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Gauge Invariance, Anomalies, and the Chiral Schwinger Model

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After having justified the gauge invariant version of the chiral Schwinger model we perform canonical quantization via Dirac brackets. The constraints are first class, exhibiting gauge invariance. As a result we find that this is the reason for the consistency of the model of Jackiw and Rajaraman. © 1987 Academic Press, Inc.

I. INTRODUCTION

Recently the interesting possibility has been proposed [1–3] that chiral gauge theories might be consistently quantized in spite of the presence of anomalies (for review, see [4]). Since up to now anomalies have been thought to break gauge (BRS) invariance of the corresponding quantum field theory[1] and hence to spoil consistency and unitarity (and renormalizability), this possibility might completely change our understanding of and relationship to anomalous models. This new development is important not only from a formal point of view, but also for very practical purposes since now anomalous models possibly do not have to be excluded from being realistic.

The proposal of ref. [1] consists of introducing by hand a scalar field in order to keep the constraints first class after quantization to maintain gauge invariance at the quantum level. This results in the addition of a Wess–Zumino term, which has been confirmed by careful path integral treatments [6–8], which improve earlier attempts [9] by reanalysing the Faddeev Popov procedure [10]. The outcome of this investigation is the fact that the Wess–Zumino term need not be introduced by hand but that it arises automatically. All previous calculations, which led to inconsistent results for anomalous gauge theories, have been based on the assumption that the Faddeev Popov trick, namely to neglect the volume of the gauge group, works in anomalous theories in the same way as in the anomaly free case. Since this assumption is not fulfilled, as will be discussed in Section II, these calculations are not reliable and should be redone in a correct framework.

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[1] At the quantum level the notion "gauge invariance" always means BRS invariance [5].

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Unfortunately, this is very difficult for realistic four-dimensional models. This led us to investigate as a first step the two-dimensional case, where the effect of chiral fermions can be explicitly calculated. This has been the reason for the growing interest in two-dimensional gauge theories. The existence of an exact solution provides us with an excellent playground to test the survival of the gauge symmetry, the anomaly cancellation and the consistency of the solution.

In a recent paper Jackiw and Rajaraman [11] studied the chiral version of the Schwinger model [12] (chiral QED in two dimensions). They interpreted the appearance of a mass term for the gauge boson as an anomalous breakdown of the gauge symmetry. Nevertheless they showed that quantization can be carried through consistently and that the quantum theory is unitary. This has been confirmed by a detailed analysis of the constraint algebra in the Hamilton formalism via Dirac bracket quantization of the bosonized theory [13, 14], showing that the constraints are second class as one would expect for a theory without gauge invariance. In the following this formulation of the chiral Schwinger model will be called "anomalous" or gauge noninvariant. In ref. [6] the correct path integral treatment was also applied to the chiral Schwinger model, they, however, obtained a gauge invariant, anomaly free quantum theory. This formulation will in the following be referred to as the anomaly free or gauge invariant chiral Schwinger model. This has been considered in the Lagrange formalism by ref. [15] (see also [16, 17]) too, these authors added the Wess–Zumino term by hand, following the proposal of ref. [1].

In the present paper we investigate in the Hamilton formalism the constraint structure of the gauge invariant formulation of the chiral Schwinger model, and we are going to explain why the so-called anomalous chiral Schwinger model is consistent and to clarify the relationship between the anomalous and the anomaly free chiral Schwinger model. In Section II we repeat the correct path integral approach in order to have the needed formulas at out disposal. In Section III the anomaly free chiral Schwinger model is quantized using the Dirac bracket procedure, Section IV will contain the discussion of the result.

II. Path Integral Approach

In this section we briefly repeat the derivation of ref. [6] in order to make our discussion more explicit. The most important observation in the path integral quantization of anomalous gauge theories is the fact that functional integration has to be performed over the complete configuration space [6–8] including the gauge volume. Hence the generating functional reads:

$$Z = \int dA\, d\psi\, d\bar{\psi}\, e^{iS[\bar{\psi},\psi,A]}, \tag{2.1}$$

where $S[\bar{\psi}, \psi, A]$ is the classical action. Now the Faddeev Popov trick [10] is repeated by inserting an appropriate unity into Eq. (2.1), namely

