equivalent to another tree in normal form which has its main root output labeled 1. We can assume that the output labeled 1 is connected to the leftmost input of the tree made out of the multiplication and BV generators. Then it suffices to check the claim for trees like the following:

But it is easy to see how to achieve this using relations 7 and 12. For example, for this tree we have:
Warning
All equivalencies before this last one sufficed in showing the desired property, i.e. the property was directly implied by the equivalence. This last equivalence is just an example for this particular tree and does not imply the property for all the trees that the property needs to be shown for. However, it is easy to see that the method above of using a sequence of applications of relations 7 and 12, not necessarily alternating, will work for all of the trees that we need to check the property for.

Thus the claim is true.

So all that we need to do is show that every tree is equivalent to a tree in normal form. We proceed as in the last section by first noting that all the green generators are equivalent to normal form. Next, we need to check that composing a normal form tree with a green generator gives a tree equivalent to a tree in normal form. But using one of $\mathbf{\nabla}_c$'s normal form representations, we automatically get trees in normal form when composing with any green generator since by above we can assume that the when we compose with an output, that output is the main root output.

Since, again, the symmetric groups act invariantly on the normal form trees, all trees are equiv. to a tree in normal form. Thus Thm. 3.1 is proved.

Sec 4: Description of $H_*(OC)$
Taking care of $H_0(OC)$ and $H_*(C)$ first makes the arguing for $H_*(OC)$ go smoothly since now we can replace any degree 0 tree with another tree which goes to the same path component and we can put any green tree in the normal form of Sec 3 with the main output labeled as we desire.

We do not need to add anymore generators to our list of 9, and there is only one more relation that we need to add:

13. $\Delta(\varphi_{O-C}(e_0)) = 0$. This relation comes from the fact that rotating the boundary component in a sphere with a closed output over the south pole and an empty boundary component over the north pole 360 degrees gives a constant map into the moduli space since boundaries are not parameterized

**Theorem 4.1**
The list of 9 generators (pg. 5-6,20) and 13 relations (pg. 6-10,20-21,23) completely describe $H_*(OC)$.

**Proof:**
Let's first discuss the homotopy type of an arbitrary path component of $OC$. First note that OC is h.e. to the moduli space of Riemann spheres with boundary, with labeled punctures in the interior and on the boundaries (no parameterizations in either case), and each puncture in the interior coming with a tangent direction.

Now consider a path component with $n$ interior punctures with directions, $k$ boundary components each having exactly one puncture, and $l$ empty boundary components. Then up to homotopy we can replace a marked boundary component with a puncture and a direction and an empty boundary component with a puncture. Thus it is h.e. to the moduli space of spheres with $n+k$ labeled punctures with directions and $l$ unlabeled punctures without directions.

For an arbitrary path component $P$ in $OC$, consider the forgetful fibre bundle $P \to \vec{P}$ where $\vec{P}$ is the moduli space obtained from $P$ by dropping all but one fixed open puncture on each non-empty boundary component. Then it is clear that the fiber is contractible.

Using the above facts and mixing the descriptions of the homologies of the framed and non-framed little disks operads, we can get a vector space isomorphism between $H_*(P)$ and the span of the equivalence classes of a set of trees in $F(G)/S$, the free dioperad on all our generators modulo all our 13
relations. We can see how the general case works while avoiding indexing messiness by assuming $P$ is of type, say, $\{1, 2, 1, 2\}, (3, 3, 4, 4), (5, 6), (5, 6), ()$. If we choose a closed output, say $1_o$, then a set for which this is true is the set of all trees of the form:

![Diagram of a tree](image1)

If we choose an open output, say $3_o$, then another set that works is trees of the form:

![Diagram of a tree](image2)
With this in mind we’ll now say that a tree is in normal form if it is or either of the 2 forms above (with any labeling).

Note that since we allow the green tree in the second form to be $e_c$, and $\varphi_{C\rightarrow O}(e_c) = e_o$, the second form contains completely red trees. Thus the above two forms give all path components.

