QUANTUM GROUPS AND THEIR APPLICATIONS IN NUCLEAR PHYSICS

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ABSTRACT

Quantum algebras are a mathematical tool which provides us with a class of symmetries wider than that of Lie algebras, which are contained in the former as a special case. After a self-contained introduction to the necessary mathematical tools (q-numbers, q-analysis, q-oscillators, q-algebras), the $su_q(2)$ rotator model and its extensions, the construction of deformed exactly soluble models ($u(3) \supset so(3)$ model, Interacting Boson Model, Moszkowski model), the 3-dimensional q-deformed harmonic oscillator and its relation to the nuclear shell model, the use of deformed bosons in the description of pairing correlations, and the symmetries of the anisotropic quantum harmonic oscillator with rational ratios of frequencies, which underly the structure of superdeformed and hyperdeformed nuclei, are discussed in some detail. A brief description of similar applications to the structure of molecules and of atomic clusters, as well as an outlook are also given.

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1. Introduction

Quantum algebras [1,2,3,4,5] (also called quantum groups) are deformed versions of the usual Lie algebras, to which they reduce when the deformation parameter q is set equal to unity. From the mathematical point of view they are Hopf algebras [6]. Details about their mathematical properties can be found in [7,8,9,10,11,12,13]. Their use in physics became popular with the introduction of the q-deformed harmonic oscillator (sec. 10) as a tool for providing a boson realization of the quantum algebra $su_q(2)$ (sec. 14), although similar mathematical structures had already been known (sec. 11). Initially used for solving the quantum Yang–Baxter equation [3], quantum algebras have subsequently found applications in several branches of physics, as, for example, in the description of spin chains[14,15], anyons[16,17,18], quantum optics[19,20,21,22,23], rotational and vibrational nuclear and molecular spectra, and in conformal field theories. By now several kinds of generalized deformed oscillators (sec. 12) and generalized deformed algebras (sec. 17) have been introduced.

It is clear that quantum algebras provide us with a class of symmetries which is richer than the class of Lie symmetries, which are contained in the former as a special case. It is therefore conceivable that quantum algebras can turn out to be appropriate for describing symmetries of physical systems which are outside the realm of Lie algebras.

Here we shall confine ourselves to applications of quantum algebras in nuclear structure physics. The structure of this review is as follows: In order to make this review self-contained, we are going first to give a brief account of the necessary tools: q-numbers and q-analysis (secs 2–9), q-deformed oscillators (secs 10–13), q-deformed algebras (secs 14–18). The remainder will be devoted to specific applications in nuclear structure problems, starting with phenomenology and advancing towards more

microscopic subjects. The $su_q(2)$ rotator model (secs 19–23) and its extensions (secs 24–26), as well as the formulation of deformed exactly soluble models (u(3)⊃so(3) model (secs 28, 29), Interacting Boson Model (secs 27, 30, 31), Moszkowski model (sec. 32)) will be covered in some detail. In addition the 3-dimensional q-deformed harmonic oscillator and its relation to the nuclear shell model (sec. 28) will be extensively discussed. Subsequently, the use of quantum algebraic techniques for the description of pairing correlations in nuclei (secs 33–35), as well as the symmetries of the anisotropic quantum harmonic oscillators with rational ratios of frequencies (sec. 36) will also be considered in some detail. The latter are of current interest in connection with the symmetries underlying superdeformed and hyperdeformed nuclear bands (sec. 36). Finally, a brief account of applications of the same techniques to the structure of molecules (sec. 37) and of atomic clusters (sec. 38), as well as an outlook (sec. 39) will be given. An earlier review of the subject has been given in [24].

2. q-numbers

The *q-number* corresponding to the ordinary number x is defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{2.1}$$

where q is a parameter. The same definition holds if x is an operator. We remark that q-numbers remain invariant under the substitution $q \to q^{-1}$.

If q is real, q-numbers can easily be put in the form

$$[x] = \frac{\sinh(\tau x)}{\sinh(\tau)}, \tag{2.2}$$

where $q = e^{\tau}$ and τ is real.

If q is a phase factor, q-numbers can be written as

$$[x] = \frac{\sin(\tau x)}{\sin(\tau)}, \tag{2.3}$$

where $q = e^{i\tau}$ and τ is real.

In both cases it is clear that in the limit $q \to 1$ (or, equivalently, $\tau \to 0$) q-numbers (or operators) tend to the ordinary numbers (or operators):

$$\lim_{q \to 1} [x] = x. \tag{2.4}$$

A few examples of q-numbers are given here:

$$[0] = 0, \qquad [1] = 1, \qquad [2] = q + q^{-1}, \qquad [3] = q^2 + 1 + q^{-2}. \tag{2.5}$$

Identities between q-numbers exist. They are, however, different from the familiar identities between usual numbers. As an exercise one can show (using the definition of q-numbers) that

$$[a][b+1] - [b][a+1] = [a-b]. \tag{2.6}$$

The *q-factorial* of an integer n is defined as

$$[n]! = [n][n-1] \ldots [2][1]. \tag{2.7}$$

The *q-binomial coefficients* are defined as

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]!}{[m-n]![n]!}, \tag{2.8}$$

while the *q-binomial expansion* is given by

$$[a \pm b]^m = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} a^{m-k} (\pm b)^k. \tag{2.9}$$

In the limit $q \to 1$ we obviously have

$$[n]! \to n! \qquad \text{and} \qquad \begin{bmatrix} m \\ n \end{bmatrix} \to \binom{m}{n}, \tag{2.10}$$

where $n!$ and $\binom{m}{n}$ are the standard factorial and binomial coefficients respectively.

It should be noticed that two-parameter deformed numbers have also been introduced

$$[x]_{p,q} = \frac{q^x - p^{-x}}{q - p^{-1}}. \tag{2.11}$$

In the special case $p = q$ they reduce to the usual q-numbers.

3. q-deformed elementary functions

In addition to q-deformed numbers and operators, q-deformed elementary functions can be introduced. The *q-exponential function* is defined as

$$e_q(ax) = \sum_{n=0}^{\infty} \frac{a^n}{[n]!} x^n, \tag{3.1}$$

while the *q-trigonometric functions* are defined as

$$\sin_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{[2n+1]!}, \qquad \cos_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{[2n]!}. \tag{3.2}$$

It should also be noticed that q-deformed polynomials, such as q-deformed Hermite polynomials [25,26,27] and q-deformed Laguerre polynomials [25] also exist (see also subsec. 36.3).

The definitions of the q-exponential function and the q-trigonometric functions given above are not unique; for a different set of definitions, based on the Tsallis statistics, see [28].

4. q-derivatives

Proceeding along this path one can build a new differential calculus, based on q-deformed quantities (see [29,30] for concise expositions). For this purpose the *q-derivative* is defined as

$$D_x^q f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}. \tag{4.1}$$

The similarity between the present definition and the one of q-numbers (eq.(2.1)) is clear.

Using the definition of the q-derivative one can easily see that

$$D_x^q (ax^n) = a[n]x^{n-1}, \tag{4.2}$$

$$D_x^q e_q(ax) = a e_q(ax). \tag{4.3}$$

¿From the definition of the q-derivative one can further derive the sum rule

$$D_x^q (f(x) + g(x)) = D_x^q f(x) + D_x^q g(x), \tag{4.4}$$

as well as the rule

$$D_{x_2}^q [ax_1 \pm bx_2]^m = \pm [m] b [ax_1 \pm bx_2]^{m-1}, \tag{4.5}$$

where a and b are costants and $[ax_1 \pm bx_2]^m$ is given by the q-binomial expansion (eq. (2.9)). One can also prove the *q-integration by parts* formula

$$D_x^q (f(x)g(x)) = \frac{f(qx)g(qx) - f(q^{-1}x)g(q^{-1}x)}{(q - q^{-1})x}. \tag{4.6}$$

From this, the following two forms of the Leibnitz rule can be derived

$$D_x^q(f(x)g(x)) = (D_x^q f(x))g(q^{-1}x) + f(qx)(D_x^q g(x)), \tag{4.7}$$

$$D_x^q(f(x)g(x)) = (D_x^q g(x))f(q^{-1}x) + g(qx)(D_x^q f(x)). \tag{4.8}$$

In addition one can show the property

$$D_x^q f(qx) = qD_x^q f(x)|_{x=qx}, \tag{4.9}$$

and the chain rules

$$D_{ax}^q f(x) = \frac{1}{a} D_x^q f(x) \qquad , \tag{4.10}$$

$$D_x^q f(x^n) = [n]x^{n-1} D_{x^n}^{q^n} f(x^n), \tag{4.11}$$

where a is a constant. Another useful result is

$$D_x^{q^n} f(x) = \frac{1}{[n]} \sum_{k=0}^{n-1} D_x^q f(q^{2k-(n-1)}x). \tag{4.12}$$

5. q-integration

The q-integration (see [29,30] for concise expositions) in the interval $[0, a]$ is defined by

$$\int_0^a f(x)d_q x = a(q^{-1} - q)\sum_{n=0}^{\infty} q^{2n+1} f(q^{2n+1}a), \tag{5.1}$$

while for the interval $[0, \infty)$ one has

$$\int_0^\infty f(x)d_q x = (q^{-1} - q)\sum_{n=-\infty}^{\infty} q^{2n+1} f(q^{2n+1}). \tag{5.2}$$

The *indefinite q-integral* is defined as

$$\int f(x)d_q x = (q^{-1} - q)\sum_{n=0}^{\infty} q^{2n+1} x f(q^{2n+1}x) + \text{constant}, \tag{5.3}$$

where $0 < q < 1$. For entire functions $f(x)$ one can easily see that this q-integral approaches the Riemann integral as $q \to 1$, and also that the operators of q-differentiation and q-integration are inverse to each other

$$D_x^q \int f(x)d_q x = f(x) = \int D_x^q f(x)d_q x. \tag{5.4}$$

One can also easily see that

$$\int ax^{n-1} d_q x = \frac{1}{[n]} ax^n + \text{constant}, \tag{5.5}$$

$$\int e_q(ax)d_q x = \frac{1}{a} e_q(ax) + \text{constant}. \tag{5.6}$$

From (4.6) one can also prove the following formulae of integration by parts

$$\int_0^a f(qx)(D_x^q g(x))d_q x = f(x)g(x)|_{x=0}^{x=a} - \int_0^a (D_x^q f(x))g(q^{-1}x)d_q x, \tag{5.7}$$

$$\int_0^a f(q^{-1}x)(D_x^q g(x))d_q x = f(x)g(x)|_{x=0}^{x=a} - \int_0^a (D_x^q f(x))g(qx)d_q x. \tag{5.8}$$

The following formulae can also be proved

$$\int f(x)d_{aq}x = a\int f(x)d_q x, \tag{5.9}$$

$$\int f(x^n) d_{q^n} x^n = [n] \int x^{n-1} f(x^n) d_q x, \tag{5.10}$$

$$\int f(x) d_q x = \frac{1}{[n]} \sum_{k=0}^{n-1} q^{2k-(n-1)} \int f(q^{2k-(n-1)} x) d_{q^n} x. \tag{5.11}$$

The *q-analogue for Euler's formula* for the function $\Gamma(x)$ is

$$\int_0^{\zeta} e_q(-x) x^n d_q x = [n][n-1][n-2]\ldots[1] = [n]!. \tag{5.12}$$

A proof of this formula can be found in [29].

6. Q-numbers

The definition of q-numbers given in sec. 2 is not the only possible one. We have already seen the two-parameter deformed numbers, defined in eq. (2.11). Furthermore, a different definition of quantum numbers (the Q-numbers) has been used in mathematics since the early ninetenth century, with rich literature existing on this subject [31,32]. *Q-numbers* are defined as

$$[x]_Q = \frac{Q^x - 1}{Q - 1}, \tag{6.1}$$

where x can be a number or an operator and Q is a deformation parameter. Q is a real number ($Q \neq 0, 1$). The notation $Q = e^T$, where T a real number, will be often used. The subscript Q will be used in this review in order to distinguish deformed numbers defined as in eq. (6.1) from these defined by eq. (2.1). It is clear that in the limit $Q \to 1$ (or, equivalently, $T \to 0$) Q-numbers become ordinary numbers, i.e. $[x]_Q \to x$.

A few examples of Q-numbers are given here:

$$[0]_Q = 0, \qquad [1]_Q = 1, \qquad [2]_Q = Q + 1, \qquad [3]_Q = Q^2 + Q + 1. \tag{6.2}$$

Q-numbers clearly do not remain invariant under the substitution $Q \to Q^{-1}$. One can easily prove that

$$[x]_Q = Q^{x-1} \, [x]_{1/Q} \, . \tag{6.3}$$

Q-numbers are connected to q-numbers through the relation [29]

$$[x] = q^{1-x} [x]_Q, \qquad \text{with} \qquad Q = q^2. \tag{6.4}$$

The definitions of Q-factorials and Q-binomial coefficients still look like the ones given in eqs. (2.7)–(2.8):

$$[n]_Q! = [n]_Q [n-1]_Q \ldots [1]_Q, \tag{6.5}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_Q = \frac{[m]_Q!}{[m-n]_Q! [n]_Q!}. \tag{6.6}$$

As it can be easily seen from eq. (6.1) under the substitution $Q \to Q^{-1}$ one obtains

$$[n]_Q! = Q^{n(n-1)/2} \, [n]_{1/Q} \, , \tag{6.7}$$

and

$$\left[\begin{array}{c} n \\ k \end{array} \right]_Q = Q^{k(n-k)} \left[\begin{array}{c} n \\ k \end{array} \right]_{1/Q} . \tag{6.8}$$

Q-factorials are connected to q-factorials by

$$[n]! = q^{-n(n-1)/2} [n]_Q!, \qquad \text{with} \qquad Q = q^2. \tag{6.9}$$

7. Q-deformed elementary functions

The definitions of Q-deformed elementary functions [31] look similar to these given in sec. 3. The *Q-deformed exponential function* is defined as

$$e_Q(ax) = \sum_{n=0}^{\infty} \frac{a^n}{[n]_Q!} x^n, \tag{7.1}$$

and satisfies the property

$$e_Q(x) e_{1/Q}(-x) = 1. \tag{7.2}$$

(Notice that $e_Q(x) e_Q(-x) \neq 1$.)

The *Q-deformed trigonometric functions* are defined as

$$\sin_Q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{[2n+1]_Q!}, \tag{7.3}$$

$$\cos_Q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{[2n]_Q!}. \tag{7.4}$$

One can easily show that

$$\sin_Q(x) = \frac{1}{2i} \left(e_Q(ix) - e_Q(-ix) \right), \tag{7.5}$$

$$\cos_Q(x) = \frac{1}{2} \left(e_Q(ix) + e_Q(-ix) \right). \tag{7.6}$$

Instead of the familiar identity $\sin^2(x) + \cos^2(x) = 1$ one has

$$\sin_Q(x) \sin_{1/Q}(x) + \cos_Q(x) \cos_{1/Q}(x) = 1. \tag{7.7}$$

The above defined Q-deformed functions are examples of Q-deformed hypergeometric functions[31,33,34]. In addition Q-deformed polynomials, which are counterparts of the ordinary non deformed polynomials, can be defined, such as Q-deformed Hermite polynomials and Q-deformed Laguerre polynomials [31].

8. Q-derivative

Given the function $f(x)$ one defines its Q-derivative D_x^Q [31] by the expression

$$D_x^Q f(x) = \frac{f(Qx) - f(x)}{(Q-1)x}. \tag{8.1}$$

The similarity between this definition and the one of Q-numbers (eq. (6.1)) is clear.

One can easily prove that

$$D_x^Q x^n \equiv \frac{Q^n x^n - x^n}{(Q-1)x} = [n]_Q x^{n-1}, \tag{8.2}$$

which looks exactly like eq. (4.2). In addition one has

$$D_x^Q e_Q(ax) = a e_Q(ax), \tag{8.3}$$

$$D_x^Q e_{1/Q}(ax) = a e_{1/Q}(aQx), \tag{8.4}$$

$$D_x^Q \sin_Q(ax) = a \cos_Q(ax), \tag{8.5}$$

$$D_x^Q \cos_Q(ax) = -a \sin_Q(ax). \tag{8.6}$$

One can also easily see that $\sin_Q(ax)$ and $\cos_Q(ax)$ are the linearly independent solutions of the Q-differential equation

$$(D_x^Q)^2 u(x) + a^2 u(x) = 0, \tag{8.7}$$

while the functions $\sin_{1/Q}(ax)$ and $\cos_{1/Q}(ax)$ satisfy the equation

$$(D_x^Q)^2 u(x) + a^2 u(Q^2 x) = 0. \tag{8.8}$$

The following *Leibnitz rules* can also be shown:

$$D_x^Q\left(f_1(x)f_2(x)\right)=\left(D_x^Q f_1(x)\right)f_2(Qx)+f_1(x)\left(D_x^Q f_2(x)\right), \tag{8.9}$$

$$D_x^Q\left(f_1(x)f_2(x)\right)=\left(D_x^Q f_1(x)\right)f_2(x)+f_1(Qx)\left(D_x^Q f_2(x)\right). \tag{8.10}$$

One can further obtain

$$D_x^Q\frac{f_1(x)}{f_2(x)}=\frac{\left(D_x^Q f_1(x)\right)f_2(x)-f_1(x)\left(D_x^Q f_2(x)\right)}{f_2(Qx)f_2(x)}. \tag{8.11}$$

For the second derivative of $f(x)$ one has

$$(D_x^Q)^2 f(x)=(Q-1)^{-2}Q^{-1}x^{-2}\left\{f(Q^2x)-(Q+1)f(Qx)+Qf(x)\right\}, \tag{8.12}$$

and by mathematical induction we obtain the general formula

$$(D_x^Q)^n f(x)=(Q-1)^{-n}Q^{-n(n-1)/2}x^{-n}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_Q(-1)^k Q^{k(k-1)/2}f(Q^{n-k}x). \tag{8.13}$$

9. Q-integration

In a way analogous to that of sec. 5 the definite Q-integral of the function $f(x)$ in the interval $[0,1]$ is defined [31] as follows

$$\int_0^1 f(x)d_Qx=(1-Q)\sum_{s=0}^{\infty}f(Q^s)Q^s, \tag{9.1}$$

assuming that Q is real and $|Q|<1$, while for the definite integral of $f(x)$ in the interval $[0,\infty]$, we have

$$\int_0^\infty f(x)d_Qx=(1-Q)\sum_{s=-\infty}^{\infty}Q^s f(Q^s). \tag{9.2}$$

For the indefinite Q-integral of $f(x)$ one has

$$\int f(x)d_Qx=(1-Q)x\sum_{s=-\infty}^{\infty}Q^s f(Q^sx)+\text{constant}. \tag{9.3}$$

One can easily check that Q-differentiation and Q-integration are operations inverse to each other

$$D^Q\int f(x)d_Qx=f(x). \tag{9.4}$$

The formula for Q-integration by parts reads

$$\int\left(D^Q f_1(x)\right)f_2(x)d_Qx=f_1(x)f_2(x)-\int f_1(Qx)\left(D^Q f_2(x)\right)d_Qx. \tag{9.5}$$

10. The q-deformed harmonic oscillator

The interest for possible applications of quantum algebras in physics has been triggered in 1989 by the introduction of the q-deformed harmonic oscillator [35,36,37], of which earlier equivalent versions existed [38,39].

The q-deformed harmonic oscillator [35,36,37,40,41,42] is defined in terms of the creation and annihilation operators $a^\dagger$ and a and the number operator N, which satisfy the commutation relations

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \tag{10.1}$$

$$aa^\dagger - q^{\mp 1} a^\dagger a = q^{\pm N}. \tag{10.2}$$

In addition the following conditions of hermitian conjugation hold (in the case of q being a real number or q being a root of unity)

$$(a^\dagger)^\dagger = a, \qquad N^\dagger = N. \tag{10.3}$$

Eq. (10.1) is the same as in ordinary quantum mechanics, while eq. (10.2) is modified by the presence of the deformation parameter q. For $q \to 1$ it is clear that eq. (10.2) goes to the usual boson commutation relation $[a, a^\dagger] = 1$. An immediate consequence of (10.2) is that

$$a^\dagger a = [N], \quad aa^\dagger = [N+1]. \tag{10.4}$$

Thus the number operator N is *not* equal to $a^\dagger a$, as in the ordinary case. The operators $a^\dagger$ and a are referred to as *q-deformed boson creation and annihilation operators* respectively.

The basis of the Fock space is defined by repeated action of the creation operator $a^\dagger$ on the vacuum state, which is annihilated by a:

$$a|0\rangle = 0, \qquad |n> = \frac{(a^\dagger)^n}{\sqrt{[n]!}}|0> . \tag{10.5}$$

The action of the operators on the basis is given by

$$N|n> = n|n>, \tag{10.6}$$

$$a^\dagger|n> = \sqrt{[n+1]}|n+1>, \tag{10.7}$$

$$a|n> = \sqrt{[n]}|n-1> . \tag{10.8}$$

We remark that these equations look very similar to the ones of the ordinary case, the only difference being that q-numbers appear under the square roots instead of usual numbers.

The Hamiltonian of the q-deformed harmonic oscillator is

$$H = \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a), \tag{10.9}$$

and its eigenvalues in the basis given above are

$$E(n) = \frac{\hbar\omega}{2}([n] + [n+1]). \tag{10.10}$$

One can easily see that for q real the energy eigenvalues increase more rapidly than the ordinary case, in which the spectrum is equidistant, i.e. the spectrum gets "expanded". In contrast, for q being a phase factor ($q = e^{i\tau}$ with τ real) the eigenvalues of the energy increase less rapidly than the ordinary (equidistant) case, i.e. the spectrum is "compressed". In particular, for q real ($q = e^\tau$) the eigenvalues can be written as

$$E(n) = \frac{\hbar\omega}{2}\frac{\sinh\left(\tau\left(n+\frac{1}{2}\right)\right)}{\sinh\frac{\tau}{2}}, \tag{10.11}$$

while for q being a phase factor ($q = e^{i\tau}$) one has

$$E(n) = \frac{\hbar\omega}{2}\frac{\sin\left(\tau\left(n+\frac{1}{2}\right)\right)}{\sin\frac{\tau}{2}}. \tag{10.12}$$

In both cases in the limit $q \to 1$ ($\tau \to 0$) the ordinary expression

$$E(n) = \hbar\omega\left(n+\frac{1}{2}\right) \tag{10.13}$$

is recovered.

In addition, the following commutation relation holds

$$[a, a^\dagger] = [N+1] - [N]. \tag{10.14}$$

For q being a phase factor, this commutation relation takes the form

$$[a, a^\dagger] = \frac{\cos\frac{(2N+1)\tau}{2}}{\cos\frac{\tau}{2}}. \tag{10.15}$$

It is useful to notice that the q-deformed boson operators $a^\dagger$ and a can be expressed in terms of usual boson operators $\alpha^\dagger$ and α (satisfying $[\alpha, \alpha^\dagger] = 1$ and $N = \alpha^\dagger\alpha$) through the relations [40,43]

$$a = \sqrt{\frac{[N+1]}{N+1}}\alpha = \alpha\sqrt{\frac{[N]}{N}}, \qquad a^\dagger = \alpha^\dagger\sqrt{\frac{[N+1]}{N+1}} = \sqrt{\frac{[N]}{N}}\alpha^\dagger. \tag{10.16}$$

The square root factors in the last equation have been called *q-deforming functionals*.

For q being a primitive root of unity, i.e. $q = e^{2\pi i/k}$ ($k = 2, 3, \ldots$), it is clear the the representation of eqs (10.5)–(10.8) becomes finite-dimensional and has dimension k, since only the vectors $|0>$, $|1>$, $\ldots$, $|k-1>$ can be present. This case has been related to the system of two anyons [44]. In what follows we are going to assume that q is *not* a primitive root of unity.

A discussion on the position and momentum operators of the q-deformed oscillator can be found in [45].

11. The Q-deformed harmonic oscillator

A different version of the deformed harmonic oscillator can be obtained by defining [43,46,47] the operators b, b^+ through the equations

$$a = q^{1/2}bq^{-N/2}, \quad a^\dagger = q^{1/2}q^{-N/2}b^\dagger. \tag{11.1}$$

Eqs. (10.1) and (10.2) then give

$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \tag{11.2}$$

$$bb^\dagger - q^2b^\dagger b = 1. \tag{11.3}$$

This oscillator has been first introduced by Arik and Coon [38] and later considered also by Kuryshkin [39]. One then easily finds that

$$b^\dagger b = [N]_Q, \quad bb^\dagger = [N+1]_Q, \tag{11.4}$$

where $Q = q^2$ and Q-numbers are defined in (6.1). The basis is defined by

$$b|0\rangle = 0, \qquad |n> = \frac{(b^\dagger)^n}{\sqrt{[n]_Q!}}|0>, \tag{11.5}$$

while the action of the operators on the basis is given by

$$N|n> = n|n>, \tag{11.6}$$

$$b^\dagger|n>=\sqrt{[n+1]_Q}|n+1>, \tag{11.7}$$

$$b|n>=\sqrt{[n]_Q}|n-1>. \tag{11.8}$$

The Hamiltonian of the corresponding deformed harmonic oscillator has the form

$$H=\frac{\hbar\omega}{2}(bb^\dagger+b^\dagger b), \tag{11.9}$$

the eigenvalues of which are

$$E(n)=\frac{\hbar\omega}{2}([n]_Q+[n+1]_Q). \tag{11.10}$$

One can easily see that for $Q=e^T$, where $T>0$ and real, the spectrum increases more rapidly than the ordinary (equidistant) spectrum, while for $Q=e^T$, with $T<0$ and real, the spectrum is increasing less rapidly than the ordinary (equidistant) case.

From the above relations, it is clear that the following commutation relation holds

$$[b,b^\dagger]=Q^N. \tag{11.11}$$

12. The generalized deformed oscillator

In addition to the oscillators described in the last two sections, many kinds of deformed oscillators have been introduced in the literature (see [48] for a list). All of them can be accommodated within the common mathematical framework of the *generalized deformed oscillator* [49,50], which is defined as the algebra generated by the operators $\{1,a,a^\dagger,N\}$ and the *structure function* $\Phi(x)$, satisfying the relations

$$[a,N]=a, \qquad [a^\dagger,N]=-a^\dagger, \tag{12.1}$$

$$a^\dagger a=\Phi(N)=[N], \qquad aa^\dagger=\Phi(N+1)=[N+1], \tag{12.2}$$

where $\Phi(x)$ is a positive analytic function with $\Phi(0)=0$ and N is the number operator. ¿From eq. (12.2) we conclude that

$$N=\Phi^{-1}(a^\dagger a), \tag{12.3}$$

and that the following commutation and anticommutation relations are obviously satisfied:

$$[a,a^\dagger]=[N+1]-[N], \qquad \{a,a^\dagger\}=[N+1]+[N]. \tag{12.4}$$

The *structure function* $\Phi(x)$ is characteristic to the deformation scheme. In Table 1 the structure functions corresponding to different deformed oscillators are given. They will be further discussed at the end of this section.

It can be proved that the generalized deformed algebras possess a Fock space of eigenvectors $|0>,|1>,\ldots,|n>,\ldots$ of the number operator N

$$N|n>=n|n>, \quad <n|m>=\delta_{nm}, \tag{12.5}$$

if the *vacuum state* $|0>$ satisfies the following relation:

$$a|0>=0. \tag{12.6}$$

These eigenvectors are generated by the formula:

$$|n>=\frac{1}{\sqrt{[n]!}}\left(a^\dagger\right)^n|0>, \tag{12.7}$$

where

$$[n]!=\prod_{k=1}^{n}[k]=\prod_{k=1}^{n}\Phi(k). \tag{12.8}$$

The generators $a^\dagger$ and a are the creation and annihilation operators of this deformed oscillator algebra:

$$a|n>=\sqrt{[n]}a|n-1>, \qquad a^\dagger|n>=\sqrt{[n+1]}a|n+1> . \tag{12.9}$$

These eigenvectors are also eigenvectors of the energy operator

$$H=\frac{\hbar\omega}{2}(aa^\dagger+a^\dagger a), \tag{12.10}$$

corresponding to the eigenvalues

$$E(n)=\frac{\hbar\omega}{2}(\Phi(n)+\Phi(n+1))=\frac{\hbar\omega}{2}([n]+[n+1]). \tag{12.11}$$

For

$$\Phi(n)=n \tag{12.12}$$

one obtains the results for the ordinary harmonic oscillator. For

$$\Phi(n)=\frac{q^n-q^{-n}}{q-q^{-1}}=[n] \tag{12.13}$$

one has the results for the q-deformed harmonic oscillator, while the choice

$$\Phi(n)=\frac{Q^n-1}{Q-1}=[n]_Q \tag{12.14}$$

leads to the results of the Q-deformed harmonic oscillator. Many more cases are shown in Table 1, on which the following comments apply:

i) Two-parameter deformed oscillators have been introduced [51,52,53], in analogy to the one-parameter deformed oscillators.

ii) Parafermionic oscillators [54] of order p represent particles of which the maximum number which can occupy the same state is p. Parabosonic oscillators [54] can also be introduced. The deformed oscillator realization of the parafermionic and parabosonic oscillator has been studied in [48], while in [55] an equivalent realization for the parabosonic oscillator has been introduced.

iii) q-deformed versions of the parafermionic and parabosonic oscillators have also been introduced [56,57].

iv) q-deformed versions of the fermionic algebra [58] have also been introduced [59,60,61,62,63,64], as well as q-deformed versions of generalized q-deformed fermionic algebras [65]. It has been proved, however, that q-deformed fermions are fully equivalent to the ordinary fermions [66,67].

13. The physical content of deformed harmonic oscillators

In order to get a feeling about the physical content of the various deformed harmonic oscillators it is instructive to construct potentials giving spectra similar to these of the oscillators.

13.1. Classical potentials equivalent to the q-oscillator

Let us consider the q-deformed harmonic oscillator first. For small values of τ one can take Taylor expansions of the functions appearing there and thus find an expansion of the q-number $[n]$ of eq. (2.1) in terms of powers of τ^2. The final result is

$$[n]=n\pm\frac{\tau^2}{6}(n-n^3)+\frac{\tau^4}{360}(7n-10n^3+3n^5)\pm\frac{\tau^6}{15120}(31n-49n^3+21n^5-3n^7)+\ldots, \tag{13.1}$$

where the upper (lower) sign corresponds to q being a phase (real). Using this expansion the energy of the q-deformed harmonic oscillator of eq. (10.10) can be rewritten as

$$E(n)/(\hbar\omega)=(n+\frac{1}{2})(1\pm\frac{\tau^2}{24})\mp\frac{\tau^2}{6}(n+\frac{1}{2})^3+\ldots \tag{13.2}$$

Table 1: Structure functions of special deformation schemes

	$\Phi(x)$	Reference
i	x	harmonic oscillator, bosonic algebra
ii	$\frac{q^x-q^{-x}}{q-q^{-1}}$	q-deformed harmonic oscillator [35,36]
iii	$\frac{q^x-1}{q-1}$	Arik–Coon, Kuryshkin, or Q-deformed oscillator [38,39]
iv	$\frac{q^x-p^{-x}}{q-p^{-1}}$	2-parameter deformed oscillator [51,52,53]
v	$x(p+1-x)$	parafermionic oscillator [54]
vi	$\frac{\sinh(\tau x)\sinh(\tau(p+1-x))}{\sinh^2(\tau)}$	q-deformed parafermionic oscillator [56,57]
vii	$x\cos^2(\pi x/2)+(x+p-1)\sin^2(\pi x/2)$	parabosonic oscillator [54]
viii	$\frac{\sinh(\tau x)}{\sinh(\tau)}\frac{\cosh(\tau(x+2N_0-1))}{\cosh(\tau)}\cos^2(\pi x/2)+$ $+\frac{\sinh(\tau(x+2N_0-1))}{\sinh(\tau)}\frac{\cosh(\tau x)}{\cosh(\tau)}\sin^2(\pi x/2)$	q-deformed parabosonic oscillator [56,57]
ix	$\sin^2 \pi x/2$	fermionic algebra [58]
x	$q^{x-1}\sin^2 \pi x/2$	q-deformed fermionic algebra [59,60,61,62,63,64]
xi	$\frac{1-(-q)^x}{1+q}$	generalized q-deformed fermionic algebra [65]
xii	x^n	[49]
xiii	$\frac{sn(\tau x)}{sn(\tau)}$	[49]

On the other hand, one can consider the potential

$$V(x) = V_0 + kx^2 + \lambda x^4 + \mu x^6 + \xi x^8 + \ldots . \tag{13.3}$$

If λ, μ, ξ are much smaller than k, one can consider this potential as a harmonic oscillator potential plus some perturbations and calculate the corresponding spectrum through the use of perturbation theory [68] (see also subsec. 34.2). In order to keep the subsequent formulae simple, we measure x in units of $(\hbar/(2m\omega))^{1/2}$. Using standard first order perturbation theory one finds that the corresponding spectrum up to the order considered is

$$E(n) = E_0 + (2\kappa + 25\mu)(n+\frac{1}{2}) + (6\lambda + 245\xi)(n+\frac{1}{2})^2 + 20\mu(n+\frac{1}{2})^3 + 70\xi(n+\frac{1}{2})^4. \tag{13.4}$$

By equating coefficients of the same powers of $(n+\frac{1}{2})$ in eqs. (13.2) and (13.4), we can determine the coefficients appearing in the expansion of the potential given in eq. (13.3). The final result for the potential, up to the order considered, is

$$V(x) = (\frac{1}{2} \pm \frac{\tau^2}{8})x^2 \mp \frac{\tau^2}{120}x^6. \tag{13.5}$$

We see therefore that to lowest order one can think of the q-oscillator with a small value of the parameter τ as a harmonic oscillator perturbed by a x^6 term. It is clear that if we go to higher order, the next term to appear will be proportional to $\tau^4 x^{10}$.