$$1 = \Delta_f[A] \int dg\, \delta(f(A^g)). \tag{2.2}$$

Here A^g is the gauge transformed vector potential, f is the gauge fixing function, Δ_f the corresponding Faddeev Popov determinant and the integration is performed over the gauge group. Then relabeling the integration variable $A \to A^{g^{-1}}$ and using the gauge invariance of dA and $\Delta_f(A)$ gives

$$Z = \int \mathscr{D}A d\psi\, d\bar{\psi}\, dg\, e^{iS[\bar{\psi}, \psi, A^{g^{-1}}]} \tag{2.3}$$

with the gauge boson integration measure

$$\mathscr{D}A = dA\, \Delta_f[A]\, \delta(f(A)). \tag{2.4}$$

The usual Faddeev Popov procedure is to argue that relabeling ψ and $\bar{\psi}$ in the same way as A and using the gauge invariance of S leads to a trivial g-integration which may be absorbed in the normalization of Z. Hence the Faddeev Popov procedure gives

$$Z = \int \mathscr{D}A\, d\psi\, d\bar{\psi}\, e^{iS[\bar{\psi}, \psi, A]}. \tag{2.5}$$

However, it is well known [9] that in the case of *chiral* gauge coupling to the fermions $d\psi\, d\bar{\psi}$ achieves a nontrivial Jacobian which gives rise to anomalies. This means that Eq. (2.5) is in general incorrect in chiral theories. The old argumentations [4] concerning anomalies are based in Eq. (2.5). Therefore they should be reanalysed using the correct generating functional, Eqs. (2.1), (2.3), and (2.8). Defining the effective gauge field action by

$$e^{iW[A]} = \int d\psi\, d\bar{\psi}\, e^{iS[\bar{\psi}, \psi, A]} \tag{2.6}$$

and the Wess–Zumino action [18] as

$$\alpha_1[A, g^{-1}] = W[A^{g^{-1}}] - W[A] \tag{2.7}$$

the generating functional, Eq. (2.3), may be rewritten according to

$$\begin{aligned} Z &= \int \mathscr{D}A\, dg\, e^{iW[A^{g^{-1}}]} \\ &= \int \mathscr{D}A\, dg\, e^{i\{W[A] + \alpha_1[A, g^{-1}]\}} \\ &= \int \mathscr{D}A\, d\psi\, d\bar{\psi}\, dg\, e^{i\{S[\bar{\psi}, \psi, A] + \alpha_1[A, g^{-1}]\}}. \end{aligned} \tag{2.8}$$

The exponent in the last line of Eq. (2.8) has been called the "standard action" in ref. [6]. From here we see that Eq. (2.5) is correct only in the case that the one-cocycle $\alpha_1[A, g^{-1}]$ vanishes, i.e., that the theory is free of anomalies in the conventional sense. This derivation clearly shows that it is not necessary to introduce the Wess–Zumino term by hand in order to restore gauge invariance, as it was proposed in ref. [1], but that it is an indispensible ingredient of the theory itself, if a gauge invariant formulation is chosen. Since Eq. (2.8) can be shown to be gauge invariant [6], anomalies do *not* spoil gauge invariance at the quantum level, as it was commonly believed. Certainly, since Z is gauge invariant and $d\psi\, d\bar{\psi}$ is not, it is clear that the action which contains the chiral fermions is not gauge invariant. This means that the action does *not* reflect the symmetry of the theory. This is just contrary to the conventional incorrect formulation based on Eq. (2.5) where the action suggests a gauge symmetry which is not present.

III. Canonical Quantization of the Chiral Schwinger Model

The chiral Schwinger model is defined by the classical action

$$S_{\text{cl}} = \int \{ -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\gamma^\mu [i\partial_\mu + e\sqrt{\pi}\, A_\mu (1 + \gamma_5)]\, \psi \}\, d^2x, \tag{3.1}$$

where we use the notation

$$\begin{gathered} \eta_{00} = -\eta_{11} = 1, \qquad \varepsilon^{01} = -\varepsilon_{01} = 1, \\ \gamma_5 = \gamma^0\gamma^1 \rightarrow \gamma_\mu \gamma_5 = \varepsilon_{\mu\nu}\, \gamma^\nu. \end{gathered} \tag{3.2}$$