**Claim 4.1**
Restricting to the span of equivalence classes of trees in normal form gives a vector space isomorphism onto $H_c(OC)$.

*Proof:*
By the above discussion, along with the $H_0(OC)$ result, all that needs to be shown is that for any tree in normal form, and for any of its outputs, it is equivalent to another tree in normal form which has this output as the main root output. But this follows from the same result for the green normal form trees of the last section and the $H_0(OC)$ result. For example, this shows the property for a tree of form two and an open output: (we can assume the red tree containing the output is on the left most green input of the green tree)
This proves Claim 4.1.

So in order to finish the proof Theorem 4, all that needs to be shown is that any tree is equivalent to a tree in normal form. This again can be done inductively as in the last two sections. But this is now straightforward to check given what has been developed so far and can be left to the reader. This concludes the proof of the main Theorem 4.1.

Sec 5: Cyclic and Semi-modular structure of operad

Restricting to the generators with one output and the relations only involving these generators, we get a complete description of the homology of the operad formed by the components of $OC$ with only one output (we’ll abuse notation and call this operad $OC$ also). This can be seen by restricting the proof above to trees with only one output. This operad is cyclic (see [8]) in the sense that there is no natural output, requiring us to label it. The action which permutes all the labels extends the action which only permutes the input labels and it does it in a composition respecting way. Thus the homology forms a 2-colored cyclic operad in the category of graded vector spaces.

The definition given in [8] of the cyclic endomorphism operad for a graded vector space $V$, finite dimensional in each degree, and a non-degenerate inner product $B$ on $V$ can be naturally extended to the colored case. $V_C$ and $V_O$ come with inner products $C_B$ and $O_B$ which are used to identify $Hom(V_{i_1} \otimes V_{i_2} \otimes \ldots \otimes V_{i_n}, V_j)$ with $HOM(V_{j_1} \otimes V_{j_2} \otimes \ldots \otimes V_{j_n}, F)$ ($F$ the base field) where $i_k, j \in \{C, O\}$.

Then since the permutation $(1, 2, 3)$ sends $m_C$ and $m_O$ to themselves, $(1, 2)$ sends $\Delta$ to itself, and $(1, 2)$ interchanges $\varphi_{C \rightarrow O}$ and $\varphi_{O \rightarrow C}$ we get that an algebra over the cyclic $H_*(OC)$ satisfies the following 4 additional relations:

1) $B_C(a, bc) = B_C(ab, c)$

2) $B_O(a, bc) = B_C(ab, c)$

3) $B_C(\Delta(a), b) = (-1)^{|a|}B_C(a, \Delta(bc))$

4) $B_O(\varphi_{C \rightarrow O}(a), b) = B_C(a, \varphi_{O \rightarrow C}(b))$

The operad $OC$ does not form a modular operad (see [9]). This is because self-sewing in general results in a surface of genus $> 0$. However, sewing two open inputs on the same boundary component gives another sphere with one more boundary component,

\[
\Psi_{ij}
\]
To see what extra algebra structure this adds, we need an appropriate endomorphism definition. In order for a map from $H_*(OC)$ to this endomorphism operad to respect the contractions

$$\Psi_{ij} : H_*(OC(n)) \rightarrow H_*(OC(n-2))$$

we need the endomorphism contractions

$$\Psi_{ij} : \text{End}(n) \rightarrow \text{End}(n-2)$$
to give zero when applied to a homomorphism which is in the image of the homology of a path component in which the inputs $i$ and $j$ are not open inputs on the same boundary component. With this in mind we make the following:

