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Average action for the Higgs model with abelian gauge symmetry

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We develop the concept of an average action for gauge theories in the continuum. The average scalar potential of the abelian Higgs model is computed in arbitrary dimensions.

1. Introduction

The concept of an effective action for averages of fields (average action) [1,2] has proven successful for a description of scalar theories in two, three or four dimensions [3]. This approach seems appropriate to settle a variety of open questions on spontaneous symmetry breaking in the standard model. It also should provide the relevant scalar potentials needed for the cosmology of phase transitions and similar issues. It may be used to overcome the infrared problems in finite-temperature field theory. The intuitive picture of the average of a scalar field $\chi(x)$ over a volume $\sim k^{-d}$ can be easily formulated in continuous space:

$$\phi_k(x) = \int d^d y\, f_k(y-x)\chi(y). \tag{1.1}$$

Here f_k is a function of $(y_\mu - x_\mu)(y^\mu - x^\mu)$ which should decrease rapidly if this quantity becomes larger than k^{-2}. The average action is a functional of the average field $\varphi(x)$. It has been constructed [2] using a gaussian constraint which enforces $\phi_k(x)$ to be approximately equal to $\varphi(x)$. The computation of the average action $\Gamma_k[\varphi]$ allows a transition from "microscopic variables" $\chi(x)$ to "macro-

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scopic variables" $\varphi(x)$. All constructions can be generalized to include chiral fermions [4].

In contrast, gauge theories pose an obvious problem: Under gauge transformations the quantity $\phi_k(x)$ (1.1) does not simply transform with a phase. As a result, the square $\phi_k^\dagger(x)\phi_k(x)$ is not gauge invariant. One can always achieve $\phi_k(x)=0$ by an appropriate space-dependent change of the phases of χ. A similar problem arises for the gauge fields: The average of a gauge field $a_\mu(x)$ makes no sense due to the inhomogeneous transformation properties of a_μ. For an abelian gauge theory one may define the average of the gauge-invariant field strength $f_{\mu\nu}=\partial_\mu a_\nu - \partial_\nu a_\mu$ in analogy to (1.1). This corresponds to the intuitive picture of the average of electric or magnetic fields. This construction cannot be generalized, however, to nonabelian gauge theories. An average of $f^a_{\mu\nu}$ according to (1.1) does not simply transform according to the adjoint representation. We need a generalization of (1.1) which adapts the concept of an average to gauge dependent fields.

The necessary generalization becomes most obvious in momentum space where (1.1) reads

$$\phi_k(q) = f_k(q)\chi(q). \tag{1.2}$$

Here $f_k(q)$ is the Fourier transform of $f_k(x)$. It is a function of q^2 which decreases rapidly for $q^2 \gg k^2$ *. Equivalently we may write (1.1) in the form

$$\phi_k(x) = f_k(-\Box)\chi(x). \tag{1.3}$$

The gauge covariant generalization of (1.3) replaces all derivatives by covariant derivatives in the "background" of the "average gauge field" $A_\mu(x)$

$$\partial_\mu \to D_\mu(A), \qquad \phi_k(x) = f_k(-D^2(A))\chi(x) \tag{1.4}$$

Here we require that the macroscopic gauge field A_μ has the same gauge-transformation properties as the microscopic field a_μ. This prescription achieves that the gauge transformations of $\phi_k(x)$ are the same as for $\chi(x)$, namely homogeneous transformations with phases. We also postulate the same gauge-transformation properties of the macroscopic and microscopic scalar fields φ and χ. It is then easy to construct gauge-invariant quantities as $(\varphi(x)-\phi_k(x)^\dagger\cdot(\varphi(x)-\phi_k(x))$. They can be used to implement a gauge-invariant constraint.
A similar prescription can be used to define the average of the (nonabelian) field strength $f^a_{\mu\nu}$:

$$F^a_{k,\mu\nu}(x) = f_k(-D^2(A))f^a_{\mu\nu}(x). \tag{1.5}$$

* We use from here on always the representation of f_k as a function of q^2. Details on f_k may be found in ref. [2].

The covariant derivative should now be taken in the adjoint representation. (For abelian gauge theories one recovers $D^2(A) = \Box$.) Defining by $F^a_{\mu\nu}$ the gauge-covariant macroscopic field strength formed from A^a_μ we can again find gauge-invariant quantities as $(F^a_{\mu\nu}(x) - F^a_{k,\mu\nu}(x))(F^{\mu\nu}_a(x) - F^{\mu\nu}_{k,a}(x))$. This allows the construction of a suitable constraint for the field strength. At this stage it only remains to implement a constraint for the longitudinal component of the gauge field. The longitudinal mode does not contribute to $f_{\mu\nu}$ and therefore remains unconstrained if we only employ a constraint for the field strength. For the "longitudinal constraint" we will exploit the fact that the difference $A^a_\mu(x) - a^a_\mu(x)$ transforms homogeneously. The quantity $D^\mu_{ac}(A)(A^c_\mu - a^c_\mu)\ D^{ab}_\nu(A)(A^\nu_b - a^\nu_b)$ with $D(A)$ in the adjoint representation is gauge invariant.

In this paper we will carry out the construction of the gauge-invariant average action explicitly for the abelian Higgs model. We perform a one loop computation of the average scalar potential and the scalar kinetic term in arbitrary dimensions. We obtain evolution equations for the dependence on the average scale k. In four dimensions these equations agree with the perturbative renormalization group equations for small couplings if all masses are small compared to the average scale k. The infrared behaviour for k smaller than the physical particle masses depends to some extent on the choice of the "averaging scheme" and will be discussed in detail. For dimensions smaller than four naive perturbation theory fails for $\varphi = 0$ due to strong infrared divergences arising from the fluctuations of the massless gauge boson. Our method allows to approach the infrared limit $k \to 0$ smoothly. The evolution equations should give a valid description of the phase transitions in two and three dimensions. In particular, this will be relevant for the understanding of finite-temperature gauge field theories in four dimensions.

2. The average action of gauge theories

We consider here the abelian gauge theory describing the interaction of a massive or massless photon with a complex scalar field. The euclidean action reads

$$S[\chi, a_\mu] = \int d^dx \left\{ \tfrac{1}{4} f_{\mu\nu} f^{\mu\nu} + |D_\mu(a)\chi|^2 + V(\chi^*\chi) \right\}, \tag{2.1}$$

where

$$D_\mu(a) \equiv \partial_\mu + i\bar{e}a_\mu, \tag{2.2}$$

$$f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu, \tag{2.3}$$

and

$$V(\chi^*\chi) = -\bar{\mu}^2(\chi^*\chi) + \tfrac{1}{2}\bar{\lambda}(\chi^*\chi)^2. \tag{2.4}$$

We shall be mainly interested in the spontaneously broken regime with $\bar{\mu}^2 > 0$. Furthermore, the dimensionality of space-time, d, will be kept arbitrary for most of the discussion. The fundamental action S, depending on the "microscopic" variables $\chi(x)$ and $a_\mu(x)$, gives rise to an average action Γ_k, depending on the average (or "macroscopic") fields $\varphi(x)$ and $A_\mu(x)$:

$$\exp\{-\Gamma_k[\varphi, A_\mu]\} = \int \mathscr{D}\chi\, \mathscr{D}a_\mu \exp\{-S_k[\chi, a_\mu; \varphi, A_\mu]\}, \tag{2.5}$$

$$S_k[\chi, a_\mu; \varphi, A_\mu] \equiv S[\chi, a_\mu] + S_{\text{constr}}[\chi, a_\mu; \varphi, A_\mu]. \tag{2.6}$$

The constraint implementing the averaging procedure is chosen as

$$\begin{aligned} S_{\text{constr}} = \int \mathrm{d}^d x \Bigg\{ & \tfrac{1}{4}\left(F_{\mu\nu} - f_k(-\Box) f_{\mu\nu}\right) \frac{Z_F(-\Box)}{1 - f_k^2(-\Box)} \left(F^{\mu\nu} - f_k(-\Box) f^{\mu\nu}\right) \\ & + \frac{1}{2\alpha} \partial_\mu (A^\mu - a^\mu) \frac{Z_G(-\Box)}{1 - f_k^2(-\Box)} \partial_\nu (A^\nu - a^\nu) \\ & - \left[\varphi - f_k(-D^2(A))\chi\right]^* \frac{D^2(A) Z_\varphi(-D^2(A))}{1 - f_k^2(-D^2(A))} \left[\varphi - f_k(-D^2(A))\chi\right] \Bigg\}. \end{aligned} \tag{2.7}$$

Here $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu(A) \equiv \partial_\mu + i\bar{e}A_\mu$ are expressed in terms of the average variable $A_\mu(x)$. The function

$$f_k(q^2) = \exp\left[-a(q^2/k^2)^\beta\right] \tag{2.8}$$

parametrizes a family of different averaging schemes depending on the constants a and β. Obviously, $S_k[\chi, a_\mu; \varphi, A_\mu]$ is invariant under a *simultaneous* gauge transformation of the microscopic variables (χ, a_μ) and the macroscopic ones, (φ, A_μ). Therefore, and because we can specify the measure $\mathscr{D}\chi\, \mathscr{D}a_\mu$ in a gauge-invariant way, $\Gamma_k[\varphi, A_\mu]$ is a gauge-invariant functional of the average fields. Our formulation can be generalized to nonabelian gauge theories. One should replace in (2.7) all derivatives acting on gauge fields by covariant derivatives in the adjoint representation. (The covariant derivatives acting on matter fields should be taken in their respective representations, of course.) We also note that no gauge fixing is needed for a computation of Γ_k. In fact, for *fixed* fields (φ, A_μ) the constraint is not invariant under gauge transformations of χ and a_μ alone. As a result, no infinite factor arises in the $\mathscr{D}\chi\mathscr{D}a$ integration from the volume of the gauge group. For given macroscopic fields the constraint acts similar to a background gauge fixing term. We finally observe that the constraint is not yet normalized here. The normalization involves a determinant depending only on A_μ. This issue will be discussed in a separate paper [5] where we will also present more formal properties of the average action for gauge theories.