The results of this subsection are corroborated by an independent study of the relation between the q-deformed harmonic oscillator and the ordinary anharmonic oscillator with x^6 anharmonicities [69].

A different interpretation of the q-deformed oscillator as the nonlinear oscillator with the frequency dependent on the amplitude has been given in [70].

13.2. Classical potentials equivalent to the Q-deformed oscillator

Similar considerations can be made for the Q-oscillator. Defining $Q = e^T$ it is instructive to construct the expansion of the Q-number of eq. (6.1) in powers of T. Assuming that T is small and taking Taylor expansions in eq. (6.1) one finally has

$$[n]_Q = n + \frac{T}{2}(n^2 - n) + \frac{T^2}{12}(2n^3 - 3n^2 + 1) + \frac{T^3}{24}(n^4 - 2n^3 + n^2) + \ldots \tag{13.6}$$

Then the corresponding expansion of the energy levels of the oscillator of eq. (11.10) is

$$E(n)/(\hbar\omega) = E_0' + (1 - \frac{T}{2} + \frac{T^2}{8} - \frac{T^3}{16} + \ldots)(n + \frac{1}{2}) + (\frac{T}{2} - \frac{T^2}{4} + \frac{5T^3}{48} - \ldots)(n + \frac{1}{2})^2$$
$$+(\frac{T^2}{6} - \frac{T^3}{12} + \ldots)(n + \frac{1}{2})^3 + (\frac{T^3}{24} - \ldots)(n + \frac{1}{2})^4. \tag{13.7}$$

Comparing this expansion to eq. (13.4) and equating equal powers of $(n + \frac{1}{2})$ we arrive at the following expression for the potential

$$V(x) = \frac{T^2}{12} + (\frac{1}{2} - \frac{T}{4} - \frac{T^2}{24} + \frac{T^3}{48})x^2 + (\frac{T}{12} - \frac{T^2}{24} - \frac{T^3}{144})x^4 + (\frac{T^2}{120} - \frac{T^3}{240})x^6 + \frac{T^3}{1680}x^8. \tag{13.8}$$

Thus to lowest order one can think of the Q-oscillator with small values of the parameter T as a harmonic oscillator perturbed by a x^4 term. A similar expression is found, for example, by Taylor expanding the modified Pöschl–Teller potential [71], which, among other applications, has been recently used in the description of hypernuclei [72,73,74,75]. The modified Pöschl–Teller potential (see also subsec. 34.2) has the form

$$V(x)_{PT} = -\frac{A}{\cosh(ax)^2}. \tag{13.9}$$

Its Taylor expansion is

$$V(x)_{PT} = A(-1 + a^2x^2 - \frac{2}{3}a^4x^4 + \frac{17}{45}a^6x^6 - \ldots). \tag{13.10}$$

We remark that this expansion contains the same powers of x as the expansion (13.8). Furthermore, the signs of the coefficients of the same powers of x in the two expansions are the same for $T < 0$.

13.3. WKB-EPs for the q-deformed oscillator

The potentials obtained above are only rough lowest order estimates. More accurate methods exist for constructing WKB equivalent potentials (WKB-EPs) giving (within the limits of WKB approximation) the same spectrum as the above mentioned oscillators. A method by which this can be achieved has been given by Wheeler [76] and is described by Chadan and Chabatier [77]. Applying this method to the q-deformed oscillator (sec. 10) with q being a phase factor one finds the potential [78,79]

$$V(x) = \left(\frac{\tau}{2\sin(\tau/2)}\right)^2 \frac{m\omega^2}{2}x^2 \left[1 - \frac{8}{15}\left(\frac{x}{2R_e}\right)^4 + \frac{4448}{1575}\left(\frac{x}{2R_e}\right)^8 - \frac{345344}{675675}\left(\frac{x}{2R_e}\right)^{12} + \ldots\right], \tag{13.11}$$

where

$$R_e = \frac{1}{\tau}\left(\frac{\hbar^2}{2m}\right)^{1/2}\left(\frac{2\sin(\tau/2)}{\hbar\omega}\right)^{1/2}, \tag{13.12}$$

while for the q-deformed oscillator with q real one has

$$V(x) = \left(\frac{\tau}{2\sinh(\tau/2)}\right)^2 \frac{m\omega^2}{2}x^2 \left[1 + \frac{8}{15}\left(\frac{x}{2R_h}\right)^4 + \frac{4448}{1575}\left(\frac{x}{2R_h}\right)^8 + \frac{345344}{675675}\left(\frac{x}{2R_h}\right)^{12} + \ldots\right], \tag{13.13}$$

where

$$R_h = \frac{1}{\tau}\left(\frac{\hbar^2}{2m}\right)^{1/2}\left(\frac{2\sinh(\tau/2)}{\hbar\omega}\right)^{1/2}. \tag{13.14}$$

The results of this subsection are corroborated by an independent study of the relation between the q-deformed harmonic oscillator and the ordinary anharmonic oscillator with x^6 anharmonicities[69].

13.4. WKB-EPs for the Q-deformed oscillator

Using the same technique one finds that the WKB equivalent potential for the Q-deformed harmonic oscillator (sec. 11) takes the form [80]

$$V(x) = V_{min} + \frac{(\ln Q)^2}{Q}\left(\frac{Q+1}{Q-1}\right)^2 \frac{1}{2}m\omega^2 x^2$$

$$\left[1 - \frac{2}{3}\left(\frac{x}{R'}\right)^2 + \frac{23}{45}\left(\frac{x}{R'}\right)^4 - \frac{134}{315}\left(\frac{x}{R'}\right)^6 + \frac{5297}{14172}\left(\frac{x}{R'}\right)^8 - \ldots\right], \tag{13.15}$$

where

$$V_{min} = \frac{\hbar\omega(\sqrt{Q}-1)}{2\sqrt{Q}(\sqrt{Q}+1)}, \tag{13.16}$$

and

$$R' = \left(\frac{\hbar\sqrt{Q}(Q-1)}{\omega m(Q+1)}\right)^{1/2}\frac{(\ln Q)^{-1}}{\sqrt{2}}. \tag{13.17}$$

We remark that this WKB-EP contains all even powers of x, in contrast to the WKB-EPs for the q-oscillator (eqs (13.11), (13.13)), which contain only the powers x^2, x^6, x^{10}, ... This is in agreement with the lowest order results obtained in subsecs 13.1 and 13.2.

In this section approximate expressions for some potentials having similar energy spectrum with some deformed oscillators have been given. Another point of view is the search for phase space coordinates having exact symmetries, like the operators involved in q-deformed algebras in the case of real q [81]. In the case of q being a root of unity the corresponding phase space is discrete [82]

14. The quantum algebra $\mathrm{su}_q(2)$

Quantum algebras are generalized versions of the usual Lie algebras, to which they reduce when the deformation parameter q is set equal to unity. A simple example of a quantum algebra is provided by $\mathrm{su}_q(2)$ [35,36], which is generated by the operators J_+, J_0, J_-, satisfying the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \tag{14.1}$$

$$[J_+, J_-] = [2J_0], \tag{14.2}$$

with $J_0^\dagger = J_0$, $(J_+)^\dagger = J_-$. We remark that eq. (14.1) is the same as in the case of the ordinary su(2) algebra, while eq. (14.2) is different, since in the usual su(2) case it reads

$$[J_+, J_-] = 2J_0. \tag{14.3}$$

In the rhs of eq. (14.3) one has the first power of the J_0 operator, while in the rhs of eq. (14.2) one has the q-operator $[2J_0]$, defined in sec. 2. Because of eqs (2.2) and (2.3) it is clear that if one writes the rhs of eq. (14.2) in expanded form all odd powers of J_0 will appear:

$$[J_+, J_-] = \frac{1}{\sinh(\tau)}\left(\frac{2\tau J_0}{1!} + \frac{(2\tau J_0)^3}{3!} + \frac{(2\tau J_0)^5}{5!} + \ldots\right) \qquad \text{for} \qquad q = e^\tau, \tag{14.4}$$

$$[J_+, J_-] = \frac{1}{\sin(\tau)}\left(\frac{2\tau J_0}{1!} - \frac{(2\tau J_0)^3}{3!} + \frac{(2\tau J_0)^5}{5!} - \ldots\right) \qquad \text{for} \qquad q = e^{i\tau}. \tag{14.5}$$

Thus $su_q(2)$ can be loosely described as a nonlinear generalization of the usual su(2): While in usual Lie algebras the commutator of two generators is always producing a linear combination of generators, in the case of quantum algebras the commutator of two generators can contain higher powers of the generators as well.

The irreducible representations (irreps) D^J of $su_q(2)$ (which have dimensionality $2J+1$) are determined by highest weight states with $J = 0, \frac{1}{2}, 1, \ldots$. The basic states $|J, M>$ (with $-J \leq M \leq J$) are connected with highest weight states $|J, J>$ as follows

$$|J, M >= \sqrt{\frac{[J+M]!}{[2J]![J-M]!}}(J_-)^{J-M}|J, J >, \tag{14.6}$$

with $J_+|J, J >= 0$ and $< J, J|J, J >= 1$. The action of the generators of the algebra on these basic vectors is given by

$$J_0|J, M >= M|J, M >, \tag{14.7}$$

$$J_\pm|J, M >= \sqrt{[J \mp M][J \pm M + 1]}|J, M \pm 1 > . \tag{14.8}$$

These expressions look similar to the ones of the usual su(2) algebra, the only difference being that q-numbers appear under the square root instead of ordinary numbers.

The second order Casimir operator of $su_q(2)$ is determined from the condition that it should commute with all of the generators of the algebra. The resulting operator is

$$C_2^q = J_-J_+ + [J_0][J_0 + 1] = J_+J_- + [J_0][J_0 - 1]. \tag{14.9}$$

Its eigenvalues in the above mentioned basis are given by

$$C_2^q|J, M >= [J][J + 1]|J, M >, \tag{14.10}$$

while for the usual su(2) the eigenvalues of the Casimir operator are $J(J+1)$. One can easily check that for real q ($q = e^\tau$ with τ real) the eigenvalues $[J][J+1]$ produce a spectrum increasing more rapidly than $J(J+1)$ (an expanded spectrum), while for q being a phase factor ($q = e^{i\tau}$ with τ real) the eigenvalues $[J][J+1]$ correspond to a spectrum increasing less rapidly than $J(J+1)$ (a compressed spectrum).

It should be noticed that the generators J_+, J_0, J_- of $su_q(2)$ are connected to the generators j_+, j_0, j_- of the usual su(2), which satisfy the commutation relations

$$[j_0, j_\pm] = \pm j_\pm, \qquad [j_+, j_-] = 2j_0, \tag{14.11}$$

through the relations [83,84]

$$J_0 = j_0, \qquad J_+ = \sqrt{\frac{[j_0 + j][j_0 - 1 - j]}{(j_0 + j)(j_0 - 1 - j)}}j_+, \qquad J_- = j_-\sqrt{\frac{[j_0 + j][j_0 - 1 - j]}{[j_0 + j][j_0 - 1 - j]}}, \tag{14.12}$$

where j is determined by the relation for the second order Casimir operator of su(2)

$$C = j_-j_+ + j_0(j_0 + 1) = j_+j_- + j_0(j_0 - 1) = j(j + 1). \tag{14.13}$$

15. Realization of $su_q(2)$ in terms of q-deformed bosons

Realizations of Lie algebras in terms of (ordinary) bosons are useful not only as a convenient mathematical tool, but also because of their applications in physics [85]. In the case of quantum algebras it turns out that boson realizations are possible in terms of the q-deformed boson operators already introduced in sec. 10.

In the case of $su_q(2)$ the generators can be mapped onto q-deformed bosons as follows [35,36]

$$J_+ = a_1^\dagger a_2, \qquad J_- = a_2^\dagger a_1, \qquad J_0 = \frac{1}{2}(N_1 - N_2), \tag{15.1}$$

where $a_i^\dagger$, a_i and N_i are q-deformed boson creation, annihilation and number operators as these introduced in sec. 10. One can easily prove that the boson images satisfy the commutation relations (14.1) and (14.2). For example, one has

$$[J_+, J_-] = a_1^\dagger a_2 a_2^\dagger a_1 - a_2^\dagger a_1 a_1^\dagger a_2 = [N_1][N_2+1] - [N_1+1][N_2] = [N_1 - N_2] = [2J_0], \tag{15.2}$$

where use of the identity (2.6) has been made.

In the q-boson picture the normalized highest weight vector is

$$|JJ>= \frac{(a_1^\dagger)^{2J}}{\sqrt{[2J]!}}|0>, \tag{15.3}$$

while the general vector $|JM>$ is given by

$$|JM>= \frac{(a_1^\dagger)^{J+M}}{\sqrt{[J+M]!}} \frac{(a_2^\dagger)^{J-M}}{\sqrt{[J-M]!}}|0> . \tag{15.4}$$

It should be noticed that it was the search for a boson realization of the $su_q(2)$ algebra that led to the introduction of the q-deformed harmonic oscillator in 1989 [35,36].

Starting from $su_q(2)$ one can formulate a q-deformed version of angular momentum theory. Some references are listed here:

i) Clebsch-Gordan coefficients for $su_q(2)$ can be found in [86,87,88,89,90,91,92,93,94].

ii) 6-j symbols for $su_q(2)$ can be found in [95,96,97,98].

iii) 9-j symbols for $su_q(2)$ can be found in [99].

iv) 3n-j symbols (with n=1, 2, 3, 4, 5) for $su_q(2)$ can be found in [100].

v) The q-deformed version of the Wigner–Eckart theorem can be found in [101,102,103].

vi) Irreducible tensor operators for $su_q(2)$ can be found in [103].

In addition, it should be noticed that a two-parameter deformation of su(2), labelled as $su_{p,q}(2)$ has been introduced [51,53,104,105,106,107,108]. Clebsch-Gordan coefficients for $su_{p,q}(2)$ have been discussed in [109,110,111,112,113].

The way in which the algebra $su_q(2)$ can be realized in terms of the Q-deformed bosons of sec. 11 is given in subsec. 35.1 (see eqs (35.13)–(35.16)).

16. The quantum algebra $su_q(1,1)$

In this section we shall give a brief account of the algebra $su_q(1,1)$ [43,114,115].

In the classical case the so(2,1) generators satisfy the commutation relations [116]

$$[K_1, K_2] = -iK_3, \qquad [K_2, K_3] = iK_1, \qquad [K_3, K_1] = iK_2, \tag{16.1}$$

which differ from the classical so(3) commutation relations in the sign of the r.h.s. of the first commutator. Defining

$$K_+ = K_1 + iK_2, \qquad K_- = K_1 - iK_2, \qquad K_3 = K_z, \tag{16.2}$$

one obtains the su(1,1) commutation relations

$$[K_z, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -2K_z, \tag{16.3}$$

which differ from the familiar su(2) commutation relations in the sign of the r.h.s. of the last commutator. The generators of su(1,1) accept the following boson representation [117]

$$K_+ = a_1^\dagger a_2^\dagger, \qquad K_- = a_1 a_2, \qquad K_z = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1), \tag{16.4}$$

where $a_1^\dagger$, a_1, $a_2^\dagger$, a_2 satisfy usual boson commutation relations.

The second order Casimir operator of so(2,1) is [116]

$$C_2[\mathrm{so}(2,1)] = -(K_1^2 + K_2^2 - K_3^2), \tag{16.5}$$

while for su(1,1) one has

$$C_2[\mathrm{su}(1,1)] = [K_0][K_0 - 1] - K_+K_- = [K_0][K_0 + 1] - K_-K_+. \tag{16.6}$$

In the quantum case, the generators of $\mathrm{su}_q(1,1)$ satisfy the commutation relations [43,114,115]

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -[2K_0]. \tag{16.7}$$

The generators of $\mathrm{su}_q(1,1)$ accept the following boson representation

$$K_+ = a_1^\dagger a_2^\dagger, \qquad K_- = a_1 a_2, \qquad K_0 = \frac{1}{2}(N_1 + N_2 + 1), \tag{16.8}$$

where the bosons $a_i^\dagger$, a_i $(i = 1, 2)$ satisfy the usual q-boson commutation relations.

The second order Casimir operator of $\mathrm{su}_q(1,1)$ is

$$C_2[\mathrm{su}_q(1,1)] = [K_0][K_0 - 1] - K_+K_- = [K_0][K_0 + 1] - K_-K_+. \tag{16.9}$$

Its eigenvalues are [114,115]

$$C_2[\mathrm{su}_q(1,1)]|\kappa\mu> = [\kappa][\kappa - 1]|\kappa\mu>, \tag{16.10}$$

where

$$\kappa = \frac{1 + |n_1 - n_2|}{2}, \quad \mu = \frac{1 + n_1 + n_2}{2}, \tag{16.11}$$

since the basis has the form $|\kappa\mu> = |n_1> |n_2>$, with

$$|n_i> = \frac{1}{\sqrt{[n_i]!}}(a_i^\dagger)^{n_i}|0> . \tag{16.12}$$

In this basis the possible values of μ are given by

$$\mu = \kappa, \kappa + 1, \kappa + 2, \ldots, \tag{16.13}$$

up to infinity, while κ may be any positive real number. The action of the generators on this basis is given by

$$K_0|\kappa\mu\rangle = \mu|\kappa\mu\rangle, \tag{16.14}$$

$$K_\pm|\kappa\mu\rangle = \sqrt{[\mu \pm \kappa][\mu \mp \kappa \pm 1]}|\kappa\mu \pm 1\rangle. \tag{16.15}$$

Clebsch-Gordan and Racah coefficients for $\mathrm{su}_q(1,1)$ can be found in [91,94,96,98,118,119]. Furthemore, a two-parameter deformed version of $\mathrm{su}_q(1,1)$, labelled as $\mathrm{su}_{p,q}(1,1)$, has been introduced [53,106,120,121,122].

17. Generalized deformed su(2) and su(1,1) algebras

In the same way that in addition to the q-deformed oscillators one can have generalized deformed oscillators, it turns out that generalized deformed su(2) algebras, containing $\mathrm{su}_q(2)$ as a special case and having representation theory similar to that of the usual su(2), can be constructed [123,124]. It has been proved that it is possible to construct an algebra

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = \Phi(J_0(J_0 + 1)) - \Phi(J_0(J_0 - 1)), \tag{17.1}$$

where J_0, J_+, J_- are the generators of the algebra and $\Phi(x)$ is any increasing entire function defined for $x \geq -1/4$. Since this algebra is characterized by the function Φ, we use for it the symbol $\mathrm{su}_\Phi(2)$. The appropriate basis $|l, m>$ has the properties

$$J_0|L, M> = M|L, M>, \tag{17.2}$$

$$J_+|L, M> = \sqrt{\Phi(L(L+1)) - \Phi(M(M+1))}|L, M+1>, \tag{17.3}$$

$$J_-|L,M>=\sqrt{\Phi(L(L+1))-\Phi(M(M-1))}|L,M-1>, \tag{17.4}$$

where

$$L=0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3,\ldots, \tag{17.5}$$

and

$$M=-L,-L+1,-L+2,\ldots,L-2,L-1,L. \tag{17.6}$$

The Casimir operator is

$$C=J_-J_+ +\Phi(J_0(J_0+1))=J_+J_- +\Phi(J_0(J_0-1)), \tag{17.7}$$

its eigenvalues indicated by

$$C|L,M>=\Phi(L(L+1))|L,M> . \tag{17.8}$$

The usual su(2) algebra is recovered for

$$\Phi(x(x+1))=x(x+1), \tag{17.9}$$

while the quantum algebra $\mathrm{su}_q(2)$

$$[J_0,J_\pm]=\pm J_\pm, \qquad [J_+,J_-]=[2J_0]_q, \tag{17.10}$$

occurs for

$$\Phi(x(x+1))=[x][x+1]. \tag{17.11}$$

The $\mathrm{su}_\Phi(2)$ algebra occurs in several cases, in which the rhs of the last equation in (17.1) is an odd function of J_0 [125]. It can be seen that other algebraic structures, like the quadratic Hahn algebra QH(3) [126] and the finite W algebra $\bar{\mathrm{W}}_0$ [127] can be brought into the $\mathrm{su}_\Phi(2)$ form, the advantage being that the representation theory of $\mathrm{su}_\Phi(2)$ is already known. It can also be proved that several physical systems, like the isotropic oscillator in a 2-dim curved space with constant curvature [128,129], the Kepler system in a 2-dim curved space with constant curvature [128,129], and the system of two identical particles in two dimensions [130] can also be put into an $\mathrm{su}_\Phi(2)$ form. More details can be found in [125].

A unified description of several deformed su(2), su(1,1) and osp(1,2) algebras has been given in [131], using Verma modules techniques. Nonlinear deformed su(2) algebras involving *two* deforming functions have been constructed in [132] and have been endowed with a double Hopf algebraic structure, referred to as a two-colour quasitriangular Hopf algebra, in [133].

Nonlinear sl(2) algebras subtending generalized angular momentum theories have been introduced in [134] and their representation theory has been developed. In particular the quadratic sl(2) and cubic sl(2) algebras have been studied in detail [134].

A general method for developing nonlinear deformations of the su(2) and su(1,1) algebras has been developed in [135,136,137]. Using this method nonlinear deformations of the su(1,1) algebra related to the Pöschl–Teller potential have been obtained in [138,139].

18. Generalized deformed parafermionic oscillators

It turns out that the generalized deformed $\mathrm{su}_\Phi(2)$ algebras mentioned in the last section are related to generalized deformed parafermionic oscillators, which we will therefore describe here.

It has been proved [140] that any generalized deformed parafermionic algebra of order p can be written as a generalized oscillator (sec. 12) with structure function

$$F(x)=x(p+1-x)(\lambda+\mu x+\nu x^2+\rho x^3+\sigma x^4+\ldots), \tag{18.1}$$

where λ, μ, ν, ρ, σ, ... are real constants satisfying the conditions

$$\lambda+\mu x+\nu x^2+\rho x^3+\sigma x^4+\ldots>0, \qquad x\in\{1,2,\ldots,p\}. \tag{18.2}$$

Considering an $\mathrm{su}_\Phi(2)$ algebra [123] with structure function

$$\Phi(J_0(J_0+1))=AJ_0(J_0+1)+B(J_0(J_0+1))^2+C(J_0(J_0+1))^3, \tag{18.3}$$

and making the correspondence

$$J_+ \to A^\dagger, \qquad J_- \to A, \qquad J_0 \to N, \tag{18.4}$$

one finds that the $\mathrm{su}_\Phi(2)$ algebra is equivalent to a generalized deformed parafermionic oscillator of the form

$$F(N) = N(p+1-N)$$
$$[-(p^2(p+1)C + pB) + (p^3C + (p-1)B)N + ((p^2-p+1)C + B)N^2 + (p-2)CN^3 + CN^4], \tag{18.5}$$

if the condition

$$A + p(p+1)B + p^2(p+1)^2C = 0 \tag{18.6}$$

holds. The condition of eq. (18.2) is always satisfied for $B > 0$ and $C > 0$.

In the special case of $C = 0$ one finds that the $\mathrm{su}_\Phi(2)$ algebra with structure function

$$\Phi(J_0(J_0+1)) = AJ_0(J_0+1) + B(J_0(J_0+1))^2 \tag{18.7}$$

is equivalent to a generalized deformed parafermionic oscillator characterized by

$$F(N) = BN(p+1-N)(-p + (p-1)N + N^2), \tag{18.8}$$

if the condition

$$A + p(p+1)B = 0 \tag{18.9}$$

is satisfied. The condition of eq. (18.2) is satisfied for $B > 0$.

Including higher powers of $J_0(J_0+1)$ in eq. (18.3) results in higher powers of N in eq. (18.5) and higher powers of $p(p+1)$ in eq. (18.6). If, however, one sets $B = 0$ in eq. (18.7), then eq. (18.8) vanishes, indicating that no parafermionic oscillator equivalent to the usual su(2) rotator can be constructed.

It turns out that several other mathematical structures, like the finite W algebras $\bar{\mathrm{W}}_0$ [127] and $\mathrm{W}_3^{(2)}$ (see subsec. 36.5) can be put into the generalized deformed parafermionic oscillator form. The same is true for several physical systems, such as the isotropic oscillator and the Kepler problem in a 2-dim curved space with constant curvature [128,129], and the Fokas–Lagerstrom [141], Smorodinsky–Winternitz [142], and Holt [143] potentials. Further details can be found in [125].

A detailed discussion of the representation theory of several deformed oscillator algebras can be found in [144,145].

19. The $\mathrm{su}_q(2)$ rotator model

It has been suggested by Raychev, Roussev and Smirnov [146] and independently by Iwao [147,148] that rotational spectra of deformed nuclei can be described by the q-deformed rotator, which corresponds to the 2nd order Casimir operator of the quantum algebra $\mathrm{su}_q(2)$, already studied in sec. 14. We shall show here that this assumption works and discuss the reasons behind this success, as well as the relation [149] between the $\mathrm{su}_q(2)$ model and the Variable Moment of Inertia (VMI) model (see sec. 20).

The q-deformed rotator corresponds to the Hamiltonian

$$H = \frac{1}{2I} C_2(\mathrm{su}_q(2)) + E_0, \tag{19.1}$$

where I is the moment of inertia and E_0 is the bandhead energy (for ground state bands $E_0 = 0$). For q real, i.e. with $q = e^\tau$ with τ real, the energy levels of the q-rotator are

$$E(J) = \frac{1}{2I}[J][J+1] + E_0 = \frac{1}{2I}\frac{\sinh(\tau J)\sinh(\tau(J+1))}{\sinh^2(\tau)} + E_0. \tag{19.2}$$

For q being a phase, i.e. $q = e^{i\tau}$ with τ real, one obtains

$$E(J) = \frac{1}{2I}[J][J+1] + E_0 = \frac{1}{2I}\frac{\sin(\tau J)\sin(\tau(J+1))}{\sin^2(\tau)} + E_0. \tag{19.3}$$

Raychev *et al.* [146] have found that good fits of rotational spectra of even–even rare earths and actinides are obtained with eq. (19.3). It is easy to check that eq. (19.2) fails in describing such spectra. In order to understand this difference, it is useful to make Taylor expansions of the quantities in the numerator of eq. (19.2) (eq. (19.3)) and collect together the terms containing the same powers of $J(J+1)$ (all other terms cancel out), finally summing up the coefficients of each power. In the first case the final result is

$$E(J) = E_0 + \frac{1}{2I}\frac{1}{(\sqrt{\frac{\pi}{2\tau}}I_{1/2}(\tau))^2}(\sqrt{\frac{\pi}{2\tau}}I_{1/2}(\tau)J(J+1) + \tau\sqrt{\frac{\pi}{2\tau}}I_{3/2}(\tau)(J(J+1))^2$$

$$+\frac{2\tau^2}{3}\sqrt{\frac{\pi}{2\tau}}I_{5/2}(\tau)(J(J+1))^3 + \frac{\tau^3}{3}\sqrt{\frac{\pi}{2\tau}}I_{7/2}(\tau)(J(J+1))^4 + \ldots \tag{19.4}$$

where $\sqrt{\frac{\pi}{2\tau}}I_{n+\frac{1}{2}}(\tau)$ are the modified spherical Bessel functions of the first kind [150].

In the second case (eq. (19.3)) following the same procedure one obtains

$$E(J) = E_0 + \frac{1}{2I}\frac{1}{(j_0(\tau))^2}(j_0(\tau)J(J+1) - \tau j_1(\tau)(J(J+1))^2$$

$$+\frac{2}{3}\tau^2 j_2(\tau)(J(J+1))^3 - \frac{1}{3}\tau^3 j_3(\tau)(J(J+1))^4 + \frac{2}{15}\tau^4 j_4(\tau)(j(j+1))^5 - \ldots), \tag{19.5}$$

where $j_n(\tau)$ are the spherical Bessel functions of the first kind [150].

Both results are of the form

$$E(J) = AJ(J+1) + B(J(J+1))^2 + C(J(J+1))^3 + D(J(J+1))^4 + \ldots. \tag{19.6}$$

Empirically it is known that nuclear rotational spectra do show such a behaviour, the coefficients A, B, C, D, ... having alternating signs (starting with A positive) and magnitudes dropping by about 3 orders of magnitude each time one moves to the next higher power of $J(J+1)$ [151,152].

It is interesting to check if the empirical characteristics of the coefficients A, B, C, D are present in the case of the expansions of eqs. (19.2), (19.3), especially for small values of τ. (Since we deal with rotational spectra, which are in first order approximation described by the usual algebra su(2), we expect τ to be relatively small, i.e. the deviation of $\mathrm{su}_q(2)$ from su(2) to be small. This is in agreement to the findings of [146], where τ is found to be around 0.03.)

One can easily see that in eq. (19.2) it is impossible to get alternating signs, while in eq. (19.3) the condition of alternating signs is readily fulfilled. This fact has as a result that the energy levels given by eq. (19.2) increase more rapidly than the levels given by the $J(J+1)$ rule, while the levels given by eq. (19.3) increase less rapidly than $J(J+1)$. In order to check the order of magnitude of the coefficients for small values of τ, it is useful to expand the spherical Bessel functions appearing in eq. (19.3) and keep only the lowest order term in each expansion. The result is

$$E(J) = E_0 + \frac{1}{2I}(J(J+1) - \frac{\tau^2}{3}(J(J+1))^2 + \frac{2\tau^4}{45}(J(J+1))^3$$

$$-\frac{\tau^6}{315}(J(J+1))^4 + \frac{2\tau^8}{14175}(J(J+1))^5 - \ldots). \tag{19.7}$$

We remark that each term contains a factor τ^2 more than the previous one. For τ in the area of 0.03, τ^2 is of the order of 10^{-3}, as it should. We conclude therefore that eq. (19.6) is suitable for fitting rotational spectra, since its coefficients have the same characteristics as the empirical coefficients of eq. (19.6).

Extended comparisons of the $\mathrm{su}_q(2)$ predictions to experimental data for ground state bands of rare earth and actinide nuclei can be found in [146,149,153,154]. More recently, the $\mathrm{su}_q(2)$ formalism has been used for the description of β- and γ-bands of deformed rare earths and actinides, with satisfactory results [155].

It is necessary for $E(J)$ to be an increasing function of J. In order to guarantee this in eq. (19.3) one must have

$$\tau(J+1) \leq \frac{\pi}{2}. \tag{19.8}$$

In the case of $\tau = 0.036$ (as in ^{232}U in [149]), one finds $J \leq 42$, this limiting value being larger than the highest observed J in ground state bands in the actinide region [156]. Similarly, for $\tau = 0.046$ (as in ^{178}Hf in [149]), one finds $J \leq 32$, this limiting value being again higher than the highest observed J in ground state bands in the rare earth region [156].

It should be mentioned that a method of fixing the deformation parameter by demanding that the relevant Hilbert space be large enough to contain all the irreps of the system has been suggested [157] in the framework of Barnett–Pegg theory, with good results.

20. Comparison of the $su_q(2)$ model to other models

20.1. The Variable Moment of Inertia (VMI) model

In lowest order approximation rotational nuclear spectra can be described by the formula

$$E(J) = \frac{J(J+1)}{2\Theta}, \tag{20.1}$$

where Θ is the moment of inertia of the nucleus, which is assumed to be constant. However, in order to get closer agreement to experimental data, one finds that he has to include higher order terms in this formula, as shown in eq. (19.6).