In this model the functional fermion integration can be done explicitly, yielding [11, 19,20]

$$W[A] = \int \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2}{2} A_\mu \left[\eta^{\mu\beta} a - (\eta + \varepsilon)^{\mu\nu} \frac{\partial_\nu \partial_\alpha}{\Box} (\eta - \varepsilon)^{\alpha\beta} \right] A_\beta \right\} d^2x, \tag{3.3}$$

where a is a free parameter which reflects the ambiguity of the regularization of the fermionic determinant. According to the statements of the preceding section there are two possibilities for quantization: first one could start from

$$Z = \int dA\, e^{iW[A]}, \tag{3.4}$$

then manifest gauge invariance is lost and anomalies are present. This approach has been investigated in refs. [11, 13]. The second alternative is to insist upon gauge invariance and to start from Eq. (2.8). If g is parametrized as $e^{-i\theta}$ then $A_\mu^{g^{-1}} = A_\mu - (1/e)\,\partial_\mu \theta$ and integration over θ gives

$$\int d\theta\, e^{i\int \bar\psi\gamma^\mu e\sqrt{\pi}\,\partial_\mu\theta(1+i\gamma_5)\psi\, d^2x} \sim \delta(\partial_\mu(\bar\psi\gamma^\mu(1+i\gamma_5)\psi)), \tag{3.5}$$

i.e., current conservation is automatically ensured at the quantum level. We note that this argument holds for abelian theories in arbitrary dimensions.

Since the fermionic action does not reflect the symmetries of the theory, the fermions have to be integrated out before canonical quantization can be performed. This results in the effective ation (cf. Eq. (2.8)):

$$W[A^{g^{-1}}] = \int \left\{ -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(a-1)\theta\,\Box\theta + e\theta\,\partial_\mu[(a-1)\,\eta^{\mu\nu} + \varepsilon^{\mu\nu}]\,A_\nu \right.$$
$$\left. + \frac{e^2}{2} A_\mu \left[\eta^{\mu\beta} a - (\eta+\varepsilon)^{\mu\nu} \frac{\partial_\nu\,\partial_\alpha}{\Box} (\eta-\varepsilon)^{\alpha\beta} \right] A_\beta \right\} d^2x. \tag{3.6}$$

According to ref. [11], the nonlocal term $A(\partial\partial/\Box)A$ can be expressed in local terms by introducing an additional field ϕ,

$$Z = \int \mathscr{D}A\, d\phi\, d\theta\, e^{i\int \mathscr{L}_{\text{eff}}\, d^2x},$$

$$\mathscr{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\phi\,\Box\phi - e\phi\,\partial_\mu(\eta-\varepsilon)^{\mu\nu}A_\nu - \frac{1}{2}(a-1)\theta\,\Box\theta$$
$$+ e\theta\,\partial_\mu[(a-1)\,\eta^{\mu\nu} + \varepsilon^{\mu\nu}]\,A_\nu + \frac{e^2}{2} a A_\mu A^\mu. \tag{3.7}$$

At this stage we note that the gauge invariance of the theory has translated to the invariance of the effective action $\int \mathscr{L}_{\text{eff}}\, d^2x$ with respect to the gauge transformation

$$A_\mu \to A_\mu - \frac{1}{e}\,\partial_\mu \Lambda, \qquad \phi \to \phi + \Lambda, \qquad \theta \to \theta - \Lambda. \tag{3.8}$$

Thus we expect the presence of first class constraints in contrast to the case of ref. [13], where no gauge invariance was present and hence all constraints have been second class. Performing the Legendre transformation, we find the primary constraint

$$\Pi_0 \approx 0 \tag{3.9}$$

and the canonical momenta

$$\Pi_1 = -F_{01} = \partial_1 A^0 + \partial_0 A^1, \tag{3.10}$$

$$\Pi_\phi = \partial_0\phi + e(A^0 + A^1), \tag{3.11}$$

$$\Pi_\theta = (a-1)\,\partial_0\theta - (a-1)\,eA^0 + eA^1. \tag{3.12}$$

In Eq. (3.9) $\approx$ denotes weak equality in Dirac's terminology [21, 22], i.e., Eq. (3.9) must not be used before all Poisson brackets of interest have been calculated. Note that Eq. (3.12) determines a one to one correspondence between Π_0 and $\partial_0\theta$ only in the case $a \neq 1$, hence we have to distinguish between $a \neq 1$ and $a = 1$.