Gauge invariance and rotation and translation symmetry allow us to write down a derivative expansion in terms of gauge-invariant field monomials ($\rho = \varphi^*\varphi$):

$$\Gamma_k[\varphi, A_\mu] = \int d^d x \Big\{ U_k(\rho) + Z_{\varphi,k}(\rho) |D_\mu(A)\varphi|^2 + \tfrac{1}{4} Z_{F,k}(\rho) F_{\mu\nu} F^{\mu\nu} + \dots \Big\}. \tag{2.9}$$

For a discussion of all relevant monomials with up to two derivatives we refer to appendix A.

In this paper we are mainly interested in the average potential $U_k(\rho)$ and the scalar wave function renormalization $Z_{\varphi,k}(\rho)$. These quantities are needed in order to determine the mass term for the vector boson. From eq. (2.9) we read off

$$M^2(k) = 2\bar{e}^2 \rho_0(k) Z_{\varphi,k}(\rho_0) Z_{F,k}^{-1}(\rho_0), \tag{2.10}$$

where ρ_0 is the (k-dependent) location of the minimum of U_k. The physical vector-boson mass in the vacuum (defined at $q^2 = 0$) obtains as $\lim_{k\to 0} M(k)$. In order to determine U_k we evaluate the path integral (2.5) for a real, x^μ-independent configuration $\varphi = \text{const.}$ along with $A_\mu \equiv 0$. Similarly, to find $Z_{\varphi,k}$, one uses $\varphi = \text{const.}$ together with a small x^μ-independent vector field $A_\mu = \text{const.}$ Then $|D_\mu(A)\varphi|^2 = \bar{e}^2 A^2 \varphi^2$, and $Z_{\varphi,k}$ can be extracted as the coefficient of the term which is quadratic both in φ and in A_μ where A_μ can be taken infinitesimal. (For details see appendix A.) Using this method we shall compute the one-loop approximation of U_k and $Z_{\varphi,k}$ in the following sections. We do not need to compute $Z_{F,k}$ here since it can be absorbed in the definition of a renormalized gauge coupling.

3. One-loop approximation of the average action

In this section we start the semiclassical evaluation of the functional integral (2.5) with

$$S_k = \int d^d x \Big\{ \tfrac{1}{2} a_\mu [-\Box \eta^{\mu\nu} + \partial^\mu \partial^\nu] a_\nu + \chi^* [-D^2(a) - \bar{\mu}^2] \chi + \tfrac{1}{2}\bar{\lambda}(\chi^*\chi)^2$$

$$+\tfrac{1}{2}(A_\mu - f_k(-\Box)a_\mu)\frac{[-\Box\eta^{\mu\nu}+\partial^\mu\partial^\nu]}{1-f_k^2(-\Box)}(A_\nu - f_k(-\Box)a_\nu)$$

$$-\frac{1}{2\alpha}(A_\mu - a_\mu)\frac{\partial^\mu\partial^\nu}{1-f_k^2(-\Box)}(A_\nu - a_\nu)$$

$$-\left[\varphi - f_k(-D^2(A))\chi\right]^*\frac{D^2(A)}{1-f_k^2(-D^2(A))}\left[\varphi - f_k(-D^2(A))\chi\right]\Big\}. \quad (3.1)$$

For the simplicity of the presentation we have set the various Z-factors in (2.7) equal to unity here. More generally, their role is completely analogous to that in ref. [3]. We expand the integration variables χ and a_μ around the configuration $(\chi^{\min}, a_\mu^{\min})$ for which S_k assumes its minimum:

$$\begin{aligned}\chi(x) &= \chi^{\min}(x) + \delta\chi(x),\\ a_\mu(x) &= a_\mu^{\min}(x) + \delta a_\mu(x).\end{aligned} \quad (3.2)$$

The first variation of S_k yields the following equations for the minimum:

$$\left[-D^2(a) - \bar{\mu}^2 + \bar{\lambda}|\chi|^2\right]\chi + \frac{D^2(A) f_k(-D^2(A))}{1-f_k^2(-D^2(A))}\left[\varphi - f_k(-D^2(A))\chi\right] = 0, \quad (3.3a)$$

$$\begin{aligned}&\left[-\Box\eta^{\mu\nu} + \partial^\mu\partial^\nu + 2\bar{e}^2|\chi|^2\eta^{\mu\nu}\right]a_\nu\\ &-\frac{f_k(-\Box)}{1-f_k^2(-\Box)}[-\Box\eta^{\mu\nu}+\partial^\mu\partial^\nu](A_\nu - f_k(-\Box)a_\nu)\\ &+\frac{1}{\alpha}\frac{\partial^\mu\partial^\nu}{1-f_k^2(-\Box)}(A_\nu - a_\nu) = \bar{e}\chi^* i\overleftrightarrow{\partial}^\mu\chi.\end{aligned} \quad (3.3b)$$

They determine $(\chi^{\min}, a_\mu^{\min})$ as a functional of (φ, A_μ) if the solution is unique. Otherwise the minimum has to be selected among the different extrema. For the one-loop approximation we also need the second variation of S_k:

$$\delta^2 S_k = \int d^d x\left\{\delta a_\mu\left[\frac{\{-\Box\eta^{\mu\nu} + (1-\alpha^{-1})\partial^\mu\partial^\nu\}}{1-f_k^2(-\Box)} + 2\bar{e}^2|\chi|^2\eta^{\mu\nu}\right]\delta a_\nu\right.$$

$$+2\delta\chi^*\left[-D^2(a) - \frac{D^2(A) f_k^2(-D^2(A))}{1-f_k^2(-D^2(A))} - \bar{\mu}^2 + 2\bar{\lambda}|\chi|^2\right]\delta\chi$$

$$+\bar{\lambda}\left[\chi^2(\delta\chi^*)^2+(\chi^*)^2(\delta\chi)^2\right]$$

$$+4\bar{e}^2a^\mu\delta a_\mu[\chi\delta\chi^*+\chi^*\delta\chi]$$

$$\left.-2i\bar{e}\delta a_\mu\left[\delta\chi^*\overleftrightarrow{\partial}^\mu\chi+\chi^*\overleftrightarrow{\partial}^\mu\delta\chi\right]\right\}. \tag{3.4}$$

Inserting the minimum $(\chi^{\min}, a_\mu^{\min})$ into $\delta^2 S_k$ we obtain the quadratic action which enters the saddle-point evaluation of the integral (2.5). The one-loop approximation of the average action reads

$$\Gamma_k[\varphi, A_\mu]=S_k[\chi^{\min}, a_\mu^{\min}]+\Gamma_k^{(1)}[\varphi, A_\mu]+\ldots, \tag{3.5}$$

with

$$\exp\left\{-\Gamma_k^{(1)}[\varphi, A_\mu]\right\}=\int\mathscr{D}(\delta\chi, \delta a_\mu)\exp\left\{-\tfrac{1}{2}\delta^2S_k\left[\chi^{\min}, a_\mu^{\min}; \delta\chi, \delta a_\mu\right]\right\}, \tag{3.6}$$

where $(\chi^{\min}, a_\mu^{\min})$ is considered as a functional of (φ, A_μ). Up to now our formulas are completely general, and no particular argument (φ, A_μ) of Γ_k has been specified. As we have explained already, for the determination of U_k it is sufficient to work with the field configuration $\varphi=$ real constant, $A_\mu\equiv 0$. In this case a solution to eqs. (3.3) is given by

$$\chi^{\min}(x)=\varphi=\text{const.},$$

$$a_\mu^{\min}(x)=0. \tag{3.7}$$

In appendix B we show that this stationary point is indeed a minimum (i.e., $\delta^2 S_k>0$ for all $\delta\chi$ and δa_μ) provided

$$\varphi^2>\frac{\mu^2}{\bar{\lambda}}-\Delta\varphi^2. \tag{3.8}$$

where $\Delta\varphi^2$ is some strictly positive quantity and $\varphi_0^2\equiv\bar{\mu}^2/\bar{\lambda}$ corresponds to the minimum of the classical potential $V(\varphi^2)=-\bar{\mu}^2\varphi^2+\frac{1}{2}\bar{\lambda}\varphi^4$. The precise value of $\Delta\varphi^2$ depends on the choice of the function f_k. Irrespective of the precise form of f_k the stationary point (3.7) is stable even for values of φ^2 (slightly) smaller than the classical minimum φ_0^2. The average potential around the minimum can therefore be reliably calculated in a loop expansion around the configuration (3.7). With $\rho=\varphi^2$ one has

$$U_k(\rho)=V(\rho)+U_k^{(1)}(\rho)+\ldots \tag{3.9}$$

This argument generalizes to all configurations which deviate only infinitesimally from (3.7), as for the case of an infinitesimally small gauge field A_μ.

In order to perform the gaussian integration in (3.6) it is necessary to diagonalize the quadratic form $\delta^2 S_k$ given in eq. (3.4). This is achieved by changing to a new field basis in which the different modes can be decoupled more easily. In appendix B we give the details of this transformation for an arbitrary background, i.e. without using a particular form of $(\chi^{\min}, a_\mu^{\min})$. Then, in appendix C, we evaluate the gaussian integrations for the special case of $\varphi = \text{const.}$, $A_\mu = \text{const.}$ with infinitesimally small A_μ.