Another way to improve agreement with experiment is to let the moment of inertia Θ to vary as a function of the angular momentum J. One thus obtains the *Variable Moment of Inertia (VMI) model* [158]. In this model the levels of the ground state band are given by

$$E(J) = \frac{J(J+1)}{2\Theta(J)} + \frac{1}{2}C(\Theta(J) - \Theta_0)^2, \tag{20.2}$$

where $\Theta(J)$ is the moment of inertia of the nucleus at the state with angular momentum J, while C and Θ_0 are the two free parameters of the model, fitted to the data. The parameter Θ_0 corresponds to the ground state moment of inertia, while instead of the parameter C it has been found meaningful to use the parameter combination

$$\sigma = \frac{1}{2C\Theta_0^3}, \tag{20.3}$$

which is related to the softness of the nucleus. The moment of inertia at given J is determined through the variational condition

$$\frac{\partial E(J)}{\partial \Theta(J)}|_J = 0, \tag{20.4}$$

which is equivalent to the cubic equation

$$\Theta(J)^3 - \Theta(J)^2\Theta_0 - \frac{J(J+1)}{2C} = 0. \tag{20.5}$$

This equation has only one real root, which can be written as

$$\Theta(J) = \sqrt[3]{\frac{J(J+1)}{4C} + \frac{\Theta_0^3}{27} + \sqrt{\frac{(J(J+1))^2}{16C^2} + \frac{\Theta_0^3 J(J+1)}{54C}}}$$

$$+\sqrt[3]{\frac{J(J+1)}{4C} + \frac{\Theta_0^3}{27} - \sqrt{\frac{(J(J+1))^2}{16C^2} + \frac{\Theta_0^3 J(J+1)}{54C}}} + \frac{\Theta_0}{3}. \tag{20.6}$$

Expanding the roots in this expression one obtains

$$\Theta(J) = \Theta_0(1 + \sigma J(J+1) - 2\sigma^2(J(J+1))^2 + 7\sigma^3(J(J+1))^3 - 30\sigma^4(J(J+1))^4 + \ldots). \tag{20.7}$$

Using eq. (20.7) in eq. (20.2) one obtains the following expansion for the energy

$$E(J) = \frac{1}{2\Theta_0}(J(J+1) - \frac{1}{2}\sigma(J(J+1))^2 + \sigma^2(J(J+1))^3 - 3\sigma^3(J(J+1))^4 + \ldots). \tag{20.8}$$

Empirically it is known [158] that for rotational nuclei the softness parameter σ is of the order of 10^{-3}. Therefore the expansion of eq. (20.8) has the same characteristics as the expansion of eq. (19.6) (alternating signs, successive coefficients falling by about 3 orders of magnitude).

20.2. Comparison of the $su_q(2)$ model to the VMI and related models

We now turn to the comparison of the expansion of eq. (19.5) to the Variable Moment of Inertia (VMI) model, discussed in the previous subsection. Comparing eqs (19.5) and (20.8) we see that both expansions have the same form. The moment of inertia parameter I of (19.5) corresponds to the ground state moment of inertia Θ_0 of (20.8). The small parameter of the expansion is τ^2 in the first case, while it is the softness parameter $1/(2C\Theta_0^3)$ in the second. However, the numerical coefficients in front of each power of $J(J+1)$ are not the same.

In [149] a comparison is made between the parameters obtained by fitting the same spectra by the $su_q(2)$ and VMI formulae. The agreement between $1/(2I)$ and $1/(2\Theta_0)$ is very good, as it is the agreement between τ^2 and σ as well. Therefore the known [158] smooth variation of Θ_0 and σ with the ratio $R_4 = E(4)/E(2)$ is expected to hold for the parameters I and τ^2 as well. This is indeed seen in [149].

The difference between the expansions of eqs (19.5) (or (19.7)) and (20.8) is also demonstrated by forming the dimensionless ratios $AC/(4B^2)$ and $A^2D/(24B^3)$ in eq. (19.6). In the case of eq. (20.8) both quantities are equal to 1, as expected, since it is known that the VMI is equivalent [159,160] to the Harris expansion [161], in which both quantities are known to be equal to 1. In the case of eq. (19.7) the corresponding values are $AC/(4B^2) = 1/10$ and $A^2D/(24B^3) = 1/280$. According to the empirical values of these ratios given in [152,162], the ratios given from eq. (19.7) are better than the ratios given by eq. (20.8), especially the second one.

20.3. The hybrid model

The hybrid model of nuclear collective motion [163,164,165,166] has been introduced in order to provide a link between the two successful ways of describing low-lying nuclear excitations: the extended form of the Bohr-Mottelson model (BMM) [167,168] and the Interacting Boson Model (IBM) [169,170,171] (see secs 27, 31 for more details). The hybrid model combines the advantages of both models, i.e. the geometrical significance of the collective coordinates inherent in the extended BMM, and the use of group theoretical concepts characterizing IBM. In the framework of the rotational limit of the hybrid model, associated with the chain u(6)$\supset$su(3), Partensky and Quesne [172,173], starting from the fact that in the geometrical description the square of the deformation is proportional to the moment of inertia of the ground state band, proved that the energy levels of the ground state band are given by

$$E(J) = \frac{A}{J(J+1)+B}J(J+1), \tag{20.9}$$

with A being a free parameter and B given by

$$B = 8N^2 + 22N - 15, \tag{20.10}$$

where N is the sum of the number of valence proton pairs (or proton-hole pairs, when more than half of the proton valence shell is filled) N_π and the number of the valence neutron pairs (or neutron-hole pairs, when more than half of the neutron valence shell is filled) N_ν. We remark that in the framework of this model the moment of inertia

$$\Theta(J,N) = \frac{J(J+1)+B}{2A} \tag{20.11}$$

depends on both the angular momentum J and the valence pair number N.

It is instructive to expand $E(J)$ in powers of J(J+1) [174]

$$E(J) = A\sum_{k=0}^{\infty}(-1)^k\left(\frac{J(J+1)}{B}\right)^{k+1}$$

$$= \frac{A}{B}(J(J+1) - \frac{1}{B}(J(J+1))^2 + \frac{1}{B^2}(J(J+1))^3 - \frac{1}{B^3}(J(J+1))^4 + \ldots). \qquad (20.12)$$

Comparing the present expansion to the one of eq. (19.7) for the $su_q(2)$ model one has

$$\tau = \sqrt{\frac{3}{B}} = \sqrt{\frac{3}{8N^2 + 22N - 15}}, \qquad (20.13)$$

obtaining thus a connection between the τ parameter of the $su_q(2)$ model and a microscopic quantity, the valence nucleon pair number. ¿From this equation it is clear that τ decreases for increasing N. Thus the minimum values of τ are expected near the midshell regions, where the best rotators are known to be located. This is a reasonable result, since τ describes the deviations from the su(2) (rigid rotator) limit. It is clear that the minimum deviations should occur in the case of the best rigid rotators.

From eq. (20.13) one can obtain for each nucleus the value of τ from the number of valence nucleon pairs present in it. These predictions for τ have been compared in [174] to the values of τ found empirically by fitting the corresponding spectra, with good results in both the rare earth and the actinide regions. In addition eq. (20.13) indicates that in a given shell nuclei characterized by the same valence nucleon pair number N will correspond to the same value of τ. Such multiplets have been studied in the framework of the hybrid model by [175].

It is worth noticing that fits of γ_1-bands in the rare earth region (Er, Yb isotopes) [155] give τ parameters very similar to the ones coming from fitting the corresponding ground state bands, in addition exhibiting the same as the one mentioned above behaviour of τ as a function of N.

Taking further advantage of the above connection between the $su_q(2)$ model and the hybrid model, the parameter τ has been connected [176] to the nuclear deformation parameter β, as well as to the electromagnetic transition probabilities B(E2:$2_1^+ \to 0_1^+$). Since both β and B(E2:$2_1^+ \to 0_1^+$) are known to increase with increasing collectivity, τ is expected to decrease with increasing β or increasing B(E2:$2_1^+ \to 0_1^+$). This expectation is corroborated by the results reported in [176].

20.4. Other models

It should be noticed that an empirical formula very similar to that of eq. (19.3) has been recently proposed by Amal'skiĭ [177] on completely different physical grounds. The formula reads

$$E(J) = A\sin^2\left(\frac{\pi J}{B}\right), \qquad (20.14)$$

where A and B are free parameters. This formula should be compared to eq. (19.3), with which it is almost identical.

A different formula, also giving very good fits of rotational spectra, has been introduced by Celeghini, Giachetti, Sorace and Tarlini [153], based on the q-Poincaré rotator.

21. Electromagnetic transitions in the $su_q(2)$ model

We have already seen that the $su_q(2)$ formalism provides an alternative to the VMI model, the deformation parameter q being connected to the softness parameter of the VMI model.

The stretching effect present in rotational energy levels, which can equally well be described in terms of the VMI model and the $su_q(2)$ symmetry, should also manifest itself in the B(E2) transition probabilities among these levels. If deviations from the su(2) symmetry are observed in the energy levels of a band, relevant deviations should also appear in the B(E2) transitions connecting them. In the case of the VMI model no way has been found for making predictions for the B(E2) transition probabilities connecting the levels of a collective band. The $su_q(2)$ symmetry naturally provides such a link. Before studying the $su_q(2)$ case, though, it is useful to recall the predictions of other models on this matter.

21.1. The collective model of Bohr and Mottelson

In rotational bands one has [178]

$$B(E2: J+2 \to J) = \frac{5}{16\pi} Q_0^2 |C^{J+2,2,J}_{K,0,-K}|^2, \tag{21.1}$$

i.e. the B(E2) transition probability depends on the relevant Clebsch-Gordan coefficient of su(2), while Q_0^2 is the intrinsic electric quadrupole moment and K is the projection of the angular momentum J on the symmetry axis of the nucleus in the body-fixed frame. For $K = 0$ bands, as the ground state bands, one then has [178]

$$B(E2: J+2 \to J) = \frac{5}{16\pi} Q_0^2 \frac{3}{2} \frac{(J+1)(J+2)}{(2J+3)(2J+5)}. \tag{21.2}$$

It is clear that the B(E2) values should saturate with increasing J.

21.2. The Interacting Boson Model (IBM)

It is also instructive to mention what happens in the case of the Interacting Boson Model (IBM) [169], the successful algebraic model of nuclear structure with which we are going to deal in more detail later (see sections 27, 31). In the case of the su(3) limit of the IBM, which is the limit applicable to deformed nuclei, the corresponding expression is [179]

$$B(E2: J+2 \to J) = \frac{5}{16\pi} Q_0^2 \frac{3}{2} \frac{(J+1)(J+2)}{(2J+3)(2J+5)} \frac{(2N-J)(2N+J+3)}{(2N+3/2)^2}, \tag{21.3}$$

where N is the total number of bosons. Instead of saturation one then gets a decrease of the B(E2) values at high J, which finally reach zero at $J = 2N$. This is a well known disadvantage of the simplest version of the model (IBM-1) due to the small number of collective bosons (s ($J = 0$) and d ($J = 2$)) taken into account. It can be corrected by the inclusion of higher bosons (g ($J = 4$), i ($J = 6$), etc), which approximately restore saturation (see [169,180] for full list of references).

Another way to avoid the problem of decreasing B(E2)s in the su(3) limit of IBM at high J is the recently proposed [181] transition from the compact su(3) algebra to the noncompact sl(3,R) algebra. The angular momentum at which this transition takes place is fitted to experiment. In this way an increase of the B(E2) values at high J is predicted, which agrees well with the experimental data for ^{236}U.

21.3. The $su_q(2)$ model

In order to derive a formula similar to (21.2) in the $su_q(2)$ case, one needs to develop an $su_q(2)$ angular momentum theory. As mentioned in sec. 15, this has already been achieved. It turns out that an equation similar to (21.1) holds in the q-deformed case, the only difference being that the Clebsch-Gordan coefficient of the $su_q(2)$ algebra must be used instead. These coefficients have the form [91]

$${}_qC^{J+2,2,J}_{K,0,-K} = q^{2K} \sqrt{\frac{[3][4][J-K+2][J-K+1][J+K+1][J+K+2]}{[2][2J+2][2J+3][2J+4][2J+5]}}. \tag{21.4}$$

For $K = 0$ bands one then has

$$B_q(E2; J+2 \to J) = \frac{5}{16\pi} Q_0^2 \frac{[3][4][J+1]^2[J+2]^2}{[2][2J+2][2J+3][2J+4][2J+5]}. \tag{21.5}$$

For $q = e^{i\tau}$ this equation takes the form

$$B_q(E2; J+2 \to J) = \frac{5}{16\pi} Q_0^2 \frac{\sin(3\tau)\sin(4\tau)}{\sin(2\tau)\sin(\tau)}$$

$$\frac{(\sin(\tau(J+1)))^2(\sin(\tau(J+2)))^2}{\sin(\tau(2J+2))\sin(\tau(2J+3))\sin(\tau(2J+4))\sin(\tau(2J+5))}. \tag{21.6}$$

It is useful to get an idea of the behaviour of this expression as a function of J, especially for the small values of τ found appropriate for the description of ground state spectra. Expanding all functions and keeping corrections of the leading order in τ only, one has

$$B_q(E2, J+2 \to J) = \frac{5}{16\pi} Q_0^2 \frac{3}{2} \frac{(J+1)(J+2)}{(2J+3)(2J+5)} (1 + \frac{\tau^2}{3}(6J^2 + 22J + 12)). \qquad (21.7)$$

We see that the extra factor, which depends on τ^2, contributes an extra increase with J, while the usual su(2) expression reaches saturation at high J and IBM even predicts a decrease.

21.4. Comparison to experiment

Is there any experimental evidence for such an increase? In order to answer this question one should discover cases in which the data will be consistent with the $su_q(2)$ expression but inconsistent with the classical su(2) expression. (The opposite cannot happen, since the classical expression is obtained from the quantum expression for the special parameter value $\tau = 0$.) Since error bars of B(E2) values are usually large, in most cases both symmetries are consistent with the data. One should expect the differences to show up more clearly in two cases:

i) In rare earth nuclei not very much deformed (i.e. with an $R_4 = E(4)/E(2)$ ratio around 3.0). These should be deformed enough so that the $su_q(2)$ symmetry will be able to describe them, having, however, at the same time values of τ not very small. Since in several of these nuclei backbending (or upbending) occurs at $J = 14$ or 16, one can expect only 5 or 6 experimental points to compare the theoretical predictions with.

ii) In the actinide region no backbending occurs up to around $J = 30$, so that this is a better test ground for the two symmetries. However, most nuclei in this region are well deformed, so that small values of τ should be expected, making the distinction between the two theoretical predictions difficult.

A few characteristic examples (4 rare earths and an actinide) are given in [182]. In all cases it is clear that the $su_q(2)$ curve follows the experimental points, while the su(2) curve has a different shape which cannot be forced to go through all the error bars. Several comments are now in place:

i) For a given nucleus the value of the parameter τ obtained from fitting the B(E2) values among the levels of the ground state band should be equal to the value obtained from fitting the energy levels of the ground state band. In [182] it is clear that both values are similar, although in most cases the value obtained from the B(E2)s is smaller than the value obtained from the spectra. It should be taken into account, however, that in most cases the number n' of levels fitted is different (larger) than the number n of the B(E2) values fitted. In the single case (^{184}W) in which $n = n'$, the two τ values are almost identical, as they should.

ii) One can certainly try different fitting procedures. Using the value of τ obtained from the B(E2) values for fitting the spectrum one gets a reasonably good description of it, although the squeezing of the spectrum is not as much as it should have been (with the exception of ^{184}W). Using the value of τ obtained from the spectrum for fitting the B(E2) values one obtains an increase more rapid than the one shown by the data (again with the exception of ^{184}W). One can also try to make an overall fit of spectra and B(E2)s using a common value of τ. Then both the squeezing of the spectrum and the rise of the B(E2)s can be accounted for reasonably well although not exactly. One should notice, however, that the experimental uncertainties of the B(E2)s are much higher than the uncertainties of the energy levels.

iii) Concerning energy levels, the rigid rotator model and the su(3) limit of the IBM predict a $J(J+1)$ increase, while the $su_q(2)$ model and the VMI model predict squeezing, which is seen experimentally.

iv) Concerning the B(E2) values, the VMI makes no prediction, the rigid rotator predicts saturation at high J, the su(3) limit of the IBM predicts decrease, while the $su_q(2)$ model predicts an increase. The evidence presented in [182] supports the $su_q(2)$ prediction, but clearly much more work, both experimental and analytical, is needed before final conclusions can be drawn. The modified su(3) limit of IBM described in [181], as well as the su(3) limit of the *sdg*-IBM [183] also support the increase of the B(E2) values at high J. Increasing BE(2) values are also predicted in the framework of the Fermion Dynamical Symmetry Model [184], as well as by a binary cluster model [185]. There is also empirical evidence for increasing B(E2) values in the recent systematics by Zamfir and Casten [186].

v) It is clear that much further work is needed as far as comparisons of the $su_q(2)$ predictions to experimental BE(2) values are concerned for safe conclusions to be reached.

vi) Since the quadrupole operator is not a member of the symmetry algebra $su_q(2)$ under consideration, it is clear that the B(E2) values studied here do not contain any dynamical deformation effects, but only the kinematical ones (through the use of the q-deformed Clebsch-Gordan coefficients). A more complete approach to the problem will be the construction of a larger algebra, of which the quadrupole operator will be a member and it will also be an irreducible tensor under $su_q(2)$ or $so_q(3)$. This task will be undertaken in sec. 29.

22. Superdeformed bands

One of the most impressive experimental discoveries in nuclear physics during the last decade was that of superdeformation [187] (see [188,189,190] for relevant reviews). The energy levels of superdeformed bands follow the $J(J+1)$ rule much more closely than the usual rotational bands. Levels with J larger than 60 have been observed. A compilation of superdeformed bands has been given in [191]. The best examples have been found in the A$\approx$150 mass region, while additional examples have been found in the A$\approx$80, A$\approx$130 and A$\approx$190 regions. It is understood that the superdeformed bands in the A$\approx$150 region correspond to elongated ellipsoidal shapes with an axis ratio close to 2:1, while in the A$\approx$130 and A$\approx$190 regions the ratios 3:2 and 1.65:1 respectively appear closer to reality.

Since the $su_q(2)$ model has been found suitable for describing normal deformed bands, it is plausible that it will also be successful in describing superdeformed bands as well. A test has been performed in [192]. The $su_q(2)$ model has been found to give good results in all mass regions, the deformation parameter τ being smaller than in the case of normal deformed bands, thus indicating smaller deviations from the $su_q(2)$ symmetry. In particular, τ has been found to obtain values about 0.01 in the A$\approx$130 and A$\approx$190 regions, while it obtains even smaller values, around 0.004, in the A$\approx$150 region, which contains the best examples of superdeformed bands observed so far. These results should be compared to the values of τ around 0.03 obtained in the case of normal deformations.

Concerning the corresponding B(E2) values, the experimental information is still quite poor for allowing a meaningful comparison of the $su_q(2)$ predictions to experiment.

A new effect, called $\Delta I = 2$ *staggering*, has been recently observed in a few superdeformed bands [193,194,195,196,197]. If $\Delta I = 2$ staggering is present, then, for example, the levels with $I = 2, 6, 10, 14, \ldots$ are displaced relatively to the levels with $I = 0, 4, 8, 12, \ldots$, i.e. the level with angular momentum I is displaced relatively to its neighbours with angular momentum $I \pm 2$. In most cases, though, the effect is smaller than the experimental error. $\Delta I = 2$ staggering has also been seen in some normal deformed bands [198,199,200]. Although many theoretical attempts for the explanation of the effect have been made (see [201] for a relevant list), it seems plausible that what is seen in most cases is simply a series of backbendings [198,202,203]. The same effect has been observed in rotational bands of diatomic molecules [204]. In this case the effect is clearly larger than the experimental error and it is also clear that it is due to interband interactions [201]. Furthermore in diatomic molecules a $\Delta I = 1$ staggering effect has been observed [205,206], i.e. a relative displacement of the levels with even I with respect to the levels with odd I. This effect is clearly larger than the experimental error. A description of the $\Delta I = 1$ staggering effect in terms of two q-deformed rotators with slightly different parameter values has been attempted [205].

23. The physical content of the $su_q(2)$ model

From the above it is clear that the $su_q(2)$ model offers a way of describing nuclear stretching, i.e. the departure of deformed nuclei from the su(2) symmetry of the rigid rotator, similar to the one of the VMI model. The parameter τ describes this departure quantitatively, vanishing in the rigid rotator limit. Therefore the deformation parameter τ should not be confused with nuclear deformation; it is in fact related to nuclear softness, as already discussed in sec. 20.

On the other hand, the increase of the moment of inertia with increasing J in the framework of the VMI model means that collectivity gets increased [178]. The $su_q(2)$ model is an alternative way for describing this increase in collectivity. But increased collectivity implies increased B(E2) transitions.

Therefore it is not surprising that the $su_q(2)$ model predicts B(E2) values increasing with J.

Given the $su_q(2)$ generators J_+, J_-, J_0, it is instructive to define as usual the operators J_x, J_y, J_z by

$$J_+ = J_x + iJ_y, \qquad J_- = J_x - iJ_y, \qquad J_0 = J_z. \tag{23.1}$$

The $su_q(2)$ commutation relations can then be rewritten in the form

$$[J_x, J_y] = \frac{i}{2}[2J_z], \qquad [J_y, J_z] = iJ_x, \qquad [J_z, J_x] = iJ_y, \tag{23.2}$$

which is a generalization of the so(3) commutation relations, obtained in the limit $q \to 1$. We remark that while in the classical so(3) case the three commutation relations have exactly the same form, in the quantum case the first commutation relation differs (in the right hand side) from the other two, thus indicating that in the framework of the problem under study the z-direction is not any more equivalent to the x- and y- directions. This is of course a phenomenological way to describe the softness of deformed nuclei by adding the appropriate perturbations to the pure su(2) Hamiltonian and has nothing to do with the isotropy of space, as implied in [207].

It is worth remarking at this point that the Casimir operator of $su_q(2)$ is also invariant under the usual su(2) [208]. Therefore the quantum number J characterizing the irreps of $su_q(2)$, and as a result the nuclear levels, is exactly the same as the quantum number J used in the case of the usual su(2). Therefore there is no reason for a "total reformulation of quantum mechanics", as implied in [207], the $su_q(2)$ generators being connected to their su(2) counterparts by the q-deforming functionals of sec. 14. In other words, one continues to believe in usual angular momentum theory and usual quantum mechanics. All what is done in the framework of the $su_q(2)$ model is to add to the usual su(2) Hamiltonian several perturbations which have a special form making them suitable to be summed up, including the original su(2) term, into the form of the $su_q(2)$ Hamiltonian.

24. The $u_{p,q}(2)$ rotator model

An extension of the $su_q(2)$ model is the $u_{p,q}(2)$ model [209,210,211,212], which is based on a two-parameter deformed algebra (see sec. 15 for a list of references). For $p = q$ (using the definition of (p, q)-numbers of eq. (2.11)) this model reduces to the $su_q(2)$ one. This model has been successfully applied to superdeformed nuclear bands [210,211]. When Taylor expanded, it becomes clear that the eigenvalues of the Hamiltonian of this model (which is the second order Casimir operator of $u_{p,q}(2)$) contain terms of the form $J(J(J+1))^n$, in addition to the $(J(J+1))^n$ ones. It is therefore closer to the modification of $su_q(2)$ which will be discussed in subsec. 26.3.

25. Generalized deformed su(2) models

Another formula giving very good results for rotational spectra has been introduced by Holmberg and Lipas [213], and rediscovered in [214]. In this case the energy levels are given by

$$E(J) = a\left[\sqrt{1 + bJ(J+1)} - 1\right]. \tag{25.1}$$

Taylor expansion of the square root immediately shows that the present formula is a special case of eq. (19.6). This formula can be derived from the collective model of Bohr and Mottelson [178].

It has been argued in [162] that the Hamiltonian of eq. (25.1) gives better agreement to rotational nuclear spectra than the one coming from the $su_q(2)$ symmetry. Using the techniques described in detail in sec. 17 one can construct a generalized deformed algebra $su_\Phi(2)$, characterized by a function $\Phi(J(J+1)$, giving the spectrum of eq. (25.1) exactly. In this particular case the algebra is characterized by the structure function

$$\Phi(J(J+1)) = a\left[\sqrt{1 + bJ(J+1)} - 1\right]. \tag{25.2}$$

It is of interest to check if this choice of structure function also improves the agreement between theory and experiment in the case of the electromagnetic transition probabilities connecting these energy levels.

In order to study this problem, one has to construct the relevant generalized Clebsch-Gordan coefficients. This problem is still open.

26. Quantum algebraic description of vibrational and transitional nuclear spectra

We have already seen that the $su_q(2)$ model describes successfully deformed and superdeformed bands. It is not surprising that the applicability of the $su_q(2)$ formalism is limited to the rotational region (where the ratio $R_4 = E(4)/E(2)$ obtains values between 3.0 and 3.33), since it is based on a deformation of the rotation algebra. For describing nuclear spectra in the vibrational ($2.0 \leq R_4 \leq 2.4$) and transitional ($2.4 \leq R_4 \leq 3.0$) regions it is clear that an extension of the model is needed. In order to be guided towards such an extension, we briefly review the existing experience of other successful models.

26.1. The Interacting Boson Model

In the rotational (su(3)) limit [179] of the Interacting Boson Model (IBM) (see secs 27, 31 for more details) the spectrum is described by a $J(J+1)$ expression, while in the vibrational (u(5)) [215] and transitional (o(6)) [216] limits expressions of the form $J(J+c)$ with $c > 1$ appear. In the u(5) limit, in particular, the energy levels are given by

$$E(N, n_d, v, n_\Delta, J, M_J) = E_0 + \epsilon n_d + \alpha n_d(n_d+4) + \beta 2v(v+3) + \gamma 2J(J+1), \tag{26.1}$$

where N is the total number of bosons, n_d is the number of d-bosons, v is the seniority, n_Δ is the "missing" quantum number in the reduction from o(5) to o(3), M_J is the third component of the angular momentum J, while E_0, ϵ, α, β, γ are free parameters. The ground state band, in particular, is characterized by quantum numbers $n_d = 0, 1, 2, \ldots$, $v = n_d$, $n_\Delta = 0$, $J = 2n_d$, so that the energy expression for it reads

$$E(J) = E_0 + \frac{\epsilon}{2}J + \frac{\alpha}{4}J(J+8) + \frac{\beta}{2}J(J+6) + 2\gamma J(J+1). \tag{26.2}$$

In the o(6) limit the energy is given by

$$E(N, \sigma, \tau, \nu_\Delta, J, M_J) = E_0 + \beta 2\tau(\tau+3) + \gamma 2J(J+1) + \eta 2\sigma(\sigma+4), \tag{26.3}$$

where σ is the quantum number characterizing the irreducible representations (irreps) of o(6), τ is the quantum number characterizing the irreps of o(5), ν_Δ is the missing quantum number in the reduction from o(5) to o(3), while E_0, β, γ, η are free parameters. The ground state band is characterized by the quantum numbers $\sigma = N$, $\tau = 0, 1, 2, \ldots$, $\nu_\Delta = 0$, $J = 2\tau$, so that the relevant energy expression takes the form

$$E(J) = E_0 + \frac{\beta}{2}J(J+6) + \gamma 2J(J+1) + \eta 2N(N+4). \tag{26.4}$$

The message from eqs (26.2) and (26.4) is that nuclear anharmonicities are described by expressions in which J and J^2 appear with different coefficients, and not with the same coefficient as in $J(J+1)$. The earliest introduction of this idea is in fact the Ejiri formula [217]

$$E(J) = kJ(J+1) + aJ, \tag{26.5}$$

which has been subsequently justified microscopically in [218].

26.2. Generalized VMI

The two-parameter VMI model is known to continue giving good fits in the transitional and even in the vibrational region. In these regions, however, the accuracy of the model is substantially improved by adding a third parameter, which essentially allows for treating J and J^2 with a different coefficient [219,220,221].

The usual VMI model has been briefly reviewed in subsec. 20.1. One of the (essentially equivalent) three-parameter extensions of the model, which give improved fits of vibrational and transitional spectra, is the generalized VMI (GVMI) model [219,220] in which the energy levels are described by

$$E(J) = \frac{J + xJ(J-2)}{\Phi(J)} + \frac{1}{2}k(\Phi(J) - \Phi_0)^2, \tag{26.6}$$

which can be easily rewritten in the form

$$E(J) = \frac{J(J+x')}{2\Phi'(J)} + \frac{1}{2}k'(\Phi'(J) - \Phi'_0)^2, \tag{26.7}$$

where $x' = x^{-1} - 2$. It is clear that for $x = 1/3$ the GVMI reduces to the usual VMI, while for transitional and vibrational nuclei x obtains lower values, so that x' becomes greater than 1. The variational condition determining the moment of inertia still has the form of eq. (20.4), while the expansion of the energy turns out to be

$$E(J) = \frac{1}{2\Phi'_0}(J(J+x') - \frac{\sigma'}{2}(J(J+x'))^2 + (\sigma')^2(J(J+x'))^3 - 3(\sigma')^3(J(J+x'))^4 + \ldots), \tag{26.8}$$

where

$$\sigma' = \frac{1}{2k'(\Phi'_0)^3}. \tag{26.9}$$

We remark that an expansion in terms of $J(J+x')$ is obtained, as compared to an expansion in terms of $J(J+1)$ in the case of the usual VMI. The physical content of the parameters is clear: the centrifugal stretching effect is accounted for by the softness parameter σ', as in the case of the usual VMI, while anharmonicities, important in the vibrational region, are introduced by $x' > 1$. Since centrifugal stretching and anharmonicities are two effects of different origins, it is reasonable to describe them by two different parameters.

26.3. Modification of the $su_q(2)$ model

The evidence coming from the IBM and the generalized VMI model described above, suggests a model in which the spectrum is given by

$$E(J) = \frac{1}{2I}[J]_q[J+c]_q, \tag{26.10}$$

which contains 3 parameters: the moment of inertia I, the deformation parameter q and the new parameter c, which is expected to be 1 in the rotational limit and larger than 1 in the vibrational and transitional regions. This energy expression can be expanded as

$$E(J) = \frac{1}{2I}\frac{1}{(j_0(\tau))^2}(j_0(\tau)J(J+c) - \tau j_1(\tau)(J(J+c))^2$$

$$+\frac{2}{3}\tau^2 j_2(\tau)(J(J+c))^3 - \frac{1}{3}\tau^3 j_3(\tau)(J(J+c))^4 + \frac{2}{15}\tau^4 j_4(\tau)(J(J+c))^5 - \ldots), \tag{26.11}$$

which is similar to eq. (19.5) with $J(J+1)$ replaced by $J(J+c)$.

It is expected that the deformation parameter τ, which plays the role of the small parameter in the expansion, as the softness parameter does in the case of the VMI, will describe the centrifugal stretching effect, while the parameter c will correspond to the anharmonicity effects. These expectations are corroborated from fits of the experimental data reported in [222]. The following comments can be made:

i) The anharmonicity parameter c is clearly decreasing with increasing R_4, i.e. with increasing collectivity. It obtains high values (8-18) in the vibrational region, while in the rotational region it stays close to 1. (It should be noted that by fixing $c = 1$ in the rotational region the fits are only very slightly changed, as expected.) In the transitional region its values are close to 3.

ii) The deformation parameter τ, which corresponds to the centrifugal stretching, is known from the $su_q(2)$ model to obtain values close to 0.3-0.4 in the rotational region, a fact also seen here. The same range of values appears in the vibrational region as well, while in the transitional region τ reaches values as high as 0.6. It is not unreasonable for this parameter, which is connected to the softness of the nucleus, to obtain its highest values in the region of γ-soft nuclei.

iii) It is worth remarking that eq. (26.10) coincides for $q = 1$ and c=integer=N with the eigenvalues of the Casimir operator of the algebra o(N+2) in completely symmetric states [223,224]. In the rotational region the fits gave N=1, which corresponds to o(3), as expected, while in the transitional region the fits gave approximately N=3, which corresponds to o(5), which is a subalgebra contained in both the u(5) and o(6) limits of the IBM. A unified description of 927 low lying states of 271 nuclei from five major shells along these lines has been given in [225].

iv) It is also worth remarking that a special case of the expression of eq. (26.10) occurs in the q-deformed version of the o(6) limit of the Interacting Boson Model, which will be reported below (see eq. (31.7)).

v) The $su_q(2)$ symmetry is known to make specific predictions for the deviation of the behaviour of the B(E2) values from the rigid rotator model (subsec. 21.3). It will be interesting to connect the spectrum of eq. (26.10) to some deformed symmetry, at least for special values of c, and examine the implications of such a symmetry for the B(E2) values. Such a study in the framework of the q-deformed version of the o(6) limit of IBM, mentioned in iv), is also of interest.

vi) It is worth noticing that an expansion in terms of $J(J+c)$ can also be obtained from a generalized oscillator (sec. 12) with a structure function

$$F(J) = [J(J+c)]_Q, \tag{26.12}$$

where $[x]_Q$ stands for the Q-numbers introduced in sec. 6 and $Q = e^T$, with T real. This is similar to an oscillator successfully used for the description of vibrational spectra of diatomic molecules (see sec. 37). It can also be considered as a deformation of the oscillator corresponding to the Morse potential (see sec. 37).