(a) *The Case* $a \neq 1$

In this case Eqs. (3.10)–(3.12) may be converted to express the velocities in terms of the momenta in order to perform the Legendre transformation. The Hamiltonian density is given by

$$\begin{aligned}\mathscr{H} &= \frac{1}{2}\Pi_1\Pi_1 + \frac{1}{2}\Pi_\phi^2 + \frac{1}{2(a-1)}\Pi_\theta^2 + A^0\,\partial\Pi_1 - e\Pi_\phi(A^0 + A^1)\\ &\quad + e\Pi_\theta\left(A^0 - \frac{1}{a-1}A^1\right) + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(a-1)(\partial\theta)^2 - e\partial\phi(A^0 + A^1)\\ &\quad + e\partial\theta[(a-1)A^1 - A^0] + \frac{e^2a^2}{2(a-1)}(A^1)^2 + \xi_1\Pi_0,\end{aligned} \tag{3.13}$$

where ξ_1 is an (up to now) undetermined Lagrange multiplier reflecting the arbitraryness of $\partial_0 A^0$ and ∂ denotes ∂_1. In terms of the Poisson brackets

$$\{f, g\} = \frac{\delta f}{\delta q}\frac{\delta g}{\delta p} - \frac{\delta f}{\delta p}\frac{\delta g}{\delta q} \tag{3.14}$$

the Hamiltonian equations of motion are determined by (f not explicitly time dependent)

$$\dot{f} \approx \{f, H\}, \qquad H = \int \mathscr{H}\, dx. \tag{3.15}$$

Consistency requires that the constraints are at least weakly stable in time, that means in our case

$$\dot{\Pi}_0 \approx -\frac{\delta H}{\delta A^0} = -(\partial\Pi_1 - e\Pi_\phi + e\Pi_\theta - e\,\partial\phi - e\,\partial\theta) \approx 0. \tag{3.16}$$

Since the Poisson bracket of this expression with the Hamiltonian vanishes, we have two constraints, namely

$$\chi_1 = \Pi_0 \approx 0, \tag{3.17}$$

$$\chi_2 = \partial\Pi_1 - e\Pi_\phi + e\Pi_\theta - e\,\partial\phi - e\,\partial\theta \approx 0, \tag{3.18}$$

which are, as expected, first class, i.e., $\{\chi_1, \chi_2\} \approx 0$. The total Hamiltonian [21] reads

$$H_T = H + \int \xi_2 \chi_2 \, dx, \tag{3.19}$$

which means that there are two Lagrange multiplier functions.

There are two methods how to get rid of the arbitrary functions ξ_1 and ξ_2. In the first one the Hilbert space is restricted to its physical subspace, where the constraints hold, the second one is the Dirac bracket quantization method. Since only in the latter case the full information of the theory can be read off from the operators alone, this is more appropriate for our purpose. The Dirac bracket quantization of gauge theories is based on the observation that the first class constraints together with appropriate auxiliary (gauge) conditions form a set of second class constraints [23]. Then quantization may be performed along the lines described by Dirac [21] for a system with second class constraints. Hence we have to fix a gauge by introducing $\chi_3 \approx 0$ and $\chi_4 \approx 0$ such that $\det\{\chi, \chi\} \neq 0$. Then the Dirac brackets are defined by

$$\{f(x_1), g(x_2)\}_D = \{f(x_1), g(x_2)\} - \iint \{f(x_1), \chi_i(y)\} \, C_{ij}(y, z) \{\chi_j(z), g(x_2)\} \, dy \, dz, \tag{3.20}$$

where $C_{ij}(y, z)$ is the inverse of $\{\chi_i, \chi_j\}$, more explicitly

$$\int C_{ij}(x, y) \{\chi_j(y), \chi_k(z)\} \, dy = \delta_{ik} \, \delta(x - z). \tag{3.21}$$

When all Dirac brackets have been calculated, the constraints may be considered as strong equations (hence ξ_1 and ξ_2 drop out, as promised) and the equations of motion can be expressed in terms of Dirac brackets

$$\dot{f} \approx \{f, H\}_D. \tag{3.22}$$

From this point on quantization is straightforward (as long as no operator ordering problems occur): all phase space functions are converted into quantum operators, the constraints become strong operator equations and the commutators are abstracted from the corresponding Dirac brackets:

$$\{f, g\}_D = h \rightarrow [\hat{f}, \hat{g}] = i\hat{h}, \tag{3.23}$$

where $\hat{f}$ denotes the operator corresponding to the phase space function f.