4. One-loop average potential

The one loop contribution to the average potential can be written as a sum

$$U_k^{(1)}(\rho) = U_t(\rho) + U_\sigma(\rho) + U_{\ell\omega}(\rho), \tag{4.1}$$

with

$$U_t(\rho) = (d-1) v_d \int \mathrm{d}x\, x^{(d/2)-1} \ln\left[P(x) + 2\bar{e}^2\rho\right], \tag{4.2}$$

$$U_\sigma(\rho) = v_d \int \mathrm{d}x\, x^{(d/2)-1} \ln\left[P(x) + 3\bar{\lambda}\rho - \bar{\mu}^2\right], \tag{4.3}$$

$$U_{\ell\omega}(\rho) = v_d \int \mathrm{d}x\, x^{(d/2)-1} \ln\Big[\left\{P(x) + \bar{\lambda}\rho - \bar{\mu}^2\right\} \times \left\{P(x) + 2\alpha\bar{e}^2\rho\right\} - 2\alpha\bar{e}^2\rho x\Big], \tag{4.4}$$

where we use the abbreviations $(x = q^2)$

$$P(x) \equiv x\left[1 - f_k^2(x)\right]^{-1} \tag{4.5}$$

and

$$v_d \equiv \left[2^{d+1}\pi^{d/2}\Gamma(d/2)\right]^{-1} \tag{4.6}$$

Here U_t and U_σ are the contributions of the transverse photon and the radial scalar mode, respectively, and $U_{\ell\omega}$ is due to the coupled system of Goldstone boson and longitudinal photon. The $x \equiv q^2$ integrations in (3.11) converge for $x \to \infty$ only after subtraction of the conventional k-independent counterterms which absorb the UV divergences. We shall not need these counterterms explicitly, since we are only interested in the k-dependence of U_k. The derivative $\partial U_k^{(1)}/\partial k$ is

UV finite. (To give a well-defined meaning to eqs. (4.2)–(4.4) themselves, and to the formal manipulations leading to them, we can adopt some conventional UV regularization scheme which respects gauge invariance, dimensional regularization, say. After having performed the derivative with respect to k, this regularization can be removed. The same remark also applies to Γ_k as a whole.)

We want to study how the shape of the average potential U_k changes as a function of the length scale k^{-1}. In particular, we are interested in the k-dependence of the location of the minimum at $\rho_0(k)$ defined by $U_k'(\rho_0(k)) = 0$. (Primes denote derivatives with respect to $\rho = \varphi^2$.) Another quantity of interest is the quartic scalar coupling defined at the minimum

$$\bar{\lambda}(k) = U_k''(\rho_0(k)), \tag{4.7}$$

which determines the mass term for the radial model $\sim 2\bar{\lambda}\rho_0$. The running of these quantities is given [3] by *

$$\bar{\delta} \equiv \frac{\partial}{\partial t}\rho_0 \equiv k\frac{\partial}{\partial k}\rho_0(k) = -\bar{\lambda}^{-1}(k)k\frac{\partial}{\partial k}U_k'(\rho_0(k)), \tag{4.8}$$

$$\bar{\beta}_\lambda \equiv \frac{\partial}{\partial t}\bar{\lambda} \equiv k\frac{\partial}{\partial k}\bar{\lambda}(k) = k\frac{\partial}{\partial k}U_k''(\rho_0(k)), \tag{4.9}$$

where $t = \ln(k/k_0)$. From eqs. (4.2)–(4.4) we obtain for the first derivatives with respect to ρ:

$$\frac{\partial}{\partial t}U_t'(\rho) = 2(d-1)v_d\bar{e}^2k^{d-2}L_1^d(2\bar{e}^2\rho), \tag{4.10}$$

$$\frac{\partial}{\partial t}U_\sigma'(\rho) = 3v_d\bar{\lambda}k^{d-2}L_1^d\big(\bar{\lambda}(3\rho-\rho_0)\big), \tag{4.11}$$

$$\frac{\partial}{\partial t}U_{\ell\omega}'(\rho) = v_d\bar{\lambda}k^{d-2}L_1^d\big(\bar{\lambda}(\rho-\rho_0)\big) + \hat{\delta}_\alpha(\rho). \tag{4.12}$$

Similarly the second derivatives read

$$\frac{\partial}{\partial t}U_t''(\rho) = -4(d-1)v_d\bar{e}^4k^{d-4}L_2^d(2\bar{e}^2\rho), \tag{4.13}$$

$$\frac{\partial}{\partial t}U_\sigma''(\rho) = -9v_d\bar{\lambda}^2k^{d-4}L_2^d\big(\bar{\lambda}(3\rho-\rho_0)\big), \tag{4.14}$$

$$\frac{\partial}{\partial t}U_{\ell\omega}''(\rho) = -v_d\bar{\lambda}^2k^{d-4}L_2^d\big(\bar{\lambda}(\rho-\rho_0)\big) + \hat{\beta}_\alpha(\rho). \tag{4.15}$$

* By using the partial derivative on the r.h.s. of (4.9) we omit an additional term $\sim \bar{\delta}U_k'''(\rho_0)$ which accounts for the k-dependence of the point of definition of $\bar{\lambda}$ [3].

Here we employ the integrals

$$L_n^d(w) \equiv k^{2n-d} \int_0^\infty \mathrm{d}x\, x^{(d/2)-1} \frac{\partial}{\partial t} [P(x) + w]^{-n}, \tag{4.16}$$

which have been evaluated in ref. [3]. We also use the renormalization group improvement [3] replacing on the r.h.s. of (4.10)–(4.15) the bare parameters $\bar{\lambda}$, $\bar{\mu}^2$ by the k-dependent quantities $\lambda(k)$, $\lambda(k)\rho_0(k)$. (This also applies to $\bar{e}$ even though we do not discuss the running of the gauge coupling in this paper.) We observe that the first term in (4.15) diverges for $\rho = \rho_0$, $k \to 0$. This reflects the strong (power law) infrared singularity due to the "Goldstone mode" which invalidates naive perturbation theory in less than four dimensions. We can deal with this problem by using a finite infrared cutoff k and compute explicitly the behaviour for $k \to 0$ by following the evolution equations.

The contributions $\hat{\delta}_\alpha$ and $\hat{\beta}_\alpha$ vanish for $\alpha = 0$. In this case $U_{\ell\omega}^{(1)}$ exactly reproduces the one loop contribution of the Goldstone boson in the theory without gauge boson. Only the transversal gauge boson contributes the term $U_{\mathrm{t}}^{(1)}$ in addition to the pure scalar theory. For general α one finds

$$\begin{aligned} \bar{\delta}_\alpha &\equiv -\bar{\lambda}^{-1}\hat{\delta}_\alpha(\rho_0) \\ &= -2v_d \alpha \bar{e}^2 \rho_0 k^{d-4} G_{1,1,0}^d(2\alpha\bar{e}^2\rho_0) \\ &\quad - 2v_d \alpha \bar{e}^2 \bar{\lambda}^{-1} k^{d-2} G_{1,0,1}^d(2\alpha\bar{e}^2\rho_0), \end{aligned} \tag{4.17}$$

$$\begin{aligned} \bar{\beta}_\alpha &\equiv \hat{\beta}_\alpha(\rho_0) \\ &= 4v_d \alpha \bar{e}^2 \bar{\lambda} k^{d-4} G_{2,-1,0}^{d+4}(2\alpha\bar{e}^2\rho_0) \\ &\quad - 4v_d \alpha^2 \bar{e}^4 k^{d-4} G_{2,0,2}^d(2\alpha\bar{e}^2\rho_0) \\ &\quad - 4v_d \alpha \bar{e}^2 \bar{\lambda}^2 \rho_0 k^{d-6} G_{1,2,0}^{d-2}(2\alpha\bar{e}^2\rho_0) \\ &\quad - 4v_d \alpha^2 \bar{e}^4 \bar{\lambda}^2 \rho_0^2 k^{d-8} G_{2,2,0}^d(2\alpha\bar{e}^2\rho_0). \end{aligned} \tag{4.18}$$

The dimensionless integrals

$$G_{n,p,r}^d(w) = k^{4n-2r-d} \int_0^\infty \mathrm{d}x\, x^{(d/2)-1} \times \frac{\partial}{\partial t} \left\{ \left(\frac{x}{P(x)} \right)^p \frac{(P(x) - x)^r}{\left[P(x)^2 + w(P(x) - x) \right]^n} \right\} \tag{4.19}$$

are evaluated in appendix D. For small values of w/k^2 (linear regime) we can expand the integrals L and G in powers of w and express $G(0)$ in terms of $L(0)$.

For large w/k^2 (Goldstone regime) only the "Goldstone boson" contributes in leading order for $\alpha = 0$:

$$\bar{\delta} = -v_d k^{d-2} L_1^d(0),$$

$$\bar{\beta}_\lambda = -v_d \bar{\lambda}^2 k^{d-4} L_2^d(0). \tag{4.20}$$

For $d > 2$ the running of $\rho_0(k)$ becomes negligible in the Goldstone regime. Then the model is in the spontaneously broken phase with a well defined "expectation value" $\langle \rho \rangle = \lim_{k \to 0} \rho_0(k)$ which determines the gauge-boson mass (2.10). This picture changes, however, for $\alpha > 0$. The leading term in $\bar{\delta}$ for $k^2 \ll 2\alpha \bar{e}^2 \rho_0$ is now given by the first term of (4.17):

$$\bar{\delta} = 4 v_d g_1^d \alpha \bar{e}^2 \rho_0 k^{d-4}. \tag{4.21}$$

In four dimensions ($g_1^4 = 1$) the minimum of the potential moves even for $k^2 \ll 2\alpha \bar{e}^2 \rho_0$ according to

$$k \frac{\partial}{\partial k} \rho_0(k) = \frac{\alpha \bar{e}^2}{8\pi^2} \rho_0(k). \tag{4.22}$$

The mixing between longitudinal gauge boson and Goldstone mode drives $\rho_0(k)$ to zero for $k \to 0$

$$\rho_0(k) \sim k^{\alpha \bar{e}^2 / 8\pi^2} \tag{4.23}$$

and the vacuum expectation value vanishes in this formulation. We will discuss the physics of these different pictures once wave function renormalization effects are included in sect. 5.

