THE STEFAN–BOLTZMANN LAW AT HIGH TEMPERATURE FOR THE GLUON GAS

I. MONTVAY
II. Institut für Theoretische Physik der Universität Hamburg [1], *Germany*

and

E. PIETARINEN [2]
Research Institute for Theoretical Physics, University of Helsinki, Finland

Received 13 November 1981

We determine the energy density of the pure SU(3) Yang–Mills system by Monte Carlo simulation. In the high-temperature regime the Stefan–Boltzmann law is satisfied within errors.

The understanding of the finite-temperature properties of Yang–Mills gauge systems has a basic significance for the description of the thermodynamical behaviour of matter. Besides, finite-temperature studies can also help in revealing some important features of the quantum field theory itself. The lattice Monte Carlo method for studying gauge field theories [1,2] turned out to be very fruitful also at non-zero temperatures. The expected [3–6] deconfining phase transition at some critical temperature T_c has been shown to occur for SU(2) [7,8] and for SU(3) [9] gauge systems. The behaviour of thermodynamical quantities was studied in the SU(2)-case both at high [10] and at low [11] temperatures. The SU(2) Yang–Mills system behaves at very high temperatures as an asymptotically free gluon gas satisfying the Stefan–Boltzmann law. This high-temperature behaviour suggests the validity of asymptotically free perturbation theory for temperatures well above T_c, in spite of the "virulent" infrared divergences [12,13]. This conclusion is strengthened by the SU(2) Monte Carlo data on the non-abelian magnetic flux [14], supporting the presence of the required non-perturbative infrared cut-off.

In quantum chromodynamics the relevant gauge group is SU(3), therefore it is important to know what are the quantitative or perhaps qualitative differences between SU(2) and SU(3). In the present paper we report on our calculations done on the finite-temperature SU(3) Yang–Mills system. The Monte Carlo results show the (approximate) validity of the Stefan–Boltzmann law for the energy density of the SU(3) gluon gas at high temperatures.

For the description of the SU(3) Yang–Mills system with finite temperature we take, as usual, a lattice of size $N^3 N_\beta$ with $N \gg N_\beta$ and periodic boundary conditions. The periodicity in the three spatial directions is imposed to diminish surface effects, whereas periodicity along the euclidean time direction introduces a non-zero temperature $T = \beta^{-1} = (\alpha a N_\beta)^{-1}$. Here a denotes the lattice spacing in the spatial directions and $\alpha a \equiv a_\beta$ is the lattice spacing in the time- (or "inverse temperature") direction. The spatial volume is $V = (aN)^3$.

The energy density (ϵ) and pressure (P) can be obtained by differentiating the partition function with respect to β and V. This could, in principle, be done on the lattice by changing N_β and N for a_β and a fixed, but a much more convenient way in Monte Carlo calculations is to fix the lattice size (N_β, N) and change the lattice spacings a_β and a. Unequal lattice spacings $a_\beta \neq a$ ($\alpha \neq 1$) mean unequal ultraviolet cut-offs for the energy and the three-momenta. Therefore, any lat-

[1] Supported by the Bundesministerium für Forschung und Technologie, Bonn, Germany.
[2] Supported by the Academy of Finland.

0 031-9163/82/0000–0000/$02.75

tice (bare) coupling constant has to depend on both a and a_β. In fact, in order to achieve (euclidean) Lorentz-invariance in the continuum limit $a \to 0$, α = fixed, one has to introduce two different coupling constants g_s and g_β in the action in front of the space-like and time-like plaquettes [15]. (The restauration of the Lorentz invariance in the continuum limit can be shown using a local effective lagrangian [16].) The euclidean lattice SU(N_c) action has in this case the form

$$S = 2g_s^{-2}\alpha \sum_{\{p_s\}} (N_c - \mathrm{Re\ tr}\ U_{p_s}) + 2g_\beta^{-2}\alpha^{-1} \sum_{\{p_\beta\}} (N_c - \mathrm{Re\ tr}\ U_{p_\beta}). \quad (1)$$