We want to perform this procedure for the special gauge which corresponds to the $\partial^\mu \theta = 0$ gauge of the Lagrange formalism

$$\chi_3 = -\partial\theta \approx 0, \tag{3.24}$$

$$\chi_4 = \Pi_\theta - eA^1 + e(a-1)A^0 \approx 0. \tag{3.25}$$

Inserting χ_3 and χ_4 into χ_2 and $\mathscr{H}$, we find precisely the constraints and Hamiltonian of ref. [13], namely

$$\begin{aligned} \chi_1 &= \Pi_0 \approx 0, \\ \chi_2' &= \partial\Pi_1 - e\Pi_\phi - e\partial\phi + e^2 A^1 - e^2(a-1)A^0 \approx 0, \end{aligned} \tag{3.26}$$

$$\begin{aligned} \mathscr{H} \approx{}& \tfrac{1}{2}(\Pi_1)^2 + A^0\partial\Pi_1 + \tfrac{1}{2}[\Pi_\phi - e(A^0 + A^1)]^2 - \tfrac{1}{2}e^2 a A_\mu A^\mu \\ &+ \tfrac{1}{2}(\partial\phi)^2 - e\,\partial\phi(A^0 + A^1). \end{aligned} \tag{3.27}$$

The difference of their theory and ours is the occurrance of the additional constraints χ_3 and χ_4 which serve to eliminate θ and Π_θ. The constraint algebra is given by

$$\{\chi(x), \chi(y)\} = \begin{pmatrix} 0 & 0 & 0 & -e(a-1) \\ 0 & 0 & -e\partial_x & 0 \\ 0 & -e\partial_x & 0 & -\partial_x \\ e(a-1) & 0 & -\partial_x & 0 \end{pmatrix} \delta(x-y). \tag{3.28}$$

The determinant of $\{\chi, \chi\}$ does not vanish (we are treating the case $a \neq 1$), hence the gauge is completely fixed by Eqs. (3.24) and (3.25). The inverse C_{ij} is given by

$$C(x, y) = \begin{bmatrix} 0 & \dfrac{-1}{e^2(a-1)} & 0 & \dfrac{1}{e(a-1)} \\ \dfrac{1}{e^2(a-1)} & 0 & \dfrac{-1}{e\partial_x} & 0 \\ 0 & \dfrac{-1}{e\partial_x} & 0 & 0 \\ \dfrac{-1}{e(a-1)} & 0 & 0 & 0 \end{bmatrix} \delta(x-y). \tag{3.29}$$

From here we find the Dirac brackets

$$\{A^0(x), A^1(y)\}_D = \frac{-1}{e^2(a-1)}\partial_x \delta(x-y), \qquad \{A^1(x), \Pi_\theta(y)\}_D = \frac{1}{e}\partial_x \delta(x-y),$$

$$\{A^0(x), \phi(y)\}_D = \frac{1}{e(a-1)}\delta(x-y), \qquad \{\phi(x), \Pi_\theta(y)\}_D = \delta(x-y),$$

$$\{A^0(x), \Pi_1(y)\}_{\rm D} = \frac{1}{a-1}\,\delta(x-y), \qquad \{A^0(x), \Pi_\theta(y)\}_{\rm D} = \frac{-1}{e(a-1)}\,\partial_x\,\delta(x-y),$$

$$\{A^0(x), \Pi_\phi(y)\}_{\rm D} = \frac{-1}{e(a-1)}\,\partial_x\delta(x-y), \qquad \{\Pi_\phi(x), \Pi_\theta(y)\}_{\rm D} = \partial_x\delta(x-y),$$

$$\{A^1(x), \Pi_1(y)\}_{\rm D} = \delta(x-y),$$

$$\{\phi(x), \Pi_\phi(y)\}_{\rm D} = \delta(x-y), \tag{3.30}$$

all other Dirac brackets vanish. Note that the corresponding commutators coincide completely with those of ref. [13], except for those involving Π_θ, which merely express the dependent nature of Π_θ. Hence this gauge reproduces exactly the system of refs. [11, 13], which was derived by usage of Eq. (2.1). This establishes onces more the equivalence of the generating functionals of Eqs. (2.1) and (2.3) in this particular theory: using the gauge noninvariant formulation is nothing else but working in a specific gauge of the gauge invariant formulation of the theory. This statement will also be valid for the case $a = 1$, see below.