We have therefore introduced an extension of the $su_q(2)$ model of rotational nuclear spectra, which is applicable in the vibrational and transitional regions as well. This extension is in agreement with the Interacting Boson Model and the Generalized Variable Moment of Inertia model. In addition to the overall scale parameter, the model contains two parameters, one related to the centrifugal stretching and another related to nuclear anharmonicities. In the rotational region the model coincides with the usual $su_q(2)$ model, while in the transitional region an approximate o(5) symmetry is seen. These results give additional motivation in pursueing the construction of a deformed version of the Interacting Boson Model. This problem will be discussed in the next section as well as in sec. 31.

27. A toy Interacting Boson Model with $su_q(3)$ symmetry

The Interacting Boson Model (IBM) [169] is a very popular algebraic model of nuclear structure. In the simplest version of IBM low lying collective nuclear spectra are described in terms of s ($J = 0$) and d ($J = 2$) bosons, which are supposed to be correlated fermion pairs. The symmetry of the simplest version of the model is u(6), which contains u(5) (vibrational), su(3) (rotational) and o(6) (γ-unstable) chains of subalgebras (see also sec. 31). A simplified version of the model, having the su(3) symmetry with su(2) and so(3) chains of subalgebras also exists [226]. It can be considered as a toy model for two-dimensional nuclei, but it is very useful in demonstrating the basic techniques used in the full IBM.

In the present section we will construct the q-deformed version of this toy model. Since this project requires the construction of a realization of $su_q(3)$ in terms of q-deformed bosons, we will also use this opportunity to study $su_q(3)$ in some detail.

27.1. The $su_q(3)$ algebra

In the classical version of the toy IBM [226] one introduces bosons with angular momentum $m = 0, \pm 2$, represented by the creation (annihilation) operators $a_0^\dagger$, $a_+^\dagger$, $a_-^\dagger$ (a_0, a_+, a_-). They satisfy usual boson commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0. \tag{27.1}$$

The 9 bilinear operators

$$\Lambda_{ij} = a_i^\dagger a_j \tag{27.2}$$

satisfy then the commutation relations

$$[\Lambda_{ij}, \Lambda_{kl}] = \delta_{jk}\Lambda_{il} - \delta_{il}\Lambda_{kj}, \tag{27.3}$$

which are the standard u(3) commutation relations. The total number of bosons

$$N = \Sigma_i \Lambda_{ii} = a_0^\dagger a_0 + a_+^\dagger a_+ + a_-^\dagger a_- \tag{27.4}$$

is kept constant. Since we are dealing with a system of bosons, only the totally symmetric irreps $\{N, 0, 0\}$ of u(3) occur.

In the quantum case one has the $u_q(3)$ commutation relations given in Table 2 [227], where A_{ij} are the generators of $u_q(3)$ and the q-commutator is defined as

$$[A, B]_q = AB - qBA. \tag{27.5}$$

Table 2: $u_q(3)$ commutation relations [210], given in the form $[A, B]_a = C$. A is given in the first column, B in the first row. C is given at the intersection of the row containing A with the column containing B. a, when different from 1, follows C, enclosed in parentheses.

	A_{11}	A_{22}	A_{33}	A_{12}	A_{23}	A_{13}
A_{11}	0	0	0	A_{12}	0	A_{13}
A_{22}	0	0	0	$-A_{12}$	A_{23}	0
A_{33}	0	0	0	0	$-A_{23}$	$-A_{13}$
A_{12}	$-A_{12}$	A_{12}	0	0	A_{13} (q)	0 (q^{-1})
A_{23}	0	$-A_{23}$	A_{23}	$-q^{-1}A_{13}(q^{-1})$	0	0 (q)
A_{13}	$-A_{13}$	0	A_{13}	$0(q)$	0 (q^{-1})	0
A_{21}	A_{21}	$-A_{21}$	0	$-[A_{11} - A_{22}]$	0	$A_{23}q^{A_{11}-A_{22}}$
A_{32}	0	A_{32}	$-A_{32}$	0	$-[A_{22} - A_{33}]$	$-q^{-A_{22}+A_{33}}A_{12}$
A_{31}	A_{31}	0	$-A_{31}$	$-q^{-A_{11}+A_{22}}A_{32}$	$-A_{21}q^{A_{22}-A_{33}}$	$-[A_{11} - A_{33}]$

	A_{21}	A_{32}	A_{31}
A_{11}	$-A_{21}$	0	$-A_{31}$
A_{22}	A_{21}	$-A_{32}$	0
A_{33}	0	A_{32}	A_{31}
A_{12}	$[A_{11} - A_{22}]$	0	$-q^{-A_{11}+A_{22}}A_{32}$
A_{23}	0	$[A_{22} - A_{33}]$	$A_{21}q^{A_{22}-A_{33}}$
A_{13}	$-A_{23}q^{A_{11}-A_{22}}$	$q^{-A_{22}+A_{33}}A_{12}$	$[A_{11} - A_{33}]$
A_{21}	0	$-qA_{31}(q)$	0 (q^{-1})
A_{32}	$A_{31}(q^{-1})$	0	0 (q)
A_{31}	0 (q)	0 (q^{-1})	0

In order to obtain a realization of $u_q(3)$ in terms of the q-bosons described in sec. 10, one starts with [228]

$$A_{12} = a_1^\dagger a_2, \quad A_{21} = a_2^\dagger a_1, \quad A_{23} = a_2^\dagger a_3, \quad A_{32} = a_3^\dagger a_2. \tag{27.6}$$

One can easily verify that the $u_q(3)$ commutation relations involving these generators are satisfied. For example, one has

$$[A_{12}, A_{21}] = [N_1 - N_2], \qquad [A_{23}, A_{32}] = [N_2 - N_3], \tag{27.7}$$

using the identity of eq. (2.6) and the identifications

$$N_1 = A_{11}, \quad N_2 = A_{22}, \quad N_3 = A_{33}. \tag{27.8}$$

One can now determine the boson realizations of A_{13} and A_{31} from other commutation relations, as follows

$$A_{13} = [A_{12}, A_{23}]_q = a_1^\dagger a_3 q^{-N_2}, \tag{27.9}$$

$$A_{31} = [A_{32}, A_{21}]_{q^{-1}} = a_3^\dagger a_1 q^{N_2}. \tag{27.10}$$

Using eq. (2.6) once more one can verify that the relation

$$[A_{13}, A_{31}] = [N_1 - N_3] \tag{27.11}$$

is fulfilled by the boson images of (27.9), (27.10). It is by now a straightforward task to verify that all commutation relations of Table 2 are fulfilled by the boson images obtained above.

Before turning to the study of the two limits of the model, we give for completeness some additional information on $\mathrm{su}_q(3)$:

i) The irreps of $\mathrm{su}_q(3)$ have been studied in [227,229,230,231,232,233,234,235,236,237].

ii) Clebsch-Gordan coefficients for $\mathrm{su}_q(3)$ have been given in [227,236,238,239].

iii) The Casimir operators of $\mathrm{su}_q(3)$ and their eigenvalues have been given explicitly in [240]. Using the Elliott quantum numbers [241,242,243]

$$\lambda = f_1 - f_2, \qquad \mu = f_2, \tag{27.12}$$

where f_i represents the number of boxes in the i-th line of the corresponding Young diagram, the irreps of $\mathrm{su}_q(3)$ are labelled as (λ, μ). The eigenvalues of the second order Casimir operator then read

$$C_2 = \left[\frac{\lambda}{3} - \frac{\mu}{3}\right]^2 + \left[\frac{2\lambda}{3} + \frac{\mu}{3} + 1\right]^2 + \left[\frac{\lambda}{3} + \frac{2\mu}{3} + 1\right]^2 - 2, \tag{27.13}$$

while the eigenvalues of the third order Casimir operator are

$$C_3 = 2\left[\frac{\lambda}{3} - \frac{\mu}{3}\right]\left[\frac{2\lambda}{3} + \frac{\mu}{3} + 1\right]\left[\frac{\lambda}{3} + \frac{2\mu}{3} + 1\right]. \tag{27.14}$$

In the limit $q \to 1$ these reproduce the ordinary results for su(3):

$$C_2 = \frac{2}{3}(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu), \tag{27.15}$$

$$C_3 = \frac{2}{27}(\lambda - \mu)(2\lambda + \mu + 3)(\lambda + 2\mu + 3). \tag{27.16}$$

iv) Irrducible tensor operators under $\mathrm{u}_q(3)$ have been considered in [244].

v) A different deformation of sl(3) has been studied in [245].

vi) Coupling (Wigner–Clebsch–Gordan) coefficients for the algebras $\mathrm{u}_q(\mathrm{n})$ have been considered in [246].

27.2. The $su_q(2)$ limit

We shall study the $\mathrm{su}_q(2)$ limit of the model first, since it is technically less demanding.

So far we have managed to write a boson realization of $\mathrm{u}_q(3)$ in terms of 3 q-bosons, namely a_1, a_2, a_3. Omitting the generators involving one of the bosons, one is left with an $\mathrm{su}_q(2)$ subalgebra. Omitting the generators involving a_3, for example, one is left with A_{12}, A_{21}, N_1, N_2, which satisfy usual $\mathrm{su}_q(2)$ commutation relations if the identifications

$$J_+ = A_{12}, \quad J_- = A_{21}, \quad J_0 = \frac{1}{2}(N_1 - N_2) \tag{27.17}$$

are made. J_0 alone forms an $\mathrm{so}_q(2)$ subalgebra. Therefore the relevant chain of subalgebras is

$$\mathrm{su}_q(3) \supset \mathrm{su}_q(2) \supset \mathrm{so}_q(2). \tag{27.18}$$

The second order Casimir operator of $\mathrm{su}_q(2)$ has been given in eq. (14.9). Substituting the above expressions for the generators one finds

$$C_2(\mathrm{su}_q(2)) = \left[\frac{N_1+N_2}{2}\right]\left[\frac{N_1+N_2}{2}+1\right]. \tag{27.19}$$

All of the above equations go to their classical counterparts by allowing $q \to 1$, for which $[x] \to x$, i.e. q-numbers become usual numbers. In the classical case [226] out of the three bosons (a_0, a_+, a_-) forming su(3), one chooses to leave out a_0, the boson with zero angular momentum, in order to be left with the su(2) subalgebra formed by a_+ and a_-, the two bosons of angular momentum two. The choice of the $\mathrm{su}_q(2)$ subalgebra made above is then consistent with the following correspondence between classical bosons and q-bosons

$$a_+ \to a_1, \quad a_- \to a_2, \quad a_0 \to a_3. \tag{27.20}$$

(We have opted in using different indices for usual bosons and q-bosons in order to avoid confusion.)

In the classical case the states of the system are characterized by the quantum numbers characterizing the irreducible representations (irreps) of the algebras appearing in the classical counterpart of the chain of eq. (27.18). For su(3) the total number of bosons N is used. For su(2) and so(2) one can use the eigenvalues of J^2 and J_0, or, equivalently, the eigenvalues of $a_+^\dagger a_+ + a_-^\dagger a_-$ and $L_3 = 4J_0$, for which we use the symbols n_d (the number of bosons with angular momentum 2) and M. Then the basis in the classical case can be written as [226]

$$|N, n_d, M> = \frac{(a_0^\dagger)^{N-n_d}}{(N-n_d)!}\frac{(a_+^\dagger)^{n_d/2+M/4}}{(n_d/2+M/4)!}\frac{(a_-^\dagger)^{n_d/2-M/4}}{(n_d/2-M/4)!}|0> . \tag{27.21}$$

In the quantum case for each oscillator one defines the basis as in sec. 10. Then the full basis in the q-deformed case is

$$|N, n_d, M>_q = \frac{(a_3^\dagger)^{N-n_d}}{[N-n_d]!}\frac{(a_1^\dagger)^{n_d/2+M/4}}{[n_d/2+M/4]!}\frac{(a_2^\dagger)^{n_d/2-M/4}}{[n_d/2-M/4]!}|0>, \tag{27.22}$$

where $N = N_1 + N_2 + N_3$ is the total number of bosons, $n_d = N_1 + N_2$ is the number of bosons with angular momentum 2, and M is the eigenvalue of $L = 4J_0$. n_d takes values from 0 up to N, while for a given value of n_d, M takes the values $\pm 2n_d$, $\pm 2(n_d-2)$, ..., ± 2 or 0, depending on whether n_d is odd or even. In this basis the eigenvalues of the second order Casimir operator of $\mathrm{su}_q(2)$ are then

$$C_2(\mathrm{su}_q(2))|N, n_d, M>_q = \left[\frac{n_d}{2}\right]\left[\frac{n_d}{2}+1\right]|N, n_d, M>_q . \tag{27.23}$$

In the case of $N = 5$ one can easily see that the spectrum will be composed by the ground state band, consisting of states with $M = 0$, 2, 4, 6, 8, 10 and $n_d = M/2$, the first excited band with states characterized by $M = 0$, 2, 4, 6 and $n_d = M/2+2$, and the second excited band, containing states with $M = 0$, 2 and $n_d = M/2+4$.

In the case that the Hamiltonian has the $\mathrm{su}_q(2)$ dynamical symmetry, it can be written in terms of the Casimir operators of the chain (27.18). Then one has

$$H = E_0 + AC_2(\mathrm{su}_q(2)) + BC_2(\mathrm{so}_q(2)), \tag{27.24}$$

where E_0, A, B are constants. Its eigenvalues are

$$E = E_0 + A\left[\frac{n_d}{2}\right]\left[\frac{n_d}{2}+1\right] + BM^2. \tag{27.25}$$

Realistic nuclear spectra are characterized by strong electric quadrupole transitions among the levels of the same band, as well as by interband transitions. In the framework of the present toy model one can define, by analogy to the classical case [226], quadrupole transition operators

$$Q_+ = a_1^\dagger a_3 + a_3^\dagger a_2, \qquad Q_- = a_2^\dagger a_3 + a_3^\dagger a_1. \tag{27.26}$$

In order to calculate transition matrix elements of these operators one only needs eqs. (10.7), (10.8), i.e. the action of the q-boson operators on the q-deformed basis. The selection rules, as in the classical case, are $\Delta M = \pm 2$, $\Delta n_d = \pm 1$, while the corresponding matrix elements are

$$_q < N, n_d + 1, M \pm 2 | Q_\pm | N, n_d, M >_q = \sqrt{[N - n_d][\frac{n_d}{2} \pm \frac{M}{4} + 1]}, \tag{27.27}$$

$$_q < N, n_d - 1, M \pm 2 | Q_\pm | N, n_d, M >_q = \sqrt{[N - n_d + 1][\frac{n_d}{2} \mp \frac{M}{4}]}. \tag{27.28}$$

From these equations it is clear that both intraband and interband transitions are possible.

In order to get a feeling of the qualitative changes in the spectrum and the transition matrix elements resulting from the q-deformation of the model, a simple calculation for a system of 20 bosons ($N = 20$) has been performed in [247]. Two cases are distinguished: i) q real ($q = e^\tau$, with τ real), ii) q a phase factor ($q = e^{i\tau}$, with τ real). The main conclusions are:

i) When q is real the spectrum is increasing more rapidly than in the classical case, while when q is a phase the spectrum increases more slowly than in the classical case, in agreement with the findings of the q-rotator model.

ii) The transition matrix elements in the case that q is real increase more rapidly than in the classical case, while they increase less rapidly than in the classical case when q is a phase.

iii) Transition matrix elements are much more sensitive to q-deformation than energy spectra. This is an interesting feature, showing that q-deformed algebraic models can be much more flexible in the description of transition probabilities than their classical counterparts.

27.3. The $so_q(3)$ limit

The classical su(3) toy model has, in addition to the above mentioned su(2) chain of subalgebras, an so(3) chain. However, the problem of constructing the $su_q(3) \supset so_q(3)$ decomposition is a very difficult one. Since this decomposition is needed in constructing the q-deformed versions of several collective models, including the Elliott model [241,242,243,248], the su(3) limit of the IBM [169,179,180], the Fermion Dynamical Symmetry Model (FDSM) [249,250], the Interacting Vector Boson Model (IVBM) [251,252,253,254], the nuclear vibron model for clustering [255], as well as the su(3) limit of the vibron model [256] for diatomic molecules, we report here the state of the art in this problem:

i) As far as the completely symmetric irreps of $su_q(3)$ are concerned, the problem has been solved by Van der Jeugt [257,258,259,260,261]. This suffices for our needs in the framework of the toy IBM, since only completely symmetric $su_q(3)$ irreps occur in it.

ii) Sciarrino [262] started from $so_q(3)$ and obtained a deformed gl(3) containing $so_q(3)$ as a subalgebra. However, it was not clear how to impose the Hopf structure on this larger algebra. Trying the other way around, he found that by starting from a $gl_q(3)$ algebra, which already possesses the Hopf structure, one loses the Hopf structure of the principal 3-dim subalgebra, which should have been $so_q(3)$.

iii) Pan [263] and Del Sol Mesa *et al.* [264] attacked the problem through the use of q-deforming functionals of secs 10, 14.

iv) Quesne [265] started with q-bosonic operators transforming as vectors under $so_q(3)$ and constructed a q-deformed u(3) by tensor coupling.

v) q-deformed subalgebras of several q-deformed algebras have recently been studied by Sciarrino [266].

vi) A simplified version of the $so_q(3)$ subalgebra of $u_q(3)$ has been constructed in [267]. Furthermore, explicit expressions for the irreducible vector and quadrupole tensor operators under $so_q(3)$ have been given and the matrix elements of the latter have been calculated. Further details on these developments will be given in sections 28 and 29.

In what follows, it suffices to use the solution given in [257], since in the model under study only completely symmetric irreps of $u_q(3)$ enter. Using the notation

$$a_+ \to a_1, \quad a_- \to a_2, \quad a_0 \to a_3, \tag{27.29}$$

the basis states are of the form

$$|n_+ n_0 n_- >= \frac{(a_+^\dagger)^{n_+}(a_0^\dagger)^{n_0}(a_-^\dagger)^{n_-}}{\sqrt{[n_+]![n_0]![n_-]!}}|0>, \tag{27.30}$$

with $a_i|0>=0$ and $N_i|n_+n_0n_->=n_i|n_+n_0n_->$, where $i=+,0,-$. The principal subalgebra $\mathrm{so}_q(3)$ is then generated by

$$L_0 = N_+ - N_-, \tag{27.31}$$

$$L_+ = q^{N_- - \frac{1}{2}N_0}\sqrt{q^{N_+}+q^{-N_+}} \quad a_+^\dagger a_0 \quad + \quad a_0^\dagger a_- \quad q^{N_+ - \frac{1}{2}N_0}\sqrt{q^{N_-}+q^{-N_-}}, \tag{27.32}$$

$$L_- = a_0^\dagger a_+ q^{N_- - \frac{1}{2}N_0}\sqrt{q^{N_+}+q^{-N_+}} + q^{N_+ - \frac{1}{2}N_0}\sqrt{q^{N_-}+q^{-N_-}}a_-^\dagger a_0, \tag{27.33}$$

satisfying the commutation relations

$$[L_0, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = [2L_0]. \tag{27.34}$$

L_0 alone generates then the $\mathrm{so}_q(2)$ subalgebra. Therefore the relevant chain of subalgebras is

$$\mathrm{su}_q(3) \supset \mathrm{so}_q(3) \supset \mathrm{so}_q(2). \tag{27.35}$$

The $\mathrm{so}_q(3)$ basis vectors can be written in terms of the vectors of eq. (27.30) as

$$|v(N,L,M)>= q^{-[(L+M)(L+M-1)]/4}\sqrt{\frac{[N+L]!![2L+1][L+M]![L-M]!}{[N-L]!![N+L+1]!}}$$

$$\sum_x q^{(2L-1)x/2} \quad s^{(N-L)/2} \quad \frac{|x, L+M-2x, x-M>}{\sqrt{[2x]!![L+M-2x]![2x-2M]!!}}, \tag{27.36}$$

where

$$s = (a_0^\dagger)^2 q^{N_+ + N_- + 1} - \sqrt{\frac{[2N_+][2N_-]}{[N_+][N_-]}}a_+^\dagger a_-^\dagger q^{-N_0 - \frac{1}{2}}, \tag{27.37}$$

x takes values from max(0,M) to $[(L+M)/2]$ in steps of 1, $L=N$, $N-2$, ..., 1 or 0, $M=-L$, $-L+1$, ..., $+L$, and $[2x]!!=[2x][2x-2]\ldots[2]$. The action of the generators of $\mathrm{so}_q(3)$ on these states is given by

$$L_0|v(N,L,M)>= M \quad |v(N,L,M)>, \tag{27.38}$$

$$L_\pm \quad |v(N,L,M)>= \sqrt{[L\mp M][L\pm M+1]} \quad |v(N,L,M\pm 1)>. \tag{27.39}$$

The second order Casimir operator of $\mathrm{so}_q(3)$ has the form

$$C_2(\mathrm{so}_q(3)) = L^2 = L_-L_+ + [L_0][L_0+1]. \tag{27.40}$$

Its eigenvalues in the above basis are given by

$$C_2(\mathrm{so}_q(3)) \quad |v(N,L,M)>= [L][L+1] \quad |v(N,L,M)>. \tag{27.41}$$

All of the above equations go to their classical counterparts by allowing $q \to 1$, for which $[x] \to x$, i.e. q-numbers become usual numbers. In the classical case [226] the states of the system are characterized by the quantum numbers characterizing the irreducible representations (irreps) of the algebras appearing in the classical counterpart of the chain of eq. (27.35). For su(3) the total number of bosons N is used. For so(3) and so(2) one can use L and M, respectively. In [226], however, the eigenvalue of $L_0' = 2L_0$ is used, which is $M' = 2M$.

Since the rules for the decomposition of the totally symmetric $\mathrm{u}_q(3)$ irreps into $\mathrm{so}_q(3)$ irreps are the same as in the classical case, it is easy to verify that for a system with $N=6$ the spectrum will be composed by the ground state band, consisting of states with $M'=0$, 2, 4, 6, 8, 10, 12 and $L=N$, the first excited band with states characterized by $M'=0$, 2, 4, 6, 8 and $L=N-2$, the second excited

band, containing states with $M = 0$, 2, 4 and $L = N - 4$, and the third excited band, containing a state with $M' = 0$ and $L = N - 6$.

In the case that the Hamiltonian has the $so_q(3)$ dynamical symmetry, it can be written in terms of the Casimir operators of the chain (27.35). Then one has

$$H = E_0 + AC_2(so_q(3)) + BC_2(so_q(2)), \tag{27.42}$$

where E_0, A, B are constants. Its eigenvalues are

$$E = E_0 + A[L][L+1] + BM^2. \tag{27.43}$$

It is then clear that in this simple model the internal structure of the rotational bands is not influenced by q-deformation. What is changed is the position of the bandheads.

We turn now to the study of electromagnetic transitions. In the present limit one can define, by analogy to the classical case, quadrupole transition operators $Q_\pm$ proportional to the $so_q(3)$ generators $L_\pm$

$$Q_\pm = L_\pm. \tag{27.44}$$

In order to calculate transition matrix elements of these operators one only needs eq. (27.39). The selection rules, as in the classical case, are $\Delta M' = \pm 2$, $\Delta L = 0$, i.e. only intraband transitions are allowed. The relevant matrix elements are

$$< v(N, L, M' + 2)|Q_+|v(N, L, M') >= \sqrt{\left[L - \frac{M'}{2}\right]\left[L + \frac{M'}{2} + 1\right]}, \tag{27.45}$$

$$< v(N, L, M' - 2)|Q_-|v(N, L, M') >= \sqrt{\left[L + \frac{M'}{2}\right]\left[L - \frac{M'}{2} + 1\right]}. \tag{27.46}$$

In order to get a feeling of the qualitative changes in the spectrum and the transition matrix elements resulting from the q-deformation of this limit of the model, a simple calculation has been done in [268]. Again the cases of q being real or q being a phase factor have been considered. The main conclusions are:

i) q-deformation influences only the position of bandheads, while it leaves the internal structure of the bands intact.

ii) When q is real the bandheads are increasing more rapidly than in the classical case, while when q is a phase the bandheads increase more slowly than in the classical case. This result is in qualitative agreement with the findings of the $su_q(2)$ model.

iii) Transition matrix elements in the case that q is real have values higher than in the classical case, while they have values lower than in the classical case when q is a phase.

28. The 3-dimensional q-deformed harmonic oscillator and the nuclear shell model

Having developed in the previous section several techniques related to the $u_q(3)$ algebra and its $so_q(3)$ subalgebra, it is instructive to see how a q-deformed version of the 3-dimensional harmonic oscillator possessing this symmetry can be built and how this oscillator can be used for reproducing the predictions of the Modified Harmonic Oscillator, first suggested by Nilsson [269] and studied in detail in [270,271,272]. The construction of the Hamiltonian of the 3-dimensional q-deformed harmonic oscillator is a non-trivial problem, since one has to construct first the square of the q-deformed angular momentum operator.

28.1. Simplified $so_q(3)$ subalgebra and $so_q(3)$ basis states

The $so_q(3)$ subalgebra of $u_q(3)$ has already been mentioned in subsec. 27.3. Here we shall derive a simplified form of it.

We shall use three independent q-deformed boson operators b_i and $b_i^\dagger$ ($i = +, 0, -$), which satisfy the commutation relations

$$[N_i, b_i^\dagger] = b_i^\dagger, \qquad [N_i, b_i] = -b_i, \qquad b_i b_i^\dagger - q^{\pm 1} b_i^\dagger b_i = q^{\mp N_i}, \tag{28.1}$$

where N_i are the corresponding number operators. (These bosons are the same as the bosons a and $a^\dagger$, with $i = +, 0, -$, used in the previous section.)

The elements of the $so_q(3)$ algebra (i.e. the angular momentum operators) acting in the Fock space of the totally symmetric representations [N,0,0] of $u_q(3)$ (which is in fact the space of the 3-dimensional q-deformed harmonic oscillator) have been derived by Van der Jeugt [257,258,259,260,261]. As it has already been mentioned in the previous section, a simplified form of the subalgebra $so_q(3) \subset u_q(3)$, can be obtained by introducing the operators [267]

$$B_0 = q^{-\frac{1}{2}N_0} b_0, \qquad B_0^\dagger = b_0^\dagger q^{-\frac{1}{2}N_0}, \qquad B_i = q^{N_i+\frac{1}{2}} b_i \sqrt{\frac{[2N_i]}{[N_i]}}, \qquad B_i^\dagger = \sqrt{\frac{[2N_i]}{[N_i]}} b_i^\dagger q^{N_i+\frac{1}{2}}, \qquad i = +, -, \tag{28.2}$$

where the q-numbers $[x]$ are defined as in eq. (2.1). We shall consider only real values for the deformation parameter q (i.e. $q = e^\tau$ with τ being real). These new operators satisfy the usual commutation relations

$$[N_i, B_i^\dagger] = B_i^\dagger, \qquad [N_i, B_i] = -B_i, \tag{28.3}$$

while in the Fock space, spanned on the normalized eigenvectors of the excitation number operators N_+, N_0, N_-, they satisfy the relations

$$B_0^\dagger B_0 = q^{-N_0+1}[N_0], \qquad B_0 B_0^\dagger = q^{-N_0}[N_0+1], \tag{28.4}$$

$$B_i^\dagger B_i = q^{2N_i-1}[2N_i], \qquad B_i B_i^\dagger = q^{2N_i+1}[2N_i+2], \qquad i = +, -, \tag{28.5}$$

and hence, the commutation relations

$$[B_0, B_0^\dagger] = q^{-2N_0}, \qquad [B_i, B_i^\dagger] = [2] q^{4N_i+1}, \qquad i = +, -. \tag{28.6}$$

It was shown in [267] that the angular momentum operators defined in [257] and reported in eqs (27.31)-(27.33), when expressed in terms of the modified operators of eq. (28.2) , take the simplified form

$$L_0 = N_+ - N_-, \qquad L_+ = q^{-L_0+\frac{1}{2}} B_+^\dagger B_0 + q^{L_0-\frac{1}{2}} B_0^\dagger B_-, \qquad L_- = q^{-L_0-\frac{1}{2}} B_0^\dagger B_+ + q^{L_0+\frac{1}{2}} B_-^\dagger B_0, \tag{28.7}$$

and satisfy the standard $so_q(3)$ commutation relations

$$[L_0, L_\pm] = \pm L_\pm, \qquad [L_+, L_-] = [2L_0]. \tag{28.8}$$

As it has been discussed in detail in [257] the algebra of eq. (28.7) is not a subalgebra of $u_q(3)$ when these algebras are considered as Hopf algebras, but it is a subalgebra when these algebras are considered as q-deformed enveloping algebras. However, in our context, working in the Fock space (constructed from three independent q-bosons), we restrict ourselves to the symmetric representation of $u_q(3)$, where the embedding $so_q(3) \subset u_q(3)$ is valid. It should also be noted that the operators L_+, L_-, L_0 given by eq. (28.7) are not invariant with respect to the replacement $q \leftrightarrow q^{-1}$, a fact that restricts us to real q.

The Casimir operator of $so_q(3)$ can be written in the form [91,96,99,103]

$$C_2^{(q)} = \frac{1}{2}\Big\{ L_+ L_- + L_- L_+ + [2][L_0]^2 \Big\} = L_- L_+ + [L_0][L_0+1] = L_+ L_- + [L_0][L_0-1]. \tag{28.9}$$

One can also define q-deformed states $|n\ell m\rangle_q$ satisfying the eigenvalue equations

$${}^q\mathbf{L}^2 |n\ell m\rangle_q = [\ell][\ell+1] |n\ell m\rangle_q, \qquad L_0 |n\ell m\rangle_q = m |n\ell m\rangle_q, \qquad N |n\ell m\rangle_q = n |n\ell m\rangle_q. \tag{28.10}$$

Here ${}^q\mathbf{L}^2 \stackrel{def}{=} C_2^{(q)}$ is the "square" of the q-deformed angular momentum and $N = N_+ + N_0 + N_-$ is the total number operator for the q-deformed bosons. These states have the form (see [267] for a derivation)

$$|n\ell m\rangle_q = q^{\frac{1}{4}(n-\ell)(n+\ell+1)-\frac{1}{2}m^2} \sqrt{\frac{[n-\ell]!![\ell+m]![\ell-m]![2\ell+1]}{[n+\ell+1]!!}} \times$$

$$\times \sum_{t=0}^{(n-\ell)/2} \sum_{p=\max(0,m)}^{\lfloor(\ell+m)/2\rfloor} \frac{(-1)^t q^{-(n+\ell+1)t}}{[2t]!![n-\ell-2t]!!} \frac{(B_+^\dagger)^{p+t}}{[2p]!!} \frac{(B_0^\dagger)^{n+m-2p-2t}}{[n+m-2p]!} \frac{(B_-^\dagger)^{p+t-m}}{[2p-2m]!!} |0\rangle, \tag{28.11}$$

where $|0\rangle$ is the vacum state, $[n]! = [n][n-1]\dots[1]$, $[n]!! = [n][n-2]\dots[2]$ or $[1]$, and, as shown in [257,259], form a basis for the most symmetric representation $[n,0,0]$ of $u_q(3)$, corrresponding to the $u_q(3) \supset so_q(3)$ chain.

28.2. Definitions of $so_q(3)$ irreducible tensor operators, tensor and scalar products

We shall recall now some definitions about q-deformed tensor operators within the framework of the algebra $so_q(3)$.

An *irreducible tensor* operator of rank j with parameter q according to the algebra $so_q(3)$ is a set of $2j+1$ operators $\mathcal{T}_{jm}^{(q)}$, satisfying the relations

$$[L_0, \mathcal{T}_{jm}^{(q)}] = m\, \mathcal{T}_{jm}^{(q)}, \qquad [L_\pm, \mathcal{T}_{jm}^{(q)}]_{q^m} q^{L_0} = \sqrt{[j \mp m][j \pm m + 1]}\, \mathcal{T}_{jm\pm 1}^{(q)}, \tag{28.12}$$

where, in order to express the adjoint action of the generators $L_0, L_\pm$ of $so_q(3)$ on the components of the tensor operator $\mathcal{T}_{jm}^{(q)}$, we use the usual notation of the *q-commutator*

$$[A, B]_{q^\alpha} = AB - q^\alpha BA. \tag{28.13}$$

By $\tilde{\mathcal{T}}_{jm}^{(q)}$ we denote the *conjugate irreducible q-tensor operator*

$$\tilde{\mathcal{T}}_{jm}^{(q)} = (-1)^{j-m} q^{-m} \mathcal{T}_{j,-m}^{(q)}\ , \tag{28.14}$$

which satisfies the relations

$$[\tilde{\mathcal{T}}_{jm}^{(q)}, L_0] = m\, \tilde{\mathcal{T}}_{jm}^{(q)}, \qquad q^{L_0} [\tilde{\mathcal{T}}_{jm}^{(q)}, L_\pm]_{q^m} = \sqrt{[j \pm m][j \mp m + 1]}\, \tilde{\mathcal{T}}_{j,m\mp 1}^{(q)}\ . \tag{28.15}$$

Then the operator

$$\mathcal{P}_{jm}^{(q)} = (\tilde{\mathcal{T}}_{jm}^{(q)})^\dagger = (-1)^{j-m} q^{-m} (\mathcal{T}_{j,-m}^{(q)})^\dagger, \tag{28.16}$$

where † denotes hermitian conjugation, transforms in the same way (given in eq. (28.12)) as the tensor $\mathcal{T}_{jm}^{(q)}$, i.e. $\mathcal{P}_{jm}^{(q)}$ also is an irreducible $so_q(3)$ tensor operator of rank j.