**Definition: OC cyclic semi-modular endomorphism operad**

For $(\mathcal{V}_C, B_C), (\mathcal{V}_O, B_O)$ as above, let $\text{End}(n) = \bigoplus_{\text{type}} \text{End}(\text{type}, n)$ where

$$\text{End}(\text{type}, n) = \text{Hom}(V_i \otimes \cdots \otimes V_n, V_j),$$

the $i_k$'s and $j$ are either $O$ or $C$ and are determined by the type, and type runs over all path components in $OC$ with $n$ inputs. We have the cyclic structure as above, and we define the compositions so that $f \circ_i g \in \text{End}(\text{type}(f) \circ_i \text{type}(g), n + m - 1)$. We use $B_C$ and $B_O$ to identify $\text{Hom}(V_i \otimes \cdots \otimes V_n, V_j)$ with $V_j \otimes V_i \otimes \cdots \otimes V_n$, and use $B_O$ to define the contractions (self-sewings) $\Psi_{ij}(i, j)$ just as in the standard definition of the modular endomorphism operad, provided the type has $i$ and $j$ together in the partition. Otherwise $\Psi_{ij}$ is defined to be zero. $\Psi_{ij}$ should take $f \in \text{End}(\text{type}, n)$ to $\text{End}(\text{type}', n-2)$, type' being the path comp. which is the image of the map $\Psi_{ij}$ restricted to the path component type in $OC$.

Considering the normal form representation of an element in $H_*(OC)$ and the fact that contractions are defined only for inputs on the same boundary component, we see that the extra structure is completely determined by the $\Psi_{ij}$'s restricted to completely red trees. So we get one more relation:

(If $i = j - 1$ then the top red tree is just $e_O$)

Algebraically, this relation says if $m_O(n) : \mathcal{V}_O \otimes^n \rightarrow \mathcal{V}_O$ is the operation given by associative multiplication, $m_O(n)(a_1, \ldots, a_n) = a_1 \cdots a_n$, then

$$\Psi_{ij}(m_O(n))(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n) = (a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n) \phi(a_{i+1} \cdots a_{j-1})$$

where $\phi = \phi_{C \rightarrow O} \circ \phi_{O \rightarrow C}$ and an empty parentheses means $(e_O)$.
Sec. 6: Obtaining operad structure in open/closed string topology

This section is an extension to an open/closed setting of the construction by Voronov [14] in which he defined the Cacti operad, h.e. to framed little disks, and showed how it produces the BV-structure on the homology of a free loop space $LM$ of a compact oriented manifold $M$ of dimension $m$ given by Chas-Sullivan [1]. The goal is to define a 2-colored version of Cacti h.e. to $OC$ and use it to realize the pair $(H_*(LM), H_*(PM_K))$ as an algebra over $H_* (OC)$ where $K$ is a fixed oriented closed submanifold of dimension $k$ and $PM_K$ is the space of paths starting and ending in $K$.

Let's first recall what the Cacti operad is and how we get $H_* (LM)$ as an algebra over its homology. Basically, Cacti is what results when you get rid of everything in $C$, the moduli space of Sec. 3, except the boundaries of the closed inputs and outputs. A typical point in $\text{Cacti}(n)$ is shown in the following picture:

![Cacti diagram]

**Explanation**: A point is given by a tree like configuration of $n$ labeled circles (lobes) of varying radii. Each circle is parameterized by marking a point on it. When 3 or more circles intersect at one point, they are given a cyclic order (can draw using counter clockwise orientation of the plane). Finally, if you start at any point on some circle and trace the picture in the counter clockwise direction, using the cyclic orderings to jump from circle to circle, then the entire picture will be traversed before returning to the starting point. Thus putting one more marked point on the picture gives it an $S^1$ parameterization and we consider the whole boundary as the output. To compose two pictures, just replace the $i_{th}$ input circle of one picture with an entire second picture via their parameterizations.

If we consider the space of maps from a point as above into our manifold $M$, then restricting to the inputs gives us an embedding from the space of maps to $LM^n$ of finite codimension $(n-1)m$. Restricting to the output gives a map into $LM$. Thus we get the following diagram:

$$\text{Cacti}(n) \times LM^n \xleftarrow{i} \text{L}^{\text{Cacti}(n)} M \longrightarrow LM$$

where $\text{L}^{\text{Cacti}(n)} M$ is the space $\{(x, f) \mid x \in \text{Cacti}(n), f \in \text{Maps}(x, M)\}$. The map $i$ is a finite codimensional embedding of codim. $(n-1)m$.