Similar as in ref. [3] we introduce the dimensionless quantities (with $Z_\varphi \equiv Z_{\varphi,k}(\rho_0)$, $Z_F \equiv Z_{F,k}(\rho_0)$)

$$\kappa(k) = k^{2-d} Z_\varphi \rho_0(k), \tag{4.24}$$

$$\lambda(k) = k^{d-4} Z_\varphi^{-2} \bar{\lambda}(k), \tag{4.25}$$

$$e^2(k) = k^{d-4} Z_F^{-1} \bar{e}^2(k). \tag{4.26}$$

In the linear regime, for $k^2 \gg 2\bar{e}^2\rho_0$, $2\alpha\bar{e}^2\rho_0$, $2\bar{\lambda}\rho_0$ we find the following evolution equations:

$$\begin{aligned}\frac{\partial}{\partial t}\kappa \equiv \beta_\kappa \\ &= (2-d-\eta)\kappa \\ &\quad + 4(d-1)v_d l_1^d e^2/\lambda - 8(d-1)v_d l_2^d (e^4/\lambda)\kappa \\ &\quad + 8v_d l_1^d - 12 v_d l_2^d \lambda\kappa \\ &\quad + 4v_d\alpha\left[l_1^d - l_2^{d+2}\right]e^2/\lambda \\ &\quad + 4v_d\alpha l_3^{d+2}e^2\kappa \\ &\quad - 8v_d\alpha^2\left[l_2^d - 2l_3^{d+2} + l_4^{d+4}\right](e^4/\lambda)\kappa,\end{aligned} \tag{4.27}$$

$$\begin{aligned}\frac{\partial}{\partial t}\lambda \equiv \beta_\lambda \\ &= (d-4+2\eta)\lambda \\ &\quad + 8(d-1)v_d l_2^d e^4 \\ &\quad + 20 v_d l_2^d \lambda^2 \\ &\quad - 8v_d\alpha l_3^{d+2}\lambda e^2 \\ &\quad + 8v_d\alpha^2\left[l_2^d - 2l_3^{d+2} + l_4^{d+4}\right]e^4.\end{aligned} \tag{4.28}$$

Here l_n^d are positive constants of order one which depend in general on the averaging scheme [3]. Only for special values of n and d are they independent of β:

$$l_n^{2n} = 1. \tag{4.29}$$

The anomalous dimension

$$\eta = -\frac{\partial}{\partial t}\ln Z_\varphi$$

will be computed in sect. 5. For $k^2 \ll 2\bar{e}^2\rho_0$, $2\alpha\bar{e}^2\rho_0$, $2\bar{\lambda}\rho_0$ one finds in leading order

$$\beta_\kappa = \left[2-d-\eta+4v_d g_1^d \alpha e^2\right]\kappa + 2v_d l_1^d. \tag{4.30}$$

Finally, we evaluate the scale dependence of the potential at the origin. This is relevant for the "symmetric regime" where the potential minimum occurs at $\rho_0 = 0$. We define

$$\overline{m}_0^2(k) = U_k'(0),$$
$$\overline{\lambda}_0(k) = U_k''(0), \tag{4.31}$$

and insert the relevant arguments in eqs. (4.10)–(4.15). One obtains

$$\frac{\partial}{\partial t}\overline{m}_0^2 = 2v_d k^{d-2}\Big\{(d-1)\bar{e}^2 L_1^d(0) + \alpha\bar{e}^2\big[L_1^d(0) - L_2^{d+2}(0)\big] + 2\overline{\lambda}_0 L_1^d\big(\overline{m}_0^2\big)\Big\}, \tag{4.32}$$

$$\frac{\partial}{\partial t}\overline{\lambda}_0 = -2v_d k^{d-4}\Big\{2(d-1)\bar{e}^4 L_2^d(0) - 2\alpha\bar{e}^2\overline{\lambda}_0 L_3^{d+2}(0) + 2(\alpha\bar{e}^2)^2\big[L_2^d(0) - 2L_3^{d+2}(0) + L_4^{d+4}(0)\big] + 5\overline{\lambda}_0^2 L_2^d\big(\overline{m}_0^2\big)\Big\}. \tag{4.33}$$

Due to the massless gauge bosons we observe in eq. (4.33) the well-known infrared power singularity for $d < 4$ and $k \to 0$. This singularity forbids the naive perturbative treatment of the origin in less than four dimensions. It constitutes one of the major problems in finite-temperature field theory in four dimensions. In our case we can smoothly follow the evolution for $k \to 0$. Defining a dimensionless coupling λ as in (4.25) we recover the evolution equation (4.28) in the limit $\overline{m}_0^2 \ll k^2$. For $\overline{m}_0^2 \gg k^2$ the term $\sim \lambda^2$ effectively drops out since it is now multiplied by the threshold function $s_2^d = L_2^d(\overline{m}^2)/L_2^d(0)$. The renormalized mass term is defined as

$$m_0^2(k) = Z_\varphi^{-1}\overline{m}_0^2(k), \tag{4.34}$$

and the corresponding dimensionless quantity is obtained by multiplication with k^{-2}.

5. Wave-function renormalization and anomalous dimensions

In this section we calculate the one-loop approximation of the renormalization constant $Z_{\varphi,k}(\rho)$ defined in eq. (2.9). We evaluate $\Gamma_k[\varphi, A_\mu]$ for constant fields φ

and A_μ, and extract $Z_{\varphi,k}$ as the coefficient of $\bar{e}^2\varphi^2A^2$. It is sufficient to consider infinitesimally-small fields A_μ. In order to determine the minimum of S_k, we have to solve the eqs. (3.3a) and (3.3b) for constant average fields. For our purposes it is sufficient to find the solution in second order in A_μ. Keeping in mind that $f_k(-D^2(A)) = f_k(\bar{e}^2A^2)$ when applied to a constant function, and that according to (2.6) the denominators $[1-f_k^2(\bar{e}^2A^2)]$ are of order $A^{2\beta}$ for small values of A^2, it is not hard to see that

$$\chi^{\min} = \varphi + \mathrm{O}(A^{2\beta}),$$

$$a_\mu^{\min} = A_\mu + \mathrm{O}(A^{2\beta}), \tag{5.1}$$

is a solution of eqs. (3.3). (The correction terms are proportional to A^4 for $\beta = 2$ and higher order in A_μ for larger values of β.) An infinitesimally-small A_μ field cannot cause instabilities and the stationary point (5.1) is a true minimum provided the condition (3.8) is met. (See also appendix B.) In order to write down the Gaussian integral for the one-loop contribution $\Gamma_k^{(1)}$ we have to insert the minimum (5.1) into the second variation $\delta^2 S_k$. The resulting quadratic form is displayed in eq. (B.7) of appendix B, and the integration is performed in appendix C. There we find

$$\Gamma_k^{(1)}[\varphi, A_\mu] = \int \mathrm{d}^d x \left\{U_k^{(1)}(\rho) + \bar{e}^2 A_\mu A^\mu \rho Z_{\varphi,k}^{(1)}(\rho) + \mathrm{O}\left(A_\mu^4\right)\right\}, \tag{5.2}$$

with

$$Z_{\varphi,k}^{(1)} = Z_\mathrm{t} + Z_\sigma + Z_{\ell\omega}, \tag{5.3}$$

where (recall that $\rho \equiv \varphi^2$)

$$Z_\mathrm{t}(\rho) = -8v_d(1-d^{-1})\bar{e}^2 \int \mathrm{d}x\, x^{(d/2)-1}\left[P(x) + 2\bar{e}^2\rho\right]^{-1}$$

$$\times \left[P(x) + 3\bar{\lambda}\rho - \bar{\mu}^2\right]^{-1}, \tag{5.4}$$

$$Z_\sigma(\rho) = v_d \rho^{-1} \int \mathrm{d}x\, x^{(d/2)-1}\left\{\dot{P}(x) + 2d^{-1}x\ddot{P}(x)\right\}$$

$$\times \left[P(x) + 3\bar{\lambda}\rho - \bar{\mu}^2\right]^{-1}, \tag{5.5}$$

$$Z_{\ell\omega}(\rho) = v_d \rho^{-1} \int dx\, x^{(d/2)-1} \Big[P^2 + 2\alpha\bar{e}^2\rho(P-x) + \big(\bar{\lambda}\rho - \bar{\mu}^2\big)\big(P + 2\alpha\bar{e}^2\rho\big)\Big]^{-1}$$

$$\times \left\{ \left(\frac{2}{d} x\dot{P} - \frac{4}{d} \frac{x\dot{P}^2}{P + 3\bar{\lambda}\rho - \bar{\mu}^2} \right) \big(P + 2\alpha\bar{e}^2\rho\big) - \frac{8}{d}\alpha\bar{e}^2\rho \frac{P - 2x\dot{P} + \bar{\lambda}\rho - \bar{\mu}^2}{P + 3\bar{\lambda}\rho - \bar{\mu}^2} \right\}. \tag{5.6}$$