Here $\Sigma_{\{p_s\}}$ (respectively $\Sigma_{\{p_\beta\}}$) denotes the summation over "space-like" plaquettes with only space-like links (respectively "time-like" plaquettes with two time-like links). The corresponding plaquette variables are $U_{p_s}, U_{p_\beta} \in \mathrm{SU}(N_c)$. Instead of the functions $g_{s,\beta}(a, a_\beta)$ one can also use $g_{s,\beta}(a, \alpha)$ or $g_{s,\beta}(g, \alpha)$, where $g = g(a)$ is the bare coupling constant in a theory with equal lattice spacings a in all directions. The two-loop renormalization group formula for the function $g(a)$ is:

$$a\Lambda_L = (11N_c g^2/48\pi^2)^{-51/121} \exp(-24\pi^2/11N_c g^2), \quad (2)$$

where Λ_L is the lattice scale parameter on the equally spaced lattice. Following ref. [15] we shall assume that in the continuum limit $a, g \to 0$ (α fixed) the two bare coupling constants g_s and g_β satisfy

$$g_s(a, \alpha)^{-2} = g(a)^{-2} + c_s(\alpha) + O(g^2),$$
$$g_\beta(a, \alpha)^{-2} = g(a)^{-2} + c_\beta(\alpha) + O(g^2). \quad (3)$$

For $\alpha = 1$ ($a_\beta = a$) we have to put, by definition, $g_s = g_\beta = g$ ($c_s = c_\beta = 0$).

The thermodynamical quantities can be derived from the action (1) in the same way as it was done in ref. [10] with a single coupling constant. In the case of SU(N_c) we obtain (N_c = number of colours = 3 in this paper):

$$\epsilon = \epsilon_a + \epsilon_g, \quad P = P_a + P_g, \quad (4)$$

where, with the notation $A_p \equiv \mathrm{Re\ tr}\ U_p$ we have

$$\epsilon_a = 2(TN_\beta)^4 (N^3 N_\beta)^{-1} \times \left\langle g_s^{-2}\alpha^4 \sum_{\{p_s\}} (N_c - A_{p_s}) - g_\beta^{-2}\alpha^2 \sum_{\{p_\beta\}} (N_c - A_{p_\beta}) \right\rangle,$$
$$P_a = \tfrac{1}{3}\epsilon_a, \quad (5)$$

and

$$\epsilon_g = 2(TN_\beta)^4 (N^3 N_\beta)^{-1} \times \left\langle \left.\frac{\partial g_s^{-2}}{\partial \alpha}\right|_a \alpha^5 \sum_{\{p_s\}} (N_c - A_{p_s}) + \left.\frac{\partial g_\beta^{-2}}{\partial \alpha}\right|_a \alpha^3 \sum_{\{p_\beta\}} (N_c - A_{p_\beta}) \right\rangle, \quad (6)$$

$$P_g = \tfrac{1}{3}\epsilon_g - \tfrac{2}{3}(TN_\beta)^4 (N^3 N_\beta)^{-1} \times \left\langle a \left.\frac{\partial g_s^{-2}}{\partial a}\right|_\alpha \alpha^4 \sum_{\{p_s\}} (N_c - A_{p_s}) + a \left.\frac{\partial g_\beta^{-2}}{\partial a}\right|_\alpha \alpha^2 \sum_{\{p_\beta\}} (N_c - A_{p_\beta}) \right\rangle. \quad (7)$$

These relations will be used below only for $\alpha = 1$ ($a_\beta = a$), i.e. for equal lattice spacings in all directions. The coupling constant derivatives needed at $\alpha = 1$ are given in the continuum limit by eq. (3) like

$$a \left.\frac{\partial g_s^{-2}}{\partial a}\right|_{\alpha=1} = a \left.\frac{\partial g_\beta^{-2}}{\partial a}\right|_{\alpha=1} = a \frac{\mathrm{d}g^{-2}}{\mathrm{d}a} \equiv -2g^{-3}\bar{\beta}(g) \equiv -2(\beta_0 + g^2\beta_1 + g^4\beta_2 + \ldots) = -11N_c/24\pi^2 + O(g^2), \quad (8)$$

and

$$\left.\frac{\partial g_{s,\beta}^{-2}}{\partial \alpha}\right|_{\alpha=1,a} = c'_{s,\beta} + O(g^2), \quad c'_{s,\beta} \equiv \left.\frac{\mathrm{d}c_{s,\beta}(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=1}. \quad (9)$$

The $O(g^2)$ corrections will be usually omitted in what follows. They can, however, be easily included. In numerical calculations the effect of $O(g^2)$ terms, for instance, in eq. (9) would show up as a weak dependence of $c'_{s,\beta}$ on g^2. We shall, however, calculate $c'_{s,\beta}$ only at a single g^2 value. (Another g^2 value would require us to go e.g. to $N_\beta = 5$ lattices, a difficult task from the point of view of computer time.)