The equivalence of the gauge invariant and the gauge noninvariant anomalous theory also implies that the statements of the latter are valid here, too, namely [11]: for $a > 1$ the quantum theory is consistent and unitary, consisting of a massive ($m^2 = e^2a^2/(a-1)$) and a massless degree of freedom, for $0 < a < 1$ or $a < 0$ the theory contains tachyons and for $a = 0$ the theory is inconsistent since the solution of the Heisenberg equations of motion is not compatible with the commutator structure [13].

The quantum system may be formulated in the unconstrained way,

$$H = \int \{\tfrac{1}{2}\Pi_1^2 + \tfrac{1}{2}[\Pi_\phi - eA^1]^2 + \tfrac{1}{2}ae^2(A^1)^2 + \tfrac{1}{2}(\partial\phi)^2 - e(\partial\phi)\,A^1$$
$$+ \tfrac{1}{2}e^2(a-1)(A^0)^2_{\rm symm}\}\,dx, \tag{3.31}$$

where A^0 is the dependent quantity

$$A^0 \equiv \frac{1}{e^2(a-1)}\,(\partial\Pi_1 - e\Pi_\phi - e\partial\phi + e^2A^1) \tag{3.32}$$

and the canonical commutators read

$$[\phi(x), \Pi_\phi(y)] = [A^1(x), \Pi_1(y)] = i\delta(x-y). \tag{3.33}$$

The operator ordering ambiguity in A^{0^2} has been resolved by symmetrization according to the general rules of nonlinear quantum mechanics [24, 25]. Furtunately the theory is not plagued by the simultaneous presence of constraints and operator ordering ambiguities, a problem which is not yet resolved for the general case [26].

Finally we want to note that the occurrence of a mass for the gauge boson has nothing to do with an explicit breakdown of a gauge symmetry since it appears also in the gauge invariant formulation. However, it is highly questionable whether this mechanism of vector boson mass generation via "anomaly cancellation" is applicable to the realistic four dimensional case, too. Even worse, this seems to be very unprobable due to the absence of dimensionful parameters in the four dimensional case.

(b) *The Case* $a=1$

Let us now turn to $a=1$, then the Lagrange density simplifies to

$$\mathscr{L}_{\text{eff}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\phi\,\Box\phi - e\phi\,\partial_\mu(\eta-\varepsilon)^{\mu\nu}A_\nu + \tfrac{1}{2}e^2A_\mu A^\mu + e\theta\partial_\mu\varepsilon^{\mu\nu}A_\nu. \tag{3.34}$$

The canonical momenta are

$$\Pi_0 = 0, \qquad \Pi_\theta = eA^1, \tag{3.35}$$

$$\Pi_1 = -F_{01}, \qquad \Pi_\phi = \dot{\phi} + e(A^0 + A^1). \tag{3.36}$$

Equations (3.35) have to be considered as constraint equations since they do not involve velocities. Hence we have two primary constraints in this case,

$$\Omega_1 = \Pi_0 \approx 0, \tag{3.37}$$

$$\tilde{\Omega}_2 = \Pi_\theta - eA^1 \approx 0. \tag{3.38}$$

The Hamiltonian density reads

$$\begin{aligned}\mathscr{H} = {} & \tfrac{1}{2}\Pi_1^2 + A^0\,\partial\Pi_1 + \tfrac{1}{2}\Pi_\phi^2 - e\Pi_\phi(A^0 + A^1) \\ & + e^2A^1(A^0+A^1) + \tfrac{1}{2}(\partial\phi)^2 - e\partial\phi(A^0+A^1) \\ & - e\partial\phi A^0 + \xi_1\Omega_1 + \xi_2\Omega_2.\end{aligned} \tag{3.39}$$

Consistency requires $\dot{\Omega}_1 \approx \dot{\tilde{\Omega}}_2 \approx 0$, this yields

$$\tilde{\Omega}_3 = \partial\Pi_1 - e\Pi_\phi + e^2A^1 - e\partial\phi - e\partial\theta \approx 0, \tag{3.40}$$

$$\tilde{\Omega}_4 = \Pi_1 \approx 0. \tag{3.41}$$

Since $\tilde{\Omega}_3$ and $\tilde{\Omega}_4$ are second class, their time derivatives do not lead to new constraints but only fix Lagrange multipliers. We have two sets of constraints

$$\Omega_1 = \Pi_0 \approx 0, \qquad \Omega_2 = \partial\Pi_1 - e\Pi_\phi + e\Pi_\theta - e\partial\phi - e\partial\theta \approx 0, \tag{3.42}$$