The subscripts t, σ and $\ell\omega$ designate the contributions of the transverse vector field, the radial scalar mode, and the coupled system of Goldstone bosons and longitudinal photons, respectively. Again, as they stand, the integrals (5.4)–(5.6) are divergent for $x \to \infty$, but they become UV finite by subtracting a set of k-independent counter terms. In practice this is not necessary because we are only interested in the anomalous dimension

$$\eta = -\frac{\partial}{\partial t} \ln Z_{\varphi,k}(\rho_0), \tag{5.7}$$

which is finite. Recalling that $Z_{\varphi,k} = 1 + Z^{(1)}_{\varphi,k} +$ higher loops, we obtain in lowest order

$$\eta = -\frac{\partial}{\partial t}\big[Z_{\mathrm{t}}(\rho_0) + Z_\sigma(\rho_0) + Z_{\ell\omega}(\rho_0)\big]$$

$$= \eta_{\mathrm{t}} + \eta_\sigma + \eta_{\ell\omega}. \tag{5.8}$$

(We neglect here the contribution $-Z'_{\varphi,k}(\rho_0)\bar{\delta}$ from the running of $\rho_0(k)$.) The t-derivative of Z_{t} and Z_σ can be expressed as

$$\eta_{\mathrm{t}} = 8v_d(1-d^{-1})\bar{e}^2 k^{d-4} L^d_{1,1}\big(2\bar{e}^2\rho_0, 2\bar{\lambda}\rho_0\big), \tag{5.9}$$

$$\eta_\sigma = -2d^{-1} v_d \rho_0^{-1} k^{d-2} M^d_{2,0}\big(2\bar{\lambda}\rho_0\big), \tag{5.10}$$

with

$$L^d_{n_1,n_2}(w_1, w_2) = k^{2(n_1+n_2)-d} \int_0^\infty dx\, x^{(d/2)-1} \times \frac{\partial}{\partial t}\Big[\big(P(x)+w_1\big)^{-n_1}\big(P(x)+w_2\big)^{-n_2}\Big], \tag{5.11}$$

$$M^d_{n_1,n_2}(w_1, w_2) = k^{2(n_1+n_2-1)-d} \int_0^\infty \mathrm{d}x\, x^{d/2}$$
$$\times \frac{\partial}{\partial t}\left[\dot{P}(x)^2 \big(P(x)+w_1\big)^{-n_1} \big(P(x)+w_2\big)^{-n_2}\right]. \quad (5.12)$$

The integrals $M^d_{n_1,n_2}$ are discussed in ref. [3] whereas the integrals $L^d_{n_1,n_2}$ are closely related to the integrals L^d_n of eq. (4.16), with $L^d_{n_1,n_2}(0) = L^d_{n_1+n_2}(0)$.

The contribution $\eta_{\ell\omega}$ can be divided into

$$\eta_{\ell\omega} = \hat{\eta}_{\ell\omega} + \eta_\alpha, \quad (5.13)$$

with $\eta_\alpha = 0$ for $\alpha = 0$. The part

$$\hat{\eta}_{\ell\omega} = 2d^{-1} v_d \rho_0^{-1} k^{d-2}\left[2M^d_{1,1}(2\bar{\lambda}\rho_0, 0) - M^d_{2,0}(0)\right] \quad (5.14)$$

can be combined with η_σ,

$$\eta_s \equiv \eta_\sigma + \hat{\eta}_{\ell\omega} = -8d^{-1} v_d \bar{\lambda}^2 \rho_0 k^{d-6} M^d_{2,2}(2\bar{\lambda}\rho_0, 0), \quad (5.15)$$

in order to reproduce the result for the pure scalar theory. For $\alpha = 0$ only the contribution from the transverse gauge bosons, η_t, has to be added. For $\alpha \neq 0$ there is an additional piece

$$\eta_\alpha = 4d^{-1} v_d \alpha \bar{e}^2 \int \mathrm{d}x\, x^{(d/2)-1}$$
$$\times \frac{\partial}{\partial t}\left\{\big(P^2 + 2\alpha\bar{e}^2\rho_0(P-x)\big)^{-1}\right.$$
$$\times \left[\frac{x(P-x)\dot{P}^2}{P^2} + 2\frac{P^2 - 2xP\dot{P} + x^2\dot{P}^2}{P(P+2\bar{\lambda}\rho_0)}\right.$$
$$\left.\left. - x\frac{P\dot{P}^2 + 2x\dot{P}^2 - P\dot{P} + 2\alpha\bar{e}^2\rho_0(\dot{P}^2 - \dot{P})}{P^2 + 2\alpha\bar{e}^2\rho_0(P-x)}\right]\right\}. \quad (5.16)$$

In the linear regime, for $k^2 \gg 2\bar{e}^2\rho_0, 2\bar{\lambda}\rho_0, 2\alpha\bar{e}^2\rho_0$ we can neglect η_s in leading approximation and find

$$\eta_t = -16(1-d^{-1}) v_d l^d_2 e^2,$$
$$\eta_\alpha = 8d^{-1} v_d \alpha\left[m^{d+2}_4 + \left(\tfrac{3}{4}d - 2\right) l^d_2\right] e^2. \quad (5.17)$$

The constants m_n^d can be found in ref. [3] with $m_n^{2n-2}=1$ $(n>2)$. Eq. (5.17) equally applies to the symmetric regime for $k^2 \gg \overline{m}_0^2$. For $d=4$ the anomalous dimension reads

$$\eta = -\frac{e^2}{8\pi^2}(3-\alpha). \tag{5.18}$$

Inserting $\eta = \eta_t + \eta_\alpha$ of (5.17) into (4.27) and (4.28) we arrive at the final result for β_κ and β_λ. One finds

$$\begin{aligned}\beta_\lambda = {} & (d-4)\lambda + 20v_d l_2^d \lambda^2 \\ & - 32(1-d^{-1})v_d l_2^d e^2\lambda + 8(d-1)v_d l_2^d e^4 \\ & + 8v_d\alpha^2\left[l_2^d - 2l_3^{d+2} + l_4^{d+4}\right]e^4 \\ & + 8v_d\alpha\left[2d^{-1}m_4^{d+2} + \left(\frac{3}{2}-\frac{4}{d}\right)l_2^d - l_3^{d+2}\right]e^2\lambda. \end{aligned} \tag{5.19}$$

In particular, β_λ becomes independent of α in four dimensions

$$\beta_\lambda = \frac{1}{8\pi^2}\left[5\lambda^2 - 6e^2\lambda + 6e^4\right] \tag{5.20}$$

and we reproduce the standard one loop result for the running of the quartic scalar coupling. The mass term for the radial scalar mode

$$m^2 = 2\lambda\kappa k^2, \tag{5.21}$$

evolves according to

$$\begin{aligned}\beta_{m^2} \equiv {} & \frac{\partial m^2}{\partial t} \\ = {} & 8v_d k^2\left\{2l_1^d\lambda + (d-1)l_1^d e^2 + \alpha\left(l_1^d - l_2^{d+2}\right)e^2\right\} \\ & + 2v_d m^2\left\{4l_2^d\lambda - 8(1-d^{-1})l_2^d e^2\right. \\ & \left. +\alpha\left[(3-8d^{-1})l_2^d + 4d^{-1}m_4^{d+2} - 2l_3^{d+2}\right]e^2\right\}. \end{aligned} \tag{5.22}$$

This allows to compute the anomalous mass dimension ω:

$$\omega = \frac{\partial}{\partial m^2}\beta_{m^2}. \tag{5.23}$$

In four dimensions, ω is again independent of α and reproduces the result of the standard perturbation theory

$$\omega = \frac{1}{8\pi^2}(2\lambda - 3e^2). \tag{5.24}$$

In contrast, the term $\sim k^2$ in (5.22) ("quadratic mass renormalization") depends on α and also on β. We finally note that the constant term in β_κ ($\lim_{\kappa\to 0}\beta_\kappa$) receives no contribution from the anomalous dimension. In two dimensions it is independent of α:

$$\beta_\kappa = 8v_2\left[1 + \frac{e^2}{2\lambda}\right] + \mathrm{O}(\kappa). \tag{5.25}$$

As a result, κ runs to zero whenever it has entered the linear regime.

For small values $k^2 \ll 2\bar{e}^2\rho_0$, $2\alpha\bar{e}^2\rho_0$, $2\bar{\lambda}\rho_0$ we can neglect η_{t}. The scalar contribution η_{s} is proportional to ρ_0^{-1}

$$\eta_{\mathrm{s}} = 4d^{-1}v_d m_{2,2}^d \kappa^{-1}, \tag{5.26}$$

where the constants $m_{2,2}^d$ can be found in ref. [3]. For nonzero α the leading contribution arises from η_α which is evaluated in appendix E:

$$\eta_\alpha = 4v_d g_1^d \alpha e^2. \tag{5.27}$$

As a result, the renormalized minimum value $Z_{\varphi,k}(\rho_0)\ \rho_0(k)$ becomes essentially independent of k for $d > 2$ (compare (4.21)), and κ evolves according to its canonical dimension (4.30):

$$\beta_\kappa = (2-d)\kappa. \tag{5.28}$$

The gauge-boson mass term (2.10) reads

$$M^2(k) = 2e^2\kappa k^2. \tag{5.29}$$

In the spontaneously broken phase in four dimensions $\lim_{k\to 0}M^2(k)$ does not vanish despite the fact that $\rho_0(k)$ runs to zero for $\alpha > 0$, cf. eq. (4.23). Different choices of α give rise to a different anomalous dimension but do not change the physical properties of the system. The symmetric and the spontaneously broken phase can be distinguished by a vanishing or nonzero value of κk^2 as $k \to 0$. The spontaneously broken phase in three dimensions is also characterized by $\lim_{k\to 0}Z_\varphi\rho_0 > 0$. No running of $\bar{e}^2 Z_{\mathrm{F}}^{-1}$ is expected for $k^2 \ll 2\bar{\lambda}\rho_0$. Again, there is a transition between the symmetric and spontaneously broken phase with massless and massive gauge boson.