Let $\mathcal{A}_{j_1 m_1}^{(q_1)}$ and $\mathcal{B}_{j_2 m_2}^{(q_2)}$ be two irreducible tensor operators. We shall define the *tensor* and *scalar product* of these tensor operators following some of the prescriptions, summarized, for example, in [91,96,99,103].

For the *tensor product* one can introduce the following operator

$$\left[\mathcal{A}_{j_1}^{(q_1)} \times \mathcal{B}_{j_2}^{(q_2)}\right]_{jm}^{(q_3)} = \sum_{m_1, m_2} {}^{q_3}C_{j_1 m_1, j_2 m_2}^{jm} \mathcal{A}_{j_1 m_1}^{(q_1)} \mathcal{B}_{j_2 m_2}^{(q_2)}, \tag{28.17}$$

where ${}^{q_3}C_{j_1 m_1, j_2 m_2}^{jm}$ are the Clebsch-Gordan coefficients corresponding to the deformation parameter q_3. In general the deformation parameters q_1, q_2 and q_3 can be arbitrary. It turns out, however, if one imposes the condition that the left hand side of eq. (28.17) transforms as an irreducible q-tensor of rank j in a way that the Wigner-Eckart theorem can be applied to eq. (28.17) as a whole, not all of the combinations of q_1, q_2 and q_3 are allowed.

If the tensors $\mathcal{A}_{j_1}^{(q_1)}$ and $\mathcal{B}_{j_2}^{(q_2)}$ depend on one and the same variable and act on a single vector, which depends on the same variable, the mentioned requirement will be satisfied only if $q_1 = q_2 = q$ and $q_3 = 1/q$, i.e. the operator

$$\left[\mathcal{A}_{j_1}^{(q)} \times \mathcal{B}_{j_2}^{(q)}\right]_{jm}^{(1/q)} = \sum_{m_1, m_2} {}^{1/q}C_{j_1 m_1, j_2 m_2}^{jm} \mathcal{A}_{j_1 m_1}^{(q)} \mathcal{B}_{j_2 m_2}^{(q)} \tag{28.18}$$

transforms as an irreducible q-tensor of rank j according to the algebra $so_q(3)$. Then, the definition of eq. (28.18) is in agreement with the property

$$\langle \alpha', \ell' \| [\mathcal{A}_{j_1}^{(q)} \times \mathcal{B}_{j_2}^{(q)}]_j^{(1/q)} \| \alpha, \ell \rangle =$$

$$= \sqrt{[2j+1]} \sum_{\alpha'',\ell''} (-1)^{\ell+j+\ell'} \left\{ \begin{array}{ccc} \ell & j & \ell' \\ j_1 & \ell'' & j_2 \end{array} \right\}_q \langle \alpha', \ell' \| \mathcal{A}^{(q)}_{j_1} \| \alpha'', \ell'' \rangle \langle \alpha'', \ell'' \| \mathcal{B}^{(q)}_{j_2} \| \alpha, \ell \rangle, \tag{28.19}$$

which is a q-analogue of the well known classical identity for the matrix elements of the tensor product of two tensor operators [273,274].

In what follows we shall consider also the scalar product of two tensor operators depending on two different variables (1) and (2) and acting on different vectors, depending on these different variables. In this particular case the *scalar product* of irreducible q and q^{-1} – tensor operators $\mathcal{A}^{(q)}_j(1)$, $\mathcal{B}^{(1/q)}_j(2)$ with the same rank, acting on *different vectors* (1) and (2), is given by means of the following definition

$$(\mathcal{A}^{(q)}_j(1) \cdot \mathcal{B}^{(1/q)}_j(2))^q = (-1)^j \sqrt{[2j+1]} \left[\mathcal{A}^{(q)}_j(1) \times \mathcal{B}^{(1/q)}_j(2) \right]^{(q)}_{00} = \sum_m (-q)^m \mathcal{A}^{(q)}_{jm}(1)\ \mathcal{B}^{(1/q)}_{j-m}(2)\ . \tag{28.20}$$

28.3. Construction of $so_q(3)$ vector operators and spherical vector operators

It should be noticed that the simlified angular momentum operators of eq. (28.7) satisfy the commutation relations of eq. (28.8), but *are not* tensor operators in the sense of definition of eq. (28.12). One can form, however, the angular momentum operators

$$\mathcal{L}^{(q)}_{\pm 1} = \mp \frac{1}{\sqrt{[2]}} q^{-L_0} L_\pm,$$

$$\mathcal{L}^{(q)}_0 = \frac{1}{[2]} \left\{ q[2L_0] + (q - q^{-1}) L_- L_+ \right\} = \frac{1}{[2]} \left\{ q[2L_0] + (q - q^{-1}) \left(C^{(q)}_2 - [L_0][L_0+1] \right) \right\}, \tag{28.21}$$

which are tensors of first rank, i.e. $so_q(3)$ vectors.

One can easily check that the operators $B^\dagger_0$, $B^\dagger_\pm$ and B_0, $B_\pm$ do not form a spherical vector (see for example [91,96,99,103]). The situation is the same as in the "standard" theory of angular momentum. Indeed, in the "standard" theory, the operators $L_0 = L_3$, $L_\pm = L_1 \pm i L_2$, do not form a spherical tensor. However, in the "standard" theory one constructs the operators $J_\pm = \mp \frac{1}{\sqrt{2}}(L_1 \pm i L_2)$, $J_0 = L_0$, which are indeed the components of a spherical tensor of first rank according the standard so(3) algebra.

Here we proceed in an analogous way. As shown in [267], one can define a vector operator $T^\dagger_m$ of the form

$$T^\dagger_{+1} = \frac{1}{\sqrt{[2]}} B^\dagger_+ q^{-2N_+ + N - \frac{1}{2}}, \qquad T^\dagger_0 = B^\dagger_0 q^{-2N_+ + N},$$

$$T^\dagger_{-1} = \frac{1}{\sqrt{[2]}} \left\{ B^\dagger_- q^{2N_+ - N - \frac{1}{2}} - (q - q^{-1}) B_+ (B^\dagger_0)^2 q^{-2N_+ + N + \frac{3}{2}} \right\}. \tag{28.22}$$

The corresponding expressions for the conjugate operators $\widetilde{T}_m = (-1)^m q^{-m} (T^\dagger_{-m})^\dagger$ are

$$\widetilde{T}_{+1} = -\frac{1}{\sqrt{[2]}} \left\{ q^{2N_+ - N - \frac{3}{2}} B_- - (q - q^{-1}) q^{-2N_+ + N + \frac{1}{2}} B^\dagger_+ (B_0)^2 \right\},$$

$$\widetilde{T}_0 = q^{-2N_+ + N} B_0, \qquad \widetilde{T}_{-1} = -\frac{1}{\sqrt{[2]}} q^{-2N_+ + N + \frac{1}{2}} B_+. \tag{28.23}$$

One can easily check, that the vector operators $\mathbf{T}^\dagger$ and $\widetilde{\mathbf{T}}$ satisfy the commutation relation

$$[\widetilde{T}_{-1}, T^\dagger_{+1}]_{q^{-2}} = -q^{2N+1}, \qquad [\widetilde{T}_0, T^\dagger_0] = q^{2N} + q^{-1}(q^2 - q^{-2}) T^\dagger_{+1} \widetilde{T}_{-1},$$

$$[\widetilde{T}_{+1}, T^\dagger_{-1}]_{q^{-2}} = -q^{2N-1} + q^{-1}(q^2 - q^{-2}) \left\{ T^\dagger_0 \widetilde{T}_0 + (q - q^{-1}) T^\dagger_{+1} \widetilde{T}_{-1} \right\}, \tag{28.24}$$

and

$$[\widetilde{T}_0, T^\dagger_{+1}] = 0, \qquad [\widetilde{T}_{+1}, T^\dagger_{+1}]_{q^2} = 0, \qquad [\widetilde{T}_{+1}, T^\dagger_0] = (q^2 - q^{-2}) T^\dagger_{+1} \widetilde{T}_0,$$

$$[\widetilde{T}_{-1}, T^\dagger_0] = 0, \qquad [\widetilde{T}_{-1}, T^\dagger_{-1}]_{q^2} = 0, \qquad [\widetilde{T}_0, T^\dagger_{-1}] = (q^2 - q^{-2}) T^\dagger_0 \widetilde{T}_{-1}. \tag{28.25}$$

Unlike the operators of eq. (28.2), the commutators $[\tilde{T}_m, \tilde{T}_n]$ and $[T^\dagger_m, T^\dagger_n]$ do not vanish, but are equal to

$$[T^\dagger_{+1}, T^\dagger_0]_{q^2} = 0, \qquad [T^\dagger_0, T^\dagger_{-1}]_{q^2} = 0, \qquad [T^\dagger_{+1}, T^\dagger_{-1}] = (q - q^{-1})(T^\dagger_0)^2,$$
$$[\tilde{T}_0, \tilde{T}_{-1}]_{q^2} = 0, \qquad [\tilde{T}_{+1}, \tilde{T}_0]_{q^2} = 0, \qquad [\tilde{T}_{+1}, \tilde{T}_{-1}] = (q - q^{-1})(\tilde{T}_0)^2, \tag{28.26}$$

which is in agreement with the results obtained in [265].

28.4. A choice for the physical angular momentum

It should be noted that the angular momentum operator $\mathcal{L}^{(q)}_M$ (considered as a vector according $so_q(3)$) can be represented in the form

$$\mathcal{L}^{(q)}_M = -\sqrt{\frac{[4]}{[2]}}\left[T^\dagger \times \tilde{T}\right]^{(1/q)}_{1M} = -\sqrt{\frac{[4]}{[2]}}\sum_{m,n} {}^{1/q}C^{1M}_{1m,1n} T^\dagger_m \tilde{T}_n. \tag{28.27}$$

Its "square" differs from $C^{(q)}_2$ and equals to [91,96,99,103]

$$(\mathcal{L}^{(q)})^2 \equiv \mathcal{L}^{(q)} \cdot \mathcal{L}^{(q)} = \sum (-q)^{-M} \mathcal{L}^{(q)}_M \mathcal{L}^{(q)}_{-M} = \frac{2}{[2]} C^{(q)}_2 + \left(\frac{q - q^{-1}}{[2]}\right)^2 (C^{(q)}_2)^2. \tag{28.28}$$

This difference, however, is not essential. Let us consider, for instance, the expectation values of the scalar operator of eq. (28.28). It has the form

$${}_q\langle \ell m | \mathcal{L}^{(q)} \cdot \mathcal{L}^{(q)} | \ell m \rangle_q = \frac{[2\ell][2\ell+2]}{[2]^2} = [\ell]_{q^2}[\ell+1]_{q^2}. \tag{28.29}$$

In this sense the replacement of ${}^q\mathbf{L}^2 \equiv C^{(q)}_2$ with $\mathcal{L}^{(q)} \cdot \mathcal{L}^{(q)}$ is equivalent to the replacement $q \to q^2$ and leads to the renormalization of some constant. For this reason we shall accept as the square of the *physical* angular momentum the quantity ${}^q\mathbf{L}^2 \equiv C^{(q)}_2$, whose eigenvalues are $[\ell][\ell+1]$.

28.5. A choice for the Hamiltonian

Now let us consider the scalar operator, constructed in terms of $T^\dagger_M$ and $\tilde{T}_M$. We have

$$X^{(q)}_0 = -\sqrt{[3]}[T^\dagger \times \tilde{T}]^{(1/q)}_{00} = -q^{-1}T^\dagger_{+1}\tilde{T}_{-1} + T^\dagger_0 \tilde{T}_0 - qT^\dagger_{-1}\tilde{T}_1. \tag{28.30}$$

We define the Hamiltonian of the three dimensional q-deformed oscillator (3-dim q-HO) as

$${}^qH_{3dim} = \hbar\omega_0 X_0 = \hbar\omega_0\left(-q^{-1}T^\dagger_{+1}\tilde{T}_{-1} + T^\dagger_0\tilde{T}_0 - qT^\dagger_{-1}\tilde{T}_1\right). \tag{28.31}$$

The motivations for such an ansatz are:

a) The operator so defined is an $so_q(3)$ scalar, i.e. it is simultaneously measurable with the physical q-deformed angular momentum square ${}^q\mathbf{L}^2$ and the z-projection L_0.

b) Only this $so_q(3)$ scalar has the property of being the sum of terms conserving the number of bosons. (Each term contains an equal number of creation and annihilation operators.)

c) In the limit $q \to 1$ the Hamiltonian of eq. (28.31) goes into

$$\lim_{q\to 1} {}^qH_{3dim} = \hbar\omega_0\left(a^\dagger_{+1}a_{+1} + a^\dagger_0 a_0 + a^\dagger_{-1}a_{-1}\right), \tag{28.32}$$

where $[a_m, a^\dagger_n] = \delta_{mn}$, i.e. in this limit the Hamiltonian of eq. (28.31) coincides with the Hamiltonian of the 3-dimensional (spherically symmetric) harmonic oscillator up to an additive constant.

What we have done so far can be summarized in the following way. We have shown that in the enveloping algebra of $so_q(3)$ there exists an $so_q(3)$ scalar operator, built out of irreducible vector operators according to the reduction $u_q(3) \supset so_q(3)$, for the case of the most symmetric irreducible representations of $u_q(3)$, which are the space of the 3-dim q-HO. In the limit $q \to 1$ this operator coincides with the

Hamiltonian of the 3-dim harmonic oscillator. It is therefore reasonable to consider this operator as the Hamiltonian of the 3-dim q-HO. (In the next subsections it will be shown that this Hamiltonian is related to the Hamiltonian of the Modified Harmonic Oscillator of Nilsson [269,270,271,272].)

It should be underlined that the Hamiltonian of eq. (28.31) does not commute with the square of the "classical" (or "standard") angular momentum $\mathbf{L}^2$. This is due to the fact that the q-deformed oscillator is really space deformed and the "standard" quantum number l is not a good quantum number for this system. On the other hand ${}^q\mathbf{L}^2$ commutes with ${}^qH_{3dim}$ and the "quantum angular momentum" ℓ can be used for the classification of the states of the q-oscillator. The projection of the standard angular momentum on the z axis, l_z, coincides however with the quantum projection, ℓ_z, i.e. it is a good quantum number.

Eq. (28.31) can be cast in a simpler and physically more transparent form. Indeed, making use of the third component of the q-deformed angular momentum (considered as an $so_q(3)$ vector)

$$\mathcal{L}_0^{(q)} = -\sqrt{[4]/[2]}\left[T^\dagger \times \tilde{T}\right]_{10}^{(1/q)} = -T_{+1}^\dagger \tilde{T}_{-1} + (q-q^{-1})T_0^\dagger \tilde{T}_0 + T_{-1}^\dagger \tilde{T}_0, \tag{28.33}$$

we obtain

$$X_0^{(q)} + q\mathcal{L}_0^{(q)} = -[2]T_{+1}^\dagger \tilde{T}_{-1} + q^2 T_0^\dagger \tilde{T}_0. \tag{28.34}$$

Since

$$T_{+1}^\dagger \tilde{T}_{-1} = \frac{[2\,N_+]}{[2]} q^{-2N_+ + 2N + 1}, \qquad T_0^\dagger \tilde{T}_0 = [N_0] q^{-4N_+ - N_0 + 2N - 1}, \tag{28.35}$$

we get upon summation

$$X_0^{(q)} = -q\mathcal{L}_{10}^{(q)} + [N + L_0] q^{N - L_0 + 1}, \tag{28.36}$$

and, after some calculations,

$${}^qH_{3dim} = \hbar\omega_0 X_0^{(q)} = \hbar\omega_0 \left\{ [N] q^{N+1} - \frac{q(q-q^{-1})}{[2]} C_2^{(q)} \right\}. \tag{28.37}$$

The eigenvalues of such a q-deformed Hamiltonian are

$${}^qE_{3dim} = \hbar\omega_0 \left\{ [n] q^{n+1} - \frac{q(q-q^{-1})}{[2]} [\ell][\ell+1] \right\}, \qquad \ell = n, n-2, \ldots, 0 \text{ or } 1. \tag{28.38}$$

In the $q \to 1$ limit we have $\lim_{q\to 1} {}^qE_{3dim} = \hbar\omega_0 n$, which coincides with the classical result.

One should note that the expression of eq. (28.38) can also be put in the form

$${}^qE_{3dim} = \hbar\omega_0 \frac{q^{n+1}}{[2]} \left\{ q^{\ell+1}[n-\ell] + q^{-\ell-1}[n+\ell] \right\}, \tag{28.39}$$

which perhaps looks more elegant. However, the form of eq. (28.38) has the advantage that the vibrational ($[n]$) and the rotational ($[\ell][\ell+1]$) degrees of freedom are clearly separated.

For small values of the deformation parameter τ, where $q = e^\tau$, one can expand the rhs of eq. (28.38) in powers of τ obtaining

$${}^qE_{3dim} = \hbar\omega_0\, n - \hbar\omega_0\tau\left(\ell(\ell+1) - n(n+1)\right) - \hbar\omega_0\tau^2 \left(\ell(\ell+1) - \frac{1}{3} n(n+1)(2n+1) \right) + \mathcal{O}(\tau^3). \tag{28.40}$$

To leading order in τ the expression of eq. (28.40) closely resembles the one giving the energy eigenvalues of the Modified Harmonic Oscillator, as we shall see in the next subsections.

28.6. The Modified Harmonic Oscillator of Nilsson

Before examining the connection between the Hamiltonian of the last subsection and the Modified Harmonic Oscillator (MHO) of Nilsson [269,270,271,272], a few words about the MHO are in place.

The basis of the shell model in finite nuclei is the assumption of an independent particle motion within a mean field. The average field in which the particles move is correctly described by the Woods–Saxon (WS) potential $V_{WS}(\mathbf{r})$. However, the corresponding eigenvalue equation can be solved only

numerically. It is therefore desirable for many purposes to have an exactly soluble model producing approximately the same spectrum as the Woods–Saxon potential. The Modified Harmonic Oscillator (MHO) Hamiltonian, first suggested by Nilsson [269] and studied in detail in [270,271,272], indeed reproduces approximately the energy spectrum of the WS potential and at the same time has a simple analytical solution. The Hamiltonian has the form

$$V_{MHO} = \frac{1}{2}\hbar\omega\rho^2 - \hbar\omega k\left\{\mu\left(\mathbf{L}^2 - \left\langle\mathbf{L}^2\right\rangle_N\right) + 2\mathbf{L}\cdot\mathbf{S}\right\}, \quad \rho = \sqrt{\frac{M\omega}{\hbar}}\, r, \tag{28.41}$$

where $\mathbf{L}^2$ is the square of the angular momentum and $\mathbf{L}\cdot\mathbf{S}$ is the spin-orbit interaction. The subtraction of the average value of $\langle\mathbf{L}^2\rangle$, taken over each N-shell,

$$\left\langle\mathbf{L}^2\right\rangle_N = \frac{N(N+3)}{2}, \tag{28.42}$$

results in avoiding the shell compression induced by the $\mathbf{L}^2$ term, leaving the "center of gravity" of each N-shell unchanged. The V_{MHO} (without the spin-orbit term $\mathbf{L}\cdot\mathbf{S}$) indeed reproduces effectively the Woods–Saxon radial potential. A better agreement with the experimental data can be achieved if the parameters of the potential are let to vary smoothly from shell to shell.

It should be noted, however, that from a mathematical point of view the introduction of an $\mathbf{L}^2$-dependent term (and of the corresponding correction $\langle\mathbf{L}^2\rangle_N$) in the MHO potential is not so innocent, as it may look at first sight, because this term depends on the state in which the particle occurs, and it this sense the potential of eq. (28.41) is "non-local" and "deformable" (with variable deformation). This effect is amplified by the fact that the constant $\mu' = k\mu$ is allowed to vary from shell to shell, i.e. depends on N.

The eigenvalues of the MHO (if one neglects the spin–orbit term) are

$$E_{nl} = \hbar\omega n - \hbar\omega\mu'\left(l(l+1) - \frac{1}{2}n(n+3)\right), \tag{28.43}$$

where μ' is allowed to vary from shell to shell.

28.7. Connection between the 3-dimensional q-deformed harmonic oscillator and the Modified Harmonic Oscillator of Nilsson

Comparing eqs (28.40) and (28.43) we see that eq. (28.40) closely resembles, to leading order in τ, eq. (28.43), thus establishing a connection between the 3-dimensional q-deformed harmonic oscillator and the Modified Harmonic Oscillator of Nilsson, with the parameter τ of the 3-dim q-HO playing the role of the parameter $\mu' = k\mu$ of the MHO. (Remember that the spin–orbit term has been omitted in both cases.)

Quantitatively, one obtains a good fit of the energy spectrum produced by the MHO potential of eq. (28.41) by choosing for $\hbar\omega_0$ and τ of the 3-dimensional q-deformed harmonic oscillator the values $\hbar\omega_0 = 0.94109$ and $\tau = 0.021948$ [275]. (It has been assumed that $\hbar\omega = 1$ for the MHO and the spin–orbit term has been neglected.) The value of τ is close to the values usually adopted for $\mu' = k\mu$. Indeed the MHO fit in the $^{208}_{82}$Pb region yields [271,272] $\mu' = 0$ for $N = 2$, $\mu' = 0.0263$ for $N = 3$, and $\mu' = 0.024$ for $N \geq 4$, while for k, fixed by the condition that the observed order of sub-shells be reproduced, the relevant values are $k = 0.08$ for $N = 2$, $k = 0.075$ for $N = 3$ and $k = 0.07$ for $N \geq 4$.

It appears surprising at first that $\hbar\omega_0$ differs slightly from 1. Indeed the correction term $n(n+1)$ in the 3-dim q-HO spectrum is slightly different from the corresponding piece $n(n+3)/2$ in the MHO. Some compression of the 3-dim q-HO spectrum is therefore inevitable. From a physical point of view, one may say that the mean radius of the deformed oscillator is slightly larger than the radius of the classical isotropic HO.

When comparing the MHO and 3-dim q-HO spectra it should be remembered that the constant μ' in the MHO potential takes different values for shells with different values of N, while in the 3-dim q-HO model the parameters τ and $\hbar\omega_0$ have the same values for all shells. The comparison shows that the q-deformation of the 3-dimensional harmonic oscillator effectively reproduces the non-locality and the "deformations" induced in the MHO model through the terms $\mathbf{L}^2$, $\langle\mathbf{L}^2\rangle_N$ and the variability of μ' with the shell number N.

28.8. 3-dimensional q-deformed harmonic oscillator with q-deformed spin–orbit term

For a full comparison with the MHO and a more realistic description of the single-particle spectrum, one needs to include a q-deformed spin–orbit term $\mathcal{L}^{(q)} \cdot \mathcal{S}^{(1/q)}$ in the q-deformed harmonic oscillator Hamiltonian ${}^{q}H_{3dim}$. To this purpose one introduces the spin operators S_+, S_0, S_-, which are elements of another (independent) $\mathrm{su}_q(2)$ algebra. These operators satisfy the commutation relations of eq. (28.8), i.e.

$$[S_0, S_\pm] = \pm S_\pm, \qquad [S_+, S_-] = [2S_0], \tag{28.44}$$

and act in the *two-dimensional* representation space of this algebra. The orthonormalized basis vectors of this space will be denoted by $|\frac{1}{2} m_s\rangle_q$, in analogy with the usual (non-deformed) case.

One can now define the q-deformed total angular momentum as [91,96,99,103]

$$J_0 = L_0 + S_0, \qquad J_\pm = L_\pm q^{S_0} + S_\pm q^{-L_0}, \tag{28.45}$$

where the q-deformed orbital angular momentum is given by eq. (28.7). The operators of eq. (28.45) satisfy the same commutation relations as the operators of the q-deformed orbital angular momentum (eq. (28.8)) and spin (eq. (28.44)), and the corresponding expression for the Casimir operator of the algebra of eq. (28.45) can be written in the form

$$C^{(q)}_{2,J} = J_- J_+ + [J_0][J_0 + 1]. \tag{28.46}$$

As in the case of the q-deformed orbital momentum we shall consider ${}^{q}J^2 \equiv C^{(q)}_{2,J}$ as the total angular momentum "square".

The common eigenvectors of $C^{(q)}_{2,J}, C^{(q)}_{2,L}$ and $C^{(q)}_{2,S}$ can be written in the usual form

$$|n(\ell\tfrac{1}{2})jm\rangle_q = \sum_{m_\ell, m_s} {}^{q}C^{jm}_{\ell m_\ell, \frac{1}{2} m_s} |n\ell m_\ell\rangle_q |\tfrac{1}{2} m_s\rangle_q, \tag{28.47}$$

where ${}^{q}C^{jm}_{\ell m_\ell, s m_s}$ are the q-deformed Clebsch-Gordan coefficients [91,96,99,103].

One can define the spin–orbit term as a scalar product, according to the definition of eq. (28.20)

$$\mathcal{L}^{(q)} \cdot \mathcal{S}^{(1/q)} = \sum_{M=0,\pm 1} (-q)^M \mathcal{L}^{(q)}_M \mathcal{S}^{(1/q)}_{-M} = \frac{1}{[2]} \left\{ C^{(q)}_{2,J} - C^{(q)}_{2,L} - C^{(q)}_{2,S} - \frac{(q-q^{-1})^2}{[2]} C^{(q)}_{2,L} C^{(q)}_{2,S} \right\}, \tag{28.48}$$

where $\mathcal{S}^{(1/q)}$ is a vector operator according to the algebra of eq. (28.44), constructed from the q-deformed spin operators taking into account the rule of eq. (28.21), but for deformation parameter $1/q$. One can then suggest the following Hamiltonian for the 3-dim q-deformed harmonic oscillator with spin–orbit interaction

$$\begin{aligned} {}^{q}H &= \hbar\omega_0 \left\{ X^{(q)}_0 - \kappa [2] \mathcal{L}^{(q)} \cdot \mathcal{S}^{(1/q)} \right\} = \\ &= \hbar\omega_0 \left\{ [N] q^{N+1} - \frac{q(q-q^{-1})}{[2]} C^{(q)}_{2,L} - \kappa \left(C^{(q)}_{2,J} - C^{(q)}_{2,L} - C^{(q)}_{2,S} - \frac{(q-q^{-1})^2}{[2]} C^{(q)}_{2,L} C^{(q)}_{2,S} \right) \right\}. \end{aligned} \tag{28.49}$$

In eq. (28.49) the various factors have been chosen in accordance with the usual convention used in the classical (spherically symmetric) shell model with spin–orbit coupling. The eigenvectors of this Hamiltonian are given by eq. (28.47), while the corresponding eigenvalues are

$$\begin{aligned} {}^{q}E_{n\ell j} = \hbar\omega_0 \Bigg\{ & [n] q^{n+1} - \frac{q(q-q^{-1})}{[2]} [\ell][\ell+1] \\ & - \kappa \left([j][j+1] - [\ell][\ell+1] - [\tfrac{1}{2}][\tfrac{3}{2}] - \frac{(q-q^{-1})^2}{[2]} [\ell][\ell+1][\tfrac{1}{2}][\tfrac{3}{2}] \right) \Bigg\}. \end{aligned} \tag{28.50}$$

In order to compare the expression for the q-deformed L–S interaction (the term proportional to κ) with the classical results one needs the expansion of the expectation value of the term given in eq.

(28.48) in powers of τ ($q = e^{\tau}$, with τ real)

$$ {}_q\langle n(\ell\tfrac{1}{2})jm|\mathcal{L}^{(q)}\cdot\mathcal{S}^{(1/q)}|n(\ell\tfrac{1}{2})jm\rangle_q = \begin{cases} \dfrac{1}{2}\ell\left(1+\dfrac{\tau^2}{6}(4\ell^2-7)+\ldots\right) & \text{if } j=\ell+1/2 \\ -\dfrac{1}{2}(\ell+1)\left(1+\dfrac{\tau^2}{6}(4\ell^2+8\ell-3)+\ldots\right) & \text{if } j=\ell-1/2 \end{cases} \tag{28.51} $$

It can be easily seen that the expression of eq. (28.48) introduces some corrections to the classical expressions for the spin–orbit interaction, which are proportional to τ^2. These corrections are small for light nuclei, where the values of ℓ are small, but they are non-negligible for heavier nuclei, where shells with higher values of ℓ are important.

One can easily see that the level scheme produced by the 3-dim q-HO with spin–orbit interaction reproduces very well the level scheme of the MHO with spin–orbit interaction [275].

29. Further development of $u_q(3)\supset so_q(3)$ models

The $u_q(3)\supset so_q(3)$ mathematical framework developed in the last section for the case of the 3-dimensional q-deformed harmonic oscillator is of more general interest, since one can use it for studying q-deformed versions of various models having the $u(3)\supset so(3)$ symmetry, such as the Elliott model [241,242,243,248] and the su(3) limit of the Interacting Boson Model. Since the quadrupole degree of freedom plays a central role in nuclear collectivity, one needs to be able to calculate matrix elements of the quadrupole operator in the $u_q(3)\supset so_q(3)$ basis. For the quadrupole operator itself two choices appear:

a) The quadrupole operator can be a q-deformed second rank tensor operator, with its matrix elements being calculated in the q-deformed basis. In this case one assumes that observables are described by q-deformed operators and physical states are described by q-deformed wavefunctions.

b) The quadrupole operator can be the usual (non-deformed) operator, with its matrix elements being calculated in the q-deformed basis. In this (less extreme) case one assumes that observables are described by the usual physical operators, while the states are described by q-deformed wavefunctions, corresponding to highly correlated eigenfunctions of the undeformed Hamiltonian.

In what follows both cases will be examined.

29.1. Matrix elements of a q-deformed quadrupole operator in the $u_q(3)\supset so_q(3)$ basis

We are going to compute the reduced matrix elements of an $so_q(3)$ quadrupole operator (second rank tensor operator under $so_q(3)$) in the $u_q(3)\supset so_q(3)$ basis (for the most symmetric $u_q(3)$ representation and for q being real).

The generalization of the Wigner–Eckart theorem in the case of the algebra $so_q(3)$ is [101,102]

$$ \langle\alpha', L'M'|T^j_m|\alpha, LM\rangle = (-1)^{2j}\,\frac{{}_qC^{L'M'}_{LM,jm}}{\sqrt{[2L'+1]}}\langle\alpha', L'\|T^j\|\alpha, L\rangle, \tag{29.1} $$

where $|\alpha, LM\rangle$ are orthonormalized basis vectors of the irreducible representation ${}_qD^L$ of the algebra $so_q(3)$ and ${}_qC^{LM}_{L_1M_1,L_2M_2}$ are the Clebsch–Gordan coefficients of the same algebra.

Our purpose is to calculate the reduced matrix elements of the q-deformed quadrupole operator Q^2, namely the quantities $\langle\lambda, L+2\|Q^2\|\lambda, L\rangle$ and $\langle\lambda, L\|Q^2\|\lambda, L\rangle$, where the operator Q^2 is

$$ Q^2_M = \sqrt{\frac{[3][4]}{[2]}}A^2_M, \qquad A^2_M = [T^\dagger\otimes\tilde{T}]^2_M = \sum_{m,n}{}_{q^{-1}}C^{2M}_{1m,1n}T^\dagger_m\tilde{T}_n. \tag{29.2} $$

It is clear that this operator is a second rank tensor under $so_q(3)$, according to the definitions given in the previous section, while in eq. (29.2) the overall constant factor guarantees the agreement between the present expression and the usual classical operator in the limit $q\to 1$. Other reduced matrix elements do not occur, since in the most symmetric representation of $u_q(3)$ only states with equal parity of λ and L occur.