An interesting consequence of eqs. (4)–(9) is that

for $\alpha = 1$ the thermodynamical quantity $\delta \equiv \frac{1}{3}\epsilon - P$ does not depend on the parameters c_s' and c_β':

$$\delta = (TN_\beta)^4 (11N_c/36\pi^2)(N^3 N_\beta)^{-1} \times \left\langle \sum_{\{p_s\}} (A_{p_s} - N_c) + \sum_{\{p_\beta\}} (A_{p_\beta} - N_c) \right\rangle . \quad (10)$$

The reason for this is that δ could be obtained without introducing unequal lattice spacings and two coupling constants at all. Namely, for fixed N and N_β we have on the equally spaced lattice

$$\delta = \tfrac{1}{3}\epsilon - P = -(3V)^{-1}\, \partial \ln Z/\partial\beta - \beta^{-1}\, \partial \ln Z/\partial V = -\frac{1}{3V\beta} a \frac{\partial \ln Z}{\partial a} = \frac{4a^{-4}}{3N^3 N_\beta} \frac{\bar{\beta}(g)}{g^3} \left\langle \sum_{\{p\}} (A_p - N_c) \right\rangle . \quad (11)$$

Using $T = (aN_\beta)^{-1}$ and eq. (8) this gives again the expression in eq. (10).

The above thermodynamical quantities still include vacuum ($T = 0$) terms. This is not surprising as even in the simplest case of a free scalar field the euclidean formulation leads to an infinite vacuum energy density term [17]. Such zero point terms can be removed in the above formulae either by subtracting the same expressions for the symmetric ($N_\beta = N$) lattice (assumed to describe $T = 0$), or by taking differences belonging to two different values of the temperature. Let us illustrate the second procedure for the quantity δ. (The treatment of ϵ_g is very similar, whereas ϵ_a does not contain a $T = 0$ term as it vanishes on the symmetric lattice.)

The relation of the temperature T with the lattice size in the time direction N_β, the coupling constant g and the ratio $\alpha = a_\beta/a$ is from eq. (2)

$$T/\Lambda_L = (\alpha N_\beta)^{-1} (11N_c g^2/48\pi^2)^{51/121} \exp(24\pi^2/11N_c g^2). \quad (12)$$

At $\alpha = 1$ for a given g, corresponding to a given lattice spacing a, the zero point term does not depend on N_β. Taking into account that the number of both space-like and time-like plaquettes is $3N^3 N_\beta$, we have from eq. (10)

$$\delta(T) - \delta(TN_\beta/N_\beta') = (TN_\beta)^4 (11N_c/12\pi^2) \times (\langle A_{p_s} + A_{p_\beta} \rangle_{g,N_\beta} - \langle A_{p_s} + A_{p_t} \rangle_{g,N_\beta'}) . \quad (13)$$

The relation between T, N_β and g is given by eq. (12) with $\alpha = 1$, therefore the temperature corresponding to the same g and different N_β' is TN_β/N_β', as indicated.

Another formula can be obtained from (13) if we put $N_\beta' = N_\beta + 1$ and sum for T, $TN_\beta (N_\beta + 1)^{-1}$, $TN_\beta^2 (N_\beta + 1)^{-2}$, ... The sum on the right-hand side can be replaced by the corresponding integral if N_β is large. This gives

$$\delta(T_1) - \delta(T_2) = 2N_\beta \Lambda_L^4 \int_{g_2^{-2}}^{g_1^{-2}} dg^{-2} \exp(96\pi^2/11N_c g^2) \times (\langle A_{p_s} + A_{p_\beta} \rangle_{g^2,N_\beta} - \langle A_{p_s} + A_{p_\beta} \rangle_{g^2,N_\beta+1}) . \quad (14)$$