Finally, we continue the evolution equations in the symmetric regime for $m_0^2 \ll k^2$. The evolution equation for λ is still given by (5.19). The running of the mass obtains from (4.32), (4.34), (5.17) by expanding $L_1^d(\overline{m}_0^2)$ in first order in $\overline{m}_0^2$:

$$\frac{\partial m_0^2}{\partial t} = -4v_d k^2\left\{2l_1^d\lambda + (d-1)l_1^d e^2 + \alpha\left(l_1^d - l_2^{d+2}\right)e^2\right\}$$
$$+ 2v_d m_0^2\left\{4l_2^d\lambda - 8(1-d^{-1})l_2^d e^2\right.$$
$$\left.+\alpha\left[(3-8d^{-1})l_2^d + 4d^{-1}m_4^{d+2}\right]e^2\right\}. \qquad (5.30)$$

With the identification $\mu^2 = -m_0^2$ for the symmetric regime and $\mu^2 = \frac{1}{2}m^2$ for the spontaneously broken regime eqs. (5.22) and (5.30) describe the same evolution of the scalar mass term on both sides of the transition between the two regimes. This transition (at $\mu^2 = 0$) is smooth and the vanishing of the gauge-boson mass does not pose a particular problem even for dimension smaller than four.

6. Conclusions

We have formulated in this paper the average action for an abelian gauge theory. We also have demonstrated how this formulation can be used for the computation of the average scalar potential. This overcomes one of the main obstacles to deal with realistic theories as the standard model with the help of an average action. Our formulation is manifestly gauge invariant and also maintains rotation and translation symmetry. No gauge fixing is needed for the computation of the average action. On the other hand, the average action depends on the "average scheme", i.e. on the choice of the averaging procedure. In particular, we have introduced a parameter α specifying the averaging of the longitudinal modes of the gauge field as compared to the transversal modes. In a certain sense this parameter mimics the role of the gauge-fixing parameter α in the conventional covariant gauges. Furthermore, our formulation is adapted for a generalization to nonabelian gauge theories.

We have derived evolution equations for the scale dependence of the average scalar potential as well as for the relevant scalar kinetic terms. Together with the evolution equation for the gauge coupling (which will be discussed in a separate paper [5]) these equations determine completely the phase structure of the system and many physical properties like particle masses, interactions or critical indices. In the limit of vanishing gauge coupling e we recover the evolution equations of the pure scalar theory with global U(1) symmetry. For $\alpha = 0$ only the transversal gauge field fluctuations add to the evolution equations. (Their contributions to the relevant β-functions can be expressed as a power series in e^2 as long as the gauge boson mass is small compared to the averaging scale k.) In the limit $\alpha = 0$ the scalar fluctuations in the Goldstone direction give exactly the same one-loop contribution as in the ungauged model. This implies a nontrivial infrared behaviour even for scales much smaller than the masses of the gauge boson and the radial scalar mode. For $\alpha > 0$ the Goldstone mode mixes with the longitudinal gauge boson. In this case the infrared effects turn out to be much stronger than for vanishing α *.

In particular, we find in four dimensions that the position of the minimum of the average scalar potential moves to zero as $k \to 0$ according $\rho_0(k) \sim k^{\alpha e^2/8\pi^2}$. The "vacuum expectation value" $\rho_0(0)$ always vanishes for $\alpha > 0$. In contrast, $\rho_0(0)$ has a nonzero limit for $\alpha = 0$ in the spontaneously broken phase **. This apparent paradox is resolved by taking into account the anomalous dimension arising from the wave function renormalization in the kinetic term. Expressed in terms of renormalized fields, $\rho_R(k) = Z_\varphi(k)\rho_0(k)$, the essential features of the average potential become independent of α. There exists always a spontaneously broken phase with nonvanishing $\rho_R(0)$. (In this case Z_φ increases $\sim k^{-\alpha e^2/8\pi^2}$ and compensates for the decreasing magnitude of ρ_0(k).) The gauge boson mass obtains then as $M^2 = 2e^2\rho_R(0)$.

The phase transition between the symmetric and spontaneously broken phase is more complicated than for the pure scalar theory since β_λ has no fixed point for $\lambda = 0$. We will address in a separate publication the question whether the transition is second or first order and discuss in more detail the form of the average potential for "Coleman–Weinberg symmetry breaking".

Appendix A

In this appendix we first list the invariants involving two derivatives in an N-component scalar theory with internal SO(N) symmetry and d-dimensional rotation and translation symmetry. They can be constructed by contractions with the tensors $\delta_{\mu\nu}$ and δ_{ab} as well as ϵ_{ab} for $N = 2$ and $\epsilon_{\mu\nu}$ for $d = 2$. Consider N even such that φ is an $N/2$ component complex field. For $N > 2$, $d > 2$ the most general invariant reads

$$Z(\varphi^\dagger\varphi)\partial_\mu\varphi^\dagger\partial^\mu\varphi + \tfrac{1}{4}Y(\varphi^\dagger\varphi)\partial_\mu(\varphi^\dagger\varphi)\partial^\mu(\varphi^\dagger\varphi). \tag{A.1}$$

* It would be interesting to study the infrared behaviour for an average scheme which resembles the unitary gauge. There should be no massless modes but one may suspect complications due to the kinetic terms.

** We note that $\rho_0(0)$ may be related to a gauge-invariant order parameter.

For SU($N/2$) symmetric models we can also write the term

$$-X(\varphi^\dagger\varphi)\triangle_\mu\triangle^\mu,$$

$$\triangle_\mu \equiv \varphi^\dagger\partial_\mu\varphi - (\partial_\mu\varphi)^\dagger\varphi. \tag{A.2}$$

For $N=2$ we may use a real basis $\varphi = (1/\sqrt{2})(\varphi_1 + i\varphi_2)$ where

$$\triangle_\mu = i\varphi^a\epsilon_{ab}\partial_\mu\varphi^b. \tag{A.3}$$

There is no independent invariant of the type (A.2) in SO(N) symmetric models, however, since for $N=2$

$$\partial_\mu(\varphi^*\varphi)\partial^\mu(\varphi^*\varphi) - \triangle_\mu\triangle^\mu = 4(\varphi^*\varphi)\partial_\mu\varphi^*\partial^\mu\varphi, \tag{A.4}$$

whereas the construction (A.3) does not exist for $N>2$. For $d=2$ and $N=2$ there exist two additional invariants

$$T(\varphi_c\varphi^c)\partial_\mu\varphi^a\partial_\nu\varphi^b\epsilon^{\mu\nu}\epsilon_{ab}, \tag{A.5}$$

$$S(\varphi_d\varphi^d)\varphi_a\partial_\mu\varphi_b\epsilon^{ab}\partial^\mu(\varphi_c\varphi^c). \tag{A.6}$$

The invariant (A.5) is parity odd, and both (A.5) and (A.6) are odd under charge conjugation. We omit in the following these two invariants which only arise in models violating C and/or P.

For a U(1) gauge theory we simply replace derivatives by covariant derivatives in (A.1): $\partial_\mu \to \partial_\mu + i\bar{e}A_\mu$. We note that the gauge invariance of the constraint ensures that the coupling of A_μ occurs always in this combination. The only renormalization of the gauge coupling arises from the kinetic term for the gauge boson ($Z_{F,k}$).

The functions $Z(\rho)$ and $Y(\rho)$ can be computed by evaluating Γ_k for appropriate configurations (φ, A_μ). For example, a real constant scalar field φ_R = const. and an infinitesimally small constant gauge field A_μ contributes

$$\Delta\Gamma_k = \bar{e}^2 Z(\varphi_R^2)\varphi_R^2 A_\mu A^\mu. \tag{A.7}$$

If we take instead of A_μ an infinitesimally small space dependent imaginary field $\delta\varphi_I(x)$ one finds

$$\Delta\Gamma_k = Z(\varphi_R^2)\partial_\mu\delta\varphi_I\partial^\mu\delta\varphi_I. \tag{A.8}$$

Evaluating Γ_k in quadratic order in A_μ in the first case, or in quadratic order in $\delta\varphi_I$ and lowest order in momentum squared Q^2 in the second, yields in both cases the function $Z(\rho)$. A space-dependent real field $\delta\varphi_R$ instead of $\delta\varphi_I$ gives the combination $Z + Y\rho$ and allows the determination of $Y(\rho)$.

Appendix B

The diagonalization of the quadratic form $\delta^2 S_k$ of eq. (3.4) and the evaluation of the gaussian integral (3.6) is greatly facilitated by decomposing the quantum fluctuations $\delta\chi$ and δa_μ according to

$$\delta\chi = 2^{-1/2}[\sigma + i\omega],$$
$$\delta a_\mu = t_\mu + \partial_\mu \ell, \tag{B.1}$$

where σ, ω and ℓ are real scalars, and t_μ is a transverse vector: $\partial^\mu t_\mu = 0$. For small fluctuations around a real constant scalar field, σ corresponds to the radial ("Higgs") mode and ω to the angular ("Goldstone") mode. Inserting (B.1) into (3.4) we obtain for the second variation in a generic background:

$$\begin{aligned}
\delta^2 S_k = \int \mathrm{d}^d x \Big\{ & t_\mu \left[P(-\Box) + 2\bar{e}^2 |\chi|^2 \right] t^\mu \\
& - \ell \left[\alpha^{-1} P(-\Box) + 2\bar{e}^2 |\chi|^2 \right] \Box \ell \\
& - 2\bar{e}^2 \ell (2t_\mu + \partial_\mu \ell) \partial^\mu |\chi|^2 \\
& + \sigma \left[\{ -D^2(a) + F(-D^2(A)) \}_{\mathrm{sym}} - \bar{\mu}^2 + \bar{\lambda} |\chi|^2 + \tfrac{1}{2}\bar{\lambda}(\chi + \chi^*)^2 \right] \sigma \\
& + \omega \left[\{ -D^2(a) + F(-D^2(A)) \}_{\mathrm{sym}} - \bar{\mu}^2 + \bar{\lambda} |\chi|^2 - \tfrac{1}{2}\bar{\lambda}(\chi - \chi^*)^2 \right] \omega \\
& + 2\omega \{ -D^2(a) + F(-D^2(A)) \}_{\mathrm{as}} \sigma \\
& - i\bar{\lambda}\omega \left[\chi^2 - \chi^{*2} \right] \sigma \\
& + 2\sqrt{2}\,\bar{e}^2 (t_\mu + \partial_\mu \ell) a^\mu [\sigma(\chi + \chi^*) - i\omega(\chi - \chi^*)] \\
& - i\sqrt{2}\,\bar{e} (t_\mu + \partial_\mu \ell) \left[\sigma \overleftrightarrow{\partial}^\mu (\chi - \chi^*) - i\omega \overleftrightarrow{\partial}^\mu (\chi + \chi^*) \right] \Big\}.
\end{aligned} \tag{B.2}$$

Here and in the following we frequently use the functions

$$F(x) \equiv x f_k^2(x) \left[1 - f_k^2(x) \right]^{-1},$$
$$P(x) \equiv x + F(x) \equiv x \left[1 - f_k^2(x) \right]^{-1}, \tag{B.3}$$

whose k-dependence will not be indicated explicitly. In eq. (B.2) we also introduced the symmetrization and antisymmetrization with respect to A_μ,

$$F(-D^2(A))_{\text{sym}} \equiv \tfrac{1}{2}\left[F(-D^2(A)) + F(-D^2(-A))\right],$$

$$F(-D^2(A))_{\text{as}} \equiv \frac{1}{2i}\left[F(-D^2(A)) - F(-D^2(-A))\right], \tag{B.4}$$

and similarly for a_μ.