Using the Wigner-Eckart theorem (eq. (29.1)) one obtains for the reduced matrix elements of the quadrupole operator Q^2 of eq. (29.2) the following results (see [267] for a proof)

$$\langle \lambda, L+2 \| Q^2 \| \lambda, L \rangle = \frac{q^{\lambda-\frac{1}{2}}}{[2]} \sqrt{\frac{[3][4]}{[2]}} \sqrt{\frac{[\lambda-L][\lambda+L+3][2L+4][2L+2]}{[2L+3]}}, \tag{29.3}$$

$$\langle \lambda, L \| Q^2 \| \lambda, L \rangle = -\frac{q^{\lambda-\frac{1}{2}}}{[2]} \sqrt{\frac{[2L][2L+1][2L+2]}{[2L-1][2L+3]}} \left\{ q^{L-\frac{1}{2}}[\lambda-L] + q^{-L+\frac{1}{2}}[\lambda+L+3] \right\}, \tag{29.4}$$

which are in agreement with the classical case in the limit $q \to 1$. Taking into account the Wigner–Eckart theorem and the symmetry properties of the q-deformed Clebsch–Gordan coefficients it can also be shown that the reduced matrix element given in eq. (29.3) has the following symmetry property

$$\langle \lambda, L+2 \| Q^2 \| \lambda, L \rangle = \langle \lambda, L \| Q^2 \| \lambda, L+2 \rangle. \tag{29.5}$$

For small values of the deformation parameter τ ($q = e^{\tau}$, with τ being real) the following Taylor expansions for the reduced matrix elements of eqs (29.3) and (29.4) are obtained

$$\langle \lambda, L+2 \| Q^2 \| \lambda, L \rangle = \sqrt{\frac{6(\lambda-L)(\lambda+L+3)(L+2)(L+1)}{2L+3}}$$

$$\times \left\{ 1 + \left(\lambda - \frac{1}{2} \right) \tau + \left(\frac{2}{3}\lambda^2 + \frac{1}{2}L^2 + \frac{3}{2}L + \frac{65}{24} \right) \tau^2 + \mathrm{O}(\tau^3) \right\}, \tag{29.6}$$

$$\langle \lambda, L \| Q^2 \| \lambda, L \rangle = -(2\lambda+3) \sqrt{\frac{L(L+1)(2L+1)}{(2L-1)(2L+3)}} \left\{ 1 + \frac{2}{2\lambda+3} \{ \lambda(\lambda+1) - L(L+1) \} \tau \right.$$

$$\left. + \frac{1}{3(2\lambda+3)} \{ (2\lambda+15)L(L+1) + (2\lambda+1)(2\lambda^2+2\lambda+3) \} \tau^2 + \mathrm{O}(\tau^3) \right\}. \tag{29.7}$$

In the limit $\tau = 0$ one obtains the classical expressions for the corresponding reduced matrix elements.

From a physical point of view of great interest are the $E2$ transition probabilities ($B(E2)$ factors), which in the classical case are expressed by means of the reduced matrix elements of the u(3) quadrupole operator. By analogy the $B(E2)$ factors corresponding to the chain $\mathrm{u}_q(3) \supset \mathrm{so}_q(3)$ can be assumed to be of the form

$$B(E2; (\lambda, L+2) \to (\lambda, L))_q = \frac{1}{[2L+5]} |\langle \lambda, L+2 \| Q^2 \| \lambda, L \rangle|^2, \tag{29.8}$$

where Q^2 is the q-deformed quadrupole operator of eq. (29.2). Again in analogy with the classical case, the reduced matrix element of eq. (29.4) is related to the deformation of the physical system in the state with angular momentum L.

It should be remembered that the results obtained here are valid only for real values of the deformation parameter q. On the other hand the comparison of the predictions of various models based on the $\mathrm{su}_q(2)$ algebra with the experimental data (sections 19–23) shows that one can achieve good agreement between theory and experiment in the case of q being a phase factor ($q = e^{i\tau}$, with τ real). It is therefore desirable to repeat the steps followed in this section for the case of q being a phase factor.

29.2. Matrix elements of the usual quadrupole operator in the $u_q(3) \supset so_q(3)$ basis

In order to be able to calculate matrix elements of the usual quadrupole operator in the q-deformed basis of eq. (28.11) we should first express the basis vectors in terms of usual (undeformed) bosons $a_i^{\dagger}$, a_i, which are connected to the q-deformed bosons $b_i^{\dagger}$, b_i through [35,36,43]

$$b_i^{\dagger} = \sqrt{\frac{[N_i]}{N_i}} a_i^{\dagger}, \qquad b_i = a_i \sqrt{\frac{[N_i]}{N_i}}, \qquad i = +, 0, -, \tag{29.9}$$

where N_i is the number operator of the undeformed bosons. In this way the operators $B_i^\dagger$, B_i of eq. (28.2) can be written in the form

$$B_0 = q^{-\frac{1}{2}N_0} a_0 \sqrt{\frac{[N_0]}{N_0}}, \qquad B_0^\dagger = \sqrt{\frac{[N_0]}{N_0}} a_0^\dagger q^{-\frac{1}{2}N_0}, \tag{29.10}$$

$$B_i = q^{N_i+\frac{1}{2}} a_i \sqrt{\frac{[2N_i]}{N_i}}, \qquad B_i^\dagger = \sqrt{\frac{[2N_i]}{N_i}} a_i^\dagger q^{N_i+\frac{1}{2}}, \qquad i = +,-. \tag{29.11}$$

As a next step we can replace $B_i^\dagger$ in eq. (28.11) with their expressions from eq. (29.11). Then, using the following identity, which holds in the Fock space,

$$\frac{(B_+^\dagger)^x}{\sqrt{[2x]!!}} \frac{(B_0^\dagger)^y}{\sqrt{[y]!}} \frac{(B_-^\dagger)^z}{\sqrt{[2z]!!}} |0\rangle = q^{\frac{1}{2}(x^2+z^2)-\frac{1}{4}y(y-1)} \frac{(a_+^\dagger)^x}{\sqrt{x!}} \frac{(a_0^\dagger)^y}{\sqrt{y!}} \frac{(a_-^\dagger)^z}{\sqrt{z!}} |0\rangle, \tag{29.12}$$

we obtain (in a slightly different notation) [259,261]

$$\left| \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q = q^{-\frac{1}{4}\{(2\lambda+M)(M-1)+L(L+1)\}} N_{\lambda LM}^{(q)} \sum_{t=0}^{(\lambda-L)/2} \sum_{x=\max(0,M)}^{\lfloor (L+M)/2 \rfloor} \frac{(-1)^t q^{(\lambda-\frac{1}{2})x-(L+\frac{3}{2})t}}{[2t]!![\lambda-L-2t]!!}$$

$$\times \frac{\sqrt{[2x+2t]!![\lambda+M-2x-2t]![2x+2t-2M]!!}}{[2x]!![L+M-2x]![2x-2M]!!} \frac{(a_+^\dagger)^{x+t}}{\sqrt{(x+t)!}} \frac{(a_0^\dagger)^{\lambda+M-2x-2t}}{\sqrt{(\lambda+M-2x-2t)!}} \frac{(a_-^\dagger)^{x+t-M}}{\sqrt{(x+t-M)!}} |0\rangle, \tag{29.13}$$

where the normalization factor $N_{\lambda LM}^{(q)}$ is

$$N_{\lambda LM}^{(q)} = \sqrt{\frac{[\lambda-L]!![L+M]![L-M]![2L+1]}{[\lambda+L+1]!!}}. \tag{29.14}$$

The expression of eq. (29.13) can be used for the calculation of the matrix elements of "standard" (or classical) tensor operators (which are built up from standard boson operators) between the states with given q-deformed angular momentum.

Let us consider now the classical quadrupole operator $Q_m^{(c)}$. Its zero component is of the form

$$Q_0^{(c)} = \sqrt{6}[a^\dagger \otimes \tilde{a}]_{20} = a_+^\dagger \tilde{a}_- + 2a_0^\dagger \tilde{a}_0 + a_-^\dagger \tilde{a}_+ = -a_+^\dagger a_+ + 2a_0^\dagger a_0 - a_-^\dagger a_- = 3N_0 - N, \tag{29.15}$$

where $a_m^\dagger, a_m (m = +, 0, -)$ are classical boson operators and $\tilde{a}_m = (-1)^m a_{-m}$.

The explicit expression for the matrix elements of $Q_0^{(c)}$ between the q-deformed states of eq. (29.13) then becomes

$$\left\langle \begin{array}{c} \lambda \\ J, M' \end{array} \right| Q_0^{(c)} \left| \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q = \delta_{M,M'} \left\langle \begin{array}{c} \lambda \\ J, M \end{array} \right| Q_0^{(c)} \left| \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q, \tag{29.16}$$

and

$$\left\langle \begin{array}{c} \lambda \\ J, M \end{array} \right| Q_0^{(c)} \left| \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q = (2\lambda+3M)\delta_{L,J} - 6\, q^{-\frac{1}{2}(2\lambda+M)(M-1)-\frac{1}{4}\{L(L+1)+J(J+1)\}} N_{\lambda LM}^{(q)} N_{\lambda JM}^{(q)}$$

$$\times \sum_{t=0}^{(\lambda-L)/2} \sum_{r=0}^{(\lambda-J)/2} \frac{(-1)^{t+r} q^{-(\lambda+L+1)t-(\lambda+J+1)r}}{[2t]!![\lambda-L-2t]!![2r]!![\lambda-J-2r]!!} \sum_x \frac{[2x]!![\lambda+M-2x]![2x-2M]!!}{[2x-2t]!![L+M-2x+2t]![2x-2t-2M]!!}$$

$$\times \frac{x\ q^{(2\lambda-1)x}}{[2x-2r]!![J+M-2x+2r]![2x-2r-2M]!!}, \tag{29.17}$$

where:

$$\max\{t, r, M+t, M+r\} \le x \le \min\{\lfloor (L+M)/2 \rfloor + t, \lfloor (J+M)/2 \rfloor + r\}. \tag{29.18}$$

Clearly the recipe described above can be applied for the calculation of matrix elements between the q-deformed angular momentum states of any classical operator. The disadvanage of this method is that one must perform special calculations for any specific operator. This can be avoided by the use of the transformation matrix between the "standard" and q-deformed states, which will be described in the next subsection.

29.3. Transformation between $so_q(3)$ and $so(3)$ basis states

For fixed λ one can expand the q-deformed basis states of eq. (29.13) in terms of classical ones (with $q = 1$), since both sets form orthonormal bases in the eigenspace $\mathcal{H}_\lambda$ of the number operator $N = N_+ + N_0 + N_-$, corresponding to the eigenvalue λ. In particular, $\dim \mathcal{H}_\lambda = \frac{1}{2}(\lambda+1)(\lambda+2)$ and the operator L_0 has the same form $L_0 = N_+ - N_-$ in both classical and q-deformed cases. In this way

$$\left| \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q = \sum_{J,M_J} \left| \begin{array}{c} \lambda \\ J, M_J \end{array} \right\rangle_c \, {}_c\left\langle \begin{array}{c} \lambda \\ J, M_J \end{array} \right| \left. \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q = \sum_J \left| \begin{array}{c} \lambda \\ J, M \end{array} \right\rangle_c \, {}_c\left\langle \begin{array}{c} \lambda \\ J, M \end{array} \right| \left. \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q , \qquad (29.19)$$

where $J = \lambda, \lambda-2, \ldots, |M|$ or $|M|+1$ and the subscript c denotes the classical (undeformed) basis states. Taking the scalar product between q-deformed basis states of eq. (29.13) and classical basis states, for the transformation matrix in eq. (29.19) one obtains

$$\left\langle \begin{array}{c} \lambda \\ J, M' \end{array} \right| \left. \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q = \delta_{M,M'} \, q^{-\frac{1}{4}\{(2\lambda+M)(M-1)+L(L+1)\}} \, N^{(c)}_{\lambda JM} \, N^{(q)}_{\lambda LM}$$

$$\times \sum_{t=0}^{(\lambda-L)/2} \sum_{r=0}^{(\lambda-J)/2} \frac{(-1)^{t+r} \, q^{-(\lambda+L+1)t}}{[2t]!![\lambda-L-2t]!!(2r)!!(\lambda-J-2r)!!} \sum_x \frac{q^{(\lambda-\frac{1}{2})x} \sqrt{[2x]!![\lambda+M-2x]![2x-2M]!!}}{[2x-2t]!![L+M-2x+2t]![2x-2t-2M]!!}$$

$$\times \frac{\sqrt{(2x)!!(\lambda+M-2x)!(2x-2M)!!}}{(2x-2r)!!(J+M-2x+2r)!(2x-2r-2M)!!}, \qquad (29.20)$$

where

$$\max\{t, r, M+t, M+r\} \le x \le \min\{\lfloor (L+M)/2 \rfloor + t, \lfloor (J+M)/2 \rfloor + r\}. \qquad (29.21)$$

It should be noted that the scalar product given in eq. (29.20) is real and has the properties

$$\left\langle \begin{array}{c} \lambda \\ J, M' \end{array} \right| \left. \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q = \delta_{M,M'} \, {}_c\left\langle \begin{array}{c} \lambda \\ J, M \end{array} \right| \left. \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q = \, {}_q\left\langle \begin{array}{c} \lambda \\ L, M \end{array} \right| \left. \begin{array}{c} \lambda \\ J, M' \end{array} \right\rangle_c , \qquad (29.22)$$

which have been used in eq. (29.19). Though looking very involved, eq. (29.20) is quite handy for computational purposes. Indeed, only few terms are contained in the sum. The same equation shows that the transformation matrix is diagonal in the third projections of J and L, and is nonvanishing even for high values of the difference $\Delta(J) = |L - J|$. However, the values are peaked around the classical value $\Delta(J) = 0$, if the parameter of deformation q tends to unity.

Let us consider now a classical tensor operator $A^{(c)}_{jm}$ of rank j according to the algebra so(3), which conserves the number of particles. Using the Wigner-Eckart theorem in the classical case we can express the matrix elements of $A^{(c)}_{jm}$ between the classical so(3) basis states in the form

$$\left\langle \begin{array}{c} \lambda \\ J, M' \end{array} \right| A^{(c)}_{jm} \left| \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_c = (-1)^{2j} \frac{C^{JM'}_{LM,jm}}{\sqrt{2J+1}} \langle \lambda J \| A^{(c)}_j \| \lambda L \rangle, \qquad (29.23)$$

where $C^{JM'}_{LM,jm}$ are the classical Clebsch-Gordan coefficients of the so(3) algebra, and we assume that the classical reduced matrix elements of $A^{(c)}_j$ in eq. (29.23) are known. In this way, from eq. (29.19) follows that

$$A^{(c)}_{jm} \left| \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q = \sum_R A^{(c)}_{jm} \left| \begin{array}{c} \lambda \\ R, M \end{array} \right\rangle_c \, {}_c\left\langle \begin{array}{c} \lambda \\ R, M \end{array} \right| \left. \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q$$

$$= \sum_R \sum_{P,M_P} \left| \begin{array}{c} \lambda \\ P, M_P \end{array} \right\rangle_c \; {}_c\left\langle \begin{array}{c} \lambda \\ P, M_P \end{array} \right| A^{(c)}_{jm} \left| \begin{array}{c} \lambda \\ R, M \end{array} \right\rangle_c \; {}_c\left\langle \begin{array}{c} \lambda \\ R, M \end{array} \right| \left. \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q , \qquad (29.24)$$

and using eq. (29.23) we have the expansion

$$ {}_q\left\langle \begin{array}{c} \lambda \\ J, M' \end{array} \right| A^{(c)}_{jm} \left| \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q = \sum_R \sum_P (-1)^{2j} \frac{C^{PM'}_{RM,jm}}{\sqrt{2P+1}}$$

$$\times \; {}_q\left\langle \begin{array}{c} \lambda \\ J, M' \end{array} \right| \left. \begin{array}{c} \lambda \\ P, M' \end{array} \right\rangle_c \langle \lambda P \| A^{(c)}_j \| \lambda R \rangle \; {}_c\left\langle \begin{array}{c} \lambda \\ R, M \end{array} \right| \left. \begin{array}{c} \lambda \\ L, M \end{array} \right\rangle_q . \qquad (29.25)$$

In particular, eq. (29.25) shows that the matrix elements of the classical tensor operator $A^{(c)}_{jm}$ between q-deformed $\mathrm{so}_q(3)$ states are nonvanishing if $M' = M + m$, as in the classical basis. It is also worth to remark that the summation expression of eq. (29.25) gives an universal way for (numerical) computation of matrix elements of different types of classical operators between the deformed basis states of eq. (29.13). For instance, it is straightforward to show that the numerical values of the quadrupole operator calculated by eq. (29.17) of the previous subsection and the method used in this subsection (involving transformation matrices) coincide. However, the method descibed in this subsection is universal and can be applied to any classical operator.

Examples of transformation brackets can be found in [276].

30. The question of complete breaking of symmetries and some applications

In the cases examined so far the q-deformed symmetries considered were close to their classical counterparts, to which they reduce for $\tau = 0$ ($q = 1$), since the values of τ were relatively small. One can then argue that results similar to the ones provided by the quantum symmetries can also be obtained from the usual Lie symmetries through the addition of suitable perturbations. What can be very useful is to start with one limiting symmetry and, through large deformations, reach another limiting symmetry. We shall refer to this as the complete breaking of the symmetry. In this way one could hope to "bridge" different Lie symmetries through the use of q-deformations, providing in addition new symmetries in the regions intermediate between the existing Lie ones.

The question of complete breaking of the symmetry in the framework of the toy IBM of sec. 27 has been studied by Cseh [277], Gupta [278], and Del Sol Mesa *et al.* [264].

Cseh [277] started with the $\mathrm{su}_q(2)$ (vibrational) limit (in a form different from the one used in sec. 27) and tried to reach the $\mathrm{so}_q(3)$ (rotational) limit. He noted that for q being a phase factor this is not possible, while for real q some rotational features are obtained, but without all of the requirements for rotational behaviour being satisfied simultaneously.

Gupta [278] started with the $\mathrm{su}_q(2)$ limit, in the form given in sec. 27. He noted that for q being a phase factor a recovery of the su(3) symmetry occurs (see also the next paragraph), while for real q, and even better for q complex ($q = e^s$ with $s = a + ib$), the $\mathrm{so}_q(3)$ limit is indeed reached.

Del Sol Mesa *et al.* [264] considered the $\mathrm{o}_q(3)$ limit of the model of sec. 27, since it corresponds to the symmetry of a q-deformed version of the spherical Nilsson Hamiltonian with spin-orbit coupling term. They found that for q being a phase factor ($q = e^{i\tau}$) and for τ obtaining values in the region $0.5 \leq \tau \leq 2$ the u(3) symmetry, which is broken in the initial model because of the presence of the spin-orbit term, is recovered. This offers a way of recovering the u(3) symmetry alternative to the one developed for the spherical Nilsson model [279,280] and the deformed Nilsson model [281,282] through the use of appropriate unitary operators.

Complex deformations have also been used in [283] in the framework of a deformed u(2) model, possessing the $\mathrm{u}(2) \supset \mathrm{u}(1)$ and $\mathrm{u}(2) \supset \mathrm{o}(2)$ chains. Again it has been found that complex deformations can bridge the two limiting symmetries.

Possible complete breaking of the symmetry has also been studied in the framework of the q-deformed version of the full Interacting Boson Model (see sec. 31).

A problem associated with complex deformations as the ones considered above is that the energy eigenvalues become complex as well. A way to avoid this problem has been introduced recently by Jannussis and collaborators [284,285].

Finally, the $o_q(3)$ limit of the model of sec. 27 has been used for describing the $^{16}O + \alpha$ cluster states in ^{20}Ne [286]. It turns out that an improved description of the energy spectrum and the α-particle spectroscopic factors occurs for $q = e^{0.124}$.

31. q-deformation of the Interacting Boson Model (IBM)

The Interacting Boson Model [179,215,216] (see [169,180] for recent overviews) is the most popular algebraic model of nuclear structure. It describes the collective properties of medium-mass and heavy nuclei away from closed shells in terms of bosons, which correspond to correlated valence fermion pairs. In its simplest form, called IBM-1, only s ($J = 0$) and d ($J = 2$) bosons are used. The overall symmetry of the model is u(6), possessing three limiting symmetries: the u(5) (vibrational) limit, corresponding to the chain of subalgebras

$$u(6) \supset u(5) \supset o(5) \supset o(3), \tag{31.1}$$

the su(3) (rotational) limit, characterized by the chain

$$u(6) \supset su(3) \supset o(3), \tag{31.2}$$

and the o(6) (γ-unstable) limit, for which the relevant chain is

$$u(6) \supset o(6) \supset o(5) \supset o(3). \tag{31.3}$$

If one of these dynamical symmetries is present, the Hamiltonian can be written in terms of the Casimir operators of the algebras appearing in the relevant chain. Thus the Hamiltonian can be analytically diagonalized in the corresponding basis. This is a great advantage of IBM and its numerous generalizations: they provide us with a large number of exactly soluble models, the predictions of which can be directly compared to experiment, without any need for lengthy numerical calculations.

From what we have already seen in sec. 27, it is worth examining if a q-deformed version of the IBM has any advantages in comparison to the standard version. In order to accomplish this, one has to construct the q-analogues of the three chains mentioned above. The difficulties associated with the su(3) chain have already been discussed in subsec. 27.3. In what follows we are going to focus attention on the o(6) chain, for which the relevant construction has been carried out [287]. The technique used is based on the notion of complementary subalgebras, which is explained in detail in [288], while here only final results will be reported. We only mention here that the notion of complementary subalgebras was introduced by Moshinsky and Quesne [289,290,291]. Two subalgebras A_1 and A_2 of a larger algebra A are complementary within a definite irrep of A, if there is an one-to-one correspondence between all the irreps of A_1 and A_2 contained in this irrep of A.

In the o(6) limit of IBM the Hamiltonian is

$$H = E_0 + \beta C_2(o(5)) + \gamma C_2(o(3)) + \eta C_2(o(6)). \tag{31.4}$$

The eigenvalues of the energy in the relevant basis have already been given in subsec. 26.1.

Using the notion of complementarity it turns out that, instead of using the o(6) chain mentioned above, it suffices to study the chain

$$su^{sd}(1,1) \otimes so(6) \supset su^{d}(1,1) \otimes so(5) \supset so(3), \tag{31.5}$$

where $su^{sd}(1,1)$ is the algebra closed by the pair operators formed out of the s and d bosons, while $su^{d}(1,1)$ is the algebra closed by the pair operators formed out of d bosons alone. (Details on the basis for symmetric irreps of su(1,1)$\otimes$o(5) can be found in [292].) The irreps of $su^{sd}(1,1)$ are characterized by the same quantum numbers as the irreps of o(6) in the o(6) chain of the IBM, while the irreps of $su^{d}(1,1)$ are characterized by the same quantum numbers as the irreps of o(5) in the o(6) limit of IBM. Therefore in the Hamiltonian one can use the Casimir operators of the $su^{sd}(1,1)$, $su^{d}(1,1)$ and su(2) subalgebras (the deformed versions of which are well known, as seen in secs 14–16) instead of the Casimir operators of o(6), o(5), o(3) respectively. Keeping the same notation as in eq. (26.3) the final result reads

$$E(N,\sigma,\tau,\nu_\Delta,J,M_J) = E_0 + \beta 8 \left[\frac{\tau}{2}\right]_q \left[\frac{\tau+3}{2}\right]_q + \gamma 2 [J]_q [J+1]_q + \eta 8 \left[\frac{\sigma}{2}\right]_q \left[\frac{\sigma+4}{2}\right]_q, \tag{31.6}$$

where the free parameters have been chosen so that the present equation reduces to its classical counterpart for $q \to 1$. For the ground state band then the analog of eq. (26.4) is

$$E(J) = E_0 + \beta' 8[J]_{q^{1/4}}[J+6]_{q^{1/4}} + \gamma 2[J]_q[J+1]_q + \eta' 8[N]_{q^{1/2}}[N+4]_{q^{1/2}}, \tag{31.7}$$

where the identities

$$\left[\frac{x}{2}\right]_q = [x]_{q^{1/2}}(q^{1/2}+q^{-1/2})^{-1}, \tag{31.8}$$

$$\left[\frac{x}{4}\right]_q = [x]_{q^{1/4}}(q^{1/2}+q^{-1/2})^{-1}(q^{1/4}+q^{-1/4})^{-1}, \tag{31.9}$$

have been used and β', η' are related to β, η and q in an obvious way. We remark that the Casimir operator of $\mathrm{su}_q^d(1,1)$, which is complementary to o(5) in the undeformed case, leads to a term of the form $[J]_{q'}[J+6]_{q'}$ with $q' = q^{1/4}$.

In refs [293,294] the question has been studied if large values of the deformation parameter can lead us from the o(6) limit to the su(3) (rotational) or u(5) (vibrational) limits, so that complete breaking of the symmetry, in the sense of sec. 30, could be obtained. It turns out that for q real the spectrum of the ground state band goes towards the rotational limit, while q being a phase factor leads towards the vibrational limit. Many more detailed studies, of both spectra and electromagnetic transition probabilities, are required before such a claim can be made.

A different method for constructing the q-deformed versions of the u(5) and o(6) limits of the IBM has been used by Pan [295]. The method is based on the use of q-deforming functionals (see secs 10, 14). The same method has been used in [263] for studying the q-deformed version of the su(3)$\supset$so(3) decomposition. The final result for the energy eigenvalues in the o(6) case is similar to the one reported in eq. (31.6), the main difference being that different deformation parameters are allowed in each of the three deformed terms in the rhs. Some comparisons of the model predictions for spectra and B(E2) values to the experimental data have been performed [295].

It is clear that several deformed versions of the IBM can be constructed, providing us with a large number of exactly soluble models. In order to demonstrate their usefulness, one has to show that by deforming the model one gets some advantages over the classical (non-deformed) version. One way to achieve this is the use of parameter-independent tests based on systematics of the data, like the ones used in [296,297,298] for the usual IBM. It is also desirable for the deformation parameter to be associated with some physical quantity, as in the case of the $\mathrm{su}_q(2)$ model. Much work is still required in these directions. Some mathematical results which can be useful in these efforts are reported below:

i) Casimir operators for $\mathrm{su}_q(\mathrm{n})$ have been given in [299], while the quadratic Casimir of $\mathrm{so}_q(5)$ can be found in [300].

ii) Raising and lowering operators for $\mathrm{u}_q(\mathrm{n})$ have been given by Quesne [301].

iii) Irreps of $\mathrm{u}_q(\mathrm{m+n})$ in the $\mathrm{u}_q(\mathrm{m})\oplus\mathrm{u}_q(\mathrm{n})$ basis have been constructed in [302], while generalized q-bosonic operators acting in a tensor product of m Fock spaces have been constructed as double irreducible tensors with respect to $\mathrm{u}_q(\mathrm{m})\oplus\mathrm{u}_q(\mathrm{n})$ in [303,304].

32. Deformed versions of other collective models

The Moszkowski model [305] is a schematic two-level model which provides a description of the phase transition from the vibrational regime to the rotational one. A q-deformed version of the model has been developed [306] and the RPA modes in it have been discussed [307]. In addition the validity of the coherent states variational method has been checked in the framework of this model and the time evolution of the system has been studied by means of a time dependent variational principle [308]. Furthermore, the q-deformed Moszkowski model with cranking has been studied in the mean field approximation and the relation between q-deformation and temperature has been discussed [309]. It should be noticed here that quantum algebraic techniques have also been found useful in describing thermal effects in the framework of the q-deformed Thouless model for superconductivity [310].

The Lipkin–Meshkov–Glick (LMG) model [311] is an exactly soluble schematic shell model. q-deformed versions of the 2-level LMG model (in terms of an $\mathrm{su}_q(2)$ algebra) [312,313,314,315,316] and of the 3-level LMG model (in terms of an $\mathrm{su}_q(3)$ algebra) [317] have been developed.

33. Fermion pairs as deformed bosons: approximate mapping

We have seen so far that several quantum algebraic phenomenological models have been proposed for the description of nuclear collective properties. These models make use of q-deformed bosons, which satisfy commutation relations differing from the standard boson commutation relations, to which they reduce in the limit $q \to 1$.

On the other hand, it is known that vibrational nuclear spectra, which are described in the simplest way by a pairing Hamiltonian, show anharmonicities (see also sec. 26), described, for example, by the *Anharmonic Vibrator Model (AVM)* [218]

$$E(J) = aJ + bJ(J-2). \tag{33.1}$$

In the framework of the single-j shell model [318,319,320,321,322], which can be extended to several non-degenerate j-shells [323,324,325] these anharmonicities are related to the fact that correlated fermion pairs satisfy commutation relations which resemble boson commutation relations but in addition include corrections due to the presence of the Pauli principle. This fact has been the cause for the development of boson mapping techniques (see the recent reviews by Klein and Marshalek [85] and Hecht [326] and references therein), by which the description of systems of fermions in terms of bosons is achieved. In recent years boson mappings have attracted additional attention in nuclear physics as a necessary tool in providing a theoretical justification for the success of the phenomenological Interacting Boson Model [169] and its various extentions, in which low lying collective states of medium and heavy mass nuclei are described in terms of bosons.

From the above observations it is clear that both q-bosons and correlated fermion pairs satisfy commutation relations which resemble the standard boson commutation relations but they deviate from them, due to the q-deformation in the former case and to the Pauli principle in the latter. A question is thus created: Are q-bosons suitable for the approximate description of correlated fermion pairs? In particular, is it possible to construct a boson mapping in which correlated fermion pairs are mapped onto q-bosons, in a way that the q-boson operators approximately satisfy the same commutation relations as the correlated fermion pair operators? In this section we show for the simple case of su(2) that such a mapping is indeed possible.

33.1. The single-j shell model

Let us consider the single-j shell model [318,319,320,321,322]. One can define fermion pair and multipole operators as

$$A^\dagger_{JM} = \frac{1}{\sqrt{2}} \sum_{mm'} (jmjm'|JM) a^\dagger_{jm} a^\dagger_{jm'}, \tag{33.2}$$

$$B_{JM} = \frac{1}{\sqrt{2J+1}} \sum_{mm'} (jmj-m'|JM)(-1)^{j-m'} a^\dagger_{jm} a_{jm'}, \tag{33.3}$$

with the following definitions

$$A_{JM} = [A^\dagger_{JM}]^\dagger, \quad B^\dagger_{JM} = [B_{JM}]^\dagger. \tag{33.4}$$

In the above $a^\dagger_{jm}$ (a_{jm}) are fermion creation (annihilation) operators and $(jmjm'|JM)$ are the usual Clebsch–Gordan coefficients.

The pair and multipole operators given above satisfy the following commutation relations:

$$[A^\dagger_{JM}, A^\dagger_{J'M'}] = 0, \tag{33.5}$$

$$[A_{JM}, A^\dagger_{J'M'}] = \delta_{JJ'}\delta_{MM'} - 2\sum_{J''} (-1)^{2j+M} \sqrt{(2J+1)(2J'+1)(2J''+1)}$$
$$(J-MJ'M'|J''M'-M) \begin{Bmatrix} J & J' & J'' \\ j & j & j \end{Bmatrix} B_{J'',M'-M}, \tag{33.6}$$

$$[B^\dagger_{JM}, A^\dagger_{J'M'}] = \sum_{J''} 2\sqrt{2J'+1}(-1)^{2j-M} (J-MJ'M'|J''M'-M)$$

$$\left\{ \begin{array}{ccc} J & J' & J'' \\ j & j & j \end{array} \right\} A^{\dagger}_{J'',M'-M} \frac{1+(-1)^{J''}}{2}, \tag{33.7}$$

$$[B_{JM}, B_{J'M'}] = \sum_{J''} (-1)^{2j-J''} [1-(-1)^{J+J'+J''}] \sqrt{2J''+1}$$

$$(JMJ'M'|J''M+M') \left\{ \begin{array}{ccc} J & J' & J'' \\ j & j & j \end{array} \right\} B_{J'',M+M'}, \tag{33.8}$$

in which the curly brackets are the usual 6-j symbols. These are the commutation relations of the so(2(2j+1)) algebra.

33.2. Fermion pairs of zero angular momentum

In the present subsection we will restrict ourselves to fermion pairs coupled to angular momentum zero. The relevant commutation relations take the form

$$[A_0, A_0^{\dagger}] = 1 - \frac{N_F}{\Omega}, \qquad [\frac{N_F}{2}, A_0^{\dagger}] = A_0^{\dagger}, \qquad [\frac{N_F}{2}, A_0] = -A_0, \tag{33.9}$$

where N_F is the number of fermions, $2\Omega = 2j+1$ is the size of the shell, and

$$B_0 = N_F/\sqrt{2\Omega}. \tag{33.10}$$

With the identifications

$$J_+ = \sqrt{\Omega} A_0^{\dagger}, \quad J_- = \sqrt{\Omega} A_0, \quad J_0 = \frac{N_F - \Omega}{2}, \tag{33.11}$$

eqs (33.9) take the form of the usual su(2) commutation relations

$$[J_+, J_-] = 2J_0, \quad [J_0, J_+] = J_+, \quad [J_0, J_-] = -J_-. \tag{33.12}$$

An exact boson mapping of the su(2) algebra is given in [85,322]

$$A_0^{\dagger} = a_0^{\dagger} \sqrt{1 - \frac{n_0}{\Omega}}, \quad A_0 = \sqrt{1 - \frac{n_0}{\Omega}} a_0, \quad N_F = 2n_0, \tag{33.13}$$

where $a_0^{\dagger}$ (a_0) are boson creation (annihilation) operators carrying angular momentum zero and n_0 is the number of these bosons.