Note that we are now neglecting everywhere, for simplicity, the $O(g^2)$ term in eq. (8) which is proportional to β_1. In the integration limits g_j ($j = 1, 2$) is given by T_j from eq. (12) with $\alpha = 1$ and N_β and the power term on the right-hand side neglected. A further possibility is to take the derivative of eq. (14) with respect to the temperature. This gives

$$d\delta/dT = T^3 N_\beta^5 (11N_c/12\pi^2) \times (\langle A_{p_s} + A_{p_\beta} \rangle_{g,N_\beta} - \langle A_{p_s} + A_{p_\beta} \rangle_{g,N_\beta+1}) . \quad (15)$$

As already mentioned, the expressions for δ and ϵ_g are very similar. The procedure which leads to eq. (15) gives for ϵ_g:

$$d\epsilon_g/dT = 6T^3 N_\beta^5 [-c_s'(\langle A_{p_s} \rangle_{g,N_\beta} - \langle A_{p_s} \rangle_{g,N_\beta+1}) - c_\beta'(\langle A_{p_\beta} \rangle_{g,N_\beta} - \langle A_{p_\beta} \rangle_{g,N_\beta+1})] . \quad (16)$$

There is, of course, also a relation for $\epsilon_g(T_1) - \epsilon_g(T_2)$ analogous to eq. (14).

The quantities ϵ_a and $d\delta/dT$ in eqs. (5) and (15) can be determined immediately from the lattice Monte Carlo data by measuring the expectation values of plaquette variables. The energy density $\epsilon = \epsilon_a + \epsilon_g$, however, can only be calculated if the values of c_s' and c_β' in (16) are known. According to eq. (9) these parameters reflect the dependence of the two coupling constants g_s and g_β on the ratio of the lattice spacings $\alpha = a_\beta/a$ in the vicinity of $\alpha = 1$. The functions $c_s(\alpha)$ and $c_\beta(\alpha)$ can be determined analytically from a perturbative calculation analogous to the one done by Hasenfratz and Hasenfratz [15]. A numerical procedure, checking at the same time the consistency of the numerical evaluation, is to consider some physical quantity on the lattice and determine the dependence of g_s and g_β on α by requiring that the numerical value of this quantity be unchanged if α is changing. An

example of such a procedure is given in a recent paper by Bhanot and Creutz [18] in a zero-temperature SU(2) lattice gauge theory with two coupling constants. The physical quantity chosen by Bhanot and Creutz is the ratio of some Wilson-loop expectation values. In thermodynamics the best physical quantity to fix [10] is the value of the deconfining phase transition temperature T_c. At this critical temperature the expectation value of the Wilson-line going through the lattice in the time direction ("order parameter") becomes non-zero [7–9].

The action S in eq. (1) depends on the combinations $G_s \equiv \alpha g_s^{-2}$ and $G_\beta \equiv \alpha^{-1} g_\beta^{-2}$, therefore first we have to locate the critical line corresponding to T_c in the (G_s, G_β)-plane. Let us measure the value of T_c in units of Λ_L, which is according to eq. (2) the scale parameter for a lattice with equal spacings in all directions. From eq. (2) or (12) we get:

$$\ln(\alpha N_\beta T_c/\Lambda_L) = 24\pi^2/11N_c g^2 + \tfrac{51}{121}\ln(11N_c g^2/48\pi^2). \tag{17}$$

From this equation it is possible to express g as a function of the variable $\alpha\nu \equiv \alpha N_\beta T_c/\Lambda_L$. Omitting, for simplicity, the $O(g^2)$ corrections in eq. (3) we have:

$$\begin{aligned} G_s &= \alpha g(\alpha\nu)^{-2} + \alpha c_s(\alpha) \equiv G_s(\alpha, \alpha\nu), \\ G_\beta &= \alpha^{-1} g(\alpha\nu)^{-2} + \alpha^{-1} c_\beta(\alpha) \equiv G_\beta(\alpha, \alpha\nu). \end{aligned} \tag{18}$$

This is a parametric representation of the critical curves in the (G_s, G_β)-plane when α is changed and ν is kept fixed.