$$\Omega_3 = \Pi_1 \approx 0, \qquad \Omega_4 = \Pi_\theta - eA^1 \approx 0, \tag{3.43}$$

the first being first class, the latter second class. This means that we have to impose two gauge conditions in order to eliminate the Lagrange multipliers associated with Ω_1 and Ω_2. We use as gauge fixing conditions

$$\Omega_5 = \partial\theta \approx 0, \tag{3.44}$$

$$\Omega_6 = e(A^0 + A^1) + \Pi_\theta - \Pi_\phi - \partial\phi \approx 0. \tag{3.45}$$

Then the matrix $\{\Omega, \Omega\}$ has a nonvanishing determinant, as required for an admissable gauge fixing. Finally we build new constraints by linear combination of the Ω's

$$\begin{aligned} &\chi_1 = \Pi_0 \approx 0, && \chi_2 = -\partial\Pi_1 + e\Pi_\phi + e\partial\phi - e^2 A^1 \approx 0, \\ &\chi_3 = \Pi_1 \approx 0, && \chi_4 = e(A^0 + 2A^1) - \Pi_\phi - \partial\phi \approx 0, \\ &\chi_5 = \partial\theta \approx 0, && \chi_6 = \Pi_\theta - eA^1. \end{aligned} \tag{3.46}$$

χ_1 to χ_4 may be recognized as Ω_1 to Ω_4 of ref. [13], χ_5 and χ_6 serve for the elimination of θ and π_θ. Using the fact that χ_2, χ_3, and χ_4 imply $A^0 = -A^1$, the Hamiltonian in this gauge can be written as

$$H = \int \left[\tfrac{1}{2}\Pi_\phi^2 + \tfrac{1}{2}(\partial\phi)^2\right] dx. \tag{3.47}$$

The final task is the evaluation of the Dirac brackets, to this aim we need

$$\{\chi(x), \chi(y)\} = \begin{pmatrix} 0 & 0 & 0 & -e & 0 & 0 \\ 0 & 0 & -e^2 & 0 & 0 & -e\partial_x \\ 0 & e^2 & 0 & -2e & 0 & e \\ e & 0 & 2e & 2\partial_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_x \\ 0 & -e\partial_x & -e & 0 & \partial_x & 0 \end{pmatrix} \delta(x-y) \tag{3.48}$$

and

$$C(x, y) = \frac{1}{e^2} \begin{bmatrix} 2\partial_k & 2 & 0 & e & 2e & 0 \\ -2 & 0 & 1 & 0 & \dfrac{-e}{\partial_x} & 0 \\ 0 & -1 & 0 & 0 & -e & 0 \\ -e & 0 & 0 & 0 & 0 & 0 \\ -2e & \dfrac{-e}{\partial_x} & e & 0 & \dfrac{-2e^2}{\partial_x} & \dfrac{e^2}{\partial_x} \\ 0 & 0 & 0 & 0 & \dfrac{e^2}{\partial_x} & 0 \end{bmatrix} \delta(x-y). \tag{3.49}$$

This leads via Dirac brackets to the nonvanishing equal time commutators

$$[A^0(x), \phi(y)] = \frac{i}{e}\,\delta(x-y),$$

$$[A^1(x), A^1(y)] = \frac{2i}{e^2}\,\partial_x \delta(x-y),$$

$$[A^1(x), \phi(y)] = \frac{1}{e}\,[\Pi_\theta(x), \phi(y)] = -\frac{i}{e}\,\delta(x-y),$$

$$[A^0(x), \Pi_\phi(y)] = -\frac{i}{e}\,\partial_x\,\delta(x-y), \tag{3.50}$$

$$[A^1(x), \Pi_\phi(y)] = \frac{1}{e}\,[\Pi_\theta(x), \Pi_\phi(y)] = \frac{i}{e}\,\partial_x\,\partial(x-y),$$

$$[\phi(x), \pi_\phi(y)] = i\delta(x-y),$$

which, together with Eq. (3.47), reproduces precisely the quantum system for $a=1$ of ref. [13].

Again we succeeded to find a gauge which translates the gauge invariant version of the theory into the so-called gauge noninvariant one. This implies that the statements on the physical content of the latter is valid in our case, too, i.e., also for $a=1$ the theory is consistent and it contains one free massless scalar degree of freedom.