Applying the Pontryagin-Thom construction to the map $i$ to get the push-forward map in homology, and then composing, we get the action $H_*(\text{Cacti}(n)) \otimes H_*(LM)^{\otimes n} \longrightarrow H_*(LM)$. This map has degree $-(n-1)m$ and gives an operad morphism, i.e. it commutes with composition and is equivariant. The operations corresponding to the generators of $H_0(\text{Cacti}(2))$ and $H_1(\text{Cacti}(1))$ are exactly the BV-operations of Chas-Sullivan.

In our situation, if we consider maps from a point in $OC$ to the manifold $M$ such that all boundary components map into the submanifold $K$, then restricting to the $S^1$ boundary of a closed input/output gives a point in $LM$ and restricting to the arc of the boundary of an open input/output gives a
point in $PM_K$. Thus we get a similar diagram as above with $OC, LM, PM_K$, and we just need to replace OC by a skeletal model which will make the left arrow a finite codimensional embedding.

Right away we see, however, that we can not get rid of everything except the boundaries of the input/outputs and get a space h.e. to OC since we have these boundary components with no open inputs on them. It is possible to define an operad which ignores these empty boundary components and gives us finite codim. embeddings, but the resulting action does not commute with composition. This can be seen by considering the degrees of the induced operations. Thus we are forced to keep the empty boundary components in our picture. This in turn forces us to keep the part of the boundaries in between two open inputs so that composition gives an empty boundary when it should.

To handle the fact that this prevents us from directly getting finite codim. embeddings, consider the following:

\[
PM_K \times PM_K \quad \leftrightarrow \quad \text{Maps}(\cdots, M,K) \to LM
\]

\[
PM_K \times PM_K \times PK \times PK \quad \leftrightarrow \quad \text{Maps}(\cdots, M,K) \to LM
\]

(The maps must send everything in black (except the main marked point) into the submanifold $K$)

Both of these maps are of finite codimension, the first of codim. $2k$ and the second of codim. $4k$. So we can get the push forward maps in homology and compose to get operations. The key observation is that if we plug the fundamental class $\in H_k(PK)$ into the $H_*(PK)'s$ then the two resulting operations are the same degree $-2k$ operation. This is the operation which is induced at the chain level by the function which takes two cells $PM_K$ and transversally intersects the endpoints of the intervals of the first cell with the beginning points of the second cell and transversally intersects the beginning points of the first cell with the endpoints of the last cell. This results in a chain in LM of dimension $2k$ less than the sum of the dimensions of the two cells.

For the next observation, consider the operations given by the following two pictures:

The left picture results in an operation of degree $-(4k + m)$ while the right one gives a $-5k$ degree operation. This is a problem since these two pictures would be in the same path component of our potential colored Cacti. To remedy this, “ghost edges” are introduced.
This map has codimension $8k + 2m$ no matter how the two circles are connected by the ghost edge. If we plug in the fundamental classes $\in H_k(PK), H_m(PM)$ after getting the push forward, then we get a degree $-(4k + m)$ operation which is the same as the operation given by the left picture above. This is the operation which takes 4 cells in $PM_K$, applies the above operation to the first two and the last two resulting in two chains in $LM$, and then takes the loop product of these two chains as in Chas-Sullivan.