In order to extract c_s' and c_β' we make a linear approximation of $\alpha c_s(\alpha)$ and $\alpha^{-1} c_s(\alpha)$ in the variable $\alpha - 1$ near $\alpha = 1$. In a numerical procedure it is not good to choose $(\alpha - 1)$ too small, because it is difficult to detect a small change in G_s and G_β. We have taken, for simplicity, $\alpha = 3/4$ for $N_\beta = 4$ and $\alpha = 4/3$ for $N_\beta = 3$. (Other choices may even be better for the optimal determination of c_s' and c_β', but these simple values are sufficient to show that c_s' and c_β' are small.) In these cases eq. (18) gives, with $\tau_c \equiv T_c/\Lambda_L$ and $G_j \equiv g(j\tau_c)^{-2}$ $(j = 3, 4)$:

$$\begin{aligned} G_s(\tfrac{3}{4}, 3\tau_c) &= \tfrac{3}{4}G_3 - \tfrac{1}{4}c_s' \equiv \bar{G}_{s4}, \\ G_\beta(\tfrac{3}{4}, 3\tau_c) &= \tfrac{4}{3}G_3 - \tfrac{1}{4}c_\beta' \equiv \bar{G}_{\beta 4}, \\ G_s(\tfrac{4}{3}, 4\tau_c) &= \tfrac{4}{3}G_4 + \tfrac{1}{3}c_s' \equiv \bar{G}_{s3}, \\ G_\beta(\tfrac{4}{3}, 4\tau_c) &= \tfrac{3}{4}G_4 + \tfrac{1}{3}c_\beta' \equiv \bar{G}_{\beta 3}. \end{aligned} \tag{19}$$

The numerical search for the critical values of G_s and G_β (where the time-like Wilson-line expectation value becomes non-zero) can be carried out by fixing the ratio G_β/G_s and changing e.g. G_s. We can, for instance, take $G_\beta = 9G_s/16$ for $\alpha = 4/3$ and $G_\beta = 16G_s/9$ for $\alpha = 3/4$. The points of the $N_\beta = 3$ and $N_\beta = 4$ critical lines in the (G_s, G_β)-plane are then $P_3 \equiv \{G_{s3}, 9G_{s3}/16\}$ and $P_4 = \{G_{s4}, 16G_{s4}/9\}$, respectively (see fig. 1). These points would coincide with $\bar{P}_3 = \{\bar{G}_{s3}, \bar{G}_{\beta 3}\}$, respectively, with $\bar{P}_4 = \{\bar{G}_{s4}, \bar{G}_{\beta 4}\}$, if in eq. (19) c_s' and c_β' would be zero. Otherwise, in the linear approximation near $\alpha = 1$, $\bar{P}_3$ is on the line connecting P_3 to the $\alpha = 1$ point $Q_3 \equiv \{G_3, G_3\}$, and $\bar{P}_4$ is on the line connecting P_4 to $Q_4 \equiv \{G_4, G_4\}$. The points Q_3 and Q_4 can be obtained by determining T_c for $\alpha = 1$ with $N_\beta = 3$ and $N_\beta = 4$, respectively. With the help of fig. 1 it is easy to show that if we put $G_{s3} = 4G_4/3 + \epsilon_3$ and $G_{s4} = 3G_3/4 + \epsilon_4$ ($\epsilon_{3,4}$ small), then the solution of eq. (19) for c_s' and c_β' is:

$$\begin{aligned} c_s' &= (25G_3G_4 - 12G_3^2 - 12G_4^2)^{-1}(3G_3 - 4G_4) \\ &\quad \times (\tfrac{27}{4}G_3\epsilon_3 + \tfrac{64}{3}G_4\epsilon_4), \\ c_\beta' &= (25G_3G_4 - 12G_3^2 - 12G_4^2)^{-1}(4G_3 - 3G_4) \\ &\quad \times (9G_3\epsilon_3 + 16G_4\epsilon_4). \end{aligned} \tag{20}$$

The Monte Carlo data for T_c, on lattices with space-like size $N = 9$, are shown in fig. 2. For the details of our Monte Carlo methods see ref. [19]. Taking the

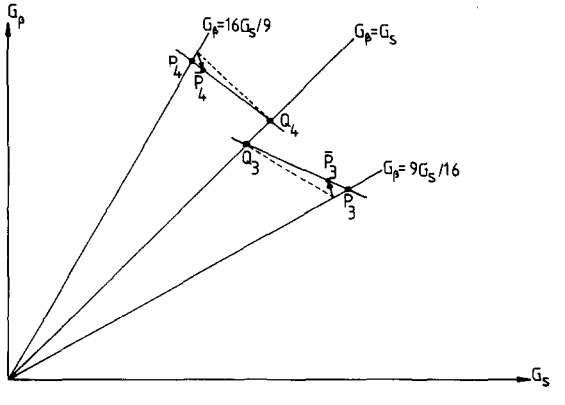

Fig. 1. The critical lines (full lines) belonging to the critical temperature T_c in the (G_s, G_β)-plane for $N_\beta = 3$ (Q_3P_3) and $N_\beta = 4$ (Q_4P_4). The dashed lines would be the critical lines for $c_s' = c_\beta' = 0$.