This finishes our analysis of the canonical quantization of the chiral Schwinger model, we explicitly showed by appropriate gauge fixing the equivalence of the gauge invariant and gauge noninvariant formulation of the model. Using the results of the latter we established besides gauge invariance at the quantum level also consistency and unitarity of the chiral Schwinger model.

There has been some discussion that the spectrum might be nonrelativistic or that Lorentz invariance has been lost [16, 27]. Unfortunately these arguments rely on the nonrelativistic gauge $A^0=0$ in a formulation without gauge invariance. The authors should have used the gauge invariant formulation as presented in this paper. Then we expect also Lorentz invariance in the $A^0=0$ gauge.

The equivalence of the gauge invariant and the noninvariant formulation has already been conjectured in ref. [28], motivated by a naive consideration of the $\theta=0$ gauge in the Lagrangian. In general, however, it is not allowed to insert a gauge condition into the action before the Euler–Lagrange equations are derived, since in this way one equation is lost (Gauss' law in the temporal gauge). The reason that the conjecture of ref. [28] is correct, is the fact that the theories are identical from the very beginning, i.e., that the generating functionals coincide.

IV. Discussion

The elaborated equivalence of the gauge invariant and so-called gauge non-invariant formulation of the chiral Schwinger model clarifies the reason of the absence of anomalies in current conservation and current commutators in the latter case. The superficial anomalous divergence of the current in the gauge noninvariant case, which is given in ref. [14],

$$\partial_\mu j^\mu = e^2[(1-a)\,\partial_\mu A^\mu - \varepsilon^{\mu\nu}\partial_\mu A_\nu], \tag{4.1}$$

can be shown to vanish if the solution of the equation of motion [11]

$$A_\mu = -\frac{1}{ea}\,[\partial_\mu\phi + (1-a)\,\varepsilon_{\mu\nu}\,\partial^\nu\phi - a\varepsilon_{\mu\nu}\,\partial^\nu h], \tag{4.2}$$

where h is a harmonic function, is inserted. The current–current Poisson bracket in the noninvariant case has an anomalous Schwinger term

$$\{j_0(x), j_0(y)\} = 2e^2\,\partial_x\,\delta(x-y). \tag{4.3}$$

This Schwinger term disappears only at the level of Dirac brackets (and hence of quantum commutators).

In the gauge invariant formulation the current is given by

$$j^\nu \equiv \partial_\mu F^{\mu\nu} = -e\partial_\mu(\eta-\varepsilon)^{\mu\nu}(\phi+\theta) + ea\,\partial^\nu\theta - ae^2 A^\nu. \tag{4.4}$$

Variation of $\mathscr{L}_{\text{eff}}$ with respect to θ yields at $\theta = 0$ as a gauge condition just that Eq. (4.1) vanishes. Here current conservation, which seems to be an accident in the noninvariant formulation requiring the complete solution of the theory, is enforced by gauge invariance (cf. Eq. (3.5)) and hence automatically fullfilled. Since $\partial_\mu j^\mu$ is gauge invariant and since the noninvariant formulation is nothing but a special gauge of the gauge invariant theory, this means that current conservation does not happen by chance but that there is a symmetry principle ensuring that the current is conserved. In the Hamilton formalism j_0 is given by

$$j_0 = e(\Pi_\phi - \Pi_\theta + \partial\phi + \partial\theta). \tag{4.5}$$

This implies that there is no Schwinger term even at the Poisson bracket level

$$\{j_0(x), j_0(y)\} = 0, \tag{4.6}$$

which certainly remains valid upon quantization. From this discussion we may conclude that the absence of an anomalous divergence of the current and of a Schwinger term can always be seen one step earlier in the gauge invariant formulation.

It is just the introduction of the field θ, i.e., the requirement of gauge invariance, which makes the absence of genuine anomalies transparent. Hence it seems to be advantageous to use also in other models and higher dimensions the gauge invariant formulation of Eq. (2.8) instead of the formulation without gauge invariance of Eq. (2.1) in order to investigate whether there are genuine anomalies or not.

Though it is not a problem in two dimensions, our final remark concerns renormalizability. In four dimensions it is the loss of gauge invariance, which usually has authors led to assume that anomalous gauge theories are not renormalizable (see, e.g., [29]). Now we learned that gauge invariance is not lost, hence the question of renormalizability of "anomalous" gauge theories seems still unanswered.

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