Eq. (B.2) is valid for any background configuration (χ, a_μ). In the present paper we shall need it only for constant fields:

$$\chi = \chi^{\min} = \varphi = \text{real constant},$$

$$a_\mu = a_\mu^{\min} = A_\mu = \text{constant}. \tag{B.5}$$

Inserting (B.5) into eq. (B.2) yields ($\rho = \varphi^2$, $\rho_0 = \bar{\mu}^2/\bar{\lambda}$):

$$\begin{aligned} \delta^2 S_k = \int \mathrm{d}^d x \Big\{ & t_\mu\left[P(-\Box) + 2\bar{e}^2\rho\right] t^\mu \\ & -\ell\left[\alpha^{-1}P(-\Box) + 2\bar{e}^2\rho\right]\Box\,\ell \\ & +\sigma\left[P(-D^2(A))_{\text{sym}} + \bar{\lambda}(3\rho - \rho_0)\right]\sigma \\ & +\omega\left[P(-D^2(A))_{\text{sym}} + \bar{\lambda}(\rho - \rho_0)\right]\omega \\ & +2\omega P(-D^2(A))_{\text{as}}\sigma \\ & +4\sqrt{2}\,\bar{e}^2\varphi(t_\mu + \partial_\mu \ell)A^\mu\sigma \\ & -2\sqrt{2}\,\bar{e}\varphi\omega\,\Box\,\ell\Big\}. \end{aligned} \tag{B.6}$$

This quadratic form will be needed for the calculation of $Z_{\varphi,k}$. To find U_k, we further put $A_\mu = 0$ and eq. (B.6) boils down to

$$\begin{aligned} \delta^2 S_k = \int \mathrm{d}^d x \Big\{ & t_\mu\left[P(-\Box) + 2\bar{e}^2\rho\right] t^\mu \\ & -\ell\left[\alpha^{-1}P(-\Box) + 2\bar{e}^2\rho\right]\Box\,\ell \\ & +\sigma\left[P(-\Box) + \bar{\lambda}(3\rho - \rho_0)\right]\sigma \\ & +\omega\left[P(-\Box) + \bar{\lambda}(\rho - \rho_0)\right]\omega \\ & -2\sqrt{2}\,\bar{e}\varphi\omega\,\Box\,\ell\Big\}. \end{aligned} \tag{B.7}$$

Finally let us address the question whether the stationary point (B.5) is actually a stable minimum of S_k, i.e., whether $\delta^2 S_k > 0$ for arbitrary excitations $(\sigma, \omega, \ell, t_\mu)$. Consider first $A_\mu = 0$ where it is sufficient to investigate the stability properties of (B.7). The transverse vector field $t_\mu(x)$ does not couple to any other field, so that its stability can be analyzed separately. Going over to momentum space, $-\Box$ becomes $q_\mu q^\mu \equiv q^2 > 0$, and the condition for $\delta^2 S_k > 0$ reads

$$P(q^2) + 2\bar{e}^2\rho > 0. \tag{B.8}$$

From (B.3) with (2.8) it is obvious that $P(q^2)$ is strictly positive. Hence (B.8) is fulfilled for any value of φ, i.e. the transverse fluctuations are always stable. Similarly, the stability of the σ-mode requires

$$P(q^2) + \bar{\lambda}(3\rho - \rho_0) > 0. \tag{B.9}$$

The most "dangerous" mode is the one with the momentum for which $P(q^2)$ assumes its minimum

$$P_{\min} \equiv \min\{P(x) \mid x > 0\} \equiv \bar{k}^2 \sim k^2. \tag{B.10}$$

the σ-modes are stable for any q^2 provided ρ is large enough:

$$\rho > \frac{1}{3\bar{\lambda}}(\bar{\mu}^2 - \bar{k}^2). \tag{B.11}$$

In particular, no instabilities occur in the interesting region near the classical minimum, $\rho \approx \bar{\mu}^2/\bar{\lambda}$. The coupled system of ω and ℓ is stable if

$$\left[P(q^2) + \bar{\lambda}\rho - \bar{\mu}^2\right] \cdot \left[P(q^2) + 2\alpha\bar{e}^2\rho\right] - 2\alpha\bar{e}^2\rho q^2 > 0. \tag{B.12}$$

In the limiting case $\alpha\bar{e}^2 = 0$ this inequality gives rise to the condition $P(q^2) + \bar{\lambda}(\rho - \rho_0) > 0$, implying

$$\rho > \frac{1}{\bar{\lambda}}(\bar{\mu}^2 - \bar{k}^2), \tag{B.13}$$

which is more stringent than (B.11) already. Moreover, allowing for $\alpha\bar{e}^2 > 0$, the domain of stability becomes even smaller. Nevertheless, it is obvious that (B.12) is equivalent to a condition of the form

$$\rho > \rho_0 - \Delta\varphi^2 \tag{B.14}$$

for some strictly positive quantity $\Delta\varphi^2$. Hence there exists a neighborhood of the classical minimum ρ_0 for which all $\ell\omega$ modes are stable. Hereby the exact value of the allowed deviation $\Delta\varphi^2$ depends on the form of the cut-off function f_k and on the parameter α.

We conclude that the stationary point (B.6) is a stable minimum if ρ is not too much smaller than the minimum ρ_0. We finally mention that for small $A_\mu \neq 0$ the lowest eigenvalue of the quadratic form characterizing $\delta^2 S_k$ depends continuously on A_μ. A finite positive mass gap remains positive for infinitesimally small A_μ. The stability condition (B.14) therefore applies to the calculation of $Z_{\varphi,k}$ as well.

Appendix C

In this appendix we evaluate the one-loop contributions $U_k^{(1)}$ and $Z_{\varphi,k}^{(1)}$ to the average potential and to the scalar wave-function renormalization, respectively.

The quadratic action $\delta^2 S_k$ resulting from the background configuration (3.7) has been given in eq. (B.7). Inserting this expression into (3.6) and performing the gaussian integrals, we obtain the average potential $U_k^{(1)} \equiv \Omega^{-1}\Gamma_k^{(1)}$, $\Omega \equiv \int d^d x$, as the sum of the following terms:

$$U_\mathrm{t}(\rho) = \tfrac{1}{2}(d-1)\Omega^{-1} \ln \det\left[P(-\square) + 2\bar{e}^2\rho\right], \tag{C.1}$$

$$U_\sigma(\rho) = \tfrac{1}{2}\Omega^{-1} \ln \det\left[P(-\square) + 3\bar{\lambda}\rho - \bar{\mu}^2\right], \tag{C.2}$$

$$\begin{aligned} U_{\ell\omega}(\rho) = \tfrac{1}{2}\Omega^{-1} \ln \det\Big[&\left\{P(-\square) + \bar{\lambda}\rho - \bar{\mu}^2\right\} \\ &\times\left\{P(-\square) + 2\alpha\bar{e}^2\rho\right\} + 2\alpha\bar{e}^2\rho\,\square\Big]. \end{aligned} \tag{C.3}$$

(We discard irrelevant constants.) Evaluating the determinants in momentum space, and making use of

$$\tfrac{1}{2}\int \frac{d^d q}{(2\pi)^d} I(q^2) = v_d \int_0^\infty dx\, x^{(d/2)-1} I(x), \tag{C.4}$$

with v_d defined in (4.6) and $x = q^2$, we finally arrive at the expressions (4.2)–(4.4) which are further discussed in sect. 4.