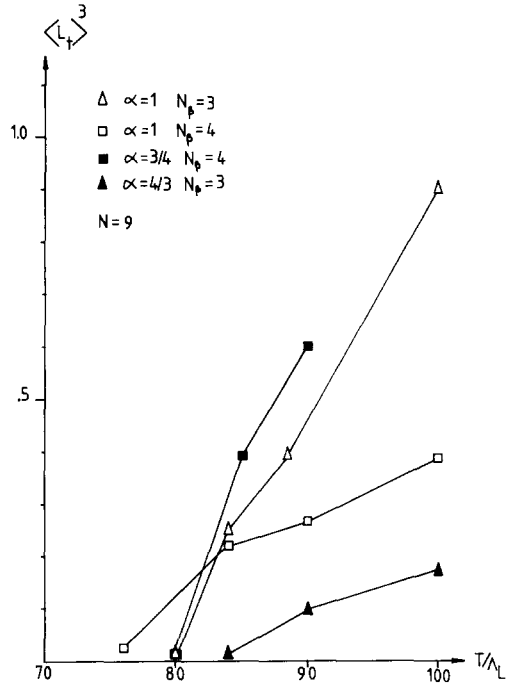

Fig. 2. The behaviour of the time-like Wilson-line expectation value $\mathrm{Re}\langle L_t \rangle^3$ as a function of the temperature calculated from eq. (12). The points represent the averages of typically 200 iterations. Statistical errors are roughly the same as the size of the points.

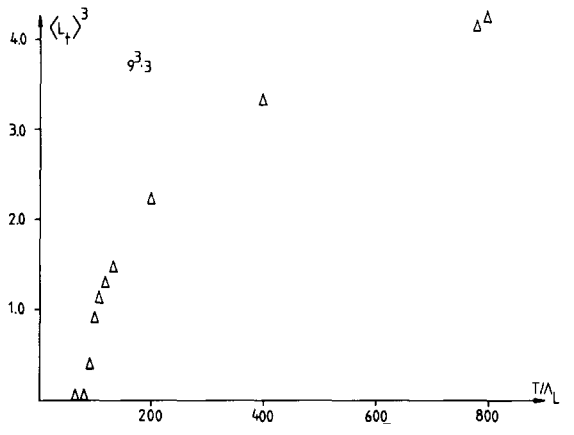

Fig. 3. The time-like Wilson-line expectation value as a function of the temperature measured in units of Λ_L. The lattice size is here $9^3 \cdot 3$ with equal spacings in all directions ($\alpha = 1$). The points are obtained from typically 500 iterations.

average of the $N_\beta = 3$ and $N_\beta = 4$ points at $\alpha = 1$, the critical temperature obtained from fig. 1 is: $T_c = (78 \pm 3)\Lambda_L$. This agrees, within the errors, with the value $T_c = 83\ \Lambda_L$ given in ref. [9]. The behaviour of the time-like Wilson-line expectation value in a larger temperature range is depicted in fig. 3. The systematic difference between the values of T_c obtained from the $N_\beta = 3$ and 4 points could be due either to finite lattice size effects or to deviations from the two-loop renormalization group formula (12) connecting g with the temperature. Assuming the second possibility, the shift of about $\Delta T_c/T_c = 0.08$ would then be due to the higher loop corrections to the β-function. In fact, regularization scheme dependent three-loop corrections were advocated recently [20] in order to explain the differences in the string tension values obtained from different lattice actions. The difference in the g^2 values belonging to $N_\beta = 3$ and $N_\beta = 4$ (for $\alpha = 1$) is about $g^2 \approx 0.04$, hence the above shift in T_c could be accounted for if for the Wilson-action [in the notation of eq. (8)] $(\beta_2 - \beta_1^2/\beta_0) \simeq 4\beta_0^2$.