The simplest pairing Hamiltonian one can consider has the form

$$H = -G\Omega A_0^{\dagger} A_0. \tag{33.14}$$

The Casimir operator of su(2) can be written as

$$\{A_0^{\dagger}, A_0\} + \frac{\Omega}{2} \left(1 - \frac{N_F}{\Omega}\right)^2 = \frac{\Omega}{2} + 1, \tag{33.15}$$

while the pairing energy takes the form

$$\frac{E}{(-G\Omega)} = \frac{N_F}{2} - \frac{N_F^2}{4\Omega} + \frac{N_F}{2\Omega}. \tag{33.16}$$

Our aim is to check if there is a boson mapping for the operators $A_0^{\dagger}$, A_0 and N_F in terms of q-deformed bosons, having the following properties:

i) The mapping is simpler than the one of eq. (33.13), i.e. to each fermion pair operator $A_0^{\dagger}$, A_0 corresponds a bare q-boson operator and not a boson operator accompanied by a square root (the Pauli reduction factor).

ii) The commutation relations (33.9) are satisfied up to a certain order.

ii) The pairing energies of eq. (33.16) are reproduced up to the same order.

33.3. Mapping using the q-deformed oscillator

In the case of the q-deformed harmonic oscillator (sec. 10), the commutation relation

$$[a, a^\dagger] = [N+1] - [N] \tag{33.17}$$

for q being a phase can be written as

$$[a, a^\dagger] = \frac{\cos\frac{(2N+1)\tau}{2}}{\cos\frac{\tau}{2}}. \tag{33.18}$$

In physical situations τ is expected to be small (i.e. of the order of 0.01). Therefore in eq. (33.18) one can take Taylor expansions of the functions appearing there and thus find an expansion of the form

$$[a, a^\dagger] = 1 - \frac{\tau^2}{2}(N^2+N) + \frac{\tau^4}{24}(N^4+2N^3-N) - \dots . \tag{33.19}$$

We remark that the first order corrections contain not only a term proportional to N, but in addition a term proportional to N^2, which is larger than N. Thus one cannot make the simple mapping

$$A_0 \to a, \quad A_0^\dagger \to a^\dagger, \quad N_F \to 2N, \tag{33.20}$$

because then one cannot get the first of the commutation relations (33.9) correctly up to the first order of the corrections. The same problem appears in the case that q is real as well. In addition, by making the simple mapping of eq. (33.20) the pairing Hamiltonian can be written as

$$\frac{H}{-G\Omega} = a^\dagger a = [N]. \tag{33.21}$$

In the case of small τ, one can take Taylor expansions of the functions appearing in the definition of the q-numbers (eq. (2.2) or (2.3)) and thus obtain the following expansion

$$[N] = N \pm \frac{\tau^2}{6}(N-N^3) + \frac{\tau^4}{360}(7N-10N^3+3N^5) \pm \frac{\tau^6}{15120}(31N-49N^3+21N^5-3N^7) + \dots , \tag{33.22}$$

where the upper (lower) sign corresponds to q being a phase factor (real). We remark that while the first order corrections in eq. (33.16) are proportional to N_F^2 and N_F, here the first order corrections are proportional to N and N^3. Thus neither the pairing energies can be reproduced correctly by this mapping.

33.4. Mapping using the Q-deformed oscillator

In the case of the Q-oscillator of sec. 11, however, the commutation relation among the bosons is

$$[b, b^\dagger] = Q^N. \tag{33.23}$$

Defining $Q = e^T$ this can be written as

$$[b, b^\dagger] = 1 + TN + \frac{T^2N^2}{2} + \frac{T^3N^3}{6} + \dots . \tag{33.24}$$

We remark that the first order correction is proportional to N. Thus, by making the boson mapping

$$A_0^\dagger \to b^\dagger, \quad A_0 \to b, \quad N_F \to 2N, \tag{33.25}$$

one can satisfy the first commutation relation of eq. (33.9) up to the first order of the corrections by determining $T = -2/\Omega$.

We should now check if the pairing energies (eq. (33.16)) can be found correctly up to the same order of approximation when this mapping is employed. The pairing Hamiltonian in this case takes the form

$$\frac{H}{-G\Omega} = b^\dagger b = [N]_Q. \tag{33.26}$$

Defining $Q = e^T$ it is instructive to construct the expansion of the Q-number of eq. (6.1) in powers of T. Assuming that T is small and taking Taylor expansions in eq. (6.1) one finally has

$$[N]_Q = N + \frac{T}{2}(N^2 - N) + \frac{T^2}{12}(2N^3 - 3N^2 + 1) + \frac{T^3}{24}(N^4 - 2N^3 + N^2) + \ldots \tag{33.27}$$

Using the value of the deformation parameter $T = -2/\Omega$, determined above from the requirement that the commutation relations are satisfied up to first order corrections, the pairing energies become

$$\frac{E}{-G\Omega} = N - \frac{N^2 - N}{\Omega} + \frac{2N^3 - 3N^2 + 1}{3\Omega^2} - \frac{N^4 - 2N^3 + N^2}{3\Omega^3} + \ldots. \tag{33.28}$$

The first two terms in the rhs of eq. (33.28), which correspond to the leading term plus the first order corrections, are exactly equal to the pairing energies of eq. (33.16), since $N_F \to 2N$. We therefore conclude that through the boson mapping of eq. (33.25) one can both satisfy the fermion pair commutation relations of eq. (33.9) and reproduce the pairing energies of eq. (33.16) up to the first order corrections.

The following comments are also in place:

i) By studying the spectra of the two versions of the q-deformed harmonic oscillator, given in eqs. (10.10) and (11.10), one can easily draw the following conclusions: when compared to the usual oscillator spectrum, which is equidistant, the spectrum of the q-oscillator is getting shrunk for q being a phase, while the spectrum of the Q-oscillator gets shrunk when $T < 0$. In a similar way, the spectrum of the q-oscillator gets expanded for q real, while the spectrum of the Q-oscillator gets expanded for $T > 0$. In physical situations (secs 19–23) it has been found that the physically interesting results are gotten with q being a phase. Thus in the case of the Q-oscillator it is the case $T < 0$ the one which corresponds to the physically interesting case. As we have already seen, it is exactly for $T = -2/\Omega < 0$ that the present mapping gives the fermion pair results.

ii) It should be recalled that the pairing model under discussion is studied under the assumptions that the degeneracy of the shell is large ($\Omega >> 1$), that the number of particles is large ($N >> 1$), and that one stays away from the center of the shell ($\Omega - N = O(N)$). The accuracy of the present mapping in reproducing the pairing energies has been checked in [327], where results for $\Omega = 11$ (the size of the nuclear fp major shell), $\Omega = 16$ (the size of the nuclear sdg major shell) and $\Omega = 22$ (the size of the nuclear pfh major shell) are reported, along with results for the case $\Omega = 50$ (as an example of a large shell). In all cases good agreement between the classical pairing model results and the Q-Hamiltonian of eq. (33.26) is obtained up to the point at which about 1/4 of the shell is filled. The deviations observed near the middle of the shell are expected, since there the expansion used breaks down.

We have thus shown that an approximate mapping of the fermion pairs coupled to angular momentum zero in a single-j shell onto suitably defined q-bosons (the Q-bosons) is possible. The su(2) commutation relations are satisfied up to the first order corrections, while at the same time the eigenvalues of a simple pairing Hamiltonian are correctly reproduced up to the same order. The small parameter of the expansion, which is T (where $Q = e^T$), turns out to be negative and inversely proportional to the size of the shell.

The present results are an indication that suitably defined q-bosons could be used for approximately describing systems of correlated fermions under certain conditions in a simplified way. The construction of q-bosons which would *exactly* satisfy the fermion pair su(2) commutation relations will be undertaken in the following section.

34. Fermion pairs as deformed bosons: exact mapping

From the contents of the previous section, the following question is created: Is it possible to construct a generalized deformed oscillator (as in sec. 12) using deformed bosons in such a way that the spectrum

of the oscillator will exactly correspond to the pairing energy in the single-j shell model, while the commutation relations of the deformed bosons will exactly correspond to the commutation relations of the correlated fermion pairs in the single-j shell under discussion? In this section we show that such an oscillator can indeed be constructed [328] by using the method of sec. 12.

34.1. An appropriate generalized deformed oscillator

We apply the procedure of sec. 12 in the case of the pairing in a single-j shell mentioned before. The boson number is half the fermion number, i.e. $N = N_F/2$. Then eq. (33.16) can be written as

$$\frac{E}{-G\Omega} = N - \frac{N^2}{\Omega} + \frac{N}{\Omega}. \tag{34.1}$$

One can then use a generalized deformed oscillator with structure function

$$F(N) = a^\dagger a = N - \frac{N^2}{\Omega} + \frac{N}{\Omega}. \tag{34.2}$$

In addition one has

$$F(N+1) = aa^\dagger = N + 1 - \frac{(N+1)^2}{\Omega} + \frac{N+1}{\Omega}. \tag{34.3}$$

What we have constructed is a boson mapping for the operators A_0, $A_0^\dagger$, N_F:

$$A_0 \to a, \quad A_0^\dagger \to a^\dagger, \quad N_F \to 2N. \tag{34.4}$$

From eq. (34.2) it is clear that this mapping gives the correct pairing energy. In addition one has

$$[a, a^\dagger] = F(N+1) - F(N) = 1 - \frac{2N}{\Omega}, \tag{34.5}$$

in agreement to the first commutation relation of eq. (33.9). Thus the correct commutation relations are also obeyed. (The last two commutation relations of eq. (33.9) are satisfied because of (12.1).)

As we have already seen in the previous section, an exact hermitian boson mapping for the su(2) algebra is known to have the form of eq. (33.13). In this mapping the Pauli principle effects are carried by the square roots accompanying the ordinary boson operators, while in the mapping of eq. (34.4) the Pauli principle effects are "built in" the deformed bosons.

The generalized oscillator obtained here has energy spectrum

$$E_N = \frac{1}{2}(F(N) + F(N+1)) = N + \frac{1}{2} - \frac{N^2}{\Omega}, \tag{34.6}$$

which is the spectrum of an anharmonic oscillator.

34.2. Related potentials

The classical potential giving the same spectrum, up to first order perturbation theory, can be easily determined (see also subsecs 13.1, 13.2). The potential

$$V(x) = \kappa x^2 + \lambda x^4, \tag{34.7}$$

is known to give in first order perturbation theory the spectrum

$$E_n = \kappa(2n+1) + \lambda(6n^2 + 6n + 3) = (2\kappa + 6\lambda)(n + \frac{1}{2}) + 6\lambda n^2. \tag{34.8}$$

Comparing eqs. (34.6) and (34.8) one finds

$$\kappa = \frac{1}{2}(1 + \frac{1}{\Omega}), \quad \lambda = -\frac{1}{6\Omega}. \tag{34.9}$$

Then the classical potential giving the same spectrum, up to first order perturbation theory, as the generalized oscillator determined here, is

$$V(x) = \frac{1}{2}(1 + \frac{1}{\Omega})x^2 - \frac{1}{6\Omega}x^4. \tag{34.10}$$

It is therefore demonstrated that the Pauli principle effects in a single-j shell with pairing interaction are equivalent to an x^4 anharmonicity.

The generalization of the results obtained in this section for the pairing hamiltonian to any anharmonic oscillator is straightforward. For example, the potential

$$V(x) = \kappa x^2 + \lambda x^4 + \mu x^6 + \xi x^8, \tag{34.11}$$

is known to give up to first order perturbation theory the spectrum

$$E_n = \kappa(2n+1)+\lambda(6n^2+6n+3)+\mu(20n^3+30n^2+40n+15)+\xi(70n^4+140n^3+350n^2+280n+105), \tag{34.12}$$

which can be rewritten in the form

$$E_n = (n+(n+1))(\kappa+5\mu)+(n^2+(n+1)^2)(3\lambda+70\xi)+(n^3+(n+1)^3)(10\mu)+(n^4+(n+1)^4)(35\xi). \tag{34.13}$$

Taking into account eq. (12.11), from eq. (34.13) one gets

$$\frac{F(N)}{2} = (\kappa + 5\mu)n + (3\lambda + 70\xi)n^2 + (10\mu)n^3 + (35\xi)n^4. \tag{34.14}$$

For $\mu = \xi = 0$ and κ, λ given from eq. (34.9), the results for the pairing problem are regained.

It is worth mentioning at this point that the energy spectrum of the generalized oscillator corresponding to the pairing correlations (eq. (34.6)) can be rewritten as

$$E_N = \frac{2}{\Omega}\left(-\frac{1}{8} + \frac{\Omega+1}{2}\left(N+\frac{1}{2}\right) - \frac{1}{2}\left(N+\frac{1}{2}\right)^2\right). \tag{34.15}$$

On the other hand, it is known that for the modified Pöschl–Teller potential (see also subsec. 13.2)

$$V(x) = D\tanh^2(x/R), \tag{34.16}$$

the energy spectrum is given by [329]

$$E_N = \frac{\hbar^2}{mR^2}\left(-\frac{1}{8} + \frac{1}{2}\sqrt{8mDR^2/\hbar^2+1} \quad \left(N+\frac{1}{2}\right) - \frac{1}{2}\left(N+\frac{1}{2}\right)^2\right). \tag{34.17}$$

It is thus clear that the energy spectrum of the generalized oscillator studied here can be obtained from the modified Pöschl–Teller potential for special values of the potential depth D.

It is also worth remarking that the "structure function" $F(N)$ of the generalized oscillator obtained here (eq. (34.2)) can be written as

$$F(N) = \frac{N}{\Omega}(\Omega + 1 - N), \tag{34.18}$$

which is similar to the one of the para-fermionic oscillator of Ohnuki and Kamefuchi [54] (see also secs 12, 18).

In summary, we have constructed a generalized deformed oscillator which satisfies the same commutation relations as fermion pair and multipole operators of zero angular momentum in a single-j shell, and, in addition, reproduces the pairing energy of this shell exactly. We have thus demonstrated that an exact hermitian boson mapping of a system of angular-momentum-zero fermion pairs in terms of bare deformed bosons can be constructed, while in the usual case the ordinary bosons are accompanied by square roots due to the Pauli principle effects. The oscillator corresponding to the pairing problem has a spectrum which can be reproduced up to first order perturbation theory by a harmonic oscillator with an x^4 anharmonicity. The construction of a generalized deformed oscillator corresponding to any anharmonic oscillator has also been achieved.

The results obtained in this section indicate that deformed bosons might be a convenient tool for describing systems of fermion pairs under certain conditions. The generalisation of the results obtained

here to fermion pairs of nonzero angular momentum, which will allow for a fuller treatment of the single-j shell in terms of deformed bosons, is a very interesting problem.

35. The seniority scheme

In the previous two sections we have seen how correlated fermion pairs of zero angular momentum can be mapped onto deformed bosons. It is however known that pairs of non-zero angular momentum play an important role in the formation of nuclear properties. In the present section a first step in the direction of describing the $J \neq 0$ pairs in terms of deformed bosons is taken.

35.1. Uncovering a dynamical symmetry

In the usual formulation of the theory of pairing in a single-j shell [330], fermion pairs of angular momentum $J = 0$ are created by the pair creation operators

$$S^\dagger = \frac{1}{\sqrt{\Omega}} \sum_{m>0} (-1)^{j+m} a^\dagger_{jm} a^\dagger_{j-m}, \tag{35.1}$$

where $a^\dagger_{jm}$ are fermion creation operators and $2\Omega = 2j + 1$ is the degeneracy of the shell. In addition, pairs of nonzero angular momentum are created by the $\Omega - 1$ operators

$$B^\dagger_J = \sum_{m>0} (-1)^{j+m} (jmj - m|J0) a^\dagger_{jm} a^\dagger_{j-m}, \tag{35.2}$$

where $(jmj - m|J0)$ are the usual Clebsch Gordan coefficients. The fermion number operator is defined as

$$N_F = \sum_m a^\dagger_{jm} a_{jm} = \sum_{m>0} (a^\dagger_{jm} a_{jm} + a^\dagger_{j-m} a_{j-m}). \tag{35.3}$$

As we have already seen, the $J = 0$ pair creation and annihilation operators satisfy the commutation relation

$$[S, S^\dagger] = 1 - \frac{N_F}{\Omega}, \tag{35.4}$$

while the pairing Hamiltonian is

$$H = -G\Omega S^\dagger S. \tag{35.5}$$

The seniority V_F is defined as the number of fermions not coupled to $J = 0$. If only pairs of $J = 0$ are present (i.e. $V_F = 0$), the eigenvalues of the Hamiltonian are (as already seen in eq. (33.16))

$$E(N_F, V_F = 0) = -G\Omega \left(\frac{N_F}{2} + \frac{N_F}{2\Omega} - \frac{N_F^2}{4\Omega} \right). \tag{35.6}$$

For non-zero seniority the eigenvalues of the Hamiltonian are

$$E(N_F, V_F) = -\frac{G}{4}(N_F - V_F)(2\Omega - N_F - V_F + 2). \tag{35.7}$$

We denote the operators N_F, V_F and their eigenvalues by the same symbol for simplicity.

In subsec. 33.4 it has been proved that the behaviour of the $J = 0$ pairs can be described, up to first order corrections, in terms of Q-bosons. In particular, making the mapping

$$S^\dagger \to b^\dagger, \qquad S \to b, \qquad N_F \to 2N, \tag{35.8}$$

the relevant pairing Hamiltonian of eq. (35.5) becomes

$$H(N, V = 0) = -G\Omega b^\dagger b = -G\Omega [N]_Q, \tag{35.9}$$

which coincides with eq. (35.6) up to first order corrections in the small parameter, which is identified as $T = -2/\Omega$. Furthermore, the Q-bosons satisfy the commutation relation of eq. (33.24), which coincides

with eq. (35.4) up to first order corrections in the small parameter, which is, consistently with the above finding, identified as $T = -2/\Omega$. Therefore the fermion pairs of $J = 0$ can be approximately described as Q-bosons, which correctly reproduce both the pairing energies and the commutation relations up to first order corrections in the small parameter.

For the case of nonzero seniority, one observes that eq. (35.7) can be written as

$$E(N_F, V_F) = G\Omega \left(\frac{V_F}{2} + \frac{V_F}{2\Omega} - \frac{V_F^2}{4\Omega} \right) - G\Omega \left(\frac{N_F}{2} + \frac{N_F}{2\Omega} - \frac{N_F^2}{4\Omega} \right), \tag{35.10}$$

i.e. it can be separated into two parts, formally identical to each other. Since the second part (which corresponds to the $J = 0$ pairs) can be adequately described by the Q-bosons b, $b^\dagger$, and their number operator N, as we have already seen, it is reasonable to assume that the first part can also be described in terms of some Q-bosons d, $d^\dagger$, and their number operator V (with $V_F \to 2V$), satisfying commutation relations similar to eqs (11.2) and (11.3):

$$[V, d^\dagger] = d^\dagger, \qquad [V, d] = -d, \qquad dd^\dagger - Qd^\dagger d = 1. \tag{35.11}$$

From the physical point of view this description means that a set of Q-bosons is used for the $J = 0$ pairs and another set for the $J \neq 0$ pairs. The latter is reasonable, since in the context of this theory the angular momentum value of the $J \neq 0$ pairs is not used explicitly. The $J \neq 0$ pairs are just counted separately from the $J = 0$ pairs. A Hamiltonian giving the same spectrum as in eq. (35.10), up to first order corrections in the small parameter, can then be written as

$$H(N, V) = G\Omega([V]_Q - [N]_Q). \tag{35.12}$$

Using eq. (33.27) it is easy to see that this expression agrees to eq. (35.10) up to first order corrections in the small parameter $T = -2/\Omega$.

Two comments concerning eq. (35.12) are in place:

i) In the classical theory states of maximum seniority (i.e. states with $N = V$) have zero energy. This is also holding for the Hamiltonian of eq. (35.12) to all orders in the deformation parameter.

ii) A landmark of the classical theory is that $E(N, V) - E(N, V = 0)$ is independent of N. This also holds for eq. (35.12) to all orders in the deformation parameter.

Knowing the Schwinger realization of the $\mathrm{su}_q(2)$ algebra in terms of q-bosons (sec. 15), one may wonder if the operators used here close an algebra. It is easy to see that the operators b^+d, d^+b and $N - V$ do not close an algebra. Considering, however, the operators (see [53] with $p = 1$)

$$J_+ = b^\dagger Q^{-V/2} d, \qquad J_- = d^\dagger Q^{-V/2} b, \qquad J_0 = \frac{1}{2}(N - V), \tag{35.13}$$

one can easily see that they satisfy the commutation relations [53,47]

$$[J_0, J_\pm] = \pm J_\pm, \qquad J_+ J_- - Q^{-1} J_- J_+ = [2J_0]_Q. \tag{35.14}$$

Using the transformation

$$J_0 = \tilde{J}_0, \qquad J_+ = Q^{(1/2)(J_0 - 1/2)} \tilde{J}_+, \qquad J_- = \tilde{J}_- Q^{(1/2)(J_0 - 1/2)}, \tag{35.15}$$

one goes to the usual $\mathrm{su}_q(2)$ commutation relations

$$[\tilde{J}_0, \tilde{J}_\pm] = \pm \tilde{J}_\pm, \qquad [\tilde{J}_+, \tilde{J}_-] = [2\tilde{J}_0], \tag{35.16}$$

where $q^2 = Q$. One can thus consider eq. (35.14) as a rewriting of the algebra $\mathrm{su}_q(2)$, suitable for boson realization in terms of Q-bosons.

It is clear that $N + V$ is the first order Casimir operator of the $\mathrm{u}_Q(2)$ algebra formed above (since it commutes with all the generators given in eq. (35.13)), while $N - V$ is the first order Casimir operator of its $\mathrm{u}_Q(1)$ subalgebra, which is generated by J_0 alone. Therefore the Hamiltonian of eq. (35.12) can be expressed in terms of the Casimir operators of the algebras appearing in the chain $\mathrm{u}_Q(2) \supset \mathrm{u}_Q(1)$ as

$$E(N, V) = G\Omega\left(\left[\frac{C_1(\mathrm{u}_Q(2)) - C_1(\mathrm{u}_Q(1))}{2} \right]_Q - \left[\frac{C_1(\mathrm{u}_Q(2)) + C_1(\mathrm{u}_Q(1))}{2} \right]_Q \right), \tag{35.17}$$

i.e. the Hamiltonian has a $u_Q(2) \supset u_Q(1)$ dynamical symmetry.

35.2. Comparison to experiment

In the construction given above we have shown that Q-bosons can be used for the approximate description of correlated fermion pairs in a single-j shell. The results obtained in the Q-formalism agree to the classical (non-deformed) results up to first order corrections in the small parameter. However, the Q-formalism contains in addition higher order terms. The question is then born if these additional terms are useful or not. For answering this question, the simplest comparison with experimental data which can be made concerns the classic example of the neutron pair separation energies of the Sn isotopes, used by Talmi [331,332].

In Talmi's formulation of the pairing theory, the energy of the states with zero seniority is given by

$$E(N)_{cl} = NV_0 + \frac{N(N-1)}{2}\Delta, \tag{35.18}$$

where N is the number of fermion pairs and V_0, Δ are constants. We remark that this expression is the same as the one in eq. (35.6), with the identifications

$$\Delta/(2V_0) = -1/\Omega, \qquad \Delta = 2G, \qquad N_F = 2N. \tag{35.19}$$

The neutron pair separation energies are given by

$$\Delta E(N+1)_{cl} = E(N+1)_{cl} - E(N)_{cl} = V_0\left(1 + \frac{\Delta}{V_0}N\right). \tag{35.20}$$

Thus the neutron pair separation energies are expected to decrease linearly with increasing N. (Notice from eq. (35.19) that $\Delta/V_0 < 0$, since $\Omega > 0$.) A similar linear decrease is predicted also by the Interacting Boson Model [169].

In our formalism the neutron pair separation energies are given by

$$\Delta E(N+1)_Q = -G\Omega([N+1]_Q - [N]_Q) = -G\Omega Q^N = -G\Omega e^{TN}. \tag{35.21}$$

Since, as we have seen, T is expected to be $-2/\Omega$, i.e. negative and small, the neutron pair separation energies are expected to fall exponentially with increasing N, but the small value of T can bring this exponential fall very close to a linear one.

The neutron pair separation energies of the even Sn isotopes from ^{104}Sn to ^{130}Sn (i.e. across the whole sdg neutron shell) have been fitted in [333] using both theories. Furthermore, in [333] a fit of the logarithms of the energies has been performed, since eq. (35.21) predicts a linear decrease of the logarithm of the energies with increasing N. Both fits give almost identical results. Eq. (35.21) (in which the free parameters are $G\Omega$ and T), gives a better result than eq. (35.20) (in which the free parameters are V_0 and Δ/V_0) for every single isotope, without introducing any additional parameter, indicating that the higher order terms can be useful.

One should, however, remark that ^{116}Sn lies in the middle of the sdg neutron shell. Fitting the isotopes in the lower half of the shell (^{104}Sn to ^{116}Sn) and the isotopes in the upper half of the shell (^{118}Sn to ^{130}Sn) separately, one finds that both theories give indistinguishably good results in both regions. Therefore Q-deformation can be understood as expressing higher order correlations which manifest themselves in the form of particle-hole asymmetry. It is also known that a strong subshell closure exists at N=64 (which corresponds to ^{114}Sn). The presence of this subshell closure can also affect the neutron pair separation energies in a way similar to the one shown by the data.

In [333] a fit of the neutron pair separation energies of the Pb isotopes from ^{186}Pb to ^{202}Pb has also been attempted. In this case both theories give indistinguishably good fits. This result is in agreement with the Sn findings, since all of these Pb isotopes lie in the upper half of the pfh neutron shell. Unfortunately, no neutron pair separation energy data exist for Pb isotopes in the lower part of the pfh neutron shell.

Concerning the values of T obtained in the case of the Sn isotopes ($T = -0.0454$, $T = -0.0447$), one observes that they are slightly smaller than the value ($T = -0.0488$) which would have been obtained

by considering the neutrons up to the end of the sdg shell as lying in a single-j shell. This is, of course, a very gross approximation which should not be taken too seriously, since it ignores the fact that most properties of nuclei can be well accounted for by the valence nucleons alone, without being affected by the closed core. In the case of the Pb isotopes mentioned above, however, the best fit was obtained with $T = -0.0276$, which is again slightly smaller than the value of $T = -0.0317$ which corresponds to considering all the neutrons up to the end of the pfh shell as lying in a single-j shell.

In summary, we have shown that pairing in a single-j shell can be described, up to first order corrections, by two Q-oscillators, one describing the $J = 0$ pairs and the other corresponding to the $J \neq 0$ pairs, the deformation parameter $T = \ln Q$ being related to the inverse of the size of the shell. These two oscillators can be used for forming an $\mathrm{su}_Q(2)$ algebra. A Hamiltonian giving the correct pairing energies up to first order corrections in the small parameter can be written in terms of the Casimir operators of the algebras appearing in the $\mathrm{u}_Q(2) \supset \mathrm{u}_Q(1)$ chain, thus exhibiting a quantum algebraic dynamical symmetry. The additional terms introduced by the Q-oscillators serve in improving the description of the neutron pair separation energies of the Sn isotopes, with no extra parameter introduced.

In the previous section a generalized deformed oscillator describing the correlated fermion pairs of $J = 0$ *exactly* has been introduced. This generalized deformed oscillator is the same as the one giving the same spectrum as the Morse potential (sec. 37), up to a shift in the energy spectrum. The use of two generalized deformed oscillators for the description of $J = 0$ pairs and $J \neq 0$ pairs in a way similar to the one of the present section is a straightforward task, while the construction out of them of a closed algebra analogous to the $\mathrm{su}_Q(2)$ obtained here is an open problem. The extension of the ideas presented here to the case of the BCS theory is an interesting open problem.

35.3. Other approaches

A q-deformed version of the pairing theory was *assumed* by Petrova [334] and Shelly Sharma [335], with satisfactory results when compared to experimental data. The present construction offers some justification for this assumption, since in both cases the basic ingredient is the modification of eq. (35.4). It should be noticed, however, that the deformed version of eq. (35.4) considered in [334,335] is different from the one obtained here (eq. (33.24)). A basic difference is that the deformed theory of [334,335] reduces to the classical theory for $q \to 1$, so that q-deformation is introduced in order to describe additional correlations, while in the present formalism the Q-oscillators involved for $Q \to 1$ reduce to usual harmonic oscillators, so that Q-deformation is introduced in order to attach to the oscillators the anharmonicity needed by the energy expression (eq. (35.6)).

Continuing along the same line Shelly Sharma and Sharma [336] derived Random Phase Approximation (RPA) equations for the pairing vibrations of nuclei differing by two nucleons in comparison to the initial one and applied their method to the study of the 0^+ states of the Pb isotopes, which offer a good example of pairing vibrations in nonsuperconducting nuclei. Furthermore, using deformed quasiparticle pairs coupled to zero angular momentum they developed a deformed version of the quasi-boson approximation for 0^+ states in superconducting nuclei and tested it against a schematic two-level shell model [336]. Another deformed two-level shell model has been developed by Avancini and Menezes [337]. q-deformed boson mappings have been developed in [338,339].

In addition a q-deformed version of the many-body BCS approximation for a pure pairing force has been developed [340], using q-deformed fermions satisfying $\mathrm{su}_q(\mathrm{n})$ anticommutation relations. A set of quantum BCS equations and a q-analog of the gap equation have been derived [340].

36. Anisotropic quantum harmonic oscillators with rational ratios of frequencies

3-dim anisotropic harmonic oscillators [341] with rational ratios of frequencies (RHOs) are of current interest because of their relevance as possible underlying symmetries of superdeformed and hyperdeformed nuclei [342,343]. In particular, it is thought [189,190] that superdeformed nuclei correspond to a ratio of frequencies of 2:1, while hyperdeformed nuclei correspond to a 3:1 ratio. In addition they have been recently connected [344,345] to the underlying geometrical structure in the Bloch–Brink α-cluster model [346], and possibly to the interpretation of the observed shell structure in atomic clusters (see sec. 38),

especially after the realization that large deformations can occur in such systems [347]. The 2-dim RHO is also of interest, since its single particle level spectrum characterizes the underlying symmetry of "pancake" nuclei [343].

RHOs are examples of maximally superintegrable systems [348,349] in N dimensions. *Superintegrable systems* in N dimensions have more than N independent integrals (constants of motion). *Maximally superintegrable systems* in N dimensions have 2N−1 independent integrals.

The two-dim [350,351,352,353,354,355,356] and three-dim [357,358,359,360,361,362,363] anisotropic harmonic oscillators have been the subject of several investigations, both at the classical and the quantum mechanical level. The special cases with frequency ratios 1:2 [143,364] and 1:3 [141] have also been considered. While at the classical level it is clear that the su(N) or sp(2N,R) algebras can be used for the description of the N-dimensional anisotropic oscillator, the situation at the quantum level, even in the two-dimensional case, is not as simple.

In this section we are going to prove that a generalized deformed u(2) algebra is the symmetry algebra of the two-dimensional anisotropic quantum harmonic oscillator, which is the oscillator describing the single-particle level spectrum of "pancake" nuclei, i.e. of triaxially deformed nuclei with $\omega_x >> \omega_y$, ω_z [343]. The method can be extended to the 3-dim RHO in a rather straightforward way.

36.1. A deformed u(2) algebra

Let us consider the system described by the Hamiltonian:

$$H = \frac{1}{2}\left({p_x}^2 + {p_y}^2 + \frac{x^2}{m^2} + \frac{y^2}{n^2}\right), \tag{36.1}$$

where m and n are two natural numbers mutually prime ones, i.e. their great common divisor is $\gcd(m,n) = 1$.

We define the creation and annihilation operators [350]

$$a^\dagger = \frac{x/m - ip_x}{\sqrt{2}}, \qquad a = \frac{x/m + ip_x}{\sqrt{2}}, \tag{36.2}$$

$$b^\dagger = \frac{y/n - ip_y}{\sqrt{2}}, \qquad b = \frac{y/n + ip_y}{\sqrt{2}}. \tag{36.3}$$

These operators satisfy the commutation relations:

$$\left[a, a^\dagger\right] = \frac{1}{m}, \quad \left[b, b^\dagger\right] = \frac{1}{n}, \quad \text{other commutators} = 0. \tag{36.4}$$

Further defining

$$U = \frac{1}{2}\left\{a, a^\dagger\right\}, \qquad W = \frac{1}{2}\left\{b, b^\dagger\right\}, \tag{36.5}$$

one can consider the enveloping algebra generated by the operators:

$$S_+ = \left(a^\dagger\right)^m (b)^n, \qquad S_- = (a)^m \left(b^\dagger\right)^n, \tag{36.6}$$

$$S_0 = \frac{1}{2}(U - W), \qquad H = U + W. \tag{36.7}$$

These genarators satisfy the following relations:

$$[S_0, S_\pm] = \pm S_\pm, \quad [H, S_i] = 0, \quad \text{for} \quad i = 0, \pm, \tag{36.8}$$

and

$$S_+ S_- = \prod_{k=1}^{m}\left(U - \frac{2k-1}{2m}\right)\prod_{\ell=1}^{n}\left(W + \frac{2\ell-1}{2n}\right), \tag{36.9}$$

$$S_- S_+ = \prod_{k=1}^{m}\left(U + \frac{2k-1}{2m}\right)\prod_{\ell=1}^{n}\left(W - \frac{2\ell-1}{2n}\right). \tag{36.10}$$

The fact that the operators S_i, $i = 0, \pm$ are integrals of motion has been already realized in [350].