In sect. 5 the determination of $Z_{\varphi,k}^{(1)}$ requires a similar calculation, but this time with $A_\mu \neq 0$. The corresponding quadratic action $\delta^2 S_k$, which has to be inserted into the functional integral (3.6), is given in eq. (B.6) of appendix B. Due to the non-diagonal terms, an exact diagonalization of this quadratic form would be rather cumbersome. The calculation simplifies considerably if we keep only terms which are second order in A_μ. In this case the modes decouple partially, and one finds for the gaussian integrals

$$\Delta\Gamma_k^{(1)}[\varphi, A_\mu] = \int d^d x\{K_\mathrm{t} + K_\sigma + K_{\ell\omega}\}, \tag{C.5}$$

where

$$K_{\mathrm{t}}=\tfrac{1}{2}\Omega^{-1}\ \ln\det{}_{\mathrm{trans}}\left[(P(-\Box)+2\bar{e}^2\rho)\delta^{\mu}_{\nu}-\frac{8\bar{e}^4\rho A^{\mu}A_{\nu}}{P(-\Box)+3\bar{\lambda}\rho-\bar{\mu}^2}\right], \tag{C.6}$$

$$K_{\sigma}=\tfrac{1}{2}\Omega^{-1}\ \ln\det\left[P(-D^2(A))_{\mathrm{sym}}+3\bar{\lambda}\rho-\bar{\mu}^2\right], \tag{C.7}$$

$$\begin{aligned}K_{\ell\omega}=\tfrac{1}{2}\Omega^{-1}\ \ln\det&\left[\left\{P(-D^2(A))_{\mathrm{sym}}+\bar{\lambda}\rho-\bar{\mu}^2+\frac{4\bar{e}^2\dot{P}^2(-\Box)A^{\mu}A^{\nu}\partial_{\mu}\partial_{\nu}}{P(-\Box)+3\bar{\lambda}\rho-\bar{\mu}^2}\right\}\right.\\ &\times\left\{(-\Box)\left[\alpha^{-1}P(-\Box)+2\bar{e}^2\rho\right]+\frac{8\bar{e}^4\rho A^{\mu}A^{\nu}\partial_{\mu}\partial_{\nu}}{P(-\Box)+3\bar{\lambda}\rho-\bar{\mu}^2}\right\}\\ &\left.-2\bar{e}^2\rho\left(\Box-\frac{4\bar{e}^2\dot{P}(-\Box)A^{\mu}A^{\nu}\partial_{\mu}\partial_{\nu}}{P(-\Box)+3\bar{\lambda}\rho-\bar{\mu}^2}\right)^2\right],\end{aligned} \tag{C.8}$$

and $\dot{P}(x)=\partial P/\partial x$. In the expression for K_{t} [the contribution of the transverse vector field $t_{\mu}(x)$] the subscript "trans" refers to the projection

$$\ln\det{}_{\mathrm{trans}}[\dots]=\mathrm{Tr}\{P_{\mathrm{T}}\ \ln[\dots]\}, \tag{C.9}$$

which is a consequence of the constraint $\partial_{\mu}t^{\mu}=0$. Here

$$P^{\mu}_{\mathrm{T}\nu}\equiv\delta^{\mu}_{\nu}-\frac{\partial^{\mu}\partial_{\nu}}{\Box} \tag{C.10}$$

is the usual transverse projector. Now it is a matter of straightforward algebra to rewrite the determinants (C.6)–(C.8) as traces in momentum space, to expand up to second order in A_{μ} and to perform the symmetric integration in q_{μ}. Finally, defining $Z_i\equiv K_i/(\bar{e}^2A^2\rho)$ for $i=\mathrm{t},\ \sigma,\ \ell\omega$, one arrives at the integrals (5.4)–(5.6) given in sect. 5.

Appendix D

In this appendix we evaluate the integrals

$$G^{d}_{n,p,r}(w)=k^{4n-2r-d}\int_0^{\infty}\mathrm{d}x\ x^{(d/2)-1}\frac{\partial}{\partial t}\left\{\left(\frac{x}{P}\right)^{p}\frac{(P-x)^{r}}{\left[P^2+w(P-x)\right]^{n}}\right\} \tag{D.1}$$

for $n > 0$, $r \geqslant 0$. They obey the relations

$$G^d_{n,p-1,r}(w) = G^d_{n,p,r}(w) + G^{d-2}_{n,p,r+1}(w), \tag{D.2}$$

$$G^d_{n,p-2,r}(w) = G^{d-4}_{n-1,p,r}(w) - (w/k^2)G^{d-4}_{n,p,r+1}(w), \tag{D.3}$$

$$\frac{\partial}{\partial w}G^d_{n,p,r}(w) = -nk^{-2}G^d_{n+1,p,r+1}(w). \tag{D.4}$$

For $w \ll k^2$ we expand

$$G^d_{n,p,r}(w) = G^d_{n,p,r}(0) - nwk^{-2}G^d_{n+1,p,r+1}(0) + \dots, \tag{D.5}$$

and use the integrals (4.16)

$$\begin{aligned} G^d_{n,p,0}(0) &= L^{d+2p}_{2n+p}(0), \\ G^d_{n,p,1}(0) &= L^{d+2p}_{2n+p-1}(0) - L^{d+2p+2}_{2n+p}(0), \\ G^d_{n,p,2}(0) &= L^{d+2p}_{2n+p-2}(0) - 2L^{d+2p+2}_{2n+p-1}(0) + L^{d+2p+4}_{2n+p}(0). \end{aligned} \tag{D.6}$$

For $k^2 \ll w$ we split the integration into three regions: (i) $x < x_0 \ll k^2$, (ii) $x \approx k^2$ and (iii) $x > x_1 \gg k^2$. Consider first the third region where $G \equiv (P-x)/x \approx \exp[-2a(x/k^2)^\beta] \ll 1$. The integral

$$G^{d\,\text{(iii)}}_{n,p,r}(w) = -2k^{4n-2r-d}\int_{x_1}^{\infty}\mathrm{d}x\frac{\partial G}{\partial x}x^{(d/2)+r-n} \times \frac{\partial}{\partial G}\{G^r(x+wG)^{-n}\} \tag{D.7}$$

is dominated for $r=0$ or $r=n$ by a small region $x \approx wG$ where we can approximate $x(G)$ by a constant $\bar{x}$ defined by

$$\begin{aligned} \bar{x} &= wG(\bar{x}) = w\exp\left[-2a(\bar{x}/k^2)^\beta\right], \\ \bar{x} &\approx k^2\left(\frac{\ln(w/k^2)}{2a}\right)^{1/\beta}. \end{aligned} \tag{D.8}$$

One obtains in leading order

$$G^{d\,\text{(iii)}}_{n,p,0}(w) \approx -2\left(\frac{\bar{x}}{k^2}\right)^{(d/2)-2n} \tag{D.9}$$

$$G^{d\,\text{(iii)}}_{n,p,n}(w) \approx 2\left(\frac{k^2}{w}\right)^n\left(\frac{\bar{x}}{k^2}\right)^{d/2} \tag{D.10}$$

The contribution from the region $x \approx k^2$, $G \approx 1$ is obviously at most of order $(k^2/w)^n$. It can be neglected for $r = 0$ or for $r = n$, $p = 0$. Finally, we use for the small-x region ($G \gg 1$) the approximations

$$x \ll P, \qquad x\frac{\partial}{\partial x}\left(\frac{P}{x}\right) \approx \frac{\beta}{\beta - 1}\frac{\partial P}{\partial x}.$$

The integral

$$\begin{aligned} G_{n,p,r}^{d\,(\mathrm{i})}(w) &= -\frac{2\beta}{\beta - 1}k^{4n-2r-d}\int_0^{x_0}\mathrm{d}x\frac{\partial P}{\partial x}x^{(d/2)+p} \\ &\quad\times\frac{\partial}{\partial P}\left\{P^{r-p-n}(P+w)^{-n}\right\} \\ &= \frac{2\beta}{\beta - 1}(2a)^{-(d+2p)/2(\beta-1)}k^{4n-2r+2p+(d+2p)/(\beta-1)} \\ &\quad\times\int_{P_0}^{\infty}\mathrm{d}P\,P^{-(d+2p)/2(\beta-1)}\frac{\partial}{\partial P}\left\{P^{r-p-n}(P+w)^{-n}\right\} \end{aligned} \tag{D.11}$$

is dominated for $p + n - r + (d + 2p)/2(\beta - 1) \gg 0$ by the integration boundary at $P_0 = P(x_0)$ and can be neglected. In summary, we find for $k^2 \ll w$

$$\begin{aligned} G_{n,p,0}^{d}(w) &= -2g_n^d(w), \\ G_{n,0,n}^{d}(w) &= 2\left(k^2/w\right)^n g_0^d(w), \end{aligned} \tag{D.12}$$

with

$$g_n^d \approx \left(\frac{\ln\left(w/k^2\right)}{2a}\right)^{(d-4n)/2\beta}. \tag{D.13}$$

Appendix E

Here we evaluate η_α of eq. (5.16) for $k^2 \ll 2\alpha\bar{e}^2\rho_0, 2\bar{\lambda}\rho_0$. We consider first the integration region (iii) of large $x > x_1 \gg k^2$ where we can approximate $P = 1$. Only the last term in (5.16) contributes in leading approximation

$$\begin{aligned} \eta_\alpha^{(\mathrm{iii})} &= -8d^{-1}v_d\alpha\bar{e}^2k^{d-4}G_{2,0,0}^{d+4\,(\mathrm{iii})}\left(2\alpha\bar{e}^2\rho_0\right) \\ &\quad - 8d^{-1}v_d\alpha^2\bar{e}^4\rho_0 H^{(\mathrm{iii})}\left(2\alpha\bar{e}^2\rho_0\right), \end{aligned} \tag{E.1}$$

with

$$H(w) \equiv \int \mathrm{d}x\, x^{d/2} \frac{\partial}{\partial t} \frac{\dot{P}^2 - \dot{P}}{\left[P^2 + w(P-x)\right]^2},$$

$$H^{(\mathrm{iii})}(w) \approx \frac{\partial}{\partial t} \int_{x_1}^{\infty} \mathrm{d}x\, x^{(d/2)-1} \dot{G}(x + wG)^{-2}. \tag{E.2}$$

This integral is dominated by a small region where $wG(x) \approx x$ and we approximate $x(G) = \bar{x}$ as in (D.8):

$$\begin{aligned} H^{(\mathrm{iii})}(w) &\approx -w^{-1} \frac{\partial}{\partial t} \bar{x}^{(d/2)-2} \\ &\approx -(d-4) w^{-1} \bar{x}^{(d/2)-2} \\ &\approx -(d-4) w^{-1} k^{d-4} g_1^d. \end{aligned} \tag{E.3}$$

It is easy to verify that (E.1) with (E.3) gives the only contributions to η_α which are not suppressed by $(k^2/2\alpha\bar{e}^2\rho_0)$. Hence we find

$$\eta_\alpha = 4 v_d \alpha g_1^d e^2. \tag{E.4}$$

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