These two observations motivate the following:

**Definition of Open/Closed Cacti**
The definition can be given by considering the following pictures which show typical points in the configuration space. First consider the case where the output is closed:
**Explanation:**
--The inputs are labeled. Green circles are closed inputs and red intervals are open inputs.
--There is a marked point somewhere on the picture giving the starting point of the output. If the marked point is on a circle, go in the counter clockwise direction. If it’s on a ghost edge, there needs to be an arrow pointing in the starting direction. Then if we take the cyclic ordering of the edges meeting at a vertex to be given by counter clockwise orientation of the plane, there is a $S^1$ parameterization of the picture as in cacti.
--The closed inputs should have a mark as in cacti.
--For a black circle (empty boundary component) with more than one vertex, mark one of the vertices. This marks where a boundary edge (black interval) “sews up” into a circle when composing.
--We can make our definition so that there is always $n - 1$ ghost edges when there are $n$ circles. We need this so that there are the same number of $PM$'s to map into as above for any two pictures in the same path component. This is done by choosing $t - 2$ of the rays emanating from an intersection vertex which is not on a circle and has $t$ ghost rays emanating from it. For the chosen $t - 2$ rays, the rays are considered as ghost edges for which this vertex is an endpoint. The other two rays are considered as one ghost edge and this vertex is just in the middle of it. For example, in the above picture, there are three ghost edges, not 5, connecting the four circles in the middle. The following picture shows the three different ways to connect three circles with 2 ghost edges which intersect off of the circles and the path in the configuration space which connects them. After seeing this example (and after seeing the picture for the case of an open output), it is not too hard to see why the path components of Open/Closed Cacti are in correspondence with the path components of $OC$:
Now let’s look at the picture showing a typical point in the config space of O/C Cacti which has an open output:

Explanation:
--Here, the number of ghost edges is equal to the number of circles.
--The interval at the bottom corresponds to boundary component containing the open output.
--If we start at the right of the interval and use the counter clock-wise orientation of the plane, we traverse the entire picture and end up at the left endpoint of the interval. This gives us a parameterization of the picture by the unit interval and will serve as the output.
--If there are no open inputs on the same boundary comp as the output, then this interval should just be a black point (which should be sent to the submanifold $K$ when we consider maps into our manifold $M$ to get an action).

We can compose pictures by identifying the parameterization of a input circle or interval with the output parameterization of an entire picture, as in Cacti. If the endpoint of a ghost edge lies on the input circle or interval, and if when we compose it happens that this endpoint is connected to the interior of a 2nd ghost edge in the picture we are replacing the input with, then this should not break the 2nd ghost edge into two edges. As described above, this endpoint just lies in the middle of this 2nd ghost edge. Also, when connecting the endpoint to a 2nd ghost edge (interior or endpoint), there is natural way to cyclically order the ghost rays emanating from this point of attachment in the result. Just “stick” the cyclic order of all the ghost edges in the input picture for which this point on the input circle (or interval) is an endpoint into the cyclic order of the ghost rays emanating from this point of attachment in the output picture in between the ray which comes right before this point in the output parameterization and the ray which comes right after this point in the output parameterization. For example, the following picture show the 3 different ways the ghost edges ending at a point on the input circle can be attached to the intersection point of two ghost edges:
Thus we see that after giving this configuration space its natural topology, as with Cacti or the Sullivan Chord Diagrams of [5], it forms a well-defined topological 2-colored operad which we’ll call Open/Closed Cacti or OC Cacti.

**Claim 6.1**

\( H_\ast(OC \text{Cacti}) \) is isomorphic to \( H_\ast(OC) \)

**Proof:**

It is clear that the path components are in bijective correspondence. Fix a path component of OC Cacti. In a similar manner as in the proof of the homotopy type of a path component of \( OC (dioperad) \) in Sec. 4, assume first that this path component has \( n \) green circles (closed inputs), \( k \) circles with exactly one open input, and \( l \) completely black circles (empty boundary components). If the path component has an open output then assume that there are no open inputs on the interval, i.e. the interval is really a point (the boundary comp. possessing the open output does not have an open input).

Then this path component is h.e. to \( Cacti(n + k + l) \) except that \( l \) of the circles are not labelled and do not have a marked point. And we know [] that \( Cacti \) is h.e. to \( JD \), the framed little disks operad. Thus it follows that this path component is h.e. to the configuration of \( n + k + l \) disks inside the unit disk such that \( n + k \) of the disks have labels and directions and \( l \) of the disks have neither directions nor labels. But this is the homotopy type of the corresponding path component in \( OC \), as described in Sec. 4.