The quantities needed in eq. (20) can be inferred from fig. 2: $G_3 = 0.917$, $G_4 = 0.948$, $\epsilon_3 = 0.020 \pm 0.010$ and $\epsilon_4 = -0.0014 \pm 0.0014$ (the errors are subjective estimates based on the statistical fluctuations of the averages during the iterations). This gives $c_s' = -0.12 \pm 0.06$ and $c_\beta' = 0.14 \pm 0.06$. The differences of plaquette expectation values needed for the calculation of $\mathrm{d}\epsilon_g/\mathrm{d}T$ in eq. (16) are, for instance, at a temperature $T = 800\ \Lambda_L$: $\langle A_{p_s} \rangle_{g,3} - \langle A_{p_s} \rangle_{g,4} = -0.0036 \pm 0.0004$ and $\langle A_{p_\beta} \rangle_{g,3} - \langle A_{p_\beta} \rangle_{g,4} = 0.0040 \pm 0.0004$. Therefore we have

$$(T^{-3}\ \mathrm{d}\epsilon_g/\mathrm{d}T)_{T \approx 800 \Lambda_L} = -0.72 \pm 0.30, \qquad (21)$$

This has a large relative error, but the important information is the order of magnitude. Namely, as we shall see later, eq. (21) implies (for this temperature) $|\epsilon_g| \ll \epsilon_a$.

The general behaviour of $|T^{-3}\ \mathrm{d}\epsilon_g/\mathrm{d}T|$ as a function of the temperature is the following: It has a high and sharp peak at T_c. Above T_c it is monotonically decreasing. The decrease in the high-temperature region ($T \gg T_c$) is compatible with being logarithmic. Therefore, in the relation

$$T^{-3}\ \mathrm{d}\epsilon_g/\mathrm{d}T = 4\epsilon_g T^{-4} + T(\mathrm{d}/\mathrm{d}T)(\epsilon_g T^{-4}), \qquad (22)$$

the second term on the right-hand side can be neglected for $T \gg T_c$. (Actually, the second term would dominate if $|\epsilon_g T^{-4}|$ would decrease faster than T^{-4}.

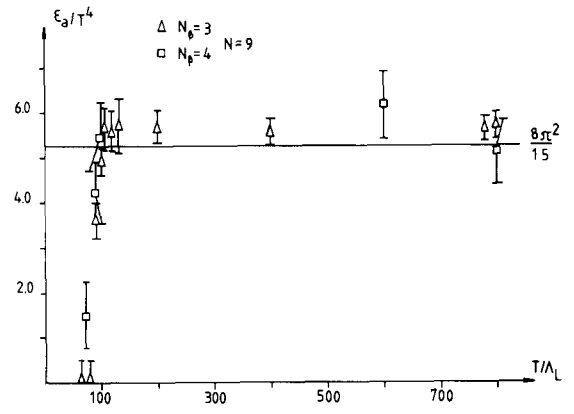

Fig. 4. The energy density as a function of the temperature obtained from the same runs as fig. 3 (triangles) and from a $9^3 \cdot 4$ lattice with $\alpha = 1$ (squares). Here $\epsilon_a T^{-4}$ is given and compared to the Stefan–Boltzmann expectation $8\pi^2/15$ (horizontal line).

But this case need not be considered at all, as $\epsilon_a T^{-4}$ is nearly constant for $T \gg T_c$, therefore $|\epsilon_g T^{-4}| \ll \epsilon_a T^{-4}$ would be trivially fulfilled.) Dropping the second term in eq. (22) we obtain from eq. (21): $(T^{-4}\,\epsilon_g)_{T=800\Lambda_L} = -0.18 \pm 0.15$. This characterizes the order of magnitude of $T^{-4}\epsilon_g$ in the high-temperature region.