The above relations mean that the harmonic oscillator of eq. (36.1) is described by the enveloping algebra of the generalization of the u(2) algebra formed by the generators S_0, S_+, S_- and H, satisfying the commutation relations of eq. (36.8) and

$$[S_-, S_+] = F_{m,n}(H, S_0 + 1) - F_{m,n}(H, S_0), \tag{36.11}$$

where

$$F_{m,n}(H, S_0) = \prod_{k=1}^{m} \left(H/2 + S_0 - \frac{2k-1}{2m} \right) \prod_{\ell=1}^{n} \left(H/2 - S_0 + \frac{2\ell - 1}{2n} \right). \tag{36.12}$$

In the case of $m = 1$, $n = 1$ this algebra is the usual u(2) algebra, and the operators $S_0, S_\pm$ satisfy the commutation relations of the ordinary u(2) algebra, since in this case one easily finds that

$$[S_-, S_+] = -2S_0. \tag{36.13}$$

In the rest of the cases, the algebra is a deformed version of u(2), in which the commutator $[S_-, S_+]$ is a polynomial of S_0 of order $m + n - 1$. In the case with $m = 1$, $n = 2$ one has

$$[S_-, S_+] = 3S_0^2 - HS_0 - \frac{H^2}{4} + \frac{3}{16}, \tag{36.14}$$

i.e. a polynomial quadratic in S_0 occurs, while in the case of $m = 1$, $n = 3$ one finds

$$[S_-, S_+] = -4S_0^3 + 3HS_0^2 - \frac{7}{9}S_0 - \frac{H^3}{4} + \frac{H}{4}, \tag{36.15}$$

i.e. a polynomial cubic in S_0 is obtained.

36.2. The representations

The finite dimensional representation modules of this algebra can be found using the concept of the generalized deformed oscillator (sec. 12), in a method similar to the one used in [365,366] for the study of quantum superintegrable systems. The operators:

$$\mathcal{A}^\dagger = S_+, \quad \mathcal{A} = S_-, \quad \mathcal{N} = S_0 - u, \quad u = \text{ constant}, \tag{36.16}$$

where u is a constant to be determined, are the generators of a deformed oscillator algebra:

$$\left[\mathcal{N}, \mathcal{A}^\dagger\right] = \mathcal{A}^\dagger, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \quad \mathcal{A}^\dagger\mathcal{A} = \Phi(H, \mathcal{N}), \quad \mathcal{A}\mathcal{A}^\dagger = \Phi(H, \mathcal{N} + 1). \tag{36.17}$$

The structure function Φ of this algebra is determined by the function $F_{m,n}$ in eq. (36.12):

$$\Phi(H, \mathcal{N}) = F_{m,n}(H, \mathcal{N} + u) = \prod_{k=1}^{m} \left(H/2 + \mathcal{N} + u - \frac{2k-1}{2m} \right) \prod_{\ell=1}^{n} \left(H/2 - \mathcal{N} - u + \frac{2\ell - 1}{2n} \right). \tag{36.18}$$

The deformed oscillator corresponding to the structure function of eq. (36.18) has an energy dependent Fock space of dimension $N + 1$ if

$$\Phi(E, 0) = 0, \quad \Phi(E, N + 1) = 0, \quad \Phi(E, k) > 0, \quad \text{for} \quad k = 1, 2, \ldots, N. \tag{36.19}$$

The Fock space is defined by:

$$H|E, k> = E|E, k>, \quad \mathcal{N}|E, k> = k|E, k>, \quad a|E, 0> = 0, \tag{36.20}$$

$$\mathcal{A}^\dagger|E, k> = \sqrt{\Phi(E, k+1)}|E, k+1>, \quad \mathcal{A}|E, k> = \sqrt{\Phi(E, k)}|E, k-1> . \tag{36.21}$$

The basis of the Fock space is given by:

$$|E, k> = \frac{1}{\sqrt{[k]!}} \left(\mathcal{A}^\dagger\right)^k |E, 0>, \quad k = 0, 1, \ldots N, \tag{36.22}$$

where the "factorial" $[k]!$ is defined by the recurrence relation:

$$[0]! = 1, \quad [k]! = \Phi(E,k)[k-1]! \quad . \tag{36.23}$$

Using the Fock basis we can find the matrix representation of the deformed oscillator and then the matrix representation of the algebra of eqs (36.8), (36.12). The solution of eqs (36.19) implies the following pairs of permitted values for the energy eigenvalue E and the constant u:

$$E = N + \frac{2p-1}{2m} + \frac{2q-1}{2n}, \tag{36.24}$$

where $p = 1, 2, \ldots, m, \qquad q = 1, 2, \ldots, n$, and

$$u = \frac{1}{2}\left(\frac{2p-1}{2m} - \frac{2q-1}{2n} - N\right), \tag{36.25}$$

the corresponding structure function being given by:

$$\Phi(E,x) = \Phi^N_{(p,q)}(x) = \prod_{k=1}^{m}\left(x + \frac{2p-1}{2m} - \frac{2k-1}{2m}\right)\prod_{\ell=1}^{n}\left(N - x + \frac{2q-1}{2n} + \frac{2\ell-1}{2n}\right)$$

$$= \frac{1}{m^m n^n}\frac{\Gamma(mx+p)}{\Gamma(mx+p-m)}\frac{\Gamma((N-x)n+q+n)}{\Gamma((N-x)n+q)}, \tag{36.26}$$

where $\Gamma(x)$ denotes the usual Gamma-function. In all these equations one has $N = 0, 1, 2, \ldots$, while the dimensionality of the representation is given by $N+1$. Eq. (36.24) means that there are $m \cdot n$ energy eigenvalues corresponding to each N value, each eigenvalue having degeneracy $N+1$. (Later we shall see that the degenerate states corresponding to the same eigenvalue can be labelled by an "angular momentum".)

It is useful to show at this point that a few special cases are in agreement with results already existing in the literature.

i) In the case $m = 1$, $n = 1$ eq. (36.26) gives

$$\Phi(E,x) = x(N+1-x), \tag{36.27}$$

while eq. (36.24) gives

$$E = N + 1, \tag{36.28}$$

in agreement with Sec. IV.A of [366].

ii) In the case $m = 1$, $n = 2$ one obtains for $q = 2$

$$\Phi(E,x) = x(N+1-x)\left(N + \frac{3}{2} - x\right), \qquad E = N + \frac{5}{4}, \tag{36.29}$$

while for $q = 1$ one has

$$\Phi(E,x) = x(N+1-x)\left(N + \frac{1}{2} - x\right), \qquad E = N + \frac{3}{4}. \tag{36.30}$$

These are in agreement with the results obtained in Sec. IV.F of [366] for the Holt potential (for $\delta = 0$).

iii) In the case $m = 1$, $n = 3$ one has for $q = 1$

$$\Phi(E,x) = x(N+1-x)\left(N + \frac{1}{3} - x\right)\left(N + \frac{2}{3} - x\right), \qquad E = N + \frac{2}{3}, \tag{36.31}$$

while for $q = 2$ one obtains

$$\Phi(E,x) = x(N+1-x)\left(N + \frac{2}{3} - x\right)\left(N + \frac{4}{3} - x\right), \qquad E = N + 1, \tag{36.32}$$

and for $q = 3$ one gets

$$\Phi(E,x) = x(N+1-x)\left(N+\frac{4}{3}-x\right)\left(N+\frac{5}{3}-x\right), \qquad E = N+\frac{4}{3}. \tag{36.33}$$

These are in agreement with the results obtained in Sec. IV.D of [366] for the Fokas–Lagerstrom potential.

In all of the above cases we remark that the structure function has forms corresponding to various versions of the generalized deformed parafermionic algebra of eq. (18.1), the relevant conditions of eq. (18.2) being satisfied in all cases. It is easy to see that the obtained algebra corresponds to this of the generalized parafermionic oscillator in all cases with frequency ratios $1:n$.

The energy formula can be corroborated by using the corresponding Schrödinger equation. For the Hamiltonian of eq. (36.1) the eigenvalues of the Schrödinger equation are given by:

$$E = \frac{1}{m}\left(n_x+\frac{1}{2}\right)+\frac{1}{n}\left(n_y+\frac{1}{2}\right), \tag{36.34}$$

where $n_x = 0,1,\ldots$ and $n_y = 0,1,\ldots$. Comparing eqs (36.24) and (36.34) one concludes that:

$$N = [n_x/m]+[n_y/n], \tag{36.35}$$

where $[x]$ is the integer part of the number x, and

$$p = \mathrm{mod}(n_x,m)+1, \quad q = \mathrm{mod}(n_y,n)+1. \tag{36.36}$$

The eigenvectors of the Hamiltonian can be parametrized by the dimensionality of the representation N, the numbers p,q, and the number $k = 0,1,\ldots,N$. k can be identified as $[n_x/m]$. One then has:

$$H\left|\begin{array}{c} N \\ (p,q) \end{array},k\right\rangle = \left(N+\frac{2p-1}{2m}+\frac{2q-1}{2n}\right)\left|\begin{array}{c} N \\ (p,q) \end{array},k\right\rangle, \tag{36.37}$$

$$S_0\left|\begin{array}{c} N \\ (p,q) \end{array},k\right\rangle = \left(k+\frac{1}{2}\left(\frac{2p-1}{2m}-\frac{2q-1}{2n}-N\right)\right)\left|\begin{array}{c} N \\ (p,q) \end{array},k\right\rangle, \tag{36.38}$$

$$S_+\left|\begin{array}{c} N \\ (p,q) \end{array},k\right\rangle = \sqrt{\Phi^N_{(p,q)}(k+1)}\left|\begin{array}{c} N \\ (p,q) \end{array},k+1\right\rangle, \tag{36.39}$$

$$S_-\left|\begin{array}{c} N \\ (p,q) \end{array},k\right\rangle = \sqrt{\Phi^N_{(p,q)}(k)}\left|\begin{array}{c} N \\ (p,q) \end{array},k-1\right\rangle. \tag{36.40}$$

36.3. The "angular momentum" quantum number

It is worth noticing that the operators $S_0, S_\pm$ do not correspond to a generalization of the angular momentum, S_0 being the operator corresponding to the Fradkin operator $S_{xx}-S_{yy}$ [128,129]. The corresponding "angular momentum" is defined by:

$$L_0 = -i\,(S_+ - S_-). \tag{36.41}$$

The "angular momentum" operator commutes with the Hamiltonian:

$$[H, L_0] = 0. \tag{36.42}$$

Let $|\ell>$ be the eigenvector of the operator L_0 corresponding to the eigenvalue ℓ. The general form of this eigenvector can be given by:

$$|\ell> = \sum_{k=0}^{N}\frac{i^k c_k}{\sqrt{[k]!}}\left|\begin{array}{c} N \\ (p,q) \end{array},k\right\rangle. \tag{36.43}$$

In order to find the eigenvalues of L_0 and the coefficients c_k we use the Lanczos algorithm [367], as formulated in [368]. From eqs (36.39) and (36.40) we find

$$L_0|\ell>=\ell|\ell>=\ell\sum_{k=0}^{N}\frac{i^k c_k}{\sqrt{[k]!}}\left|\begin{array}{c}N\\(p,q)\end{array},k\right\rangle=$$

$$=\frac{1}{i}\sum_{k=0}^{N-1}\frac{i^k c_k\sqrt{\Phi^N_{(p,q)}(k+1)}}{\sqrt{[k]!}}\left|\begin{array}{c}N\\(p,q)\end{array},k+1\right\rangle-\frac{1}{i}\sum_{k=1}^{N}\frac{i^k c_k\sqrt{\Phi^N_{(p,q)}(k)}}{\sqrt{[k]!}}\left|\begin{array}{c}N\\(p,q)\end{array},k-1\right\rangle \tag{36.44}$$

From this equation we find that:

$$c_k=(-1)^k 2^{-k/2}H_k(\ell/\sqrt{2})/\mathcal{N},\quad \mathcal{N}^2=\sum_{n=0}^{N}2^{-n}H_n^2(\ell/\sqrt{2})/n! \tag{36.45}$$

where the function $H_k(x)$ is a generalization of the "Hermite" polynomials (see also [82,369]), satisfying the recurrence relations:

$$H_{-1}(x)=0,\quad H_0(x)=1, \tag{36.46}$$

$$H_{k+1}(x)=2xH_k(x)-2\Phi^N_{(p,q)}(k)H_{k-1}(x), \tag{36.47}$$

and the "angular momentum" eigenvalues ℓ are the roots of the polynomial equation:

$$H_{N+1}(\ell/\sqrt{2})=0. \tag{36.48}$$

Therefore for a given value of N there are $N+1$ "angular momentum" eigenvalues ℓ, symmetric around zero (i.e. if ℓ is an "angular momentum" eigenvalue, then $-\ell$ is also an "angular momentum" eigenvalue). In the case of the symmetric harmonic oscillator ($m/n=1/1$) these eigenvalues are uniformly distributed and differ by 2. In the general case the "angular momentum" eigenvalues are non-uniformly distributed. For small values of N analytical formulae for the "angular momentum" eigenvalues can be found [82]. Remember that to each value of N correspond $m\cdot n$ energy levels, each with degeneracy $N+1$.

In order to have a formalism corresponding to the one of the isotropic oscillator, let us introduce for every N and (m,n,p,q) an ordering of the "angular momentum" eigenvalues

$$\ell_\mu^{L,m,n,p,q},\quad \text{where}\quad L=N\quad\text{and}\quad \mu=-L,-L+2,\ldots,L-2,L, \tag{36.49}$$

by assuming that:

$$\ell_\mu^{L,m,n,p,q}\le\ell_\nu^{L,m,n,p,q}\quad\text{if}\quad \mu<\nu, \tag{36.50}$$

the corresponding eigenstates being given by:

$$|L,\mu;m,n,p,q\rangle=\sum_{k=0}^{L}\frac{(-i)^k H_k(\ell_\mu^{L,m,n,p,q}/\sqrt{2})}{\mathcal{N}\sqrt{2^{k/2}[k]!}}\left|\begin{array}{c}N\\(p,q)\end{array},k\right\rangle=\sum_{k=0}^{L}d_{k+1}\left|\begin{array}{c}N\\(p,q)\end{array},k\right\rangle \tag{36.51}$$

The above vector elements constitute the analogue corresponding to the basis of "spherical harmonic" functions of the usual oscillator. The calculation of the "angular momentum" eigenvalues of eq. (36.49) and the coefficients $d_1,d_2,\ldots,d_{L+1}$ in the expansion of eq. (36.51) is a quite difficult task. The existence of general analytic expressions for these quantities is not obvious. The first few "angular momentum" eigenvalues are given by:

$$\ell_{\pm1}^{1,m,n,p,q}=\pm\sqrt{\frac{1}{m^m n^n}\frac{\Gamma(m+p)}{\Gamma(p)}\frac{\Gamma(n+q)}{\Gamma(q)}}, \tag{36.52}$$

and

$$\ell_0^{2,m,n,p,q}=0, \tag{36.53}$$

$$\ell_{\pm2}^{2,m,n,p,q}=\pm\sqrt{\frac{1}{m^m n^n}\left(\frac{\Gamma(m+p)}{\Gamma(p)}\frac{\Gamma(2n+q)}{\Gamma(n+q)}+\frac{\Gamma(2m+p)}{\Gamma(m+p)}\frac{\Gamma(n+q)}{\Gamma(q)}\right)} \tag{36.54}$$

For $L > 2$ the analytic expressions of the angular momentum eigenvalues and the coefficients d_k are longer, but their calculation is a straightforward task. Numerical results for these quantities in the cases of frequency ratios 1:2 and 1:3 are given in [370,371].

After working out a few examples (see [370,371,372] for details) one finds out the following points:

i) In the basis described by eqs. (36.16)-(36.19) it is a trivial matter to distinguish the states belonging to the same irrep for any $m : n$ ratio, while in the Cartesian basis this is true only in the 1:1 case.

ii) In the 1:2 case the irreps have degeneracies 1, 1, 2, 2, 3, 3, 4, 4, ..., i.e. "two copies" of the u(2) degeneracies 1, 2, 3, 4, ... are obtained.

iii) In the 1:3 cases the degeneracies are 1, 1, 1, 2, 2, 2, 3, 3, 3, ..., i.e. "three copies" of the u(2) degeneracies are obtained.

iv) It can be easily seen that the 1:n case corresponds to "n copies" of the u(2) degeneracies.

v) Cases with both m, n different from unity show more complicated degeneracy patterns, also correctly reproduced by the above formalism. In the 2:3 case, for example, the degeneracy pattern is 1, 1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 2, 3, 3, ...

vi) The only requirement for each energy eigenvalue to correspond to one irrep of the algebra is that m and n have to be mutually prime numbers. If m and n possess a common divisor other than 1, then some energy eigenvalues will correspond to sums of irreps, i.e. to reducible representations.

vii) The difference between the formalism used in [357,359,360,363] and the one used here is that in the former case for given m and n appropriate operators have to be introduced separately for each set of (p, q) values, while in the present case only one set of operators is introduced.

36.4. Multisections of the isotropic oscillator

In [373] the concept of bisection of an isotropic harmonic oscillator has been introduced. One can easily see that multisections (trisections, tetrasections, ...) can be introduced in a similar way [374]. The degeneracies of the various anisotropic oscillators can then be obtained from these of the isotropic oscillator by using appropriate multisections.

Using the Cartesian notation (n_x, n_y) for the states of the isotropic harmonic oscillator we have the following list:

N=0: (00)
N=1: (10) (01)
N=2: (20) (02) (11)
N=3: (30) (03) (21) (12)
N=4: (40) (04) (31) (13) (22)
N=5: (50) (05) (41) (14) (32) (23),

where $N = n_x + n_y$. The corresponding degeneracies are 1, 2, 3, 4, 5, 6, ..., i.e. these of u(2).

A bisection can be made by choosing only the states with n_y=even. Then the following list is obtained:

N=0: (00)
N=1: (10)
N=2: (20) (02)
N=3: (30) (12)
N=4: (40) (04) (22)
N=5: (50) (14) (32).

The degeneracies are 1, 1, 2, 2, 3, 3, ..., i.e. these of the anisotropic oscillator with ratio of frequencies 1:2. The same degeneracies are obtained by choosing the states with n_y=odd. Therefore a **bisection** of the isotropic oscillator, distinguishing states with $\mathrm{mod}(n_y, 2) = 0$ and states with $\mathrm{mod}(n_y, 2) = 1$, results in two interleaving sets of levels of the 1:2 oscillator.

By analogy, a **trisection** can be made by distinguishing states with $\mathrm{mod}(n_y, 3) = 0$, or $\mathrm{mod}(n_y, 3) = 1$, or $\mathrm{mod}(n_y, 3) = 2$. One can easily see that in this case three interleaving sets of states of the 1:3 oscillator, having degeneracies 1, 1, 1, 2, 2, 2, 3, 3, 3, ..., occur.

Similarly a **tetrasection** can be made by distinguishing states with $\mathrm{mod}(n_y, 4) = 0$, or $\mathrm{mod}(n_y, 4) = 1$, or $\mathrm{mod}(n_y, 4) = 2$, or $\mathrm{mod}(n_y, 4) = 3$. The result is four interleaving sets of states of the 1:4 oscillator,

having degeneracies 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, ...

By bisecting n_x and trisecting n_y one is left with six interleaving sets of states with degeneracies 1, 1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 2, 3, 3, ..., i.e. degeneracies of the 2:3 oscillator.

By bisecting (or trisecting, tetresecting, etc) both n_x and n_y one is obtaining the original u(2) degeneracies of the isotropic oscillator.

It is therefore clear that the degeneracies of all $m:n$ oscillators can be obtained from these of the isotropic oscillator by appropriate multisections. In particular:

i) The degeneracies of the $1:n$ oscillator can be obtained from these of the 1:1 (isotropic) oscillator by n-secting n_y or n_x.

ii) The degeneracies of the $m:n$ oscillator can be obtained from these of the 1:1 oscillator by m-secting n_x and n-secting n_y.

36.5. Connection to $W_3^{(2)}$

For the special case $m=1$, $n=2$ it should be noticed that the deformed algebra received here coincides with the finite W algebra $W_3^{(2)}$ [375,376,377,378]. The commutation relations of the $W_3^{(2)}$ algebra are

$$[H_W, E_W] = 2E_W, \qquad [H_W, F_W] = -2F_W, \qquad [E_W, F_W] = H_W^2 + C_W, \tag{36.55}$$

$$[C_W, E_W] = [C_W, F_W] = [C_W, H_W] = 0, \tag{36.56}$$

while in the $m=1$, $n=2$ case one has the relations

$$[\mathcal{N}, \mathcal{A}^\dagger] = \mathcal{A}^\dagger, \qquad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \qquad [\mathcal{A}, \mathcal{A}^\dagger] = 3S_0^2 - \frac{H^2}{4} - HS_0 + \frac{3}{16}, \tag{36.57}$$

$$[H, \mathcal{A}^\dagger] = [H, \mathcal{A}] = [H, S_0] = 0, \tag{36.58}$$

with $S_0 = \mathcal{N} + u$ (where u a constant). It is easy to see that the two sets of commutation relations are equivalent by making the identifications

$$F_W = \sigma \mathcal{A}^\dagger, \qquad E_W = \rho \mathcal{A}, \qquad H_W = -2S_0 + kH, \qquad C_W = f(H), \tag{36.59}$$

with

$$\rho\sigma = \frac{4}{3}, \qquad k = \frac{1}{3}, \qquad f(H) = -\frac{4}{9}H^2 + \frac{1}{4}. \tag{36.60}$$

36.6. Discussion

In conclusion, the two-dimensional anisotropic quantum harmonic oscillator with rational ratio of frequencies equal to m/n, is described dynamically by a deformed version of the u(2) Lie algebra, the order of this algebra being $m+n-1$. The representation modules of this algebra can be generated by using the deformed oscillator algebra. The energy eigenvalues are calculated by the requirement of the existence of finite dimensional representation modules. An "angular momentum" operator useful for labelling degenerate states has also been constructed. The algebras obtained in the special cases with $1:n$ ratios are shown to correspond to generalized parafermionic oscillators. In the special case of $m:n=1:2$ the resulting algebra has been identified as the finite W algebra $W_3^{(2)}$. Finally, it is demonstrated how the degeneracies of the various $m:n$ oscillators can be obtained from these of the isotropic oscillator by appropriate multisections.

The extension of the present method to the three-dimensional anisotropic quantum harmonic oscillator is already receiving attention [379,380], since it is of clear interest in the study of the symmetries underlying the structure of superdeformed and hyperdeformed nuclei [342,343].

37. The use of quantum algebras in molecular structure

The techniques developed in this article can be applied in very similar ways in describing properties of diatomic and polytomic molecules. A brief list will be given here.

1) Rotational spectra of diatomic molecules have been described in terms of the $su_q(2)$ model [147,381,382,383,384,385]. As in the case of nuclei, q is a phase factor ($q = e^{i\tau}$). In molecules τ is of the order of 0.01. The use of the $su_q(2)$ symmetry leads to a partial summation of the Dunham expansion describing the rotational–vibrational spectra of diatomic molecules [381]. Molecular backbending (bandcrossing) has also been described in this framework [386]. Rotational spectra of symmetric top molecules have also been considered [387,388] in the framework of the $su_q(2)$ symmetry. Furthermore, two q-deformed rotators with slightly different parameter values have been used [205] for the description of $\Delta I = 1$ staggering effects in rotational bands of diatomic molecules (see also [206]). (For a discussion of $\Delta I = 2$ staggering effects in diatomic molecules see [201,204]. For a discussion of staggering effects in nuclei see sec. 22.)

2) Vibrational spectra of diatomic molecules have been described in terms of q-deformed anharmonic oscillators having the $su_q(1,1)$ [389] or the $u_q(2) \supset o_q(2)$ [390,391] symmetry, as well as in terms of generalized deformed oscillators similar to the ones used in sec. 26 [392,393,394]. These results, combined with 1), lead to the full summation of the Dunham expansion [389,390]. A two-parameter deformed anharmonic oscillator with $u_{qp}(2) \supset o_{qp}(2)$ symmetry has also been considered [395].

3) The physical content of the anharmonic oscillators mentioned in 2) has been clarified by constructing WKB equivalent potentials (WKB-EPs) [79,396] and classical equivalent potentials [68], similar to the ones of sec. 13. The results have been corroborated by the study of the relation between $su_q(1,1)$ and the anharmonic oscillator with x^4 anharminicities [397]. The WKB-EP corresponding to the $su_q(1,1)$ anharmonic oscillator has been connected to a class of Quasi-Exactly Soluble Potentials (QESPs) [398].

4) Generalized deformed oscillators giving the same spectrum as the Morse potential [399] and the modified Pöschl–Teller potential [400], as well as a deformed oscillator containing them as special cases [401,402] have also been constructed. In addition, q-deformed versions of the Morse potential have been given, either by using the $so_q(2,1)$ symmetry [403] or by solving a q-deformed Schrödinger equation for the usual Morse potential [404]. For the sake of completeness it should be mentioned that a deformed oscillator giving the same spectrum as the Coulomb potential has also been constructed [50].

5) A q-deformed version of the vibron model for diatomic molecules [405,406] has been constructed [288,407,408,409], in a way similar to that described in sec. 31.

6) For vibrational spectra of polyatomic molecules a model of n coupled generalized deformed oscillators has been built [410], containg the approach of Iachello and Oss [411,412] as a special case. Furthermore, a system of two Q-deformed oscillators coupled so that the total Hamiltonian has the $su_Q(2)$ symmetry (in the way described in [413]) has been proved to be appropriate for the description of vibrational spectra of triatomic molecules [414]. In addition, a 3-dimensional anisotropic q-deformed harmonic oscillator has been used [415] for the description of vibrational spectra of triatomic molecules. A description of a tetrahedral molecule (methane) in terms of q-deformed oscillators has been given in [416].

7) Quasi-molecular resonances in the systems $^{12}C+^{12}C$ and $^{12}C+^{16}O$ have been described in terms of a q-deformed oscillator plus a rigid rotator [417].

A review of several of the above topics, accompanied by a detailed and self-contained introduction to quantum algebras, has been given by Raychev [418]. A recent review on algebraic methods in molecular spectroscopy has been given by Kellman [419], while more extensive presentations of algebraic techniques applied in molecular spectroscopy can be found in the books by Iachello and Levine [405] and by Frank and Van Isacker [406].

38. The use of quantum algebras in the structure of atomic clusters

Metal clusters have been recently the subject of many investigations (see [420,421,422] for relevant reviews). One of the first fascinating findings in their study was the appearance of magic numbers [423,424,425,426,427,428,429], analogous to but different from the magic numbers appearing in the shell structure of atomic nuclei [430]. This analogy led to the early description of metal clusters in terms of the Nilsson–Clemenger model [431], which is a simplified version of the Nilsson model [269,270,271,272] of atomic nuclei, in which no spin-orbit interaction is included. Further theoretical investigations in terms of the jellium model [432,433] demonstrated that the mean field potential in the case of simple metal clusters bears great similarities to the Woods–Saxon potential of atomic nuclei, with a slight modification of the "wine bottle" type [434]. The Woods–Saxon potential itself looks like a harmonic oscillator truncated at a

certain energy value and flattened at the bottom. It should also be recalled that an early schematic explanation of the magic numbers of metallic clusters has been given in terms of a scheme intermediate between the level scheme of the 3-dimensional harmonic oscillator and the square well [420]. Again in this case the intermediate potential resembles a harmonic oscillator flattened at the bottom.

On the other hand, as we have seen in sec. 28, a q-deformed version of the 3-dimensional harmonic oscillator has been constructed [275], taking advantage of the $u_q(3) \supset so_q(3)$ symmetry. The spectrum of this 3-dimensional q-deformed harmonic oscillator has been found [275] to reproduce very well the spectrum of the modified harmonic oscillator introduced by Nilsson [269,272], without the spin-orbit interaction term. Since the Nilsson model without the spin orbit term is essentially the Nilsson–Clemenger model used for the description of metallic clusters [431], it is worth examining if the 3-dimensional q-deformed harmonic oscillator can reproduce the magic numbers of simple metallic clusters.

It has indeed been found [435] that the 3-dimensional q-deformed harmonic oscillator with $u_q(3) \supset so_q(3)$ symmetry correctly predicts all experimentally observed magic numbers of alkali metal clusters up to 1500, which is the expected limit of validity for theories based on the filling of electronic shells [423]. This indicates that $u_q(3)$, which is a nonlinear deformation of the u(3) symmetry of the spherical (3-dimensional isotropic) harmonic oscillator, is a good candidate for being the symmetry of systems of alkali metal clusters.

39. Outlook

Nobody likes binding himself by statements concerning the future. However, we attempt to give here a partial list of open problems, roughly following the order of the material in this review:

1) The list of physical systems which can be classified under a generalized deformed su(2) symmetry (sec. 17) or under a generalized deformed parafermionic oscillator scheme (sec. 18) can be enlarged. Self-similar potentials and isospectral oscillator Hamiltonian systems could probably be related to these symmetries.

2) The description of B(E2) values in terms of the $su_q(2)$ model attempted in sec. 21 takes into account only the kinematical deformation effects. In order to take into account dynamical deformation effects, one has to build a larger algebra, of which the quadrupole operators will be members. The case of the $u_q(3) \supset so_q(3)$ algebra has been considered in secs 28, 29. A detailed study is needed in order to see if the experimental data support the modifications predicted by these models. In particular the $su_q(2)$ prediction about B(E2) values increasing with increasing angular momentum J, supported by the predictions of other models as well (see sec. 21), requires further testing against detailed experimental data.

3) The construction of Clebsch-Gordan coefficients for the subclass of generalized deformed su(2) algebras for which this could be possible (see secs 17, 25) is an open problem.

4) The $su_q(3) \supset so_q(3)$ decomposition for $su_q(3)$ irreps other than the completely symmetric ones (see secs 27–29 for the current state of the art) remains an open problem, the solution of which is necessary for developing a deformed version of the su(3) limit of the Interacting Boson Model.

5) Realizations of multi-level shell models in terms of deformed bosons (see secs 33–35 for some one- and two-level cases) should be further pursued.

6) The symmetry algebras of the various 3-dim anisotropic harmonic oscillators with rational ratios of frequencies should be worked out, since they are of interest in relation to superdeformed and hyperdeformed nuclei, and possibly to deformed atomic clusters (see secs 36, 38 for references). A deformed u(3) algebra should occur in this case, which could serve as the basis for building a deformed analog of the Elliott model [241,242,243,248] suitable for superdeformed nuclei.

7) In molecular physics (sec. 37) the study of vibrations of highly symmetric polyatomic molecules (including fullerenes) by these techniques is of interest.

8) In the structure of atomic clusters (sec. 38) the 3-dimensional q-deformed harmonic oscillator (sec. 28) appears to be an appropriate basis for further investigations.

In addition the following final comments are also in place:

a) The usefulness of any "new theory" in nuclear structure is usually judged by its ability to predict some novel excitation mode. For example, the prediction of the low-lying (around 3 MeV) 1^+ mode in even nuclei has been a success of the Interacting Boson Model-2 [169]. In the present case, the $su_q(2)$

model prediction that B(E2) values in deformed nuclei increase as a function of angular momentum, and do not reach a saturation value as the su(2) symmetry suggests, is an interesting point which requires further experimental testing, as already mentioned in sec. 21.

b) Any "new theory" in nuclear structure oughts to acquire a microscopic justification, by establishing links to the underlying fermionic degrees of freedom. The mappings of secs 33 and 34 are a first step in this direction, although they deal only with fermion pairs of zero angular momentum in a single-j shell. The generalization of these mappings to pairs of nonzero angular momentum and to multi-j level schemes is a formidable but challenging task.

c) Quantum algebras, being nonlinear extensions of Lie algebras, are specially suited for describing small perturbations in systems characterized by Lie symmetries. Several examples in nuclei, molecules, and atomic clusters have already been mentioned. Future work should clarify if the success of quantum algebras is limited within these bounds or if there is some deeper physical reason behind their success. For the time being, quantum algebras have been proved to be a useful tool for the description of small deviations from the usual Lie symmetries.

d) An interesting question is if quantum algebras are not only suitable for describing small deviations from Lie symmetries, but in addition can bridge different Lie symmetries. Preliminary work in this direction has been reported in sec. 30.

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