Finally, just as in \( OC \), the forgetful map from an arbitrary path component to the space obtained from this path comp. by dropping all but one fixed open input on each of the circles with 2 or more open inputs and dropping all inputs on the interval. Then this is a fibre bundle with contractible fibre.

Thus \( H_\ast(OC \text{Cacti}) \) is isomorphic to \( H_\ast(OC) \) as a vector space. But it is clear that the map from \( H_\ast(OC \text{Cacti}) \) to \( H_\ast(OC) \), given by the vector space isomorphisms induced on each path component by the homotopy equivalencies above, is equivariant and commutes with composition. So they are isomorphic as colored operads.

\[ \blacksquare \]
Next we discuss how to get an action of $H_s(OC\text{ Cacti})$ on $(H_s(LM), H_s(PM_K))$. First, let’s consider the diagram associated to a fixed point in $c \in OC\text{ Cacti}$. If $c_1$ is the number of closed inputs, $c_2$ the number of open inputs, $c_3$ the number of black intervals + the number of black circles, and $c_4$ the number of ghost edges, then we have:

$$LM^{c_1} \times PM^{c_2}_K \times PK^{c_3} \times PM^{c_4} \xleftarrow{\text{in}} \text{Maps}(c, M, K) \xrightarrow{\text{out}} LM \text{ or } PM_K$$

The maps should send all black intervals and circles into the submanifold $K$. If the output is open, then the endpoints of the interval should also be sent into $K$ (if this interval is really a point, then this point should go to $K$). Remember that one of the vertices of a black circle is marked so that a map restricted to this circle gives us a point in $LK \subseteq PK$.

The map $\text{in}$ is an embedding of finite codimension. So we can get the push forward map in homology and then plug the fundamental classes into the $H_s(PK)$ and $H_s(PM)$’s to obtain an operation $H_s(LM) \otimes_{c_1} H_s(PM_K) \otimes_{c_2} \rightarrow H_s(LM) \text{ or } H_s(PM_K)$. The codimension can be computed and thus the degree of this operation. In the case of a closed output, the degree is $-[(\# \text{ of circles} - 1) \cdot m + (\# \text{ of open inputs}) \cdot k]$. In the case of an open output the degree is $-[(\# \text{ of circles}) \cdot m + (\# \text{ of open inputs} - 1) \cdot k]$. These degrees are exactly what they should be. That is, if we build the path component of $c$ out of the generators of $H_0(OC\text{ Cacti})$ via a tree in normal form, then the degree of the operation given by $c$ agrees with the degree of the operation given by composing the operations in $\text{End}(H_*(LM), H_*(PM_K))$ corresponding the generators in the same way.

Next, consider the path component $P$ where $c \in P$. We “ALMOST” have the following diagram: $P \times LM^{c_1} \times PM^{c_2}_K \times PK^{c_3} \times PM^{c_4} \xleftarrow{\text{in}} \xrightarrow{\text{out}} \text{Maps}(P, M) \rightarrow LM \text{ or } PM$. The issue is that there are choices that would have to be made for each $c \in P$ in order to get a map $\text{in}$ and it is not clear that this can be done in a continuous way. For the black intervals of the picture, there is a canonical way to choose the starting point of the interval and which $PK$ it corresponds to. But for the ghost edges, there is no canonical way to choose which endpoint is its starting point and which $PM$ it corresponds to. Also, there is no canonical way to choose which $PK$ a black circle corresponds to.

There is no way to add these labels to our operad such that it doesn’t matter the order in which we compose $n$ pictures into the $n$ inputs of a picture. However, it is not necessary to change the definition of $OC\text{ Cacti}$. Let $\overline{P}$ be the space obtained from $P$ by labeling the ghost edges and black circles, and marking one endpoint on each ghost edge. Then $\overline{P} \rightarrow P$ is a quotient map by a finite and free action so that the induced map $H_*(\overline{P}) \rightarrow H_*(P)$ is onto (remember we are over a field of characteristic 0). In fact, it should give an isomorphism when restricted to any path component in $\overline{P}$ (to see $\overline{P}$ might not be path connected, just consider the case of two green circles with one ghost edge connecting them).