The other part of the energy density is, according to eqs. (4), (5), for $\alpha = 1$:

$$\epsilon_a = 6(TN_\beta)^4 g^{-2} \langle A_{p_\beta} - A_{p_s} \rangle_{g,N_\beta}. \qquad (23)$$

The Monte Carlo results for this quantity on $9^3 \cdot 3$ and $9^3 \cdot 4$ lattices are given in fig. 4, where the values of the $N_\beta = 3$ points are divided by a factor 1.7 and those of $N_\beta = 4$ by a factor 1.35. These factors take into account the finite-size effects derived from the lattice thermodynamics of a free scalar field [21]. The same factors were necessary also in the SU(2) case [10] in order to obtain agreement with the expected free gas behaviour of the energy density at high temperatures. In our case the Stefan–Boltzmann law predicts for the energy density $\epsilon = 8\pi^2 T^4/15$. As shown by fig. 4 this prediction is fulfilled for $T > 120\,\Lambda_L$ by $\epsilon = \epsilon_a$ within the errors of the numerical calculation. The small negative value of $\epsilon_g T^{-4}$ shifts the points in fig. 4 a little bit downwards, therefore it makes the agreement with the Stefan–Boltzmann value even slightly better. But this improvement is insignificant due to the numerical errors on both $\epsilon_a T^{-4}$ and $\epsilon_g T^{-4}$. A more precise numerical determination of $\epsilon = \epsilon_a + \epsilon_g$ would require considerably higher Monte Carlo statistics.

The behaviour of the quantity $T^{-3}\,\mathrm{d}\delta/\mathrm{d}T$ given in eq. (15) is very similar to the behaviour of $|T^{-3}\,\mathrm{d}\epsilon_g/\mathrm{d}T|$. It has a sharp and high peak at T_c and decreases monotonously for $T > T_c$. (We shall give detailed Monte Carlo data on δ and ϵ_g in a forthcoming publication [22].) This is another argument for the conclusion that for $T \gg T_c$ the pure SU(3) Yang–Mills system behaves as an asymptotically free gas, in spite of the infrared singularities seen in perturbation theory. For an ultrarelativistic ideal gas δ vanishes. Its deviation from zero gives a measure of the effective strength of interaction manifested in the decrease of the pressure if compared to the energy density. Note that a finite effective gluon mass implies also $\delta > 0$.

E.P. wants to thank the DESY theory group for hospitality during the preparation of this work. We are indebted to P. Hasenfratz, H. Joos, M. Lüscher, G. Mack, H. Satz and K. Symanzik for useful discussions.

References

[1] K. Wilson, Phys. Rev. D10 (1974) 2445.
[2] M. Creutz, Phys. Rev. D21 (1980) 2308.
[3] D.A. Kirzhnits and A.D. Linde, Phys. Lett. 42B (1972) 471.
[4] J.C. Collins and M.J. Perry, Phys. Rev. Lett. 34 (1975) 1353.
[5] A.M. Polyakov, Phys. Lett. 72B (1978) 477.
[6] L. Susskind, Phys. Rev. D20 (1979) 2610.
[7] L.D. McLerran and B. Svetitsky, Phys. Lett. 98B (1981) 195.
[8] J. Kuti, J. Polónyi and K. Szlachányi, Phys. Lett. 98B (1981) 199.
[9] K. Kajantie, C. Montonen and E. Pietarinen, Z. Phys. C9 (1981) 253.
[10] J. Engels, F. Karsch, I. Montvay and H. Satz, Phys. Lett. 101B (1981) 89.
[11] J. Engels, F. Karsch, I. Montvay and H. Satz, Phys. Lett. 102B (1981) 332.
[12] A.D. Linde, Phys. Lett. 96B (1980) 289.
[13] D.J. Gross, R.D. Pisarski and L.G. Yaffe, Rev. Mod. Phys. 53 (1981) 43.
[14] A. Billoire, G. Lazarides and Q. Shafi, Phys. Lett. 103B (1981) 450.
[15] A. Hasenfratz and P. Hasenfratz, KFKI Budapest preprint 1981-15.

[16] K. Symanzik, DESY preprint 81-068, and private communication.
[17] C. Bernard, Phys. Rev. D9 (1974) 3312.
[18] G. Bhanot and M. Creutz, Brookhaven preprint BNL 29640 (1981).
[19] E. Pietarinen, Nucl. Phys. B190 (1981) 349.
[20] C.B. Lang, C. Rebbi, P. Salomonson and B.S. Skagerstam, Göteborg preprint 81-22 (1981).
[21] J. Engels, F. Karsch and H. Satz, Bielefeld preprint BI-TP-8128, to be published.
[22] I. Montvay and E. Pietarinen, in preparation.