Now, we do have the above diagram for $\overline{P}$. So we can get the push forward and obtain $H_*(\overline{P}) \otimes H_*(LM) \otimes_{c_1} H_*(PM_K) \otimes_{c_2} \rightarrow H_*(LM) \text{ or } H_*(PM_K)$. But it is fairly clear that all the elements in $H_*(\overline{P})$ which are in the preimage of one element in $H_*(P)$ give the same operation since we are plugging in fundamental classes. Thus we obtain our desired action.

**Theorem 6.2:** This action is an operad morphism.

**Proof:**

It is clearly equivariant so we need to see that it respects composition. An appropriate commutative diagram could probably be set up, but it is more enlightening to argue directly as follows:
We can give a description of the operation corresponding to any element in $H_s(OC\text{ Cacti})$.

First consider the operation corresponding to the following path component:

It is clear what operation this gives. It transversally intersects the right endpoint of the first input with the left endpoint of the second input (degree $-k$), transversally intersects the right endpoint of the second with left of the third, ..., and does the same with the $(n-1)st$ and $nth$. Finally, it takes the resulting chain of paths and transversally intersects the starting points with the ending points (degree $-k$) giving a chain of loops. This is a degree $-(nk)$ operation with output in $H_*(LM)$.

Putting this together with the result for regular Cacti, we can see what operation corresponds to any homology class of a path component with a closed output and no completely black circles. Take a homology class corresponding to a tree in normal form. Then the corresponding operation is exactly the one obtained by composing the operations corresponding to the generators $m_c$, $m_o$, and $\varphi_{o \to c}$ as prescribed by the tree in normal form.

Next consider a homology class of a path component with an open output and no black circles corresponding to some tree in normal form. Notice that there exists a chain representing this homology class such that its image in the path component consists only of pictures which have a picture of the type with a closed output connected to the interval by one ghost edge. The ghost edge and interval can remain constant. Thus it suffices to consider the operation given by the following path component:
Since the operation is well defined on a path component, we can assume the ghost edge is connected to the leftmost endpoint of the interval. Then we see that the operation can be described as first taking the 1st input and transversally intersects its marked point with $K$, then multiplying it by the other inputs in $H_*(PM_K)$. If the interval is just a point, then this picture is $\varphi_{C \to O}$ and its image is the operation which transversally intersects the marked points of the loops with $K$.

So putting this together with the case of a closed output above, we see that again the image of this homology class is the operation given by composing the operations corresponding to the generators via the tree in normal form.

Finally we see what effect the empty boundary components have. Let’s consider what operation is given by the path component which corresponds to a sphere with a closed output, one closed input, and one empty boundary component:

We get an operation $H_* (LM) \otimes H_* (PK) \to H_* (LM)$. This operation takes an element of $H_* (PK)$ and transversally intersects the endpoints giving an element of $H_* (LM)$ and then loop multiplies this element by the input $e_0 \in H_* (LM)$. Thus after plugging in $e_0$, the fundamental class of $H_* (PK)$, we get that this operation is $a\varphi_{O \to C} (e_0)$ (where $\varphi_{O \to C}$ here means the image in $End(H_* (LM), H_* (PM_K)$ of the operation $\varphi_{O \to C} \in H_* (OC Cacti)$. A black circle by itself would just correspond to the point $\varphi_{O \to C} (e_0) \in H_0 (LM)$, as it should since this is how we get a black circle in OC Cacti.

Putting this together, we now see that the image of any homology class in $H_* (OC Cacti)$ corresponding to a tree in normal form is exactly the operation given by composing the images of the generators as prescribed by the tree.

Finally, we can just check that the operations corresponding to the generators satisfy the relations of $OC Cacti$.

These two facts together imply that the action respects composition.
Bibliography

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